

Star subdivisions and connected even factors in the square of a graph

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Abstract.

For any positive integer s , a $[2, 2s]$ -factor in a graph G is a connected even factor with maximum degree at most $2s$. We prove that if every induced $S(K_{1,2s+1})$ in a graph G has at least 3 edges in a block of degree at most two, then G^2 has a $[2, 2s]$ -factor. This extends the results of Hendry and Vogler and of Abderrezzaq et al.

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1 Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and we consider only finite undirected simple graphs, unless otherwise stated.

Let $G = (V, E)$ be a graph with vertex set V and edge set E . Let $\alpha(G)$ denote the independence number of G , i.e., the cardinality of a largest independence set in G . For any vertex x of G , let $d_G(x)$ denote the degree of x in G , $N_G(x)$ the set of all neighbors of x in G , $N_G[x] = N_G(x) \cup \{x\}$. The square of a graph G , denoted by G^2 , is the graph with $V(G^2) = V(G)$ in which two vertices are adjacent if their distance in G is at most two. Thus $G \subseteq G^2$.

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For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S . For a positive integer s , the graph $S(K_{1,2s+1})$ is obtained from the complete bipartite graph $K_{1,2s+1}$ by subdividing each edge once. The graph G is said to be $S(K_{1,2s+1})$ -free if it does not contain any induced copy of $S(K_{1,2s+1})$.

A connected graph that has no cut vertices is called a *block*. A *block of a graph G* is a subgraph of G that is a block and is maximal with respect to this property. The *degree of a block B* in a graph G , denoted by $d(B)$, is the number of cut vertices of G belonging to $V(B)$.

A *factor* in a graph G is a spanning subgraph of G . A connected even factor in G is a connected factor in G with all vertices of even degree. A $[2, 2s]$ -factor in G is a connected even factor in G in which degree of every vertex is at most $2s$. A graph is *hamiltonian* if it has a spanning cycle. In other word, a graph is hamiltonian if and only if it has a $[2, 2]$ -factor.

The following result concerns the existence of a $[2, 2]$ -factor in the square of a 2-connected graph.

Theorem A [3]. *Let G be a 2-connected graph. Then G^2 is hamiltonian.*

Gould and Jacobson in [4] conjectured that for the hamiltonicity of G^2 , the connectivity condition can be relaxed for $S(K_{1,3})$ -free graphs. Their conjecture was proved by Hendry and Vogler in [5].

Theorem B [5]. *Let G be a connected $S(K_{1,3})$ -free graph. Then G^2 is hamiltonian, i.e., has a $[2, 2]$ -factor.*

Moreover, Abderrezzak, Flandrin and Ryjáček in [1] proved the following result in which graphs may contain an induced $S(K_{1,3})$ of a special type.

Theorem C [1]. *Let G be a connected graph such that every induced $S(K_{1,3})$ in G has at least three edges in a block of degree at most two. Then G^2 is hamiltonian, i.e., has a $[2, 2]$ -factor.*

It is a natural question if there exists a $[2, 2s]$ -factor in the square of a graph if one replaces $S(K_{1,3})$ by $S(K_{1,2s+1})$ in Theorems B and C. In this paper, we will give a positive answer to this question; we will extend Theorems B and C as follows.

Theorem 1. *Let G be a connected $S(K_{1,2s+1})$ -free graph of order at least three and s a positive integer. Then G^2 has a $[2, 2s]$ -factor.*

Since the square of an $S(K_{1,2s+1})$ itself has no $[2, 2s]$ -factor, Theorem 1 is the best possible in a sense.

Theorem 2. Let s be a positive integer and G be a connected graph such that every induced $S(K_{1,2s+1})$ has at least three edges in a block of degree at most two. Then G^2 has a $[2, 2s]$ -factor.

Note that Theorem 2 is a strengthening of Theorem 1, but we state Theorem 1 separately because it will be used in the proof of Theorem 2.

2 Preliminaries and auxiliary results

As noted in Section 1, for graph-theoretic notation not explained in this paper, we refer the reader to [2].

A graph G is *even* if every vertex of G has even degree. In the subsequent sections, we frequently take the symmetric difference of two subgraphs of a graph. Let H, H' be subgraphs of a graph G . The graph $H \Delta H'$ has vertex set $V(H) \cup V(H')$ and its edge set is the symmetric difference of $E(H)$ and $E(H')$. Note that if H and H' are both even graphs, then $H \Delta H'$ is also an even graph.

