Notes on exact meets and joins

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Abstract

An exact meet in a lattice is a special type of infimum characterized by, inter alia, distributing over finite joins. In frames, the requirement that a meet is preserved by all frame homomorphisms makes for a slightly stronger property. In this paper these concepts are studied systematically, starting with general lattices and proceeding through general frames to spatial ones, and finally to an important phenomenon in Scott topologies.

Introduction

The notion of an exact meet probably first appeared in [7] (1970), under the name *admissible meet*. It was used as a technical device for the study of injective hulls of semilattices. In [2] (1984), the first author arrived at the concept by extrapolating the following characterization of meets in Boolean algebras. A lower bound b of a subset A in a Boolean algebra B is the infimum of A iff, for all differences $d-c$ in B,

$$\forall\, a \in A\ (a \geq d-c \ \Rightarrow\ b \geq d-c.$$

Modeling $x \geq d-c$ in a general lattice by $x \vee c \geq d$, and restricting this to the pairs $c < d$, one obtains the definition of a special type of meet called an *exact meet*, coinciding with the notion of an admissible meet, as it turned out. Such meets then proved useful for various purposes: injective hulls (again) and essential extensions, and in the study of completions of lattice ordered groups.

A very similar notion, with a very different motivation, appeared under the name *free meet* in [18] (1994). Free meets are the meets in frames which are preserved by all frame homomorphisms. The property of being free is stronger than exactness (and we use the term *strong exactness* here) but it is closely related.

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To illustrate the situation, consider the lattice $\Omega(X)$ of all open subsets of a T_D-topological space X. Here, the exact meets are those intersections which happen to be open, in other words the meets that coincide with intersections. Because such meets are intersections, they distribute over all finite joins with elements of $\Omega(X)$ and in fact, this is the characteristic feature of exact meets in general. In a general space X, strongly exact meets of open sets are always open intersections, but if X is not T_D then there are intersections that are not open such that their meet is exact nevertheless.

Open intersections of systems of open sets also appeared in another context, in an important step in proving the Hofmann-Lawson duality. Namely, in the Scott topology of a continuous frame, the intersection of certain open sets was shown to be open iff the open sets were indexed by a compact index set. (For a precise formulation see 5.2 below.)

In this paper we present a systematic study of these phenomena. We start with exactness in a general, not necessarily distributive, lattice. Then we proceed to frames, where we are dealing with the exactness of the meets only, since all joins are exact in frames. (The latter feature, by the way, distinguishes frames among complete lattices.) We obtain characterizations in terms of the behavior of closed and open sublocales; in fact, the characterizations for general lattices in the preceding section can be viewed as describing the behavior of 'generalized closed sublocales'. The discrepancy between the characterizations in terms of the open sublocales as opposed to the closed ones then leads to the reappearance of the aforementioned free meets of Wilson. Furthermore, the situation is analyzed in the case of spatial frames, where the T_D-spatiality makes the two notions coincide, and is, in fact, characterized by this fact. Finally, we discuss the open intersections in Scott topologies. We conclude with a brief discussion of the preservation of exact meets by homomorphisms.

1 Preliminaries and problem setting

1.1. Although some of the statements may be formulated for more general posets, the most general setting we will consider will be lattices L without special completeness or distributivity properties. For a subset A and element x of a lattice L, we shall write, as usual,

$$\uparrow A = \{y \mid y \geq a \in A\}, \quad \text{and} \quad \uparrow x = \{y \mid y \geq x\}.$$

We express the fact that x is an upper bound of A by writing

$$A \leq x \quad \text{if} \quad \forall a \in A,\ a \leq x.$$

Similarly we write $x \leq A$ if x is a lower bound of A, and we make use of the abbreviations

$$A{\vee}b = \{a \vee b \mid a \in A\}\,, \quad A{\wedge}b = \{a \wedge b \mid a \in A\}\,, \quad \text{and} \quad A{\to}x = \{a{\to}x \mid a \in A\}\,.$$

From [2] we adopt the operations

$$a \downarrow b = \{x \mid x \wedge b \leq a\} \quad \text{and} \quad a \uparrow b = \{x \mid x \vee a \geq b\}\,.$$

We write

$$\bigvee A, \quad \text{resp.} \quad \bigwedge A,$$

for the supremum (join), resp. infimum (meet) of A if it exists, so that use of the symbol entails the assertion that the supremum or infimum exists.

1.2. Recall that a *frame* is a complete lattice L satisfying the distributivity law

$$\Big(\bigvee A\Big) \wedge b = \bigvee (A \wedge b) = \bigvee_{a \in A} (a \wedge b)$$

for all $A \subseteq L$ and $b \in L$. We speak of a *co-frame* if we have the distributivity law $(\bigwedge A) \vee b = \bigwedge (A \vee b)$ instead. *Frame homomorphisms* are maps preserving all joins and all finite meets; the resulting category will be denoted by **Frm**. A typical frame is the lattice $\Omega(X)$ of all open sets of a topological space; if $f\colon X \to Y$ is a continuous map then $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$ is obviously a frame homomorphism.

Every frame is a Heyting algebra; the Heyting operation will be denoted by $a{\to}b$. In particular, a frame has pseudocomplements

$$a^* = a{\to}0 = \bigvee \{x \mid x \wedge a = 0\}\,.$$

Similarly, a co-frame has *pseudosupplements*, i.e., co-pseudocomplements $a^\# = \bigwedge \{x \mid x \vee a = 1\}$. For more about frames see, e.g., [13, 17], or the more recent [15].

1.3. Sublocales. Frames can be viewed as generalized spaces. Subspaces of a frame L are then represented as *sublocales*, that is, as subsets $S \subseteq L$ such that

- for all $M \subseteq S$, the meet $\bigwedge M$ lies in S, and
- for every $x \in L$ and $s \in S$, $x{\to}s$ lies in S.

The sublocale S is a frame in the order inherited from L, and there is a natural frame surjection $L \to S$ (the representation of a "subspace" is contravariant), namely the left Galois adjoint of the embedding $j\colon S \to L$, which is a localic map in the sense of 1.3.2 below. The family of all sublocales of L constitutes a co-frame

$$\mathcal{S}\ell(L)$$

with intersection for meet, and join defined by

$$\bigvee_J S_i = \left\{\bigwedge M \mid M \subseteq \bigcup\nolimits_J S_i\right\}.$$

Equivalently, a sublocale S can be represented by the frame congruence induced by the frame surjection $j^*\colon L \to S$ adjoint to the embedding $j\colon S \to L$.

