

CATEGORICAL GEOMETRY AND INTEGRATION WITHOUT POINTS

IGOR KRIZ AND ALEŠ PULTR

(dedicated to the memory of Irving Segal)

ABSTRACT. Theory of integration over infinite-dimensional spaces is known to encounter serious difficulties. To avoid some of them it is natural to attempt the categorical thinking. Such an approach was suggested and initiated by Segal in his pioneering article [24]. In our paper we follow his ideas from a different perspective, slightly more categorical, and strongly inspired by the point-free topology.

First, we develop a general (point-free) concept of measurability (extending the standard Lebesgue integration when applying to the classical σ -algebra). Second (and here we have a major difference from the classical theory), we prove that every finite-additive function μ with values in $[0, 1]$ can be extended to a measure on an abstract σ -algebra; this correspondence is functorial and yields uniqueness. As an example we show that the Segal space can be characterized by completely canonical data. Furthermore, from our results it follows that a satisfactory point-free integration arises everywhere where we have a finite-additive probability function on a Boolean algebra.

1. INTRODUCTION

The basic concept of a σ -algebra, meaning a system of subsets of a given set closed under complements and countable unions, and the accompanying concept of σ -additive measure [19], are the cornerstones of the modern theory of integration. Yet, these concepts are tested to the extreme (and sometimes beyond) in contexts of stochastic analysis [20] on one hand, and quantum field theory (see [5] for a relatively recent introduction) on another. In these cases, one often desires a

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theory of integration over infinite-dimensional spaces where σ -additive measures are either not present or difficult to construct. A classical example is provided by the case of attempting to define a Gaussian measure on the σ -algebra generated by cylindrical sets [18, 20] in an infinite-dimensional Hilbert space H . This is impossible, because the “Gaussian measure” on cylindrical sets themselves is not σ -additive. One known solution of this problem [10, 4] uses a method of “enlarging” the space H . This became known as “Radonification”. Its most famous example is the Wiener measure [26, 27]. The disadvantage of this method is that the “enlargement” is intrinsically non-canonical (in particular not functorial in the Hilbert space with respect to isometries). A different solution [11] became known as the so called Segal space. This construction can be made functorial, but still involves an arbitrary σ -algebra: functoriality is achieved by forgetting a part of the data.

Irving Segal, actually, very much believed in the idea of *algebraic integration*, i.e. a theory of integration which would start with an algebraic structure *without* an underlying space of points. In his 1965 discursive article [24], he laid out the perspectives of such a theory around that time. At the beginning of the article, he mentions early efforts of describing Lebesgue integration in terms of Boolean rings by Carathéodory and his associates, and the ultimate snag of this direction due to the fact that functions only enter that picture in a circumlocutory fashion, forestalling a comprehensive development of integration theory along these lines. He went on to describe many more sophisticated efforts to define point-free integration in more modern times. A prominent example is an approach to quantum field theory of completing the algebra of bounded continuous cylinder functions on a Hilbert space H , completing it with respect to the *sup*-norm, and applying the Gelfand construction and the Riesz representation theorem to produce a compact Hausdorff space X . The 1987 book by Glimm and Jaffe [9] describes many applications and developments in that direction. In [24], Segal also goes on to describe other “point-free” developments in integration theory, including his own 1954 paper [25] on the Kolmogoroff theorem, and integration theories based on Von Neumann algebras. Such efforts continued to more modern times: to give just one example, in the 2002 paper [3], T.Coquand and E.Palmgren described a Daniell-type integration, considering measures on Boolean algebras with a “strong apartness relation”, and extending them to a metric completion.

The space X , however, is hard to get one's hands on in the context of analysis. The large mathematical physics community preferred to use a non-canonical Schwartz distribution space to support a countably additive version of this measure rather than that compact Hausdorff space. Leonard Gross invented the notion of Abstract Wiener space in order to develop potential theory on a Hilbert space H . The norm on the abstract Wiener space, invented by L.Gross, is essential for formulating some natural theorems in potential theory. Thus, we come full circle, and points seem to force their way back into the theory.

In this paper, we approach Segal's idea of 'point-free integration' from a different, surprisingly naive and fundamental point of view, going, in a way, back to Carathéodory, but with a new take provided by category theory. During the first author's Prague years, the authors of this paper collaborated on a series of papers in category theory, developing analogues of certain constructions and facts of point set topology in the point-free context. In the theory of locales, a topological space is replaced by a "completely $\bigvee - \bigwedge$ distributive lattice" (frame) which models the algebraic structure of the set of open sets. Point-free topology has a long history. It boomed since the late fifties (out of the many authors and articles from the early time let us mention e.g. [6] and [12]). The development of the first decades culminated in Johnstone's monograph [14] (see also his excellent surveys [16] and [15]), and continues still, in the more recent decades particularly in the theory of enriched point-free structures. Both authors also published in the field (e.g. [13, 17, 22]). For more about frames and further references see e.g. [14], [15], [16], [21], [22]. It should be mentioned that our investigations in this paper are also related to a special branch of pointfree topology, the theory of σ -frames (motvated by the so called Alexandroff spaces) where one assumes countable suprema and "countable $\bigvee - \bigwedge$ distributivity" only (see e.g. [8] or [2]).