A *trail* between vertices u_0 and u_r is a finite sequence $T = u_0e_1u_1e_2u_2 \cdots e_ru_r$, whose terms are alternately vertices and edges, with $e_i = u_{i-1}u_i$, $1 \leq i \leq r$, where the edges are distinct. A trail T is *closed* if $u_0 = u_r$, and it is *spanning* if $V(T) = V(G)$. An s -trail between u_0 and u_r is a trail starting at u_0 , ending at u_r and in which every vertex is visited at most s times. In other words, a $[2, 2s]$ -factor in a graph G can be viewed as a spanning closed s -trail in G and vice versa. We define the degree of a vertex x in an s -trail as the number of edges incident with x in the corresponding $[2, 2s]$ -factor.

We use the following fact (see [6], Corollary 2.3.1 for a proof).

Theorem D [6]. Let $k \geq 2$ be an integer and G a k -connected graph. If $\alpha(G) > k$ then $V(G)$ can be covered with $\alpha(G) - k$ disjoint paths.

From the proof of this Theorem it follows that the statement is true without the restrictions on k , in particular for $k = 0$.

Corollary 3. Let G be a graph. Then there are at most $\alpha(G)$ disjoint paths covering $V(G)$.

Let G_1, G_2 be graphs such that $V(G_1) \cap V(G_2) = \{x\}$. The symbol $G = G_1xG_2$ denotes a graph G with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

Given a subgraph K of a graph H , we define $\partial_H(K)$ as the set of all edges of H with exactly one endvertex in $V(K)$. Thus $\partial_H(K)$ is a (not necessarily minimal) edge-cut.

Lemma 4. Let H be a connected graph and $P = xyz$ a path of length two such that $V(H) \cap V(P) = \{x\}$. If $(HxP)^2$ has a $[2, 2s]$ -factor, then one of the following holds:

- (a) H^2 contains a spanning closed s -trail T such that the degree of x in T is at most $2s - 2$, or
- (b) H^2 contains a spanning s -trail T between x and some $x' \in N_H(x)$.

Proof. Let F be a $[2, 2s]$ -factor of $(HxP)^2$ and let K_0, \dots, K_ℓ be all the components of $F \setminus \{y, z\}$, where $x \in V(K_0)$. Furthermore, define $W = N_F(y) \setminus \{z\}$ and $W_i = W \cap V(K_i)$ ($i = 0, \dots, \ell$). Observe that each W_i is nonempty. Clearly, the induced subgraph Q of H^2 on $W \cup \{x\}$ is complete.

Since F covers z , it includes the edges yz and xz . For $0 \leq i \leq \ell$, every edge in $\partial_F(K_i)$ is incident with y , except for the edge $xz \in \partial_F(K_0)$. Since

$$\partial_F(K_i) = \partial_H(K_i) \cap E(F)$$

and the intersection of any edge-cut with an eulerian subgraph has even cardinality, we conclude that for $0 \leq i \leq \ell$,

$$|W_i| \text{ is odd if and only if } i = 0.$$

If $w \in W_i$ and $w \neq x$, then the degree of w in K_i is odd and does not exceed $2s - 1$. The same is true for $w = x$ provided that $x \notin W$, since then xz is the only edge of $\partial_F(K_0)$ incident with x . On the other hand, if $x \in W$, then both xz and xy have this property, so the degree of x in K_0 is even and does not exceed $2s - 2$.

For each i , $0 \leq i \leq \ell$, choose a matching M_i that covers all except one or two vertices of W_i (one if $i = 0$, two otherwise) and uses as few edges as possible from F . We argue that the symmetric difference $K_i \Delta M_i$ is connected. We may assume that M_i uses at least one edge of F , otherwise there is nothing to prove. For a fixed i , let $X \subseteq W_i$ be the set consisting of vertices incident with edges in $E(M_i) \cap E(F)$, together with the vertices of W_i left uncovered by M_i . By the choice of M_i , $K_i[X]$ must be complete and $|X| \geq 3$. All the edges of K_i that are removed as a result of taking the symmetric difference are edges of $K_i[X]$. Since any graph obtained by removing a matching from a complete graph on at least 3 vertices is connected, the claim follows.

