ADJOINT FUNCTORS IN GRAPH THEORY

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Abstract. We survey some uses of adjoint functors in graph theory pertaining to colourings, complexity reductions, multiplicativity, circular colourings and tree duality. The exposition of these applications through adjoint functors unifies the presentation to some extent, and also raises interesting questions.

1. Introduction

We will motivate our subject with known examples from the literature, which the reader may recognize. Relevant definitions are postponed to the following section.

Theorem 1.1 (Geller, Stahl). *Let G be a graph. Then the lexicographic product $G[K_2]$ is n-colourable if and only if G admits a homomorphism to the Kneser graph $K(n, 2)$.*

Theorem 1.2 (El-Zahar, Sauer). *For two graphs G and H, the categorial product $G \times H$ is n-colourable if and only if G admits a homomorphism to the exponential graph K_n^H.*

Theorem 1.3 (Hell, Nešetřil). *For a graph G, let $G^{1/3}$ be the graph obtained by replacing each edge of G by a path with three edges. Then there exists a homomorphism of $G^{1/3}$ to the 5-cycle C_5 if and only if G is 5-colourable.*

Date: July 20, 2013.

2010 *Mathematics Subject Classification.* 05C15, 18B35.

Key words and phrases. graph products, adjoint functors, finite duality.

The second author's research is supported by grants from NSERC and ARP.

An n-colouring of a graph G is a homomorphism of G to the complete graph K_n, so the above results all state that the existence of some homomorphism is equivalent to the existence of some other homomorphism. More precisely, there exists a homomorphism of some $\Lambda(G)$ to a target K if and only if there exists a homomorphism of G to some $\Gamma(K)$. In Theorem 1.1, we have $\Lambda(G) = G[K_2]$ and $\Gamma(K_n) = K(n, 2)$. In Theorem 1.2, $\Lambda(G) = G \times H$ and $\Gamma(K_n) = K_n^H$ for some fixed H. In Theorem 1.3, $\Lambda(G) = G^{1/3}$ and $\Gamma(C_5) = K_5$. It may not yet be clear that these various Γ all generalize to well-defined functors. We present one more example where the presentation of $\Gamma(K_n)$ is cryptic, and even the existence of an appropriate Λ is not obvious.

Theorem 1.4 (Gyárfás, Jensen, Stiebitz). *A graph G admits an n-colouring in which the neighbourhood of each colour class is an independent set if and only if G admits a homomorphism to the graph $U(n)$.*

It is time to introduce some relevant terminology.

2. Pultr templates and functors

We refer the reader to [12] for an introduction to graph homomorphisms.

Definition 2.1.

(i) A *Pultr template* is a quadruple $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ where P, Q are graphs and ϵ_1, ϵ_2 homomorphisms of P to Q such that Q admits an automorphism q with $q \circ \epsilon_1 = \epsilon_2$ and $q \circ \epsilon_2 = \epsilon_1$.

(ii) Given a Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, the *left Pultr functor* $\Lambda_{\mathcal{T}}$ is the following construction: For a graph G, $\Lambda_{\mathcal{T}}(G)$ contains one copy P_u of P for every vertex u of G, and for every edge $[u, v]$ of G, $\Lambda_{\mathcal{T}}(G)$ contains a copy $Q_{u,v}$ of Q with $\epsilon_1(P)$ identified with P_u and $\epsilon_2(P)$ identified with P_v.

(iii) Given a Pultr template $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$ the *central Pultr functor* $\Gamma_{\mathcal{T}}$ is the following construction: For a graph K, the vertices of $\Gamma_{\mathcal{T}}(K)$ are the homomorphisms $g : P \to K$, and the edges of $\Gamma_{\mathcal{T}}(K)$ are the pairs $[g_1, g_2]$ such that there exists a homomorphism $h : Q \to K$ with $g_1 = h \circ \epsilon_1$, $g_2 = h \circ \epsilon_2$.

Note that the automorphism q of Q interchanging ϵ_1 and ϵ_2 in (i) makes the conditions in (ii) and (iii) symmetric, so that $\Lambda_{\mathcal{T}}(G)$ is well defined, and $\Gamma_{\mathcal{T}}(K)$ is a graph rather than a digraph. The following two examples model Theorems 1.1 and 1.2.

Examples.

- Let $\mathcal{T} = (K_2, K_4, \epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 are homomorphism mapping K_2 to two non-incident edges of K_4. Then $\Lambda_{\mathcal{T}}(G)$ is the lexicographic product $G[K_2]$. The vertices of $\Gamma_{\mathcal{T}}(K_n)$ are essentially the arcs of K_n, that is, the couples (i, j) with $1 \leq i, j \leq n$, $i \neq j$. Its edges are the pairs $[(i, j), (i', j')]$ such that $\{i, j\} \cap \{i', j'\} = \emptyset$. Thus $\Gamma_{\mathcal{T}}(K_n)$ is the Kneser graph $K(n, 2)$ with all its vertices doubled; it is homomorphically equivalent to $K(n, 2)$. Thus Theorem 1.1 states that $\Lambda_{\mathcal{T}}(G)$ admits a homomorphism to K_n if and only if G admits a homomorphism to $\Gamma_{\mathcal{T}}(K_n)$.
- Let H be a fixed graph and $\mathcal{T}_H = (H \times K_1, H \times K_2, \epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2, are the two natural injections of the independent set $H \times K_1$ into the categorial product $H \times K_2$. Then $\Lambda_{\mathcal{T}_H}(G)$ is the categorial product $G \times H$, and $\Gamma_{\mathcal{T}_H}(K)$ is the exponential graph K^H. Theorem 1.1 states that $\Lambda_{\mathcal{T}_H}(G)$ admits a homomorphism to K_n if and only if G admits a homomorphism to $\Gamma_{\mathcal{T}_H}(K_n)$.