1.3.1. Open and closed sublocales. The open subspace of L associated with the element $a \in L$ is represented by the *open* sublocale

$$\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid a \to x = x\},$$

which can be represented by the congruence $\Delta_a = \{(x, y) \mid x \wedge a = y \wedge b\}$.

The complement in $\mathcal{S}\ell(L)$ of $\mathfrak{o}(a)$ is the *closed* sublocale

$$\mathfrak{c}(a) = \uparrow a,$$

with the associated congruence $\nabla_a = \{(x, y) \mid x \vee a = y \vee b\}$. Note that the *closure* of a sublocale S, the smallest closed sublocale containing S, is given by a particularly simple formula

$$\overline{S} = \mathsf{cl}\,(S) = \mathfrak{c}(\bigwedge S).$$

We recall from [15] the following equations in the co-frame $\mathcal{S}\ell(L)$:

$$\bigvee_J \mathfrak{o}(a_i) = \mathfrak{o}\left(\bigvee\nolimits_J a_i\right), \quad \mathfrak{o}(a) \wedge \mathfrak{o}(b) = \mathfrak{o}(a \wedge b),$$
$$\bigwedge_J \mathfrak{c}(a_i) = \mathfrak{c}\left(\bigvee\nolimits_J a_i\right), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b).$$

1.3.2. Localic maps. Following the development of the second and third authors in their treatise [15], we will use the term *localic map* to refer to the right Galois adjoint $f\colon L \to M$ of a frame homomorphism $h\colon M \to L$; that is, we have $h(x) \leq y$

iff $x \le f(y)$.[1] That is, they are the meet-preserving maps $f\colon L \to M$ whose left adjoints $f^*\colon M \to L$ preserve binary meets. Alternatively, they may be described as the meet-preserving maps $L \to M$ which satisfy

$$f(f^*(a) \to b) = a \to f(b).$$

See [15] for proofs and additional details. From this same text we shall also require several technical results in the sequel.

1.3.3. Lemma. *Let $f\colon L \to M$ be a localic map with left adjoint $f^*\colon M \to L$.*

(1) *The image function $f[-] = (S \mapsto f[S])$ maps $\mathcal{S}\ell(L)$ into $\mathcal{S}\ell(M)$ and preserves joins.*

(2) *For each sublocale T of M there is a unique largest sublocale contained in $f^{-1}[T]$, designated $f_{-1}[T]$.*

(3) *The function $f_{-1}\colon \mathcal{S}\ell(M) \to \mathcal{S}\ell(L)$ is right adjoint to the image function in* (1)*, and therefore preserves all meets in $\mathcal{S}\ell(M)$.*

(4) *$f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$ for all $a \in M$.*

Proof. See II.2.3, III.4.1, and III.6.3 in [15]. □

1.3.4. The Booleanization sublocale. By the formula for closure, a sublocale $S \subseteq L$ is *dense*, that is, $\overline{S} = L$, iff $0 \in S$. Thus, every frame L contains the minimal dense sublocale

$$\mathfrak{B}L = \{a^{**} \mid a \in L\}$$

([12]). $\mathfrak{B}L$ is a Boolean frame, called the *Booleanization* of L.

1.4. Exact meets and joins. Recall [2]. An element b is the *exact meet* of a subset A of a lattice L if

- b is a lower bound of A, and
- for any $c < d$, if $A \subseteq c \uparrow d$ then $b \in c \uparrow d$.

(The latter in detail: if $a \vee c \ge d$ for all $a \in A$ then $b \vee c \ge d$.)

Dually, b is the *exact join* of a subset A if

- b is an upper bound of A, and

[1] The reader is cautioned of the widespread use of the term 'localic map' or 'localic morphism' for the formal morphism in the category **Loc** of locales, the formal dual of the category **Frm**. Here we have a concrete representation of these 'inverted arrows' as maps.

- for any $c < d$, if $A \subseteq c \downarrow d$ then $b \in c \downarrow d$.

Exact meets and joins have various motivations. For instance, the exact joins and meets in a distributive lattice L are precisely those which remain valid in the injective Boolean hull ρL of L ([2, Proposition 1.10]). Here we will be particularly interested in the distributivity aspects of exact joins. In [7], subsets A with exact joins are called *admissible*.

1.4.1. Proposition. *An exact meet is a meet.*

Proof. Suppose b is an exact meet of A and consider an arbitrary $x \leq A$. Then $A \subseteq b \uparrow (b \vee x)$ implies $b \in b \uparrow (b \vee x)$, which is to say that $b \geq x$. $\square$

1.4.2. Theorem. [2, Lemma 1.8] *The following statements about an element b and a subset A of a lattice L are equivalent.*

(1) *b is an exact meet of A.*

(2) *$\bigwedge (A \vee x) = b \vee x$ for each x.*

Proof. (1) $\Rightarrow$ (2): $b \vee x$ is certainly a lower bound of $A \vee x$, and if it is not the greatest such then there is some $y \leq A \vee x$ such that $y \not\leq b \vee x$. Then we would have $b \vee x < y \vee b \vee x \leq A \vee x$, i.e., $A \subseteq (b \vee x) \uparrow (y \vee b \vee x)$, which would imply the contradiction $b \vee b \vee x = b \vee x \geq y \vee b \vee x$.
(2) $\Rightarrow$ (1): Suppose $\bigwedge(A \vee x) = b \vee x$ for all x. For each $a \in A$ we get $x \leq a$ by setting $x = a$. If for $c < d$ we have $a \vee c \geq d$ for all $a \in A$ then $b \vee c = \bigwedge(A \vee c) \geq d$. This shows that b is the exact meet of A. $\square$

It is a surprising fact that the property that every existing meet in a lattice is exact is expressible in the first order language of lattice theory.

1.4.3. Proposition. [2, Proposition 1.10] *The following statements about a lattice L are equivalent.*

(1) *Every meet which exists in L is exact.*

(2) *If $\bigwedge A$ exists for some subset $A \subseteq L$ then $\bigwedge(A \vee b)$ exists for every $b \in L$, and $\bigwedge(A \vee b) = (\bigwedge A) \vee b$.*

(3) *For all $a < b < c$ in L there exists some $d \in L$ such that $a < d \leq c$ and $b \uparrow c \subseteq a \uparrow d$.*

Consequently, such a lattice is distributive, and a co-frame if complete.