Observe that for $i \geq 1$, each $K_i \Delta M_i$ contains exactly two vertices of odd degree (and the degree does not exceed $2s - 1$). The same is true for $i = 0$ unless $x \in W$ and x

is not incident with M_0 , in which case $K_0 \Delta M_0$ is eulerian and the degree of x in this graph is at most $2s - 2$. It follows that if $\ell = 0$, then we can set $T := K_0 \Delta M_0$ and we are done (T satisfies condition (a) if $x \in W \setminus V(M_0)$ and condition (b) otherwise).

If $\ell \geq 1$, then let u_0 be the vertex of $W_0 \setminus V(M_0)$, and for $i \geq 1$, let $W_i \setminus V(M_i) = \{u_i, v_i\}$. Taking the union of all the graphs $K_i \Delta M_i$ and adding the edges $u_0v_1, u_1v_2, \dots, u_{\ell-1}v_\ell$, we obtain a connected graph T in which the only vertices of odd degree are x and u_ℓ , and which satisfies condition (b) in the lemma. ■

Using a similar argument as in the proof of Lemma 4, one can prove the following.

Lemma 5. Let H be a connected graph and $P = xy$ an edge such that $V(H) \cap V(P) = \{x\}$. If $(HxP)^2$ has a $[2, 2s]$ -factor, then H^2 has a spanning s -trail T between $x' \in N_H[x]$ and some vertex $x'' \in N_H(x)$.

The following theorem will be used in the proof of Theorem 2.

Theorem E [3]. *Let y and z be arbitrarily chosen vertices of a 2-connected graph G . Then G^2 has a hamiltonian cycle C such that the edges of C incident with y are in G and at least one of the edges of C incident with z is in G . If y and z are adjacent in G , then these are three different edges.*

3 Proofs

The purpose of this section is to prove Theorem 2. As mentioned in Section 1, the proof makes use of Theorem 1 which we derive next.

Proof of Theorem 1. This proof is inspired by the proof in [5]. We prove our result by induction on $|V(G)|$. Clearly G^2 is hamiltonian (hence has a $[2, 2]$ -factor) for graphs with $|V(G)| \leq 6$, since G is $S(K_{1,3})$ -free. By Theorem A, we may assume that G has cut vertices. If all cut vertices have degree two, then G is a path and hence G^2 is hamiltonian. So we may assume that there is a cut vertex u such that $d_G(u) = d \geq 3$. Since G is connected, we may take a spanning tree S of G such that S contains all edges of G incident with u . We label the neighbors of u by u_1, u_2, \dots, u_d in such a way that $d_G(u_i) \geq 2$ for $1 \leq i \leq m$ and $d_G(u_i) = 1$ for $m + 1 \leq i \leq d$. For $i \leq m$, let G_i be the subgraph of G induced by the vertices in the component of the forest $S - u$ containing u_i ; we fix a neighbour u'_i of u that is not contained in the same component of $G - u$ as u_i (note that there must be such a vertex since u is a cut vertex of G), and let $H_i = G[V(G_i) \cup \{u, u'_i\}]$. Then H_i is a proper $S(K_{1,2s+1})$ -free subgraph of G since H_i is

an induced subgraph of G and $d_G(u) \geq 3$. Note that H_i is connected. By the inductive hypothesis, H_i^2 has a $[2, 2s]$ -factor. Note that $d_{H_i^2}(u'_i) = 2$.

By Lemma 4 it follows that at least one of the following facts holds.

- (a) there exists a spanning closed s -trail T_i in G_i^2 such that $d_{T_i}(u_i) \leq 2s - 2$;
- (b) there exists a spanning s -trail T_i in G_i^2 between u_i and some $z_i \in N_{G_i}(u_i)$.

Without loss of generality we may assume that $\{u_1, u_2, \dots, u_{m'}\} \subseteq \{u_1, u_2, \dots, u_m\}$ is the set of all vertices u_i such that G_i has an s -trail of type (b), for a suitable $m' \leq m$. Construct the graph H from $G[\{u_1, u_2, \dots, u_{m'}, z_1, z_2, \dots, z_{m'}\}]$ by contracting edges $u_i z_i$ to a vertex w_i for $i = 1, \dots, m'$. Since G is $S(K_{1, 2s+1})$ -free, $\alpha(H) \leq 2s$. By Corollary 3, there are $\ell \leq \alpha(H)$ vertex-disjoint paths P_1, P_2, \dots, P_ℓ covering $V(H)$. Without loss of generality, we may assume that $P_i = w_{s_{i-1}+1} w_{s_{i-1}+2} \dots w_{s_i}$, for $i = 1, \dots, \ell$ (where $s_0 = 0$ and $s_\ell = m'$). Since we contracted edges $u_j z_j$ to vertices w_j , both u_j and z_j have a neighbor in $\{u_{j+1}, z_{j+1}\}$ in G^2 for $i = 1, \dots, \ell$, and $j = s_{i-1} + 1, \dots, s_i - 1$. Hence from the paths P_i ($i = 1, \dots, \ell$) and s -trails T_j ($j = 1, \dots, m'$) we can obtain the following s -trails F_i in G^2 :

