Locally Constrained Graph Homomorphisms — Structure, Complexity, and Applications

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Abstract

A graph homomorphism is an edge preserving vertex mapping between two graphs. Locally constrained homomorphisms are those that behave well on the neighborhoods of vertices — if the neighborhood of any vertex of the source graph is mapped bijectively (injectively, surjectively) to the neighborhood of its image in the target graph, the homomorphism is called locally bijective (injective, surjective, respectively). We show that this view unifies issues studied before from different perspectives and under different names, such as graph covers, distance constrained graph labelings, or role assignments. Our survey provides an overview of applications, complexity results, related problems, and historical notes on locally constrained graph homomorphisms.

1 Introduction

Homomorphisms are standard mathematical transformations that preserve given algebraic structures. Stemming from classical algebraic approaches in the theory of groups, semigroups, modules, etc., the notion of homomorphisms (or shortly morphisms) became the cornerstone of the category theory, as well as the core ingredient of modern combinatorics, as a tool used for the study of relational structures.

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In this paper we focus on graph homomorphisms. A *graph* is a pair $G = (V_G, E_G)$ where V_G is the set of *vertices* of G and E_G is a set of pairs of vertices, referred to as the *edges* of G. We use a standard graph-theoretical notation. Namely, the edge containing vertices u and v is denoted by (u, v). In most cases we consider *undirected graphs*, and then (u, v) denotes the unordered pair (i.e., two-element set) $\{u, v\}$. When considering *directed graphs*, (u, v) means an ordered pair, and then we talk about a directed edge starting in vertex u and ending in vertex v. Graphs are typically displayed graphically — drawn — in the plane so that vertices are represented by points (or disks or boxes to allow better distinguishing them) and edges are drawn as simple curves joining their endvertices. Drawings are also convenient when displaying *multigraphs*, i.e., when multiple edges joining the same pair of vertices and loops (edges whose endvertices are identical) are allowed. In most of this paper we deal with *simple undirected graphs*, and any deviation to directed graphs or multigraphs is properly announced (and happens mainly in Section 3.5). The number of edges a vertex u is incident with is called the *degree* of the vertex (beware — loops are counted twice) and it is denoted by $\deg_G(u)$. In an undirected graph G, two vertices are called *adjacent* if $(u, v) \in E_G$. The set of vertices adjacent to u is called the *(open) neighborhood* of u and it is denoted by $N_G(u)$. A sequence of vertices such that any two consecutive ones are adjacent is called a *walk*. A walk is a *path* if it passes through every vertex at most once. A closed walk (i.e., the first and last vertices are also adjacent) which passes through every vertex at most once is called a *cycle*. When *loops* (i.e., edges with identical starting and ending vertices) are allowed, a loop counts as a cycle of length 1. A *tree* is a connected graph that contains no cycles. A *matching* is a disjoint union of edges.

We mostly consider finite graphs, i.e., graphs with finite vertex sets. We will especially announce when allowing infinite graphs, but often stress the finiteness requirement when we find this useful to avoid possible confusion.

Graph homomorphisms are mappings between sets of vertices of two graphs having the following property: If two vertices form an edge of the source graph then their images form an edge in the target graph (which may be the same graph). In the paper we mostly keep the symbols so that G stands for the source graph and H for the target one. Then a mapping $f : V_G \to V_H$ is a graph homomorphism if and only if $(f(u), f(v)) \in E_H$ for all pairs $(u, v) \in E_G$. We write $G \to H$ if a graph homomorphism from G to H exists.

The following two general questions related to graph homomorphisms are the leitmotifs of the recent monograph by Hell and Nešetřil [33]:

- the existence of a homomorphism between two graphs and the computational complexity of the associated decision problem (which could be parameterized by various restrictions on both graphs),
- the structure of the quasiorder determined by the existence of such homomorphisms.