For any standard graph product $\star$ (see [13]) and any graph H, we can similarly form a template $\mathcal{T} = (H \star K_1, H \star K_2, \epsilon_1, \epsilon_2)$, such that $\Lambda_{\mathcal{T}}(G) = G \star H$, and $\Gamma_{\mathcal{T}}(K)$ is an "exponential structure" in the sense that $G \star H$ admits a homomorphism to K if and only if G admits a homomorphism to $\Gamma_{\mathcal{T}}(K)$. Geller and Stahl [6] proved that $\chi(G[H])$ equals $\chi(G[K_{\chi(H)}])$, and used an extension of Theorem 1.1 to prove that the latter is measured by homomorphisms into Kneser graphs. El-Zahar and Sauer [1] used the correspondence of Theorem 1.2 to prove that the chromatic number of a categorial product of 4-chromatic graphs is 4.

The next examples involve Pultr functors outside the mould of graph products.

- Let P_3 denote the path with three edges and ϵ_1, ϵ_2 be the homomorphisms mapping K_1 to the endpoints of P_3. Let $\mathcal{T}_3 = (K_1, P_3, \epsilon_1, \epsilon_2)$. Then $\Lambda_{\mathcal{T}_3}(G)$ is obtained from G by replacing each edge by a path with three edges, that is, $G^{1/3}$. $\Gamma_{\mathcal{T}_3}(K)$ is obtained from K by adding edges between vertices joined by a walk of length 3 in K. In particular, $\Gamma_{\mathcal{T}_3}(K)$ contains loops if and only if K contains triangles or loops, and $\Gamma_{\mathcal{T}_3}(C_5) = K_5$.Theorem 1.3 states that $\Lambda_{\mathcal{T}_3}(G)$ admits a homomorphism to C_5 if and only if G admits a homomorphism to $\Gamma_{\mathcal{T}_3}(C_5)$).

Theorem 1.3 is an adaptation by Hell and Nešetřil of a result by Maurer, Sudborough and Welzl [15]. It is an example of a reduction among homomorphisms problems. With similar reductions, Hell and Nešetřil [11] eventually proved that for any fixed non-bipartite graph H, the problem of determining whether an input graph G admits a homomorphism into H is NP-complete.

Theorems 1.1, 1.2, 1.3 are particular manifestations of the following general property:

Theorem 2.2 (Pultr [21]). *For any Pultr template $\mathcal{T}$ and any graphs G, K, there exists a homomorphism of $\Lambda_{\mathcal{T}}(G)$ to K if and only if there exists a homomorphism of G to $\Gamma_{\mathcal{T}}(K)$.*

3. Adjoint functors and categories

Two functors Λ and Γ are said to be respectively *left* and *right adjoints* of each other if there is a natural correspondence between the morphisms of $\Lambda(X)$ to Y and the morphisms of X to $\Gamma(Y)$. Note that this correspondence between morphisms is apparently a stronger statement than the existential statement of Theorem 2.2, but this depends on the precise categorial context.

- **(i)** In the usual category of graphs, it can be shown that for the templates $\mathcal{T}$ modelling Theorems 1.1 and 1.2, $\Lambda_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$ are left and right adjoints in the sense above. However, the template $\mathcal{T}$ modelling Theorem 1.3 does not give rise to left and right adjoints: The number of homomorphisms of $\Lambda_{\mathcal{T}}(G)$ to K is not always equal to the number of homomorphisms of G to $\Gamma_{\mathcal{T}}(K)$
- **(ii)** In the "thin" category (or preorder) of graphs, the morphisms between given graphs is not distinguished: there is at most one generic morphism from one graph to another. In this context,

Theorem 2.2 states that for any Pultr template $\mathcal{T}$, $\Lambda_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$ are left and right adjoints of each other.

(iii) In the category of multigraphs, where morphisms must specify images of vertices and also of edges, $\Lambda_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$ are always left and right adjoints. Furthermore Pultr [21] has shown that all pairs of adjoint functors in this category are of the form $\Lambda_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$.

Though the first context is the most commonly understood, our applications so far and those to come are existential. Consequently we work in the thin category of graphs, and call a pair of functors Λ, Γ left and right adjoints of each other if the existence of a homomorphism of $\Lambda(G)$ to K is equivalent to the existence of a homomorphism of G to $\Gamma(K)$.

Any Pultr template $\mathcal{T}$ gives rise to the adjoints $\Lambda_{\mathcal{T}}$ and $\Gamma_{\mathcal{T}}$. But unlike the case of the category of multigraphs, other pairs of adjoint functors exist. In particular, $\Gamma_{\mathcal{T}}$ is called a "central" rather than a "right" functor because in some significant cases $\Gamma_{\mathcal{T}}$ itself admits a right adjoint $\Omega_{\mathcal{T}}$. For instance, in the next section we interpret Theorem 1.4 in terms of the right adjoint of a central Pultr functor.