- for a trivial (one-vertex) path P_i , $F_i = T_i$,
- for a nontrivial path P_i , we construct F_i by joining the trails $T_{s_{i-1}+1}, \dots, T_{s_i}$ with the edges $x_j x_{j+1}$, where $x_j \in \{u_j, z_j\}$ and $x_{j+1} \in \{u_{j+1}, z_{j+1}\}$ with respect to P_i . Clearly $d_{F_i}(u_{s_{i-1}+1}) < 2s$, $d_{F_i}(x_{s_i}) < 2s$ and F_i spans all the vertices of $G_{s_{i-1}+1} \cup \dots \cup G_{s_i}$.

Note that the number of s -trails F_i is $\ell \leq 2s$.

Let $T = u_{m'+1} T_{m'+1} u_{m'+1} u_{m'+2} T_{m'+2} u_{m'+2} \dots u_m T_m u_m u_{m+1} \dots u_d$ be an s -trail containing all vertices of $G_{m'+1} \cup \dots \cup G_m$ and all neighbours of u of degree one in G . We set $F' = u_1 F_1 x_{s_1} u x_{s_2} F_2 u_{s_1+1} u_{s_2+1} F_3 \dots x_{s_\ell} F_\ell u_{s_{\ell-1}+1} u_{m'+1}$ for even ℓ and $F' = u_1 F_1 x_{s_1} u x_{s_2} F_2 u_{s_1+1} u_{s_2+1} F_3 \dots u_{s_{\ell-1}+1} F_\ell x_{s_\ell} u u_{m'+1}$ for odd ℓ . In both cases, F' is an s -trail containing all vertices of $G_1 \cup \dots \cup G_{m'}$. Finally, we construct a trail $F = u_1 F' u_{m'+1} T u_d u_1$. Clearly, $d_F(u) = \ell \leq 2s$ and F corresponds to a $[2, 2s]$ -factor in G^2 . ■

Corollary 6. Let G be a simple connected graph with $\Delta(G) \leq 2s$. Then G^2 has a $[2, 2s]$ -factor.

Before we present the proof of Theorem 2, we give some additional definitions. Let x be a cut vertex of G , and H' be a component of $G - x$. Then the subgraph

$H = G[V(H') \cup \{x\}]$ is called a *branch* of G at x . Let F be a connected subgraph of G and x some vertex of F . Let $P_i(x)$ denote a path on i vertices with end vertex x . The subgraph F is called to be *nontrivial* at x if it contains a $P_3(x)$ as a proper induced subgraph (i.e., F is trivial at x if $F = P_3(x)$ or $V(F) \subseteq N[x]$).

Now we present the proof of Theorem 2.

Proof of Theorem 2. We prove this theorem by contradiction. Suppose that Theorem 2 is not true and choose a graph G in such a way that

- (1) G is connected and every induced $S(K_{1,2s+1})$ in G has at least three edges in a block of degree at most two;
- (2) G^2 has no $[2, 2s]$ -factor;
- (3) $|V(G)|$ is minimized with respect to (1) and (2).

The following fact is necessary for our proof.

Claim 1. *Let x be a cut vertex of G and F_1, F_2 two connected subgraphs of G such that F_1, F_2 belong to different branches of G at x . Assume that F_2 is nontrivial at x , i.e., F_2 contains an induced $P_3(x) = xyz$ as a proper induced subgraph. Then the graph $G' = F_1xP_3(x)$ also satisfies (1).*

Proof of Claim 1. If not, there exists in G' some $S(K_{1,2s+1})$ that has no connected part of order at least 4 in a block of degree at most two. But if so, it is the same in G , since any $S(K_{1,2s+1})$ in G' is also an induced $S(K_{1,2s+1})$ of G . \square

Since in our proof we have assumed that G^2 has no $[2, 2s]$ -factor, we know from Theorem 1 that G contains some $S(K_{1,2s+1})$ as an induced subgraph. By (1), the $S(K_{1,2s+1})$ has at least 3 edges in some block H of G of degree at most 2. Notice that $|V(H)| \geq 5$.