In our survey we follow these two major approaches on more specific graph homomorphisms, namely on *locally constrained graph homomorphisms*. The introduction of the following definition is motivated by the fact that $f(N_G(u)) \subseteq N_H(f(u))$ holds for every vertex $u \in V_G$ whenever $f : V_G \longrightarrow V_H$ is a homomorphism from G to H (i.e., the image of the neighborhood of a vertex of the source graph is contained in the neighborhood of the image of this vertex in the target graph). Thus locally constrained graph homomorphisms are those that act well between the neighborhood of every vertex of the source graph and the neighborhood of its image. We distinguish three types of locally constrained homomorphisms:

Definition 1. A graph homomorphism $f : V_G \to V_H$ is called *locally bijective* if for every vertex u of G, the restriction of the mapping f to the domain $N_G(u)$ and range $N_H(f(u))$ is a bijection.

Analogously, a graph homomorphism is *locally injective* (*locally surjective*) if its restriction to any $N_G(u)$ and $N_H(f(u))$ is injective (surjective, respectively).

In accordance with the notation used for ordinary graph homomorphisms we write $G \xrightarrow{B} H$, $G \xrightarrow{I} H$ and $G \xrightarrow{S} H$ when a homomorphism with a particular local constraint exists. Examples of these three kinds of homomorphisms are depicted in Fig. 1.

Though the classification of locally constrained homomorphisms looks as a very natural extension of the theory of graph homomorphisms as a whole, it is quite interesting to note that all three kinds of them have been intensively studied under different names and in different connections. Our survey brings this unifying view, as well as an overview of applications, related problems, and historical notes.

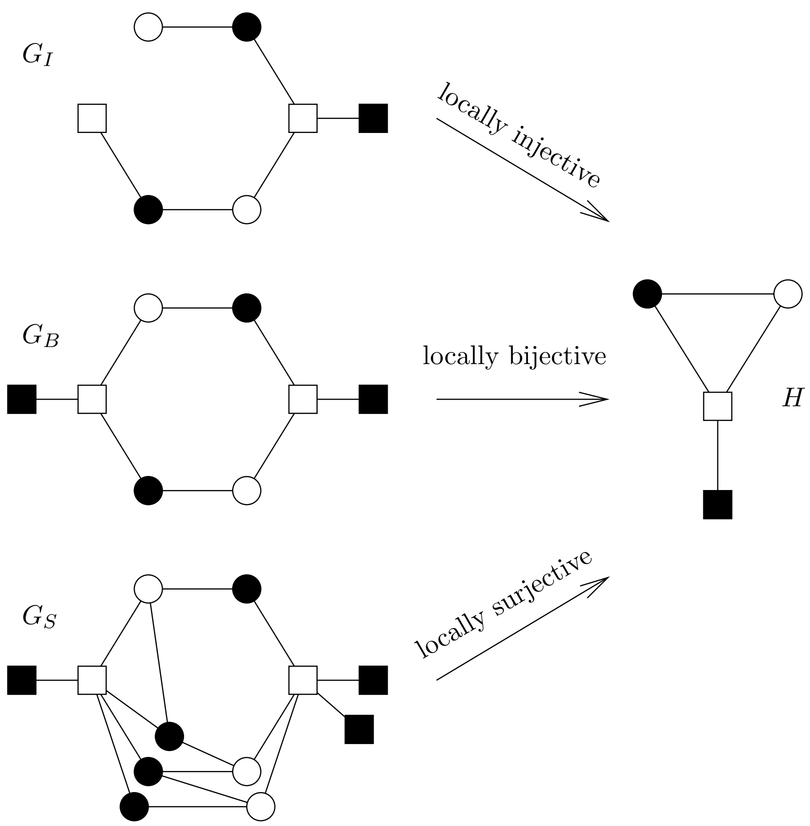


Figure 1: Examples of locally constrained homomorphisms. Homomorphisms are indicated by vertex shapes. E.g., all white circles on the lefthand side are mapped on the white circle on the righthand side, etc.

2 Genealogy of locally constrained homomorphisms

2.1 Locally bijective homomorphisms as graph covers

The concept of locally bijective homomorphisms is well established in combinatorial topology. Reidemeister in his classical monograph from 1932 [56, pages 109–114][1] shows several basic facts about locally bijective homomorphisms (called here *"Isomorphismus von Streckenkomplex $\mathfrak{C}$ zu Streckenkomplex $\mathfrak{C}^*$"*). Among others we find the following propositions.