The lesson of point-free topology is that techniques of category theory can often be used to supply concepts which seemingly need points. In the context of this paper, the key point is that we find a categorical version of the appropriate concept of a *measurable function*, since such objects can be no longer defined in the present context by values on points. Our definition generalizes the notion of measurable function in the case of a classical σ -algebra, and a theory of integration which gives the same results as Lebesgue integration in this classical case. The first main result of the present paper is that measure and

Lebesgue integration theory can indeed be generalized to the context of abstract σ -algebras.

The second main result of this paper (Theorem 13) shows a major difference from the classical theory: For every finite-additive function μ on a Boolean algebra \mathcal{B} with values in $[0, 1]$, there exists a measure on an abstract σ -algebra extending μ . Moreover, by using appropriate conditions, one can obtain *uniqueness* (and hence functoriality). We show as an example how from a point-free point of view, the Segal space can be characterized by data completely canonical (up to canonical isomorphism). More generally, by the results of the present paper, a point-free measure, and point-free integration theory arises everywhere where we have a finite-additive probability function on a Boolean algebra.

To place the present paper in the context of existing work, we refer to the ultimate authority, namely D.H. Fremlin's book [7]. While that book documents many ideas relevant to the topic of the present paper, astonishingly, the basic and naive geometric view which we present here seems to have been bypassed by the field. The Stone representation theorem [7], p.70, asserts that every abstract σ -algebra with a measure is isomorphic to the σ -algebra of some measure space. Maharam's theorem (Chapter 33 of [7]) gives yet another characterization of the possible isomorphism types of abstract σ -algebras with measure (under some mild assumptions). Although this is not stated explicitly in [7], the statement of our Theorem 13 can actually be deduced from the material covered there (see [7], Exercise 325Y (b) on p. 103). All these results use complicated pointed measure constructions as intermediaries. In contrast, our direct categorical "Grothendieck-style" treatment of abstract σ -algebras develops point-free integration as an intrinsic geometry (in analogy, for example, with the intrinsic categorically-geometric treatment of super-manifolds in [5] Chapter 3, which is of great benefit even though all super-manifolds are in fact "spatial").

The present paper is organized as follows: the development of integration theory for measures on abstract σ -algebras is done in Section 2. The extension of finite-additive to σ -additive measures is handled in Section 3 below.

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2. MEASURE AND INTEGRATION ON ABSTRACT σ -ALGEBRAS

Definition 1. *By an abstract σ -algebra, we mean a Boolean algebra B in which there exist countable joins. By a morphism of abstract σ -algebras we mean a map which preserves order, $0, 1$, complements, and countable joins (and therefore meets).*

This is an obvious definition, but let us review a few useful facts. We automatically have the distributivity

$$(1) \quad \left(\bigvee_{i=0}^{\infty} a_i \right) \wedge b = \bigvee_{i=0}^{\infty} (a_i \wedge b).$$

This is because the operation $(\setminus a)\vee?$ is right adjoint to $a\wedge?$ considered as self-functors of the POSET B . On the other hand, it is *not* automatic for morphisms of Boolean algebras $B \rightarrow C$ where B, C are abstract σ -algebras to preserve countable joins (take, for example, the morphism from the Boolean algebra of all subsets on \mathbb{N} to $\{0, 1\}$ given by an ultrafilter which does not correspond to a point).

Recall that Boolean algebras can be characterized as commutative associative unital rings satisfying the relation

$$a^2 = a.$$

In this identification, the product corresponds to the meet, and $+$ to symmetric difference \oplus . (Note that since join and meet are symmetric in a Boolean algebra, another symmetric identification is possible; however, this is the usual convention.) From the point of view of this identification, the coproduct of Boolean algebras is the tensor product.

Abstract σ -algebras can be similarly characterized as universal algebras with operations of at most countable arity, for example as commutative associative unital rings satisfying the identity

$$a^2 = a$$

with an operation of countable arity

$$\prod_{i=0}^{\infty} a_i$$

which satisfies infinite commutativity, infinite associativity

$$\prod_{i=0}^{\infty} \left(\prod_{j=0}^{\infty} a_{ij} \right) = \prod_{i,j=0}^{\infty} a_{ij},$$

$$\prod_{i=0}^{\infty} a_i = \prod_{i=0}^n a_i \text{ when } a_{n+1} = a_{n+2} = \dots = 1,$$

and

$$\prod_{i=0}^{\infty} (a_i + a_i b + b) = \prod_{i=0}^{\infty} a_i + b \prod_{i=0}^{\infty} a_i + b.$$