Proof. The equivalence of (1) and (2) is an immediate application of 1.4.2. So suppose these conditions hold in L, and consider $a < b < c$ for which no element d can be found to satisfy (3). We claim that $\bigwedge A = a$, where $A = a \vee (b \uparrow c)$. For the claim could fail only if $a < d \le A$ for some $d \in L$, in which case $b \uparrow c \subseteq a \uparrow d$, contrary to assumption. But a is not the exact meet of A, for $A \subseteq b \uparrow c$ while $a \notin b \uparrow c$. This cannot happen by (2), so we are forced to conclude that an element d can be found satisfying (3).

Suppose (1) fails, say $\bigwedge A = a$ but $A \vee b \ge c > a \vee b$. By replacing b with $a \vee b$ if necessary, we may assume that $a < b < c$. Since $A \subseteq b \uparrow c$ and $\bigwedge A = a$, no element d satisfying (3) can be found. $\square$

1.4.4. Remark. The concept of an exact join is, in a way, dual to, or, rather, orthogonal to, that of being a *linear* element of a lattice L. In [12], a is called linear in L if, for each $B \subseteq L$,

$$a \wedge \bigvee B = \bigvee (a \wedge B)\,.$$

Dually, a is termed *co-linear* if $a \vee \bigwedge B = \bigwedge (a \vee B)$ for all $B \subseteq L$. One of the notable facts is that, in a subfit frame, a is co-linear if and only if it is complemented (see, e.g. [12, 15]).

2 Exact meets in general lattices

In the sequel we will concentrate on exact meets. The results are easily dualized, and, more specifically, we will be interested in the phenomena in frames and particularly in topological spaces, where the joins are automatically exact and hence the exactness of meets is what is of interest.

2.1. In spaces and, as we shall see, in frames, the exactness of a meet is connected with the openness of an intersection of open objects and the closedness of the union of closed objects. For instance, if U_i are open in X and it so happens that $\bigcap U_i$ is open, then, for any open V, $(\bigcap U_i) \cup V = \bigcap (U_i \cup V)$ and hence $\bigwedge U_i = \bigcap U_i$ is an exact meet in $\Omega(X)$.

The general phenomena go in this direction. Therefore we will imitate the sublocale terminology and speak of the subsets of a general lattice of the form

$$\mathfrak{c}(a) = {\uparrow} a$$

as being the *closed* ones.

More generally, we define a *geometric subset* of a lattice L (the lattices will be assumed bounded, but this may not be necessary) as a subset $S \subseteq L$ such that

$$\text{if } M \subseteq S \text{ and } \bigwedge M \text{ exists then } \bigwedge M \in S.$$

The set of all geometric subsets of a lattice L will be denoted by

$$\mathcal{G}(L).$$

2.2. Proposition. *For any lattice L, $\mathcal{G}(L)$ is a complete lattice in the inclusion order, with the join given by*

$$\bigvee_J S_i = \left\{ \bigwedge M \mid M \subseteq \bigcup\nolimits_J S_i,\ \bigwedge M \text{ exists} \right\}.$$

Consequently, if L is a frame then the sublocale frame $\mathcal{S}\ell(L)$ is a subset of $\mathcal{G}(L)$ closed under all joins.

Proof. Let N be a subset of $\bigvee S_i$ and let $\bigwedge N$ exist. For each $n \in N$ we have an $M_n \subseteq \bigcup S_i$ such that $\bigwedge M_n$ exists and is equal to n. Now it is a standard fact that then $\bigwedge(\bigcup\{M_n \mid n \in N\})$ exists and is equal to $\bigwedge N$. If, in addition, L is a frame and the S_i's are sublocales then for $x \in \bigvee S_i$, say $x = \bigwedge N$ for $N \subseteq \bigcup S_i$, and for $y \in L$ we have

$$y \to x = y \to \bigwedge N = \bigwedge_N (y \to n).$$

Since, for each $n \in N$, $y \to n \in \bigcup_J S_i$ because $n \in \bigcup_J S_i$, it follows that $y \to x \in \bigvee_J S_i$ and consequently that $\bigvee_J S_i \in \mathcal{S}\ell(L)$. $\square$

2.3. Proposition. *Let $\bigvee_A \mathfrak{c}(a)$ be closed in $\mathcal{G}(L)$. Then $\bigwedge A$ exists, and $\bigvee_A \mathfrak{c}(a) = \mathfrak{c}\left(\bigwedge A\right)$.*

Proof. Suppose $\bigvee_A \mathfrak{c}(a) = {\uparrow} b$. Then, first, all the a's are in ${\uparrow} b$ and hence $b \leq A$. If $x \leq A$ then x is a lower bound of each subset $M \subseteq {\uparrow} A$. In particular, $b \in \bigvee_A \mathfrak{c}(a)$ entails $b = \bigwedge M$ for some $M \subseteq {\uparrow} A$, hence $x \leq b$. $\square$

2.4. Theorem. *A meet $\bigwedge A$ is exact in L if and only if the join $\bigvee_A \mathfrak{c}(a)$ is closed in $\mathcal{G}(L)$.*

Proof. Let u be the exact meet of A in L. Surely ${\uparrow} u \supseteq {\uparrow} a$ for all $a \in A$, so that ${\uparrow} u \geq \bigvee_A \mathfrak{c}(a)$. To demonstrate the opposite inequality, consider an arbitrary $x \geq u$. Then we have by the exactness

$$x = u \vee x = \left(\bigwedge A\right) \vee x = \bigwedge (A \vee x).$$

Since $A \vee x \subseteq {\uparrow} A = \bigcup_A {\uparrow} a$, this yields $x = \bigwedge(A \vee x) \in \bigvee_A \mathfrak{c}(a)$.

Let $\bigvee_A {\uparrow} a$ be closed, i.e., equal to ${\uparrow} u$ for some $u \in L$. Then $u = \bigwedge A$ by 2.3. To show this meet exact, consider an arbitrary $x \in L$. Then

$$\left(\bigwedge A\right) \vee x = u \vee x \in {\uparrow} u$$

and hence $u \vee x = \bigwedge B$ for some $B \subseteq \uparrow A$. That means that for each $b \in B$ there is some $a_b \in A$ such that $b \geq a_b$, and since $u \vee x \leq b$ we have $b \geq x$ and finally $b \geq a_b \vee x$. Thus,

$$u \vee x = \bigwedge B \geq \bigwedge_B (a_b \vee x) \geq \bigwedge (A \vee x) \geq u \vee x. \square$$

3 Exact and strongly exact meets in frames

From now on, L will be a frame.