We do not know which central Pultr functors admit right adjoints. It would be interesting to characterize all pairs of adjoint functors in the thin category of graphs, though this objective may be out of reach. We will instead use the known examples to show the type of applications that the search for new adjoint functors may yield.

4. The right adjoint of $\Gamma_{\mathcal{T}_3}$

Recall that the Pultr template $\mathcal{T}_3$ is $(K_1, P_3, \epsilon_1, \epsilon_2)$, where ϵ_1 and ϵ_2 map K_1 to the endpoints of P_3. Any graph G is a spanning subgraph of $\Gamma_{\mathcal{T}_3}(G)$. Therefore any proper n-colouring of $\Gamma_{\mathcal{T}_3}(G)$ is a fortiori a proper n-colouring of G. Let $c : G \to K_n$ be a proper n-colouring. Then c is *not* a proper n-colouring of $\Gamma_{\mathcal{T}_3}(G)$ if and only if G contains a path of length three whose end vertices are identically coloured. The middle points of this path are then adjacent neighbours of a colour class. Therefore, the proper n-colourings of $\Gamma_{\mathcal{T}_3}(G)$ are precisely the proper n-colourings of G such that the neighbourhood of each colour class is an independent set.

Thus Theorem 1.4 states that there exists a homomorphism of $\Gamma_{\mathcal{T}_3}(G)$ to K_n if and only if there exists a homomorphism of G to some $U(n)$.

Definition 4.1. For a graph H, let $\Omega_{\mathcal{T}_3}(H)$ be the graph constructed as follows. The vertices of $\Omega_{\mathcal{T}_3}(H)$ are the couples (u, U) such that $u \in V(H)$ and $U \subseteq N_H(u)$, the neighbourhood of u in H. Two couples (u, U), (v, V) are joined by an edge of $\Omega_{\mathcal{T}_3}(H)$ if $u \in V$, $v \in U$, and every vertex in U is adjacent to every vertex in V.

Theorem 4.2 ([23])**.** *For any graphs G and H, there exists a homomorphism of $\Gamma_{\mathcal{T}_3}(G)$ to H if and only if there exists a homomorphism of G to $\Omega_{\mathcal{T}_3}(H)$.*

With $H = K_n$ and $U(n) = \Omega_{\mathcal{T}_3}(K_n)$, this is the statement of Theorem 1.4. The purpose of Gyárfás, Jensen and Stiebitz [7] was to answer affirmatively a question of Harvey and Murty by showing that there exists, for every n, a n-chromatic graph with "strongly independent colour classes", that is, a n-chromatic graph G_n such that $\Gamma_{\mathcal{T}_3}(G_n)$ is n-chromatic. By Theorem 1.4, such a graph exists if and only if $G_n = \Omega_{\mathcal{T}_3}(K_n)$ has this property. They prove that indeed $\chi(\Omega_{\mathcal{T}_3}(K_n)) = n$.

The purpose in [23] was to find multiplicative graphs.

Definition 4.3. A graph K is *multiplicative* if whenever a product $G \times H$ admits a homomorphism to K, one of the factors G or H admits a homomorphism to K.

For a long time, only K_2 and the odd cycles were known to be multiplicative. Adjoint functors help to find new multiplicative graphs from known ones: For any Pultr template $\mathcal{T}$, we have $\Gamma_{\mathcal{T}}(G \times H) \simeq \Gamma_{\mathcal{T}}(G) \times \Gamma_{\mathcal{T}}(H)$. Using this property, it is not hard to show that if $\Gamma_{\mathcal{T}}$ admits a right adjoint $\Omega_{\mathcal{T}}$, then for any multiplicative graph K, $\Omega_{\mathcal{T}}(K)$ is multiplicative. In the case of $\mathcal{T}_3$, we have $\Gamma_{\mathcal{T}_3}(\Omega_{\mathcal{T}_3}(G))$ homomorphically equivalent to G for any graph G, and this allows to prove the following.

Theorem 4.4 ([23])**.** *For any graph K, K is multiplicative if and only if $\Omega_{\mathcal{T}_3}(K)$ is multiplicative.*

For relatively prime positive integers m, n such that $2m \leq n$, the *circular complete graph* $K_{n/m}$ is the graph whose vertices are the elements of the cyclic group $\mathbb{Z}_n$, where u and v are joined by an edge if $u - v \in \{m, \ldots, n-m\}$. Note that $K_{(2m+1)/m}$ is the odd cycle C_{2m+1}, and for $n/m < 3$, $\Gamma_{\mathcal{T}_3}(K_{n/m}) \simeq K_{n/(3m-n)}$. It can be shown that for $n/m < 12/5$, $\Omega_{\mathcal{T}_3}(K_{n/(3m-n)})$ is homomorphically equivalent to $K_{n/m}$. Using these results, it was possible to show that the circular complete graphs $K_{n/m}$ with $n/m < 4$ are all multiplicative. For a while, it looked like the same method would yield many new discoveries of multiplicative graphs. None have yet been found, but the results of Hajiabolhassan and Taherkhani, which we present next, have exhibited more links between similar functors and circular complete graphs.