From this point of view, the coproduct of abstract σ -algebras B_1, B_2 can also be characterized as a kind of “completed tensor product”, concretely the quotient of the free abstract σ -algebra on the set

$$\{b_1 \otimes b_2 \mid b_i \in B_i\}$$

by the relations

$$\bigvee_{n=0}^{\infty} (a_n \otimes b) \sim \left(\bigvee_{n=0}^{\infty} a_n \right) \otimes b,$$

$$\bigvee_{n=0}^{\infty} (b \otimes a_n) \sim \left(b \otimes \bigvee_{n=0}^{\infty} a_n \right),$$

$$0 \otimes a \sim a \otimes 0 \sim 0.$$

In this paper, we will describe a theory of measure and Lebesgue integration on abstract σ -algebras. A *measure* on an abstract σ -algebra (resp. an *additive function* on a Boolean algebra) B is a function

$$\mu : B \rightarrow [0, \infty]$$

which satisfies $\mu(0) = 0$ (in both cases) and

$$\mu \left(\bigvee_{i=0}^{\infty} a_i \right) = \sum_{i=0}^{\infty} \mu(a_i)$$

when $i \neq j \Rightarrow a_i \wedge a_j = 0$ (resp.

$$\mu(a \vee b) = \mu(a) + \mu(b)$$

when $a \wedge b = 0$).

A abstract σ -algebra (resp. Boolean algebra) B with a measure (resp. additive function μ) is called *reduced* if

$$\mu(a) = 0 \Rightarrow a = 0.$$

An *ideal* I on an abstract σ -algebra (resp. Boolean algebra B) is a subset $I \subseteq B$ such that $0 \in I$, when $a \in I$ and $b \leq a$ then $b \in I$ (in both cases) and

$$\bigvee_{i=0}^{\infty} a_i \in I \text{ if } a_i \in I$$

(resp.

$$a \vee b \in I \text{ if } a, b \in I.$$

Lemma 2. *Let B be an abstract σ -algebra (resp. a Boolean algebra). Let $I \subseteq B$ be an ideal. Define an equivalence relation \sim on B by*

$$a \sim b \text{ if } a \oplus b \in I.$$

Then $B/I := B/\sim$ with operations induced from B is an abstract σ -algebra (resp. Boolean algebra).

□

Example: If μ is a measure (resp. additive function) on an abstract σ -algebra (resp. Boolean algebra) B , then

$$I_\mu := \{a \in B \mid \mu(a) = 0\}$$

is an ideal and B/I_μ has an induced measure (resp. additive function) from B with respect to which it is reduced.

By a σ -*frame* we shall mean a lattice which contains $0, 1$, finite meets and countable joins which are distributive. By a *universal abstract σ -algebra* on a σ -frame we shall mean the left adjoint of the forgetful functor from abstract σ -algebras to σ -frames.

Lemma 3. *The following abstract σ -algebras are canonically isomorphic:*

- (1) *The quotient of the free abstract σ -algebra on the set of intervals $[0, t]$, $0 \leq t \leq \infty$ by the relations $[0, s] \leq [0, t]$ for $s \leq t$ and $\bigwedge [0, s_n] = [0, t]$ when $s_n \searrow t$*
- (2) *The universal abstract σ -algebra on the σ -frame of open sets with respect to the analytic topology on $[0, \infty]$.*
- (3) *The quotient of the free abstract σ -algebra on the set of intervals $[t, \infty]$, $0 \leq t \leq \infty$ by the relations $[t, \infty] \leq [s, \infty]$ for $s \leq t$ and $\bigwedge [s_n, \infty] = [t, \infty]$ when $s_n \nearrow t$*

Proof: Using universality, one constructs maps of abstract σ -algebras between (1) and (2) in both directions: a map from (1) to (2) is given

by

$$[0, t] \mapsto \setminus (t, \infty].$$

A map from (2) to (1) is given by

$$(s, t) \mapsto (\bigvee [0, t_i]) \setminus [0, s]$$

where $t_i \nearrow t$. Correctness follows from compactness of closed intervals in $[0, \infty]$. The maps are obviously inverse to each other. Inverse maps of abstract σ -algebras between (2) and (3) are constructed analogously. \square

Let \mathcal{B} be the abstract σ -algebra characterized by the equivalent properties of Lemma 3.

Problem: Is \mathcal{B} isomorphic to the abstract σ -algebra of Borel sets in $[0, \infty]$?

We will also denote by $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}(\mathbb{C})$) the universal abstract σ -algebras on the σ -frame of open sets in \mathbb{R} (resp. \mathbb{C}) with respect to the analytic topology. A *non-negative measurable function* on an abstract σ -algebra Σ is a morphism of abstract σ -algebras

$$f : \mathcal{B} \rightarrow \Sigma.$$

For measurable functions $f, g : \mathcal{B} \rightarrow \Sigma$, we write

$$f \leq g$$

if for every $t \in [0, \infty]$,

$$f[0, t] \geq g[0, t].$$

Addition

$$+ : [0, \infty] \times [0, \infty] \rightarrow [0, \infty]$$

induces a map of abstract σ -algebras

$$+ : \mathcal{B} \rightarrow \mathcal{B} \amalg \mathcal{B}$$

(here \amalg denotes categorical coproduct). The *sum* of measurable functions $f, g : \mathcal{B} \rightarrow \Sigma$ is the composition

$$\mathcal{B} \xrightarrow{+} \mathcal{B} \amalg \mathcal{B} \xrightarrow{f \amalg g} \Sigma \amalg \Sigma \xrightarrow{\nabla} \Sigma$$

where ∇ is the categorical co-diagonal. A similar construction clearly works with $+$ replaced by any continuous operation on $[0, \infty]$, such as multiplication, or the min or max function.