3.1. Recall that for the pseudosupplement $x^\#$ in a co-frame we have

$$y \geq \left(\bigwedge_J x_i\right)^\# \iff y \vee \left(\bigwedge_J x_i\right) = \bigwedge_J (y \vee x_i) = 1 \iff \forall i \ (y \vee x_i = 1)$$
$$\iff \forall i \ \left(y \geq x_i^\#\right)$$

and hence

$$\left(\bigwedge_J x_i\right)^\# = \bigvee_J x_i^\#. \tag{3.1.1}$$

3.2. Lemma. *In the co-frame $\mathcal{S}\ell(L)$ we have $\bigvee_J \mathfrak{c}(a_i) = \mathfrak{c}(a)$ if and only if $(\bigcap_J \mathfrak{o}(a_i))^{\#\#} = \mathfrak{o}(a)$.*

Proof. Set $S = \bigcap_J \mathfrak{o}(a_i)$ and let $S^{\#\#} = \mathfrak{o}(a)$. Then by (3.1.1)

$$\bigvee \mathfrak{c}(a_i) = \bigvee \mathfrak{o}(a_i)^\# = \left(\bigwedge \mathfrak{o}(a_i)\right)^\# = S^\# = S^{\#\#\#} = \mathfrak{o}(a)^\# = \mathfrak{c}(a).$$

On the other hand, if $\bigvee \mathfrak{c}(a_i) = \mathfrak{c}(a)$ then, again by (3.1.1),

$$\left(\bigwedge \mathfrak{o}(a_i)\right)^{\#\#} = \left(\left(\bigwedge \mathfrak{o}(a_i)\right)^\#\right)^\# = \left(\bigvee \mathfrak{o}(a_i)^\#\right)^\# = \left(\bigvee \mathfrak{c}(a_i)\right)^\# = \mathfrak{c}(a)^\# = \mathfrak{o}(a). \square$$

¿From 1.4.2, 2.2, 2.4, and 3.2 we immediately obtain

3.3. Theorem. *The following facts about a meet $a = \bigwedge_J a_i$ in a frame L are equivalent.*

(1) *The meet $a = \bigwedge_i a_i$ is exact.*

(2) *For every $b \in L$, $\bigwedge_i (a_i \vee b) = a \vee b$.*

(3) $\bigvee_i \mathfrak{c}(a_i) = \mathfrak{c}(a)$ *in* $\mathcal{S}\ell(L)$.

(4) $\bigvee_i \mathfrak{c}(a_i)$ *is a closed sublocale of* L.

(5) *If* $x \geq a$ *then there exist* $x_i \geq a_i$ *such that* $x = \bigwedge_i x_i$.

(6) $(\bigwedge_i \mathfrak{o}(a_i))^{\#\#} = (\bigcap_i \mathfrak{o}(a_i))^{\#\#} = \mathfrak{o}(a)$ *in* $\mathcal{S}\ell(L)$.

(7) $(\bigwedge_i \mathfrak{o}(a_i))^{\#\#} = (\bigcap_i \mathfrak{o}(a_i))^{\#\#}$ *is an open sublocale of* L. □

((5) is just $\bigvee_j {\uparrow} a_i = {\uparrow} a$ explicitly rewritten.)

3.4. Characterizing P-frames. Of some importance in general topology are the P-spaces, i.e., the Tychonoff spaces on which a continuous real-valued function must be constant in some neighborhood of each point. Zero sets are obviously open in such spaces, meaning cozero sets are clopen. Indeed, this is taken as the frame definition: a completely regular frame L is said to be a *P-frame* if each cozero elements is complemented, i.e., if its cozero part $\operatorname{coz} L$ is a Boolean σ-frame. (See [3] for several characterizations of P-frames, together with information on their role in the general theory.)

Perhaps the handiest of the several well-known characterizations of P-spaces is that a countable intersection of open sets remains open. But this attribute has resisted a pointfree formulation, and for good reason. To say of an open set that it is the set-theoretic intersection of some countable family in the frame of open sets is much stronger than to say that it is their meet. In fact, this strict sort of meet would appear at first glance to be an inherently pointed notion.

Notice, however, the meets which are actually set-theoretic intersections are just those that commute with the joins. On this basis, one might therefore hope to capture the P-frame property by requiring countable meets to be exact. Such is not the case, unfortunately, but a slightly weaker condition does work. We shall say that a meet $a = \bigwedge A$ is *cozero exact* if $b \vee \bigwedge A = b \vee a$ for all $b \in \operatorname{coz} L$.

3.4.1. Theorem. *A completely regular frame is a P-frame iff each countable meet is cozero exact.*

Proof. Suppose each countable meet in L is cozero exact. Consider a cozero element $a \in L$, and write $a = \bigvee_n a_n$ for cozero elements $a_n \prec a$. Let $b_n \in \operatorname{coz} L$ witness $a_n \prec a$, i.e., $a_n \wedge b_n = 0$ and $a \vee b_n = 1$. We claim that $b = \bigwedge_n b_n$ is the complement of a. For

$$a \wedge b = \left(\bigvee_n a_n\right) \wedge b = \bigvee_n (a_n \wedge b) \leq \bigvee_n (a_n \wedge b_n) = 0, \quad \text{and}$$
$$a \vee b = a \vee \bigwedge_n b_n = \bigwedge_n (a \vee b_n) = 1.$$

Now suppose that L is a P-frame, i.e., $a \vee a^* = 1$ for all $a \in \operatorname{coz} L$. Consider an arbitrary subset $\{a_n\}$ and a cozero element b of L. Since clearly $b \vee \bigwedge_n a_n \le \bigwedge_n (b \vee a_n)$, we need only establish the opposite inequality. For that purpose, consider arbitrary $c \in \operatorname{coz} L$ such that $c \le \bigwedge_n (b \vee a_n)$. Now for each n, the fact that $c \le b \vee a_n$ is equivalent to $b \wedge c^* \le a_n$. Consequently, $b \wedge c^* \le \bigwedge_n a_n$, with the result that $b \vee \bigwedge_n a_n \ge c$. $\square$