5. Odd powers and roots

In this section we present generalizations of the functors $\Lambda_{\mathcal{T}_3}, \Gamma_{\mathcal{T}_3}$ and $\Omega_{\mathcal{T}_3}$ studied by Hajiabolhassan and Taherkhani [8]. For an integer m, let P_m denote the path with m edges. For odd m, let $\mathcal{T}_m = (K_1, P_m, \epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 are the homomorphisms mapping K_1 to the endpoints of P_m. Then $\Lambda_{\mathcal{T}_m}(G)$ is the graph obtained from G by replacing each edge by a copy of P_m, that is, the m-subdivision $G^{1/m}$ of G. $\Gamma_{\mathcal{T}_m}(H)$ is the "m-th power of H", obtained from H by adding edges between pairs of vertices connected by a walk of length m. (In particular, $\Lambda_{\mathcal{T}_1}(G) = \Gamma_{\mathcal{T}_1}(G) = G$.) We now describe a right adjoint of $\Gamma_{\mathcal{T}_m}$:

Definition 5.1. For an odd integer $m = 2k+1$ and a graph H, let $\Omega_{\mathcal{T}_m}(H)$ be the graph constructed as follows. The vertices of $\Omega_{\mathcal{T}_m}(H)$ are the $(k+1)$-tuples $(u, U_1, \ldots, U_k)$ such that $u \in V(H)$, $U_1 \subseteq N_H(u)$, $U_i \subseteq V(H)$ and U_i is completely joined to U_{i-1} for $i = 2, \ldots, k$. Two k-tuples $(u, U_1, \ldots, U_k)$, $(v, V_1, \ldots, V_k)$ are joined by an edge of $\Omega_{\mathcal{T}_m}(G)$ if $u \in V_1$, $v \in U_1$, $U_{i-1} \subseteq V_i$ and $V_{i-1} \subseteq U_i$ for $i = 2, \ldots, k$, and U_k is completely joined to V_k. (Here, $N_H(u)$ is the set of all vertices of H adjacent to u, and two sets of vertices are called *completely joined* if every vertex in one is adjacent to every vertex in the other.)

Theorem 4.2 generalizes as follows.

Theorem 5.2 ([8]). *For two graphs G and H, there exists a homomorphism of $\Gamma_{\mathcal{T}_m}(G)$ to H if and only there exists a homomorphism of G to $\Omega_{\mathcal{T}_m}(H)$.*

For odd s and r, define $P_r^s(G) = \Gamma_{\mathcal{T}_s}(\Lambda_{\mathcal{T}_r}(G))$ and $R_s^r(H) = \Gamma_{\mathcal{T}_r}(\Omega_{\mathcal{T}_s}(H))$. There exists a homomorphism of $P_r^s(G) = \Gamma_{\mathcal{T}_s}(\Lambda_{\mathcal{T}_r}(G))$ to H if and only if there exists a homomorphism of $\Lambda_{\mathcal{T}_r}(G)$ to $\Omega_{\mathcal{T}_s}(H)$, that is, if and only if there exists a homomorphism of G to $\Gamma_{\mathcal{T}_r}(\Omega_{\mathcal{T}_s}(H)) = R_s^r(H)$. Thus P_r^s and R_s^r are right and left adjoint of each other, though they are not necessarily left, central or right functors associated to Pultr templates. These are "ordered" as follows.

Theorem 5.3 ([8]). *Let s, r, s', r' be odd integers such that $\frac{s}{r} \leq \frac{s'}{r'}$. Then for any graph G, $P_r^s(G)$ admits a homomorphism to $P_{r'}^{s'}(G)$ and $R_{s'}^{r'}(G)$ admits a homomorphism to $R_s^r(G)$.*

The *circular chromatic number* $\chi_c(G)$ of a graph G is the minimum value n/m such that G admits a homomorphism to the circular complete graph $K_{n/m}$. Note that for odd $s = 2i+1$, $\Omega_{\mathcal{T}_s}(K_3)$ is homomorphically equivalent to the $3s$-cycle $K_{(6i+3)/(3i+1)}$, and for $r = 2j+1$, $R_s^r(K_3) = \Gamma_{\mathcal{T}_r}(\Omega_{\mathcal{T}_s}(K_3))$ is homomorphically equivalent to $K_{(6i+3)/(3i+1-j)}$. Using the fact that P_r^s and R_s^r are right and left adjoint of each other, we get the following.

Theorem 5.4 ([8]). *For a graph G, $\chi_c(G)$ is the supremum of the values $(6i+3)/(3i+1-j)$ such that $P_{2j+1}^{2i+1}(G)$ is 3-colourable.*

6. Oriented paths as templates

In this and the following sections we change the setting and consider digraphs rather than undirected (symmetric) graphs. To this end, we need to modify slightly the definition of a Pultr template.

Definition 6.1. In the setting of digraphs, a *Pultr template* is a quadruple $\mathcal{T} = (P, Q, \epsilon_1, \epsilon_2)$, where P, Q are digraphs and ϵ_1, ϵ_2 are homomorphisms of P to Q.

Thus we no longer require the existence of a special automorphism q of Q, whose purpose was to ensure that $\Gamma_{\mathcal{T}}(G)$ would be an undirected graph for any undirected graph G.