Let S be a set. Denote by 2^S the abstract σ -algebra of all subsets of S .

Lemma 4. *Every morphism of abstract σ -algebras $\phi : B \rightarrow 2^{\{0\}}$ is induced by a point in $[0, \infty]$, i.e. a map of sets $\{0\} \rightarrow [0, \infty]$.*

Proof: Clearly, we cannot have $\phi(\{x\}) = \phi(\{y\}) = 1$ for two points $x \neq y$, since then

$$0 = \phi(0) = \phi(\{x\} \wedge \{y\}) = \phi(\{x\}) \wedge \phi(\{y\}) = 1.$$

Thus, there is at most one point $x \in [0, \infty]$ with

$$(2) \quad \phi(\{x\}) = 1.$$

We claim that an x satisfying (2) exists. In effect, assuming $\phi(\{\infty\}) = 0$, we must have $\phi[n_i/2^i, (n_i + 1)/2^i) = 1$ for some $n_i \in \mathbb{N}$ by countable additivity. Then

$$\phi \left(\bigwedge_{i=0}^{\infty} [n_i/2^i, (n_i + 1)/2^i) \right) = 1,$$

but the argument on the left hand side has at most one point. On the other hand, it cannot be empty because $\phi(0) = 0$. \square

A *simple function* on an abstract σ -algebra Σ is a morphism of abstract σ -algebras $F : \mathcal{B} \rightarrow \Sigma$ for which there exists a finite Boolean algebra (hence abstract Σ -algebra) $\mathcal{F} = 2^{\{0, \dots, n\}}$ and morphisms of abstract σ -algebras $\chi : \mathcal{B} \rightarrow \mathcal{F}$, $s : \mathcal{F} \rightarrow \Sigma$ such that the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\chi} & \mathcal{F} \\ & \searrow f & \downarrow s \\ & & \Sigma \end{array}$$

commutes.

By Lemma 4, χ is induced by

$$i \mapsto x_i \in [0, \infty], \quad i = 0, \dots, n.$$

Let $\mu : \Sigma \rightarrow [0, \infty]$ be a measure. Then define

$$\int f d\mu := \sum_{i=0}^n \mu(s(\{i\}))x_i.$$

For an arbitrary function $f : \mathcal{B} \rightarrow \Sigma$, define

$$\int f d\mu = \sup \left\{ \int g d\mu \mid g \leq f \text{ and } g \text{ is simple} \right\}.$$

Remark: Clearly, when (X, Σ, μ) is a measurable space, $\phi : X \rightarrow [0, \infty]$ and is (Borel) measurable, then defining $f : \mathcal{B} \rightarrow \Sigma$ by

$$f(a) := \phi^{-1}(a),$$

we have by definition

$$\int f d\mu = \int \phi d\mu.$$

Lemma 5. *Let Σ be an abstract σ -algebra with measure μ . If $s_1, s_2 : \mathcal{B} \rightarrow \Sigma$ are simple functions, then $s_1 + s_2$ is a simple function and*

$$\int (s_1 + s_2) d\mu = \left(\int s_1 d\mu \right) + \left(\int s_2 d\mu \right).$$

□

For an abstract σ -algebra Σ and functions $f, f_n; \mathcal{B} \rightarrow \Sigma$, $n \in \mathbb{N}$, we write $f_n \nearrow f$ if

$$f_0 \leq f_1 \leq f_2 \leq \dots \leq f$$

and for all $t \in [0, \infty]$,

$$f([0, t]) = \bigwedge_n f_n([0, t]).$$

(Note: The non-trivial inequality is \leq .) It is worth noting that by characterization (1) of Lemma 3, whenever

$$f_0 \leq f_1 \leq f_2 \leq \dots$$

for non-negative measurable functions $f_i : \mathcal{B} \rightarrow \Sigma$, there exists a unique non-negative measurable function $f : \mathcal{B} \rightarrow \Sigma$ such that $f_i \nearrow f$. Analogously, one defines for non-negative measurable functions

$$f_0 \geq f_1 \geq f_2 \dots$$

$f_n \searrow f$ when

$$f([t, \infty]) = \bigwedge_n f_n([t, \infty]).$$

Using characterization (3) of Lemma 3, one shows that such non-negative measurable function f always exists.