Case 1: $d(H) = 1$. Let c be the cut vertex of G belonging to H and let R be the union of all branches of G at c which intersect H only at c .

If H is trivial at c , then $V(H) - \{c\} = \{b_1, b_2, \dots, b_h\} \subseteq N(c)$. The graph $G' = Rc(cb_1)$ satisfies condition (1). So by minimality of G , the graph G^2 has a $[2, 2s]$ -factor and, by Lemma 5, R^2 has a spanning s -trail T between some $c' \in N_R[c]$ and some $c'' \in N_R(c)$. Let $F = c'Tc''b_1\dots b_hc'$. It is easy to see that F is a $[2, 2s]$ -factor in G^2 , a contradiction.

Hence H is nontrivial at c , i.e., it contains a proper induced path $P_3(c) = cb_1b_2$. By Theorem E, H^2 contains a hamiltonian path $b_1P_{H^2}c$ connecting b_1 and c . On the other

hand the graph $G'' = RcP_3(c)$ is connected and, by Claim 1, G'' satisfies condition (1). Since $|V(G'')| < |V(G)|$, $(G'')^2$ has a $[2, 2s]$ -factor and by Lemma 4, one of the following subcases occur.

If the graph R^2 has a spanning closed s -trail T' in which $d_{T'}(c) \leq 2s - 2$, then $F = cT'cb_1P_{H^2}c$ is a $[2, 2s]$ -factor in G^2 , a contradiction.

If the graph R^2 has a spanning s -trail T'' between c and some neighbor $c''' \in N_R(c)$, then $F = cT''c'''b_1P_{H^2}c$ is a $[2, 2s]$ -factor in G^2 , contradicting condition (2).

Case 2: $d(H) = 2$. Let c_1 and c_2 be two cut vertices of G belonging to H and let B_i , $i = 1, 2$, be the union of all branches of G at c_i not containing H . This means that $G = (B_1c_1H)c_2B_2$. The subgraph H is a block and thus, by Theorem E, $V(H)$ can be covered by two vertex-disjoint paths $a_1P_H^1a_2$ and $c_2P_H^2c_1$ in H^2 , where $a_1 \in N(c_1)$ and $a_2 \in N(c_2)$. We distinguish, up to symmetry, the following three subcases.

Subcase 2.1: B_1 is trivial at c_1 and B_2 is trivial at c_2 .

If $V(B_1) = \{b_1, b_2, \dots, b_k, c_1\} \subseteq N[c_1]$, $k \geq 1$, and $B_2 = P_3(c_2) = c_2d_1d_2$, then $F = c_1b_1b_2\dots b_k a_1P_H^1a_2d_1d_2c_2P_H^2c_1$ is even a hamiltonian cycle in G^2 , which contradicts the fact that G^2 has no $[2, 2s]$ -factor.

The proof is similar if $B_1 = P_3(c_1)$ and $V(B_2) \subseteq N[c_2]$.

If $V(B_1) = \{b_1, b_2, \dots, b_k, c_1\} \subseteq N[c_1]$ and $V(B_2) = \{d_1, d_2, \dots, d_l, c_2\} \subseteq N[c_2]$, then $F = c_1b_1b_2\dots b_k a_1P_H^1a_2d_1d_2\dots d_l c_2P_H^2c_1$ is also a hamiltonian cycle in G^2 , contradicting (2).

Finally, if $B_1 = P_3(c_1) = c_1b_1b_2$ and $B_2 = P_3(c_2) = c_2d_1d_2$, then again the cycle $F = c_1b_2b_1a_1P_H^1a_2d_1d_2c_2P_H^2c_1$ gives a similar contradiction.

Subcase 2.2: B_1 is nontrivial at c_1 and B_2 is trivial at c_2 .