3.5. Strongly exact (free) meets. Points (3) and (4) in 3.3, compared with (6) and (7), give rise naturally to the question of what happens if we require open meets (intersections) of the open sublocales instead of closed joins of the corresponding closed sublocales, i.e.,

$$\bigwedge_J \mathfrak{o}(a_i) = \bigcap_J \mathfrak{o}(a_i) = \mathfrak{o}(a). \qquad \text{(s-exact)}$$

This property is stronger (see 3.5.4, 3.6.2 and 4.3.1 below), although it does coincide with exactness in a broad class of topological spaces, as we shall see in 4.2.4. For our purposes, we will refer to the s-exact property as *strong exactness*, and immediately obtain

3.5.1. Theorem. *The following facts about a meet $a = \bigwedge_J a_i$ in a frame L are equivalent.*

(1) *The meet $a = \bigwedge_i a_i$ is strongly exact.*

(2) $\bigwedge_i \mathfrak{o}(a_i) = \bigcap_i \mathfrak{o}(a_i)$ *is an open sublocale of L.*

(3) *If $a_i \to x = x$ for all $i \in J$ then $(\bigwedge_J a_i) \to x = x$.*

(3) is just the s-exact condition written explicitly. Furthermore, if $\bigcap_i \mathfrak{o}(a_i) = \mathfrak{o}(a)$ then, necessarily, $a = \bigwedge a_i$. This follows immediately from the fact that $\mathfrak{o}(x) \subseteq \mathfrak{o}(y)$ iff $x \le y$.

3.5.2. Viewed from another perspective, strongly exact meets appeared under the name

free meets

in the unpublished thesis of Todd Wilson [18]. There they were defined as the meets which are *preserved by all frame homomorphisms.* Wilson characterized freeness by means of several interesting conditions, one of which was s-exactness. Here is a variant of Wilson's characterization.

3.5.3. Theorem. *The following statements about a meet $\bigwedge_J a_i$ in a frame L are equivalent.*

(1) $\bigwedge_i a_i$ *is strongly exact.*

(2) *For every frame homomorphism* $h\colon L\to M$, $h(\bigwedge_i a_i)=\bigwedge_i h(a_i)$, *and* $\bigwedge_i h(a_i)$ *is strongly exact.*

(3) *For every frame homomorphism* $h\colon L\to M$, $h(\bigwedge_i a_i)=\bigwedge_i h(a_i)$.

(4) *For every* $x\in L$, $\bigwedge((A\to x)\to x)=((\bigwedge A)\to x)\to x$.

Proof. (1)⇒(2): Let f be the localic map adjoint to a frame homomorphism $h\colon L\to M$. Consider $f_{-1}\colon \mathcal{S\ell}(L)\to\mathcal{S\ell}(M)$. By 1.3.3 we have

$$\bigcap \mathfrak{o}(h(a_i))=\bigcap f_{-1}(\mathfrak{o}(a_i))=f_{-1}(\mathfrak{o}(a))=\mathfrak{o}(h(a)).$$

(2)⇒(3) is trivial.

(3)⇒(4): Consider the mapping $h_x\colon L\to\mathfrak{Bc}(x)$ defined by

$$h_x(a)=(a\to x)\to x.$$

Since $a\to x$ is the pseudocomplement of $c(x)(a)=x\vee a$ in $\mathfrak{c}(x)=\uparrow x$, it is clear that h_x is the frame map $b_{\mathfrak{c}(x)}\circ c(x)$. Thus h_x preserves the meet of A by assumption, and, since the meets in the frame $\mathfrak{Bc}(x)$ coincide with the meets in L, the conclusion follows.

(4)⇒(1): By 3.5.1(3) it suffices to check that $a\to x=x$ for every $a\in A$ implies $(\bigwedge A)\to x=x$, i.e., $(\bigwedge A)\to x\le x$, since the other inequality is always true. So let $a\to x=x$ for every $a\in A$. Then, by hypothesis, $((\bigwedge A)\to x)\to x=x\to x=1$ and hence $(\bigwedge A)\to x\le x$. □

3.5.4. Example. The poset $L=\{1>2>\cdots>n>\cdots>0\}$ is obviously both a frame and a co-frame so that *all meets in L are exact.* On the other hand we have the frame homomorphism $h\colon L\to\{0,1\}$ with $h(0)=0$ and $h(n)=1$ otherwise. Now $h(\bigwedge_{n\neq 0}n)=h(0)=0\neq 1=\bigwedge_{n\neq\infty}h(n)$ and hence $\bigwedge_{n\neq\infty}n$ *is not strongly exact.*

It is worth remarking that $\mathfrak{o}(n)=\{k\mid k>n\}\cup\{0,1\}$ for $n\neq 0$, while $\mathfrak{o}(0)=\{1\}$. Therefore $\bigcap_{n\neq 0}\mathfrak{o}(n)=\{0,1\}\neq\mathfrak{o}(0)$, consistent with 3.5.1.

3.6. Conservative subsets. In [9], the authors use *conservative* subsets of frames to study paracompactness. Translated into our language, a subset $A\subseteq L$ is conservative if $\bigwedge B$ is exact for every $B\subseteq A$. Chen [8] also uses conservative sets to present some new characterizations of paracompact frames. In particular, he proves characterization (4) in our 3.3 using congruences ([8, Lemma 2.3]).

Exact meets are also related to the concepts of *interior-preserving* and *closure-preserving* families of sublocales of Plewe ([16]). Recall that a family $\mathcal{S}=\{S_i\mid i\in I\}\subseteq\mathcal{S\ell}(L)$ is *closure-preserving* if for all $J\subseteq I$, $\mathsf{cl}(\bigvee_J S_i)=\bigvee_J\mathsf{cl}(S_i)$. Dually, $\mathcal{S}$ is *interior-preserving* if for all $J\subseteq I$, $\mathsf{int}(\bigwedge_J S_i)=\bigwedge_J\mathsf{int}(S_i)$. Then, a subset A

of L is said to be *interior-preserving* (resp. *closure-preserving*) if $\{\mathfrak{o}(a) \mid a \in A\}$ is interior-preserving (resp. $\{\mathfrak{c}(a) \mid a \in A\}$ is closure-preserving). Interior-preserving covers play a decisive role in the construction of canonical examples of transitive quasi-uniformities for frames ([10]). Of course, any interior-preserving cover of L is closure-preserving but, somewhat surprisingly and contrary to what happens in spaces, the converse does not hold in general.