Lemma 6. *For an abstract σ -algebra Σ and a non-negative measurable function $f : \mathcal{B} \rightarrow \Sigma$, there exist simple functions $s_n : \mathcal{B} \rightarrow \Sigma$ such that $s_n \nearrow f$.*

Proof: Let

$$S_n(k) := \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) \text{ if } k = 0, \dots, (n2^n) - 1,$$

$$S_n(n2^n) := [n, \infty).$$

Then let, for $a \in \mathcal{B}$,

$$s_n(a) := \bigvee_{\{k \in \{0, \dots, n2^n\} \mid k/2^n \in a\}} f(S_n(k)).$$

□

Theorem 7. (*Levi's theorem, Lebesgue monotone convergence theorem*) let Σ be an abstract σ -algebra with measure μ and let $f, f_n : \mathcal{B} \rightarrow \Sigma$ be functions, $f_n \nearrow f$. Then

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Proof: The \leq inequality is immediate. Now for a simple function $s \leq f$, let

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{g} & 2^{\{0, \dots, n\}} \\ & \searrow s & \downarrow \tilde{s} \\ & & \Sigma \end{array}$$

let $\tilde{s}(i) := a_i$ and let $x_i \in [0, \infty]$ be such that

$$g(a) = \{i \mid x_i \in a\}$$

(see Lemma 4). Let

$$b_{ink} := f_n([y_{ik}, \infty))$$

where $y_{ik} = x_i - 1/k$ if $x_i \neq \infty$, and $y_{ik} = k$ if $x_i = \infty$. Then $f_n \nearrow f$ implies

$$(3) \quad \bigvee_n b_{ink} \leq a_i.$$

This implies

$$\lim_{n \rightarrow \infty} \int f_n d\mu \geq \lim_{n \rightarrow \infty} \sum_{i=0}^m \mu(b_{ink} \wedge a_i) y_{ik} = \sum_{i=0}^m \mu(a_i) y_{ik}.$$

By (3), the limit of the right hand side as $k \rightarrow \infty$ is

$$\int s d\mu.$$

Since $s \leq f$ was an arbitrary simple function, the \geq inequality follows. \square

Lemma 8. *If, for an abstract σ -algebra Σ and functions $f_1, f_2, g_1, g_2 : \mathcal{B} \rightarrow \Sigma$, $f_1 \leq f_2$ and $g_1 \leq g_2$, then*

$$f_1 + f_2 \leq g_1 + g_2.$$

Proof: Follows from the definition and the fact that there exists a countable sequence of pairs $0 \leq r_i, s_i, r_i + s_i \leq t$, such that

$$+([0, t]) = \bigvee_i [0, r_i] \otimes [0, s_i].$$

\square

Lemma 9. *For an abstract σ -algebra Σ and non-negative measurable functions $f, f_n : \mathcal{B} \rightarrow \Sigma$, $n \in \mathbb{N}$, if we have $f_n \nearrow f$, $g_n \nearrow g$, then $f_n + g_n \nearrow f + g$.*

Proof:

$$f_0 + g_0 \leq f_1 + g_1 \leq \dots \leq f + g$$

follows from Lemma 8. But there exist $0 \leq r_{i1}, \dots, r_{in_i}, s_{i1}, \dots, s_{in_i}$ such that

$$(4) \quad +([0, t]) = \bigwedge_{i=0}^{\infty} \bigvee_{j=1}^{n_i} [0, r_{ij}] \otimes [0, s_{ij}].$$

Then

$$(5) \quad (f \amalg g)(S) = \bigwedge_n (f_n \amalg g_n)(S)$$

is true for $S = [0, r_{ij}] \otimes [0, s_{ij}]$, hence by distributivity for

$$S = \bigvee_{j=1}^{n_i} [0, r_{ij}] \otimes [0, s_{ij}].$$

To see this, compute

$$\begin{aligned}
 (f \amalg g)(S) &= \bigvee_{j=1}^{n_i} \bigwedge_n (f_n \amalg g_n)([0, r_{ij}] \otimes [0, s_{ij}]) \\
 &= \bigwedge_{m_1, \dots, m_{n_i}} \bigvee_{j=1}^{n_i} (f_{m_j} \amalg g_{m_j})([0, r_{ij}] \otimes [0, s_{ij}]) \\
 &= \bigwedge_n (f_n \amalg g_n)(S).
 \end{aligned}$$

Hence, (5) holds for $S = +([0, t])$ by (4). \square

Theorem 10. *For an abstract σ -algebra Σ with measure μ and $f, g : \mathcal{B} \rightarrow \Sigma$ non-negative measurable functions,*

$$\int (f + g) d\mu = \int f d\mu + \int g d\mu.$$

Proof: By Lemma 6, choose simple functions $s_n \nearrow f$, $s'_n \nearrow g$. We have

$$\begin{aligned}
 \int (f + g) d\mu &= \text{by Lemma 9 and Lebesgue's theorem 7} \\
 &= \lim_{n \rightarrow \infty} \int (s_n + s'_n) d\mu = \text{by Lemma 5} \\
 &= \lim_{n \rightarrow \infty} \int s_n d\mu + \lim_{n \rightarrow \infty} \int s'_n d\mu = \text{by theorem 7} = \\
 &= \int f d\mu + \int g d\mu.
 \end{aligned}$$

\square

Since we showed that minima, maxima, and limits of increasing and decreasing sequences of non-negative measurable functions are defined as non-negative measurable functions, one can define \liminf and \limsup among non-negative measurable functions on an abstract σ -algebra.