Proposition 2 (Reidemeister [56])**.** *Every locally bijective homomorphism between two trees is an isomorphism.*

Proof. We proceed by a contradiction: If two vertices u and v of the source tree G had the same image in $f : G \xrightarrow{B} H$, then the image of the unique path

[1]See also a more accessible paper by Bodlaender [8].

from u to v would induce a closed walk in H, i.e. a cycle. This contradicts the assumption that H is a tree. Hence f is globally injective.

Now take any u from G and consider an arbitrary vertex x of H. Take preimage of the path from $f(u)$ to x is a path from u to a uniquely determined vertex v, which moreover satisfies that $f(v) = x$. Hence f is also globally surjective, i.e., a bijection between the vertex sets of G and H.

Now, every edge of G has a unique image under f, i.e., the mapping is an isomorphism. □

Note that it suffices to assume either that G is a tree, or that H is a tree and G is connected to obtain the same result. Note also that the result holds true even for infinite trees.

Proposition 3 (Reidemeister [56]). *If the graph H is connected, then every graph G allowing a locally bijective homomorphism to H has the property that the cardinality of V_G is a multiple of $|V_H|$.*

In this context it is often written that G is a *k-fold cover* of H, where k is the (integer) ratio $\frac{|V_G|}{|V_H|}$.

Proof. Let $f : G \xrightarrow{B} H$ and consider an arbitrary edge $e = xy$ of H. Then the preimage of e induces (in G) a matching between the sets of vertices that map onto x and those that map onto y. Since H is connected, we get that the preimage of any vertex has the same cardinality and the claim follows. □

Reidemeister gave also a construction of all k-fold covers of a fixed connected graph in terms of permutations over edges not belonging to a spanning tree of H. (This construction was independently rediscovered in 1977 by Gross and Tucker [31] in terms of permutation voltage assignments in a symmetric group of k elements.)

The construction is illustrated in Fig. 2. The preimage of the fixed spanning tree of H is a spanning forest of identical trees, each determining a single layer of G. Every edge e in H outside the spanning tree is assigned a direction, say xy, and a permutation π_{xy} from the permutation group S_k. The preimage of such e in G is the bipartite graph of π_{xy}, i.e., the graph where the i-th copy of x is joined to the $\pi(i)$-th copy of y.

Locally bijective homomorphisms are a natural discrete variant of the notion of covering spaces in topology. In topological terms, a covering projection is a mapping that is continuous and also bijective on a suitable small

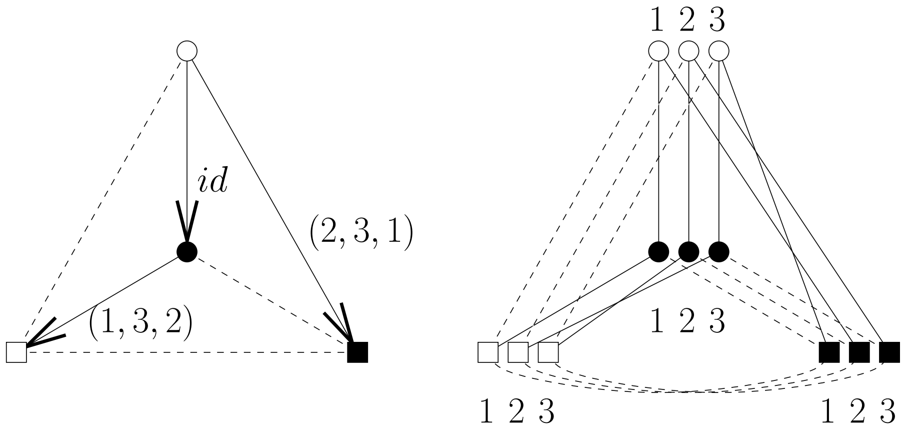


Figure 2: A construction of a 3-fold cover of the graph K_4. The spanning tree of K_4 and its preimage in G are indicated by dashed edges.

neighborhood of any point of the domain set. A prime example of a covering projection is the mapping that identifies the antipodal points of the two-dimensional sphere — the resulting space is the projective plane.