To define the left Pultr functor and the central Pultr functor corresponding to a Pultr template $\mathcal{T}$, simply replace the word "edge(s)" with "arc(s)" in Definition 2.1 (ii), (iii).

In the rest of this section, we present an oriented analogue of the construction of $\Omega_{\mathcal{T}}$ and of Theorem 5.2.

Let Q be an orientation of a path and consider the Pultr template $\mathcal{T} = (K_1, Q, \epsilon_1, \epsilon_2)$, where ϵ_1, ϵ_2 are the homomorphisms mapping K_1 to the end-points of Q. Similarly to the situation in Section 5, $\Lambda_{\mathcal{T}}(G)$ is the digraph obtained from G by replacing each arc with a copy of the path Q. $\Gamma_{\mathcal{T}}(H)$ is the digraph on the same vertex set as H in which there is an arc from u to v if and only if there exists an oriented walk from u to v in H whose steps are oriented according to the orientations of the arcs of Q. The right adjoint of $\Gamma_{\mathcal{T}}$ is as follows:

Definition 6.2. Suppose Q has vertices $0, 1, \dots, m$. For a digraph H, let $\Omega_{\mathcal{T}}(H)$ be the following digraph: The vertices of $\Omega_{\mathcal{T}}(H)$ are all the $(m+1)$-tuples $(u, U_1, \dots, U_m)$ such that $u \in V(H)$ and $U_i \subseteq V(H)$ for $i = 1, 2, \dots, m$, with $u \Rrightarrow U_m$. There is an arc in $\Omega_{\mathcal{T}}(H)$ from $(u, U_1, \dots, U_m)$ to $(v, V_1, \dots, V_m)$ if and only if

- (1a) $u \in V_1$ if $0 \to 1$ in Q,
- (1b) $v \in U_1$ if $1 \to 0$ in Q; and
- (2) for each $i = 1, \dots, m-1$:
  - (a) $U_i \subseteq V_{i+1}$ if $i \to i+1$ in Q,
  - (b) $V_i \subseteq U_{i+1}$ if $i+1 \to i$ in Q.

(The notation $a \to b$ means that there is an arc from a to b and $u \Rrightarrow V$ means that there is an arc from u to every element of V.)

Theorem 6.3 ([4]). *For any two digraphs G and H, there exists a homomorphism of $\Gamma_{\mathcal{T}}(G)$ to H if and only if there exists a homomorphism of G to $\Omega_{\mathcal{T}}(H)$.*

With oriented paths, the structure and ordering of functors analogous to P_r^s and R_s^r (see Section 5) gets much more complex. Possible applications are currently unknown (see also Section 10).

In the next section, we consider Pultr functors for which, on the other hand, an abundance of applications can be found in the literature.

7. Shift graphs

Definition 7.1. Let H be a digraph. The *arc graph* of H is the digraph $\delta(H)$ whose vertices are the arcs of H and $(u, v) \to (x, y)$ in $\delta(H)$ if and only if $v = x$.

Observe that δ is the central Pultr functor given by the template $(\vec{P}_1, \vec{P}_2, \epsilon_1, \epsilon_2)$, where

$$\vec{P}_1 = 0 \to 1,$$
$$\vec{P}_2 = 0 \to 1 \to 2,$$
$$\epsilon_1 : i \mapsto i, \qquad \epsilon_2 : i \mapsto i + 1.$$

A convenient fact about arc graphs is that we know good bounds on the chromatic number of $\delta(H)$ in terms of the chromatic number of H (see [9, 20]). Given a proper k-colouring of $\delta(H)$, we can construct a proper 2^k-colouring of H: let the colour of a vertex u of H be the set of all colours used on the outgoing arcs from u in the proper k-colouring of $\delta(H)$. This shows that $\chi(\delta(H)) \geq \log \chi(H)$. (In fact, we have $\chi(\delta(H)) = \Theta(\log \chi(H))$.)

This fact was used by Poljak and Rödl [20] to discuss the possible boundedness of the "Poljak-Rödl function": Define $f : \mathbb{N} \to \mathbb{N}$ by letting $f(n)$ be the minimum possible chromatic number $\chi(G \times H)$ of a product of n-chromatic digraphs G and H. (Here, the chromatic number of a digraph is defined to be the chromatic number of its symmetrization.) It is not known whether f is bounded or unbounded, but Poljak and Rödl were able to limit the possible upper bounds on f. One element of the proof is the bound $\chi(\delta(G)) \geq \log \chi(G)$ and the other is the fact that δ, like all central Pultr functor, commutes with the categorial product, that is, the identity $\delta(G \times H) \simeq \delta(G) \times \delta(H)$. The best result in this direction is the following:

Theorem 7.2 ([19]). *Either $f(n) \leq 3$ for all n, or $\lim_{n\to\infty} f(n) = \infty$.*

Let g be the undirected analogue of f. Hedetniemi's conjecture states that $g(n) = n$ for all n. Using the results on directed graphs, it is possible to prove that g is either bounded above by 9 or unbounded (see [19]). In fact, g is unbounded if and only if f is unbounded (see [25]).

Properties of the arc graph construction are also used in the analysis of the "shift graphs" that are the folklore examples of graphs with large odd girth and large chromatic number.