Lemma 11. (*Fatou's lemma*) If $f_n : \mathcal{B} \rightarrow \Sigma$ are non-negative measurable functions for $n \in \mathbb{N}$, then

$$(6) \quad \int \left(\liminf_{n \rightarrow \infty} f_n \right) d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

Proof: (following [23]): Put

$$g_k := \inf_{i \geq k} f_i,$$

and apply Lebesgue monotone convergence theorem to g_k . \square

Now for an abstract σ -algebra Σ , by a *complex (resp. real) measurable function on Σ* we mean a map of abstract σ -algebras from $\mathcal{B}(\mathbb{C})$ (resp. $\mathcal{B}(\mathbb{R})$) to Σ . Since absolute value is continuous, by Lemma 3 we have for a complex measurable function f on Σ a non-negative measurable function $|f|$. If Σ is equipped with a measure, we say that a complex measurable function f is *integrable* if

$$\int |f| < \infty.$$

The σ -frame of open sets of \mathbb{C} is the coproduct of two copies of the σ -frame of open sets of \mathbb{R} . Thus, we may write a complex measurable function on Σ as

$$f = u + iv$$

where u, v are real measurable functions. We then define

$$u^+ = \max(u, 0), \quad u^- = \min(u, 0),$$

and when f is integrable,

$$\int f d\mu = \int u^+ d\mu - \int u^- d\mu + i \int v^+ d\mu - i \int v^- d\mu.$$

Linearity of the integral on complex integrable functions follows from Theorem 10.

Theorem 12. (*Lebesgue dominated convergence theorem - following Rudin [23]*) Suppose f_n are complex measurable functions on an abstract σ -algebra Σ such that

$$f = \lim_{n \rightarrow \infty} f_n$$

and there exists an integrable non-negative function g on Σ such that

$$|f_n| \leq g.$$

Then f is integrable,

$$\lim_{n \rightarrow \infty} \int |f_n - f| d\mu = 0$$

and

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Proof: Apply Fatou's lemma to the function $2g - |f_n - f|$. □

3. ABSTRACT RADONIFICATION

There is an obvious forgetful functor from abstract σ -algebras to Boolean algebras. Moreover, the functor obviously preserves limits, and, in fact, a left adjoint exists by the well known general principle for (possibly infinitary) universal algebras. We will denote the left adjoint by $CB(?)$.

Theorem 13. *Let B be a Boolean algebra and $\mu_0 : B \rightarrow [0, 1]$ an additive function. Then there exists a unique measure $\mu : CB(B) \rightarrow [0, 1]$ such that the following diagram commutes:*

$$\begin{array}{ccc} B & \xrightarrow{\mu_0} & [0, 1] \\ \eta \downarrow & \nearrow \mu & \\ CB(B) & & . \end{array}$$

Proof: By transfinite induction, we shall construct Boolean algebras B_α and additive functions

$$\mu_\alpha : B_\alpha \rightarrow [0, 1].$$

Put $B_0 := B$. Pick a countable subset

$$S_\alpha = \{s_{\alpha 0}, s_{\alpha 1}, \dots\} \subset B_\alpha$$

such that

$$S_\alpha \neq S_\gamma \text{ for any } \gamma < \alpha.$$

Then let

$$B_{\alpha+1} := B_\alpha \amalg 2^{\{x_\alpha, y_\alpha\}}$$

and define

$$\mu_{\alpha+1}(a \otimes 1) := \mu_{\alpha}(a),$$

$$(7) \quad \mu_{\alpha+1}(a \otimes x_{\alpha}) := \lim_{n \rightarrow \infty} \mu_{\alpha}(a \wedge (s_{\alpha 1} \vee \dots \vee s_{\alpha n})).$$

Additivity of limits clearly implies that this extends uniquely to an additive function

$$\mu_{\alpha+1} : B_{\alpha+1} \rightarrow [0, 1].$$

For a limit ordinal α , we define B_{α} be the union of B_{β} over $\beta < \alpha$, and we let μ_{α} be the common extension of μ_{β} , $\beta < \alpha$.

Now let γ be a cardinal number $> |B|$ of cofinality $> \omega$ such that

$$\alpha < \gamma \Rightarrow \alpha^{\aleph_0} < \gamma.$$

Then $\{S_{\alpha} | \alpha < \gamma\}$ is the set of *all* countable subsets of B_{γ} . Let

$$\tilde{B} := B_{\gamma} / I_{\mu_{\gamma}}$$

and let $\tilde{\mu} : \tilde{B} \rightarrow [0, 1]$ be the induced additive function.