Intuitively, any graph can be viewed as the set of points of its drawing with no edge crossing. Observe that in such a point set the vertices of degree two are topologically irrelevant — they cannot be distinguished from the inner edge points, because their surroundings are topologically identical. In contrary, the other vertices can be easily identified in the point set as well as their degrees.

Moreover, any covering projection between point sets of two graphs without vertices of degree two translates straightforwardly to a locally bijective homomorphism between the same graphs — it is enough to restrict the mapping only onto vertices. In the opposite direction, any locally bijective homomorphism transforms to many covering projections — there are uncountably many ways to define the mapping along curves representing edges, even though these mappings are homeomorphically equivalent.

Boldi and Vigna [9] traced the first occurrence of locally bijective homomorphisms to Grothendieck [32], who — under the name of *graph fibration* — in late 1950's translated the notion of fibration in homotopy theory to categorical terms.

Sachs [58] established in 1964 the notion of *graph divisor*. In our current terminology $G \xrightarrow{B} H$ is equivalent with Sachs's definition of H being a divisor of G. The name was chosen to reflect the property of division of the associated characteristic polynomials:

Theorem 4 (Sachs [58]). *If H is a divisor of G, then the characteristic* *polynomial of H divides the characteristic polynomial of G.*

Idea of the proof. Recall that the characteristic polynomial of a graph G is the characteristic polynomial of its adjacency matrix A_G.

The idea is to transform an eigenvector $\mathbf{y}$ of $\mathbf{A}_H$ to an eigenvector $\mathbf{x}$ of $\mathbf{A}_G$ as follows: Let us denote the vertices of G by $u_1, \ldots u_n$ and those of H by $v_1, \ldots v_k$. If u_i maps onto v_j in $G \xrightarrow{B} H$, then we set $x_i = y_j$. The dependency between these eigenvectors of $\mathbf{A}_G$ is the same as between the original ones for $\mathbf{A}_H$, which implies the statement for characteristic polynomials. See e.g. a textbook of Godsil and Royle [29, page 197] for a formal proof. □

The concept of graph divisors was then intensively used in the study of characteristic polynomials of various graph classes, see e.g. a monograph of Cvetković, Doob and Sachs [14].

Another occurrence of the notion of locally bijective homomorphisms can be found in the monograph of Biggs [7, page 130]. Here he credits Conway for a construction of an infinite family of connected 5-transitive cubic graphs. In 1974 Djoković [15] extended this construction and obtained an infinite class of finite 4-regular 7-arc-transitive graphs. In the same year Gardiner [28] used a similar approach involving locally bijective homomorphisms to construct antipodal distance-regular graphs.

2.2 Negami's conjecture

An interesting role of locally bijective homomorphisms appeared in a classification of projective planar graphs. A graph G is called *projective planar* if it can be drawn in the projective plane with no edge-crossings. With a little help of the definition of the projective plane mentioned earlier we get an alternative description: projective planar graphs are the graphs that can be drawn in the plane in such a way that the only edge-crossings appear in a single disc and every pair of edges that pass through this disc cross exactly once. Such a disc is called a *crosscap*.

In 1988, Negami [53] observed that for every projective planar graph it is possible to construct a 2-fold cover which itself is a planar graph. He then conjectured that the corresponding two graph classes are equal:

Conjecture 5 (Negami [53])**.** *A finite graph H can be embedded in the projective plane if and only if it has a finite* planar cover*, i.e., if there exists a finite planar graph G such that $G \xrightarrow{B} H$.*