3.6.1. Lemma. *Let $A \subseteq L$. Then:*

(1) *A is interior-preserving iff $\bigwedge_B \mathfrak{o}(b) = \mathfrak{o}(\bigwedge B)$ for every $B \subseteq A$.*

(2) *A is closure-preserving iff $\bigvee_B \mathfrak{c}(b) = \mathfrak{c}(\bigwedge B)$ for every $B \subseteq A$.*

Proof. We only prove (a), the proof for (b) is similar. A is interior-preserving iff $\{\mathfrak{o}(a) \mid a \in A\}$ is interior-preserving iff $\mathsf{int}\,(\bigwedge_B \mathfrak{o}(b)) = \bigwedge_B \mathsf{int}\,(\mathfrak{o}(b))$ for every $B \subseteq A$ iff $\mathsf{int}\,(\bigwedge_B \mathfrak{o}(b)) = \bigwedge_B(\mathfrak{o}(b))$ for every $B \subseteq A$ iff $\bigwedge_B \mathfrak{o}(b)$ is open for every $B \subseteq A$.

¿From Lemma 3.6.1 and Theorem 3.3 we immediately obtain

3.6.2. Corollary. *A subset A of a frame L is conservative if and only if it is closure-preserving.* □

4 Exact meets in spaces and spatial frames

4.1. T_D and $T_{D\text{-}0}$. Recall that a space X is T_D if

$$\forall x \in X\ \exists U \ni x \text{ open such that } U \smallsetminus \{x\} \text{ is open.} \tag{T_D}$$

(This concept goes back to 1963, see [1] and [6].) More generally, a space X is $T_{D\text{-}0}$ if its T_0-modification X^0 is T_D. The T_0-modification X^0 of a space X is obtained by factoring X by the equivalence

$$x \sim y \quad \equiv_{\mathrm{df}} \quad \overline{\{x\}} = \overline{\{y\}}.$$

We will need the notion of a $\sim$-set in X, namely a subset $A \subseteq X$ such that

$$x \in A \text{ and } x \sim y \quad \Rightarrow \quad y \in A.$$

Obviously each open set is a $\sim$-set.

4.2. Proposition. *A space is T_D iff the following equivalence holds.*

$$(\forall A \text{ open},\ \mathsf{int}\,(U \cup A) = \mathsf{int}\,U \cup A\ ) \quad \textit{iff} \quad U \textit{ is open.} \tag{4.2.1}$$

Proof. Let X be T_D and let $\mathsf{int}\,(U \cup A) = \mathsf{int}\,U \cup A$ for all open A. Let $x \in U$. Choose an open A such that $x \notin A$ and $A \cup \{x\}$ is open. Then $\mathsf{int}\,(U \cup A) = \mathsf{int}\,U \cup A$ and hence $x \in \mathsf{int}\,U$.

Let the implication hold and let $x \in X$. If $U \cup \{x\}$ is open for every open U there is nothing to prove. Else choose an open U such that $U \cup \{x\}$ is not open. By the implication there is an open A such that $\mathsf{int}\,(U \cup \{x\} \cup A) \neq \mathsf{int}\,(U \cup \{x\}) \cup A = U \cup A$. Obviously x is the only element in which the two sets can differ, and hence $x \in \mathsf{int}\,(U \cup \{x\} \cup A)$ and there is an open V such that $x \in V \subseteq U \cup \{x\} \cup A$. Then $V \smallsetminus \{x\} = V \cap (U \cup A)$ is open. $\square$

4.2.1. Corollary. *A space is $T_{D\text{-}0}$ iff the following equivalence holds for every $\sim$-set U.*

$$(\forall A \text{ open},\ \mathsf{int}\,(U \cup A) = \mathsf{int}\,U \cup A\ ) \quad \textit{iff} \quad U \textit{ is open.}$$

4.2.2. Lemma. *In any space X,*

$$\mathsf{int}\,U = \bigwedge \left\{ X \smallsetminus \overline{\{x\}} \mid x \notin U \right\}.$$

Proof. An open $V \subseteq U$ is a subset of each $X \smallsetminus \overline{\{x\}}$ with $x \notin U$; hence $V \subseteq \bigwedge \left\{ X \smallsetminus \overline{\{x\}} \mid x \notin U \right\}$. On the other hand, $\bigwedge_{x \notin U} \left(X \smallsetminus \overline{\{x\}} \right)$ is open and we have

$$\bigwedge_{x \notin U} \left(X \smallsetminus \overline{\{x\}} \right) \subseteq \bigcap_{x \notin U} \left(X \smallsetminus \overline{\{x\}} \right) \subseteq \bigcap_{x \notin U} (X \smallsetminus \{x\}) = U.$$

Thus, $\bigwedge \left\{ X \smallsetminus \overline{\{x\}} \mid x \notin U \right\}$ is the largest open set contained in U. $\square$

4.2.3. Theorem. *The following statements are equivalent for a topological space X.*

(1) *X is $T_{D\text{-}0}$.*

(2) *A meet $\bigwedge U_i$ is exact in $\Omega(X)$ iff $\bigcap U_i$ is open.*

Proof. (1)⇒(2): Obviously if $\bigcap U_i$ is open then $\bigwedge U_i$ is exact. Now let $\bigwedge U_i$ be exact. Set $U = \bigcap U_i$ and take any open A. We have

$$\begin{aligned} \mathsf{int}\,U \cup A = \left(\bigwedge U_i \right) \cup A = \bigwedge (U_i \cup A) = \mathsf{int} \bigcap (U_i \cup A) = \\ = \mathsf{int}\left(\left(\bigcap U_i \right) \cup A \right) = \mathsf{int}\,(U \cup A) \end{aligned}$$

and hence, since U is obviously a $\sim$-set (all the U_i are), U is open by 4.2.1.