Lemma 14. *\tilde{B} is an abstract σ -algebra and $\tilde{\mu} : \tilde{B} \rightarrow [0, 1]$ is a measure.*

Proof: First, we claim that for every $\alpha < \gamma$,

$$(8) \quad x_{\alpha} = \bigvee S_{\alpha}.$$

In effect,

$$\mu_{\gamma}(s_{\alpha i} \wedge y_{\alpha}) = \mu_{\alpha+1}(s_{\alpha i} \wedge y_{\alpha}) = \mu_{\alpha+1}(s_{\alpha i}) - \mu_{\alpha+1}(s_{\alpha i} \wedge x_{\alpha}) = 0,$$

and hence $s_{\alpha i} \wedge y_{\alpha} = 0$, so $x_{\alpha} \geq s_{\alpha i}$. Now suppose $z \geq s_{\alpha i}$ for all $i = 0, 1, \dots$. Let $u := x_{\alpha} \setminus z$. Then $x_{\alpha} \geq u$ and $u \wedge s_{\alpha i} = 0$ for all i .

This means that

$$\tilde{\mu}(u) + \mu_{\alpha} \left(\bigvee_{i=0}^n s_{\alpha i} \right) \leq \mu_{\alpha+1}(x_{\alpha}) = \lim_{n \rightarrow \infty} \mu_{\alpha} \left(\bigvee_{i=0}^{\infty} s_{\alpha i} \right)$$

(the last equality is by (7)). Hence, $\tilde{\mu}(u) = 0$ so $u = 0$, which proves (8). Note that we have also proved

$$(9) \quad \lim_{n \rightarrow \infty} \tilde{\mu} \left(\bigvee_{i=0}^n s_{\alpha i} \right) = \tilde{\mu} \left(\bigvee_{i=0}^{\infty} s_{\alpha i} \right),$$

which concludes the proof of the Lemma. \square

To continue the proof of the Theorem, we have constructed a diagram

$$\begin{array}{ccc} B & \xrightarrow{\mu_0} & [0, 1] \\ \beta \downarrow & \nearrow \tilde{\mu} & \\ \tilde{B} & & \end{array}$$

where \tilde{B} is an abstract σ -algebra, the inclusion β is a morphism of Boolean algebras, and $\tilde{\mu}$ is a measure. By adjunction, we have a unique morphism of abstract σ -algebras $\tilde{\beta}$ completing the following diagram

$$\begin{array}{ccc} B & & \\ \downarrow & \searrow \beta & \\ CB(B) & \xrightarrow{\tilde{\beta}} & \tilde{B}. \end{array}$$

Then $\tilde{\mu}\tilde{\beta}$ is a measure, proving the existence statement of the Theorem. To prove uniqueness, note that \tilde{B} is generated by B as an abstract σ -algebra and therefore the measure of each element of $CB(B)$ can be computed recursively from μ_0 by transfinite induction. \square

Remark: From our setup, one may perhaps expect that Theorem 13 would have a simple proof using the universal property of $CB(?)$. However, at present we don't know such a proof.

Corollary 15. *Under the assumptions of Theorem 13, there exists a unique (up to unique isomorphism) morphism of Boolean algebras*

$$\beta : B \rightarrow \tilde{B}$$

where \tilde{B} is a reduced abstract σ -algebra with a measure $\tilde{\mu}$, such that \tilde{B} is generated by $Im(\beta)$, and $\tilde{\mu}\beta = \mu_0$.

Proof: For existence, we may take \tilde{B} , $\tilde{\mu}$ as constructed in the proof of Theorem 13. For uniqueness, if we pick \tilde{B} , $\tilde{\mu}$ as in the statement of the Corollary, Theorem 13 gives a morphism of abstract σ -algebras

$$(10) \quad \tilde{\beta} : CB(B) \rightarrow \tilde{B}$$

such that $\tilde{\mu}\tilde{\beta} = \mu$. Additionally, the assumption that \tilde{B} is generated by the image of β implies that (10) is onto. Now since \tilde{B} is reduced, (10) factors through a unique morphism of abstract σ -algebras

$$(11) \quad \bar{\beta} : CB(B)/I_{\mu} \rightarrow \tilde{B}.$$

Then (11) is onto because (10) is, and

$$\bar{\beta}(x) = 0 \Rightarrow \mu(x) = \tilde{\mu}\tilde{\beta}(x) = 0 \Rightarrow x = 0,$$

so $\bar{\beta}$ is injective and hence an isomorphism. \square

Remark: Note that to construct, say, a non-negative measurable function on \tilde{B} , all we need is an order-preserving map

$$f : [0, \infty] \rightarrow CBB$$

(on the left hand side, we identify t with $[0, t]$) such that for $t_n \searrow t$,

$$\lim_{n \rightarrow \infty} \mu(f(t_n)) = \mu(f(t)).$$

Since μ can be (theoretically) computed from σ -additivity and μ_0 , measurable maps are, in principle, in abundant supply.

Example: An (irreducible) Gaussian probability space consists of a Hilbert space H , a probability space (X, Σ, μ) and a Hilbert space embedding

$$(12) \quad \alpha : H \subset L^2(X)$$

such that all the variables in the image of (12) have centered Gaussian law, and they generate Σ .