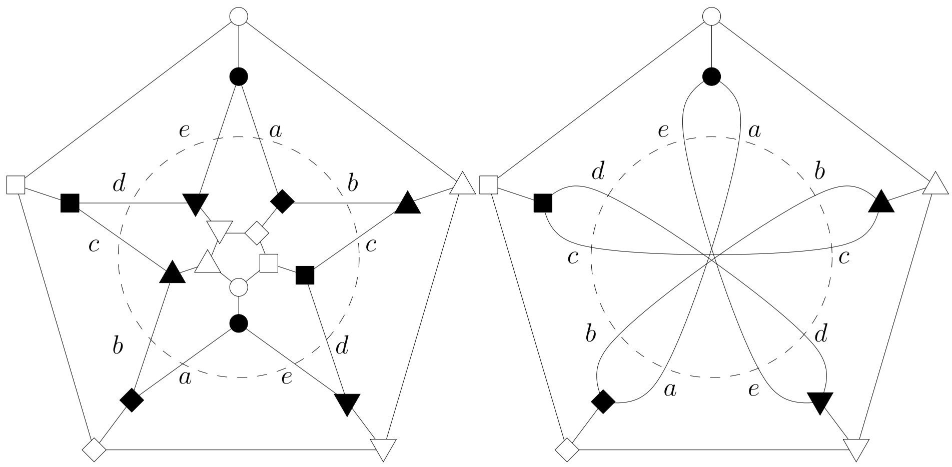


Figure 3: The icosahedron as a 2-fold cover of the Petersen graph. On the righthand side the dashed circle indicates the crosscap. On the lefthand side it marks the circle where the outer parts of two Petersen graphs are glued together.

For example, Mohar and Thomassen [52, page 201] pointed out that both the Petersen graph and K_6, the complete graph on six vertices, have planar covers. These are e.g. the icosahedron and the dodecahedron. The locally bijective homomorphism from the icosahedron to the Petersen graph is depicted in Fig. 3. Note that this particular homomorphism has the property that it identifies the antipodal pairs of vertices; an analogous mapping on the dodecahedron gives K_6. Hence, according to Negami's conjecture, the Petersen graph as well as K_6 should be projective planar, and they indeed are.

The inclusion

$$\{H \text{ is projective planar}\} \subseteq \{H \text{ has a finite planar cover}\}$$

is simple. It suffices to take two drawings of the projective planar graph, remove both crosscaps, invert one copy inside the disc of the crosscap and finally glue these two parts along the boundary of the crosscap as indicated in Fig. 3 for the Petersen graph.

The opposite inclusion seems difficult, and is still not settled. However, the theory of graph minors and minor closed graph classes gives a promising approach[2]. The class of projective planar graphs is minor closed, as any class

[2]A graph H is a minor of a graph G if an isomorphic copy of H can be obtained from a subgraph of G by a sequence of edge-contractions. A class of graphs is *minor closed* if it contains all minors of all of its graphs. In their Graph Minor project [57], Robertson and

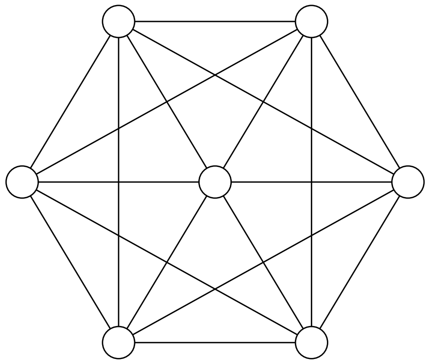


Figure 4: The unresolved forbidden graph $K_{1,2,2,2}$.

of graphs embeddable in a fixed surface. For the other class we get the same property:

Proposition 6. *The class of graphs with finite planar covers is minor closed.*

Proof. If H allows a finite k-fold planar cover $G \xrightarrow{B} H$, it is enough to consider whether this property is maintained for $H \circ e$, i.e., for the graph obtained from H by the contraction of some edge e.

In G we contract k edges in the preimage of e. If e was contained in no triangle, then the resulting graph G' is a planar cover of $H \circ e$.

In the other case, we identify all vertices z adjacent to both endpoints of the edge $e = xy$ in H and for each such z we proceed as follows: Observe that the preimage of the triangle x, y, z induces in G a disjoint union of cycles, and the length of every cycle is divisible by three. Consequently, in G' every third edge of these cycles was contracted, so now the cycles have even length. When we remove every second edge from the contracted cycles, we get a matching (a set of disjoint edges) of size k. By repeating this procedure for all z we obtain the desired finite planar cover of $H \circ e$. □

Hence to prove Negami's conjecture it is possible to involve the Robertson-Seymour theory of graph minors. Archdeacon showed in 1981 that the class of projective planar graphs can be characterized by 35 forbidden minors [4]. Hence it would be enough to show that none of these 35 forbidden graphs has a finite planar cover.