(2)⇒(1): We will prove that the equivalence from the display in 4.2.1 holds. Let U be a $\sim$-set such that for any open A

$$\mathsf{int}\,(U \cup A) = \mathsf{int}\,U \cup A.$$

Recall 4.2.2. We have

$$\bigwedge\left\{X \smallsetminus \overline{\{x\}} \mid x \notin U\right\} \cup A = \mathsf{int}\,U \cup A = \bigwedge\left\{X \smallsetminus \overline{\{x\}} \mid x \notin U \cup A\right\} =$$
$$= \bigwedge\{\left(X \smallsetminus \overline{\{x\}}\right) \cup A \mid x \notin U \cup A\} \supseteq \bigwedge\left\{\left(X \smallsetminus \overline{\{x\}}\right) \cup A \mid x \notin U\right\}.$$

(The last equality holds since $X \smallsetminus \overline{\{x\}} \supseteq A$ if $x \notin A$.) This makes $\mathsf{int}\,U = \bigwedge\{X \smallsetminus \overline{\{x\}} \mid x \notin U\}$ an exact meet and consequently makes U open. □

4.2.4. Corollary. *In a $T_{D\text{-}0}$ space, a meet is exact iff it is strongly exact.* □

4.2.5. A frame is *spatial* if it is isomorphic to an $\Omega(X)$; it is *T_D-spatial* if the X can be chosen to be T_D. In [5] it was shown that not every spatial frame is T_D-spatial, and T_D-spatiality was characterized. Here we have a new characterization.

Proposition. *A spatial frame L is T_D-spatial iff each exact meet in L is strongly exact.*

4.3. Proposition. *Let X be a general topological space and let $\bigwedge_i U_i$ be strongly exact in $\Omega(X)$. Then $\bigcap_i U_i$ is open.*

Proof. Consider the congruences Δ_{U_i}, Δ_U from 1.3.1. If $\mathfrak{o}(U) = \bigcap \mathfrak{o}(U_i)$ then Δ_U is the supremum of the system of congruences Δ_{U_i} in the lattice of congruences on L, which is dually isomorphic to $\mathcal{S}\ell(L)$. Set $A = \bigcap_i U_i$ and consider the congruence

$$E = \{(V, W) \mid V, W \in \Omega(X),\ V \cap A = W \cap A\}.$$

If $V \cap U_i = W \cap U_i$ then $V \cap A = V \cap U_i \cap A = W \cap U_i \cap A = W \cap A$, hence $\Delta_{U_i} \subseteq E$ for all i, and hence $\Delta_U \subseteq E$. In particular $U \Delta_U X$ and hence UEX, that is, $U \cap A = A$ and $A \subseteq U$, and since U is the interior of A, $A = U$. □

4.3.1. Note. On the other hand, $\bigcap U_i$ can be open without $\bigwedge U_i$ being strongly exact. The lattice L in 3.5.4. can be represented as

$$X = (\{1, 2, \ldots, n, \ldots\}, \{\widetilde{n} \mid n \in L\})$$

with $\widetilde{n} = \{k \mid k \leq n \text{ in } L\}$ and $\widetilde{0} = \emptyset$. In this representation, $\bigcup_{n\geq 1} \widetilde{n} = \widetilde{0}$ while $\bigwedge_{n\geq 1} \widetilde{n}$ is not strongly exact.

Thus, the property of $\bigcap U_i$ being open in $\Omega(X)$ is in general strictly between exactness and strong exactness. Here is a large class of spaces in which the open intersections and strongly exact meets do not coincide.

Observation. Let X be a non-empty T_1-space without isolated points. Then $\bigcap_{x\in X}(X \smallsetminus \{x\}) = \emptyset$ is open but $\bigwedge_{x\in X}(X \smallsetminus \{x\}) \supseteq \mathfrak{B}\Omega(X)$ (recall 1.3.4: all the $X \smallsetminus x$ are dense) and hence it is not strongly exact.

5 Open intersections in Scott topology

Scott topologies are typically not T_D and hence the first part of Section 4 does not apply. We will discuss the open intersections only.

5.1. The set of all up-sets (that is, the $M \subseteq X$ such that $\uparrow M = M$) of a poset X will be denoted by

$$\mathfrak{U}(X).$$

Recall that the *Scott topology* σ_X on a poset X with suprema of directed sets consists of the $U \in \mathfrak{U}(X)$ such that

$$\text{for any directed } D \subseteq X, \quad \bigvee D \in U \ \Rightarrow\ D \cup U \neq \emptyset.$$

In this section the spectrum of a frame L will be represented as the set $\Sigma' L$ of all completely prime filters P in L endowed with the topology consisting of the open sets $\Sigma'_a = \{P \mid a \in P\}$, $a \in L$. It is a well known (and very easy) fact that each $P \in \Sigma' L$ is Scott open in L.

More generally, in a general lattice L we will consider the pre-topology

$$\Sigma'_L = \{\Sigma'_x \mid x \in L\}, \quad \Sigma'_x = \{U \in \mathfrak{U}(L) \mid x \in U\}.$$

5.2. One of the important facts needed in the proof of the Hofmann-Lawson duality ([11], see also [13, 17, 15]) is that

> *an intersection $\bigcap \mathcal{P}$ of a set of completely prime filters is Scott open iff $\mathcal{P}$ is a compact subset of $\Sigma' L$.*

In this section we will show that this is part of a more general fact.

5.3. A subset $\mathcal{U}$ of $\mathfrak{U}(L)$ will be called *d-compact* if one can choose in every directed cover of $\mathcal{U}$ by the element of Σ'_L an element covering $\mathcal{U}$.

Note. In a topology, d-compactness coincides with compactness. Further, compactness with respect to a pretopology T coincides with the compactness of the topology generated by T, by Alexander's lemma. But with reducing d-compactness we would have troubles and hence we keep this concept in the pretopology context.

5.3.1. Proposition. *Let a set $\mathcal{U}$ of Scott open sets be d-compact in Σ'_L. Then $\bigcap \mathcal{U}$ is Scott open.*

Proof. Take an $s = \bigvee D \in \bigcap \mathcal{U}$ with D directed. Then $s \in U$ for each $U \in \mathcal{U}$ and hence there is a $d(U) \in D$ such that $d(U) \in U$, that is, $U \in \Sigma'_{d(U)}$.