Since $|V(H) \cup V(B_2)| > 3$, there exists some vertex in $V(H) \cup V(B_2)$ (for example each vertex in $V(B_2) \setminus \{c_2\}$) nonadjacent to c_1 , the subgraph $G' = Hc_2B_2$ is nontrivial. Then G' contains a path $P_3(c_1) = c_1n_1n_2$ as a proper induced subgraph. Now let $G_1 = B_1c_1n_1n_2$. By Claim 1, G_1 satisfies condition (1). By minimality of G , the graph G_1^2 has a $[2, 2s]$ -factor and thus, by Lemma 4, we have the following two possibilities.

a) The graph B_1^2 has a spanning closed s -trail T in which $d_T(c_1) \leq 2s - 2$.

If $V(B_2) = \{b_1, b_2, \dots, b_k, c_2\} \subseteq N[c_2]$, $k \geq 1$, then $F = c_1Tc_1a_1P_H^1a_2b_1b_2\dots b_k c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 , a contradiction with (2).

If $B_2 = P_3(c_2) = c_2d_1d_2$, then $F = c_1Tc_1a_1P_H^1a_2d_1d_2c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 , which contradicts condition (2).

b) The graph B_1^2 has a spanning s -trail T' between c_1 and some neighbor $c'_1 \in N_{B_1}(c_1)$.

If $V(B_2) = \{b_1, b_2, \dots, b_k, c_2\} \subseteq N[c_2]$, $k \geq 1$, then $F = c_1T'c'_1a_1P_H^1a_2b_1b_2\dots b_k c_2P_H^2c_1$

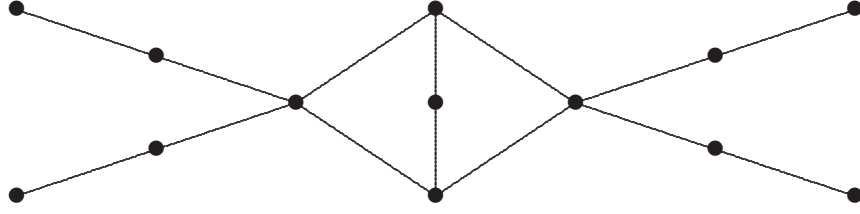


Figure 1: An example showing that a condition in Theorem 2 cannot be relaxed.

is a $[2, 2s]$ -factor in G^2 and contradicts (2).

If $B_2 = P_3(c_2) = c_2d_1d_2$, then $F = c_1T'c'_1a_1P_H^1a_2d_1d_2c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 , a contradiction with (2).

Subcase 2.3: B_1 is nontrivial at c_1 and B_2 is nontrivial at c_2 .

Let G_1 be the same graph as in Subcase 2.2 and in a similar way as in Subcase 2.2 let $G_2 = B_2c_2m_1m_2$, where a path $c_2m_1m_2$ is a proper induced subgraph of Hc_1B_1 . Then, by Claim 1, both G_1 and G_2 satisfy condition (1). By minimality of G , the graphs G_1^2 and G_2^2 have a $[2, 2s]$ -factor and thus, by Lemma 4, we have the following two possibilities.

a) The graph B_1^2 has a spanning closed s -trail T in which $d_T(c_1) \leq 2s - 2$.

If the graph B_2^2 has a spanning closed s -trail T' in which $d_{T'}(c_2) \leq 2s - 2$, then $F = c_1Tc_1a_1P_H^1a_2c_2T'c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 and contradicts (2).

If the graph B_2^2 has a spanning s -trail T'' between c_2 and some neighbor $c'_2 \in N_{B_2}(c_2)$, then $F = c_1Tc_1a_1P_H^1a_2c'_2T''c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 , contradicting condition (2).

b) The graph B_1^2 has a spanning s -trail T^* between c_1 and some neighbor $c'_1 \in N_{B_1}(c_1)$.

If the graph B_2^2 has a spanning closed s -trail T^{**} in which $d_{T^{**}}(c_2) \leq 2s - 2$, then $F = c_1T^*c'_1a_1P_H^1a_2c_2T^{**}c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 , a contradiction.

If the graph B_2^2 has a spanning s -trail T^\bullet between c_2 and some neighbor $c'_2 \in N_{B_2}(c_2)$, then $F = c_1T^*c'_1a_1P_H^1a_2c'_2T^\bullet c_2P_H^2c_1$ is a $[2, 2s]$ -factor in G^2 and contradicts (2). ■

The graph G in Figure 1 shows that (for $s = 1$) the constant 3 in Theorem 2 cannot be decreased. Although every induced $S(K_{1,2s+1})$ in G has at least two edges in a block of degree at most two, G^2 has no $[2, 2s]$ -factor.

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