Following this idea, a joint effort of Negami [53], Fellows [18], and Archdeacon [5] led in 1989 to a proof that 33 of the forbidden minors for

Seymour proved the so called Wagner conjecture, stating that every minor closed class of graphs can be described by a finite number of non-isomorphic minor-minimal forbidden graphs.

the class of projective planar graphs have no planar cover (though the results of Archdeacon have only been published in 2002). Hence, it remained to verify this property for only two graphs, namely for $K_{4,4} - e$ and $K_{1,2,2,2}$.

Hliněný found an affirmative argument for $K_{4,4} - e$ in 1998 [34]. Since then, only a single forbidden minor — namely the graph $K_{1,2,2,2}$ — resists to be shown that it allows no planar cover. As the conjecture is not proved yet, Hliněný and Thomas [35] showed in 2004 that the conjecture can allow at most 16 possible counterexamples (upto obvious constructions).

If Negami's conjecture is proved, it will provide an alternative definition of projective planar graphs as those graphs that have a 2-fold planar cover.

2.3 Distributed networks and common covers

Locally bijective homomorphisms have several applications in computer science.

The first one appeared in early 1980's in the problem of recognizing uniform synchronous computer networks. In such networks, computers are placed on the nodes of the network and all are of the same type. It means that they execute the same algorithm in a synchronized way. Communication between computers is done along the edges of the network. The recognition problem asks, whether there exists an algorithm, that — if run on the nodes of the network — recognizes the topology of the network, i.e., determines the underlying graph.

Angluin [2] and also Angluin and Gardiner [3] observed that whenever $G \xrightarrow{B} H$, then these two graphs can not be recognized by such algorithms. The intuitive reason is simple — the state of each computer placed on a vertex $u \in V_G$ is the same as the state of the computer placed on $f(u)$ in H, where f is the assumed locally bijective homomorphism. In fact they proved in a slightly more general setting that classes of graphs closed under taking covers cannot be recognized with a finite fixed set of processor types. To prove the complete characterization of graphs which cannot be recognized in this way, Angluin and Gardiner conjectured that two graphs have a *finite common cover* if and only if they have the same *degree refinement matrix*, which will be discussed in more detail in the next section.

The conjecture of Angluin and Gardiner was proved by Leighton in 1982 in the affirmative way. As its statement requires further concepts we will state and discuss it in more detail later in Section 3.4.

In 1986 Mohar [51] adjusted Leighton's construction to classify the surfaces where the common covers can be embedded (depending on the surfaces hosting the underlying graphs).

Following the work of Angluin [2], Litovsky, Métivier and Zielonka [45] showed in 1993 that the families of series parallel graphs and planar graphs cannot be recognized by means of local computations. Courcelle and Métivier [13] proved in 1994 that the only nontrivial minor-closed graph classes that can be recognized by local computations are those that are formed from graphs with at most one cycle in each component.

Further models of local computations involving also locally injective and locally surjective homomorphisms were considered by Chalopin and Paulusma [10] in 2006.

2.4 Locally injective homomorphisms and the Frequency Assignment Problem

The other two kinds of local constraints have also interesting history and provide a wide spectrum of applications.

Nešetřil [54] showed already in 1971 that every locally injective mapping $G \xrightarrow{I} G$ of a connected finite graph G to itself is an isomorphism of G.

In his tutorial from 1983, Stallings [59] mentioned that every locally injective homomorphism $G \xrightarrow{I} H$ can be extended to a locally bijective homomorphism $G' \xrightarrow{B} H$ for G' being a supergraph of G.

Locally injective homomorphisms were applied in a hardness proof for the existence of distance constrained labelings of graphs [23], a notion stemming from a highly practical problem of interference-free frequency assignment for wireless networks. The classical concept of $L(p,q)$-labeling introduced by Roberts (according to Griggs and Yeh [30]) asks for an assignment $l : V_G \to \{0, 1, \dots\}$ of labels to the vertices of a given graph G such that:

- if u and v are adjacent then $|l(u) - l(v)| \geq 2$,
- if u and v share a common neighbor then $l(u) \neq l(v)$.