Theorem 16. *Let B denote the set of cylindrical Borel measurable subsets of H , and let $\mu_0 : B \rightarrow [0, 1]$ be the Gaussian additive function. Then there is a canonical isomorphism*

$$\tilde{B} \cong \Sigma / I_\mu.$$

Proof: We define a map of Boolean algebras

$$f : B \rightarrow \Sigma$$

as follows; for $a \in H$, $b \in \mathcal{B}(\mathbb{R})$ (the Borel σ -algebra on \mathbb{R}),

$$f(\langle a, ? \rangle^{-1}(b)) := \alpha^{-1}(b).$$

Clearly, this extends uniquely to a map of Boolean algebras, and we have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\mu_0} & [0, 1] \\ f \downarrow & \nearrow \mu & \\ \Sigma & & \end{array}$$

Additionally, by definition, Σ is generated by $f(B)$ as a σ -algebra (and hence an abstract σ -algebra). Thus, our statement follows from Corollary 15. \square

A *radonification* of H is a bounded injective dense linear map ι from H into a Banach space \overline{H} , a bounded linear map $\bar{?} : H \rightarrow \overline{H}$ such that $\bar{a}\iota = \langle a, ? \rangle$ and a σ -algebra Σ on \overline{H} with a probability measure μ such that \bar{a} are measurable functions and for $b \in \mathcal{B}(\mathbb{R}^n)$,

$$\mu\left(\bigcap_{i=1}^n \bar{a}_i^{-1}(b)\right) = \mu_0\left(\bigcap_{i=1}^n a_i^{-1}(b)\right).$$

Proposition 17. *In this situation, there is a unique isomorphism of abstract σ -algebras*

$$\tilde{B} \cong \Sigma/I_\mu$$

which sends μ to $\tilde{\mu}$.

Proof: $a^{-1}(b) \mapsto \bar{a}^{-1}(b)$ clearly defines an embedding of Boolean algebras

$$f : B \rightarrow \Sigma$$

such that $\mu f = \mu_0$, so again, our statement follows from Corollary 15. \square

A non-Gaussian example: Using Corollary 15, we may obtain an infinite-dimensional point-free abstract integration theory for any sequence of absolutely continuous independent random variables X_1, X_2, \dots . If H is a real Hilbert space with Hilbert basis e_1, e_2, \dots , and if f_n is the probability density of X_n , and

$$\pi_n : H \rightarrow \langle e_1, \dots, e_n \rangle$$

is the orthogonal projection, we may define, for a Borel subset $S \subseteq \langle e_1, \dots, e_n \rangle$,

$$\mu(\pi^{-1}(S)) = \int_S f_1(x_1) \cdots f_n(x_n) dx_1 \cdots dx_n.$$

This is a finitely additive function from the set B of cylindrical Borel measurable subset of H with values in $[0, 1]$, so it extends to a σ -additive function $\tilde{\mu} : \tilde{B} \rightarrow [0, 1]$.

Most point-free measures $\tilde{\mu}$ one obtains in this way are, however, unnatural in the sense that they are highly dependent on the coordinate

system we choose. It is interesting, in this context, to search for finite-dimensional joint distributions which, with respect to onto linear maps, transform (contravariantly) in families with finitely many parameters in each dimension. Such non-Gaussian examples are hard to come by.

A beautiful example of this kind is the multivariate skew-normal distribution introduced by A. Azzalini and A. Dalla Valle [1]. For the most general finite-dimensional version of this distribution, consider an $(n+1) \times (n+1)$ positive-definite real symmetric matrix $A = (a_{ij})_{0 \leq i, j \leq n}$ such that $a_{11} = 1$. Let $(X_0, X)^T \in \mathbb{R} \times \mathbb{R}^n$ be a centered normally distributed random variable with covariance matrix A . Then let Z be the random variable give by

$$Z = \begin{cases} X & \text{if } X_0 > 0 \\ -X & \text{if } X_0 < 0. \end{cases}$$

The random variable Z is absolutely continuous in \mathbb{R}^n with probability density function which we will denote by ϕ_A . This is the skew-normal distribution (see [1], Proposition 6).

For our purposes, it suffices to consider the case when $(a_{ij})_{1 \leq i, j \leq n}$ is the identity matrix, since one can always reach this case by linear transformation. Let, then $\delta_1, \delta_2, \dots \in \mathbb{R}$ be such that

$$(13) \quad \sum_{n=1}^{\infty} \delta_n^2 < 1.$$

Let

$$A_n = \begin{pmatrix} 1 & \delta_1 & \delta_2 & \dots & \delta_n \\ \delta_1 & 1 & 0 & \dots & 0 \\ \delta_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \delta_n & 0 & 0 & \dots & 1 \end{pmatrix}$$

((13) assures that A_n is positive-definite). Define an additive function with values in $[0, 1]$ on the set B of cylindrical Borel-measurable subsets of the Hilbert space H by setting, for a Borel-measurable subset $S \subseteq \langle e_1, \dots, e_n \rangle$,

$$\mu(\pi_n^{-1}S) = \int_S \phi_{A_n}(x_1, \dots, x_n) dx_1 \dots dx_n.$$

Applying Corollary 15, we obtain a reduced abstract σ -algebra \tilde{B} with a measure $\tilde{\mu}$. This is the infinite-dimensional skew-normal distribution. As far as we know, a pointed version of this distribution has not been considered.

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