Definition 7.3. Let n, k be positive integers, $k \geq 2$. The *directed shift graph* $R(n,k)$ is the digraph with vertex set

$$V(R(n,k)) = \{(u_1, \dots, u_k) \colon 1 \leq u_1 < u_2 < \cdots < u_k \leq n\}$$

where $(u_1, \dots, u_k) \to (v_1, \dots, v_k)$ if and only if $u_2 = v_1$, $u_3 = v_2$, $\dots$, $u_k = v_{k-1}$.

The *undirected shift graph* $R'(n,k)$ is the symmetrization of $R(n,k)$.

Theorem 7.4 ([17]). *Let $c, k \geq 2$ and put $n = 2^{2^{\cdot^{\cdot^{\cdot^{2^c}}}}}$, where the tower of powers has height k. Then the undirected shift graph $R'(n,k)$ has chromatic number at least c and odd girth at least $2k+1$.*

We will show that both properties of high chromatic number and high odd girth are related to properties of adjoint functors. First note that $R(n,k) = \Gamma_{\mathcal{T}_k}(\vec{T}_n)$ for the Pultr template $\mathcal{T}_k = (\vec{P}_k, \vec{P}_{k+1}, \epsilon_1, \epsilon_2)$ with

$$\vec{P}_k = 0 \to 1 \to \cdots \to k-1,$$
$$\vec{P}_{k+1} = 0 \to 1 \to \cdots \to k,$$
$$\epsilon_1 : i \mapsto i, \qquad \epsilon_2 : i \mapsto i+1.$$

At the same time, $R(n,2) = \delta(\vec{T}_n)$ and for $k \geq 3$ we have $R(n,k) \simeq \delta(R(n,k-1))$. Hence we can get all shift graphs by iterating the arc graph functor, starting with a transitive tournament. The bound $\chi(\delta(G)) \geq \log \chi(G)$ directly implies that $\chi(R(n,k)) \geq \log^{k-1} n$, where $\log^{k-1}$ means the binary logarithm iterated $k-1$ times.

Next we are going to show that the odd girth of the undirected shift graphs is large, namely the odd girth of $R'(n,k)$ is at least $2k+1$. Suppose that some odd cycle C admits a homomorphism to $R'(n,k)$. Then there exists an orientation $\vec{C}$ of C which admits a homomorphism to $R(n,k) \simeq \delta(R(n,k-1))$. Therefore there exists a homomorphism of $\delta_L(\vec{C})$ to $R(n,k-1)$, where δ_L is the left adjoint of δ. By construction, $\delta_L(\vec{C})$ contains an arc $u_0 \to u_1$ for every vertex u of $\vec{C}$, with u_1 identified to v_0 for every arc $u \to v$ of $\vec{C}$. Thus the number of vertices of $\delta_L(\vec{C})$ is the same as that of $\vec{C}$. Also, $\delta_L(\vec{C})$ is not bipartite since a homomorphism of $\delta_L(\vec{C})$ to K_2 would correspond to a homomorphism of $\vec{C}$ to $\delta(K_2) \simeq K_2$, which is impossible since C is an odd cycle. Therefore, $\delta_L(\vec{C})$ contains an odd cycle. However, $R(n,k)$ has no directed cycles (since the projection on the first coordinate is a homomorphism to a transitive tournament) hence $\vec{C}$ has at least one source u and one sink v. The two vertices u_0 and v_1 are then respectively a source and a sink in $\delta_L(\vec{C})$. Hence the odd girth of (the symmetrization of) $\delta_L(\vec{C})$ is smaller than that of C. Since the odd girth of $K_n = R(n,1)$ is 3, this implies that the odd girth of $R(n,k)$ is at least $2k+1$. (In fact, the odd girth $R(n,k)$ of is exactly $2k+1$, unless $n < 2k+1$.)

8. Pultr functors and tree duality

Definition 8.1. A set $\mathcal{F}$ of digraphs is a *complete set of obstructions* for a digraph H if for any digraph G there exists *no* homomorphism of G to H if and only if there exists a homomorphism of *some* $F \in \mathcal{F}$ to G. We also say that $(\mathcal{F}, H)$ is a *homomorphism duality.*

If H admits a finite complete set of obstructions, then we say that *H has finite duality*; in this case, by [14, 18], H admits a finite complete set of obstructions all of whose elements are oriented trees. Conversely, every finite set $\mathcal{F}$ of oriented trees is a complete set of obstructions for some digraph H. If H admits a (not necessarily finite) complete set of obstructions all of whose elements are oriented trees, we say that *H has tree duality.*

In [3] we proved that if H has tree duality, then so does its arc graph $\delta(H)$. Furthermore we gave an explicit description of a complete set of tree obstructions for $\delta(H)$, provided we are given a complete set of tree obstructions for H.

Definition 8.2. Let T be a tree. For every vertex u of T, let $F(u)$ be a tree that admits a homomorphism to $\vec{P}_1$; fix such a homomorphism $\phi_u : F(u) \to P_1$, so that for any arc (x, y) of $F(u)$ we have $\phi_u(x) = 0$, $\phi_u(y) = 1$. For each arc e of T, incident with u, fix a vertex $v(u, e)$ of $F(u)$ in such a way that if e is outgoing from u, then $\phi_u(v(u, e)) = 1$, and if e is incoming to u, then $\phi_u(v(u, e)) = 0$. Construct a tree S by taking all the trees $F(u)$ for all the vertices u of T, and by identifying the vertex $v(u, e)$ with $v(u', e)$ for every arc $e = (u, u')$ of T. Any tree S constructed from T by the above procedure is called a *sproink* of T.