For $(U_1, \ldots, U_n)$ choose $d(U_1, \ldots, U_n) \in D$, $d(U_1, \ldots, U_n) \geq U_i$, $i = 1, \ldots, n$. Then $U_i \in \Sigma'_{d(U_1,\ldots,U_n)}$ for all i, and

$$C = \{\Sigma'_{d(U_1,\ldots,U_n)} \mid U_1, \ldots, U_n \in \mathcal{U}\}$$

is a directed cover of $\mathcal{U}$. By d-compactness we have a $d \in C$ $(\subseteq D)$ such that $U \subseteq \Sigma'_d$ for all $U \in \mathcal{U}$ so that $d \in \bigcap \mathcal{U}$. □

5.3.2. Proposition. *Let $X = (X, \leq)$ be a complete lattice. Let $\mathcal{U}$ be a set of Scott open sets in X and let $\bigcap \mathcal{U}$ be Scott open. Then $\mathcal{U}$ is d-compact in Σ'_X.*

Proof. Let $\mathcal{U} \subseteq \bigcup \{\Sigma'_d \mid d \in D\}$ with D directed such that $U_d \subseteq U_e$ whenever $d \leq e$. Then for each $U \in \mathcal{U}$ there is a $d \in D$ such that $U \in \Sigma'_d$, hence $d \in U$ so that $\bigvee D \in U$ and finally $\bigvee D \in \bigcap \mathcal{U}$. Since $\bigcap \mathcal{U}$ is Scott open there is a $d \in D$ such that $d \in \bigcap \mathcal{U}$, and hence $U \subseteq \Sigma'_d$ for all $U \in \mathcal{U}$. □

5.4. Proposition. *Let L be a complete lattice. Then the intersection $\bigcap \mathcal{U}$ in the Scott topology σ_L is open iff $\mathcal{U}$ is d-compact in the pretopology Σ'_L on $\mathfrak{U}(L)$.* □

6 More about exactness and maps

In 3.4 we saw that each frame homomorphism preserves all strongly exact meets. Indeed this fact characterized strong exactness. Consequently, no such universal behaviour can be expected from plain exactness. In this section we will present two special facts.

First, however, we will apply 3.5 to the T_D case. From 4.2.4 we immediately obtain

6.1. Corollary. *If L is T_D-spatial then a frame homomorphism $h\colon L \to M$ sends all exact meets in L to strongly exact meets in M.* □

6.2. Co-weakly open homomorphisms. Recall from [4] that a frame homomorphism h is *weakly open* if $h(x^{**}) \leq h(x)^{**}$. We will say that h is *co-weakly open* if, for the associated co-frame homomorphism f_{-1} and the pseudosupplement $S^{\#}$, one has

$$f_{-1}(S)^{\#\#} \subseteq f_{-1}(S^{\#\#}).$$

6.2.1. Proposition. *A co-weakly open homomorphism preserves all exact meets.*

Proof. Let $a = \bigwedge_i a_i$ be an exact meet. By 3.3(6), $\mathfrak{o}(a) = (\bigwedge_i \mathfrak{o}(a_i))^{\#\#}$. Writing ϕ for f_{-1} and S for $\bigwedge_i \mathfrak{o}(a_i)$, we obtain

$$\phi(S)^{\#\#} \subseteq \phi(S^{\#\#}) = \phi(\mathfrak{o}(a)) = \mathfrak{o}(h(a)) \subseteq \bigcap_i \mathfrak{o}(h(a_i)) = \bigcap_i \phi(\mathfrak{o}(a_i)) = \phi(S),$$

in short

$$\phi(S)^{\#\#} \subseteq \mathfrak{o}(h(a)) \subseteq \phi(S).$$

Since for a complemented C, $C^{\#\#} = C$, this makes $\phi(S)^{\#\#} = \mathfrak{o}(h(a))$ and the statement follows. □

6.3. Recall from III.7.3 in [15] that a localic map $f\colon L \to M$ is *closed* if the image of each closed sublocale is closed (this concept captures the closedness of continuous maps) and that when this is so then $f[\mathfrak{c}(a)] = \mathfrak{c}(f(a))$ for each $a \in L$. Further, f is closed if and only if, for its left adjoint h,

$$c \le f(a) \vee b \text{ iff } h(c) \le a \vee h(b) \text{ for every } a \in L \text{ and } b, c \in M. \tag{6.3.1}$$

6.3.1. Proposition. *A closed localic map preserves all exact meets.*

Proof. Let $f\colon L \to M$ be a localic map. The image function $f[-]\colon \mathcal{S}\ell(L) \to \mathcal{S}\ell(M)$ is a left adjoint and hence it preserves suprema. Thus we obtain

$$\bigvee \mathfrak{c}(f(a_i)) = \bigvee f[\mathfrak{c}(a_i)] = f[\bigvee \mathfrak{c}(a_i)] = f[\mathfrak{c}(a)] = \mathfrak{c}(f(a)). \quad \square$$

6.3.2. An interesting consequence of this fact is the extension to frames of the result of Michael [14, Corollary 1] that the image of a (normal or regular) paracompact space under a continuous closed mapping is paracompact (see also [9, Corollary to Theorem 2]). For observing that, recall from [9] that a subset U of L is a *closed covering* if $x = \bigwedge_{u\in U}(x \vee u)$ for every $x \in L$. A closed covering is a *dual-refinement* [8] of a cover A if for each $u \in U$ there exists $a \in A$ such that $u \vee a = 1$. By Theorem 1 of [9] (cf. [8, Theorem 3.3]) a frame L is paracompact and normal iff each cover A of L has a conservative dual-refinement.

Corollary. *The image of a normal paracompact frame under a closed localic mapping is paracompact.*

Proof. Let L be a normal paracompact frame and f be a closed localic onto mapping from L onto a frame M; we denote by h its left adjoint. To prove that M is paracompact, it suffices, by the mentioned result of Dowker-Strauss [9], to show that every cover of M has a conservative dual-refinement. Let C be a cover of M. Then $A = h[C]$ is a cover of L and by hypothesis there is a conservative dual-refinement U of A. Let $V = f[U]$. By 6.3.1, V is conservative. Moreover, it is a closed covering of M: for each $y \in M$,

$$y = f(x) = f\left(\bigwedge_{u\in U}(x \vee u)\right) = \bigwedge_{u\in U} f(x \vee u) \ge \bigwedge_{u\in U} f(x) \vee f(u) = \bigwedge_{v\in V}(y \vee v).$$

Finally, it is a dual-refinement. Indeed, for each $v = f(u) \in V$, let $a \in A$ be such that $a \vee u = 1$ and consider $c \in C$ such that $a = h(c)$. Since f is closed, we may conclude by (6.3.1) that $1 \le v \vee c$ iff $1 \le u \vee h(c) = u \vee a$, hence $v \vee c = 1$. □

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