Theorem 8.3 ([3]). *If $\mathcal{F}$ is a complete set of tree obstructions for some digraph H, then the set of all sproinks of all the trees in $\mathcal{F}$ is a complete set of tree obstructions for its arc graph $\delta(H)$.*

Example. Let $\vec{P}_k$ be the directed path with k arcs (that is, the path $0 \to 1 \to 2 \to \cdots \to k$) and let $\vec{T}_k$ be the transitive tournament on k vertices. By [16], $\{\vec{P}_k\}$ is a complete set of tree obstructions for $\vec{T}_k$. To get a complete set of obstructions for $\delta(\vec{T}_k)$, we can take just the *minimal* sproinks of $\vec{P}_k$ (minimal with respect to the ordering by existence of homomorphisms). In the minimal sproinks, $F(0)$ and $F(k)$ will each be the one-vertex graph K_1. All the other $F(u)$'s will be alternating paths ("zigzags") that will connect at their end-points. So all the minimal sproinks of $\vec{P}_k$ for $k \geq 3$ can be described by the regular expression

$$\uparrow(\uparrow(\downarrow\uparrow)^*)^{k-3}\uparrow.$$

Note that if $(\mathcal{F}, H)$ and $(\mathcal{F}', H')$ are dualities, then so is $(\mathcal{F} \cup \mathcal{F}', H \times H')$. Hence starting with graphs with finite duality, whose structure is rather well understood, and taking iterated arc graphs and products yields a fairly large class of graphs with tree duality as well as a complete set of obstructions for each of them. In particular, the knowledge of a complete set of obstructions of the directed shift graphs was used in [24] to prove the density of the lattices $K_n^{\mathcal{D}}$ of directed graph powers of K_n (under homomorphic equivalence).

There is nothing special about the arc graph, however. In fact, all central Pultr functors and all their right adjoints preserve tree duality:

Theorem 8.4 ([3, 4]). *Let $\mathcal{T}$ be a Pultr template and let H be a digraph with tree duality. Then $\Gamma_{\mathcal{T}}$ has tree duality. Furthermore, if there exists a digraph $\Omega_{\mathcal{T}}(H)$ such that, for any digraph G, $G \to \Omega_{\mathcal{T}}(H)$ iff $\Gamma_{\mathcal{T}}(G) \to H$, then $\Omega_{\mathcal{T}}(H)$ has tree duality.*

The tree obstructions for $\Omega_{\mathcal{T}}(H)$ have a neat description using the left adjoint $\Lambda_{\mathcal{T}}$. On the other hand, an explicit description of the tree obstructions for $\Gamma_{\mathcal{T}}(H)$ for a general Pultr template $\mathcal{T}$ is currently unknown. The knowledge of the obstructions in some special cases can have interesting applications, as we show next.

9. Circular Gallai–Roy theorem

The following well-known theorem is usually credited to Gallai [5] and Roy [22], even though it had independently been proved earlier by Vitaver [26] and Hasse [10].

Theorem 9.1 (Vitaver, Hasse, Roy, Gallai). *A graph G is k-colourable if and only if it admits an orientation $\vec{G}$ such that there is no homomorphism of $\vec{P}_k$ to $\vec{G}$.*

As we have already mentioned, $\{\vec{P}_k\}$ is a complete set of obstructions for $\vec{T}_k$, the transitive tournament on k vertices. That is, for any digraph $\vec{G}$ we have $\vec{P}_k \nrightarrow \vec{G}$ if and only if $\vec{G} \to \vec{T}_k$. Observe that G is k-colourable if and only if it admits an orientation $\vec{G}$ such that $\vec{G} \to \vec{T}_k$ and you get Theorem 9.1.

We are now interested in finding an analogous condition for circular colourability. Recall that for relatively prime integers m, n such that $2m \leq n$, the circular complete graph $K_{n/m}$ is the graph whose vertices are the elements of the cyclic group $\mathbb{Z}_n$, where u and v are joined by an edge if $u - v \in \{m, \ldots, n-m\}$; the circular chromatic number $\chi_c(G)$ of a graph G is the minimum value n/m such that G admits a homomorphism to the circular complete graph $K_{n/m}$.

Definition 9.2. Let $m \geq 1$ be an integer. The *m-th interleaved adjoint* of a digraph H is the digraph $\iota_m(H)$ whose vertices are all the m-tuples of vertices of H, and $(u_1, \ldots, u_m) \to (v_1, \ldots, v_m)$ in $\iota_m(H)$ if $u_i \to v_i$ in H for all $i = 1, \ldots, m$ and $v_i \to u_{i+1}$ in H for all $i = 1, \ldots, m-1$.

It turns out that ι_m is a central Pultr functor with the template $(P_m, Q_m, \epsilon_{m,1}, \epsilon_{m,2})$, where P_m has vertices $1, 2, \ldots, m$ and no arcs; Q_m has vertices $1_1, 1_2, 2_1, 2_2, \ldots, m_1, m_2$ and arcs $u_1 \to u_2$ for $u = 1, 2, \ldots, m$ and $u_2 \to (u+1)_1$ for $u = 1, 2, \ldots, m-1$, and $\epsilon_{m,1}(u) = u_1$, $\epsilon_{m,2}(u) = u_2$ for all $u = 1, 2, \ldots, m$.

Let λ_m be the left adjoint of ι_m. Hence in particular $\lambda_m(G) \to \vec{T}_n$ if and only if $G \to \iota_m(\vec{T}_n)$. By homomorphism duality, $\lambda_m(G) \to \vec{T}_n$ if and only if $\vec{P}_n \nrightarrow \lambda_m(G)$. Combining these two equivalences and using the explicit description of $\lambda_m(G)$ (given by Definition 2.1(ii)) we can get the following.

Proposition 9.3 ([2])**.** *Let $\mathcal{P}_{n,m-1}$ be the family of oriented paths obtained from the directed path $\vec{P}_n$ by reversing at most $m-1$ arcs. For a digraph G, there exists a homomorphism of G to $\iota_m(\vec{T}_n)$ if and only if there exists no homomorphism to G from any path in $\mathcal{P}_{n,m-1}$.*

A surprising connection between circular colourings and interleaved adjoints has been discovered by Yeh and Zhu [27].

Theorem 9.4 ([27])**.** *For integers m, n such that $n \geq 2m$, there exist homomorphisms both ways between $K_{n/m}$ and $B(n, m)$, the symmetrization of $\iota_m(\vec{T}_n)$.*

Thus, for an undirected graph G there exists a homomorphism of G to $K_{n/m}$ if and only if there exists an orientation $\vec{G}$ of G that admits a homomorphism to $\iota_m(\vec{T}_n)$. Hence we get the sought analogue of Theorem 9.1.

Theorem 9.5. *Let $\mathcal{P}_{n,m-1}$ be the family of oriented paths obtained from the directed path $\vec{P}_n$ by reversing at most $m-1$ arcs. A graph G has circular chromatic number at most n/m if and only if it admits an orientation $\vec{G}$ such that there exists no homomorphism to $\vec{G}$ from any path in $\mathcal{P}_{n,m-1}$.*

10. Open problems

Finally, we present several open problems hoping to stimulate interest in the topic. The problem we currently find quite intriguing and at the same time within reach is this:

Problem 10.1. For which Pultr templates $\mathcal{T}$ does the central Pultr functor $\Gamma_{\mathcal{T}}$ admit a right adjoint? This problem is open in both the directed and the undirected case, and has a different flavour in each.

The proof of Theorem 5.3 makes use of the fact that $\Gamma_{\mathcal{T}_m}(\Lambda_{\mathcal{T}_m}(G))$ is homomorphically equivalent to G for any graph G, for the path templates $\mathcal{T}_m$ of Section 5 with odd m. In fact, any G admits a homomorphism to $\Lambda(\Gamma(G))$ for any pair of adjoint functors Λ, Γ. So the important property of the path templates is that $\Gamma_{\mathcal{T}_m}(\Lambda_{\mathcal{T}_m}(G)) \to G$ for any graph G. This leads to the following question:

Problem 10.2. For what Pultr templates $\mathcal{T}$ does $\Gamma_{\mathcal{T}}(\Lambda_{\mathcal{T}}(G))$ admit a homomorphism to G for any G?

By Theorem 6.3, the central Pultr functor $\Gamma_{\mathcal{T}}$ admits a right adjoint $\Omega_{\mathcal{T}}$ for path templates $\mathcal{T}$ also in the setting of digraphs. Thus we may consider directed analogues of the functors P_r^s and R_s^r of Section 6. However, the ordering of the path templates is no longer linear, nor is the ordering of the corresponding P_r^s's and R_s^r's.

Problem 10.3. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4$ be Pultr templates for digraphs such that each P is K_1, each Q is an oriented path and each ϵ_1, ϵ_2 map K_1 to the end points of Q. Then by Theorem 6.3 there exists a right adjoint $\Omega_{\mathcal{T}_i}$ for each i. Define $P_i^j(G) = \Gamma_{\mathcal{T}_j}(\Lambda_{\mathcal{T}_i}(G))$; put $R_j^i(H) = \Gamma_{\mathcal{T}_i}(\Omega_{\mathcal{T}_j}(H))$ for $i, j \in \{1, 2, 3, 4\}$. Under what conditions on the templates do we get an analogue of Theorem 5.3?

The following problem is motivated by the chromatic properties of arc graphs, see Section 7.

Problem 10.4. Characterize Pultr templates $\mathcal{T}$ for which there exists an unbounded function $c : \mathbb{N} \to \mathbb{N}$ such that $\chi(\Gamma_{\mathcal{T}}(H)) \geq c(\chi(H))$ for any digraph H.

Finally, we would like to see a construction similar to the sproinks of Theorem 8.3, for arbitrary Pultr templates.

Problem 10.5. Describe a complete set of tree obstructions for $\Gamma_{\mathcal{T}}(H)$ in terms of the tree obstructions for H, for a general Pultr template $\mathcal{T}$.

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