The difference between the smallest and the largest label used is the *span* of the labeling.

Observe that an $L(2,1)$-labeling of span k can be also viewed as a locally injective homomorphism to the complement of the path of length k, see Fig. 5 for an example. In particular, the second condition of the definition of

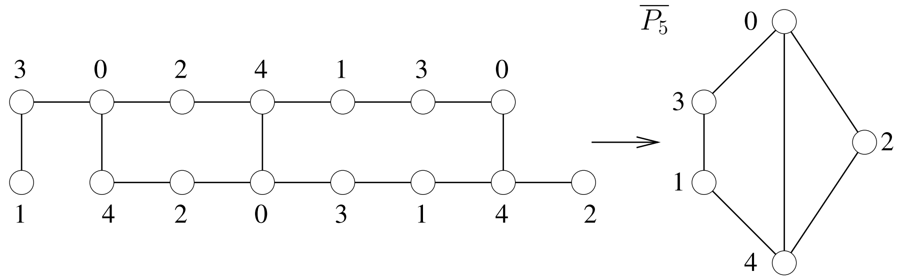


Figure 5: An $L(2,1)$-labeling of span five viewed as a locally injective homomorphism $G \xrightarrow{I} \overline{P_5}$.

the $L(2,1)$-labeling forces that the homomorphism is locally injective. This gives us an opportunity to explore this labeling as well as similar labelings (e.g., those with circular metric) in the framework of locally constrained homomorphisms.

This observation also leads to a natural generalization of $L(2,1)$-labelings. If the interference metric in the frequency space can be modeled by a graph, say H, then an $H(2,1)$-*labeling* of G is a mapping $f : V_G \to V_H$ such that $\mathrm{dist}_H(f(u), f(v)) \geq 2$ whenever $(u,v) \in E_G$, and $f(u) \neq f(v)$ whenever $\mathrm{dist}_G(u,v) = 2$ (here dist denotes the distance of vertices measured by the length of a shortest path between them). It is easy to see that an $H(2,1)$-labeling of a graph G is exactly a locally injective homomorphism from G to $\overline{H}$, the complement of H — if $f(u) \neq f(v)$, then requiring $\mathrm{dist}_H(f(u), f(v)) \geq 2$ is equivalent to saying that $(f(u), f(v)) \in E_{\overline{H}}$. Apart from the linear metric (i.e., $L(2,1)$-labelings), circular metric (which is equivalent to locally injective homomorphisms into complements of cycles) has been considered by Leese and Noble [42], and by Liu and Zhu [46].

2.5 Locally surjective homomorphisms and the Role Assignment Problem

Locally surjective homomorphisms were introduced by Everett and Borgatti [16], who called them *role colorings*. They originated in the theory of social behavior. The target graph H, called in this case the *role graph*, models roles and their relationships in a society. The source graph G represents relations between particular individuals of some group. The task is to assign roles to individuals so that each person given a particular role has, among its neighbors, every role prescribed by the role graph at least once, while no other roles may appear in the neighborhood.

For example take the role graph H as the path on three vertices with roles "producer", "reseller" and "consumer". We now ask whether these three roles can be assigned to the vertices of the source graph such that the neighbors of each producer and each consumer are only resellers and each reseller is adjacent to at least one producer and at least one consumer but to no reseller.

By nature of this motivation it is reasonable to consider also cases when the role graph contain some loops, while the source graph is usually assumed to be simple.

From the above description it is clear that role assignments are equivalent to locally surjective homomorphisms. In addition to classical problems of asking whether some role assignment exists (when the role graph is known), a related problem asks whether a given graph G admits a role assignment using at most k roles (i.e., the role graph has to be determined). Not very surprisingly both these problems are computationally difficult as it will be discussed later.

A wider class of role assignments and vertex partitions analogous to equitable partitions (e.g., for directed graphs) was considered in 1994 by Everett and Borgatti for further models in social network theory. A recent survey on results in this direction was given by Lerner [44] in 2005.

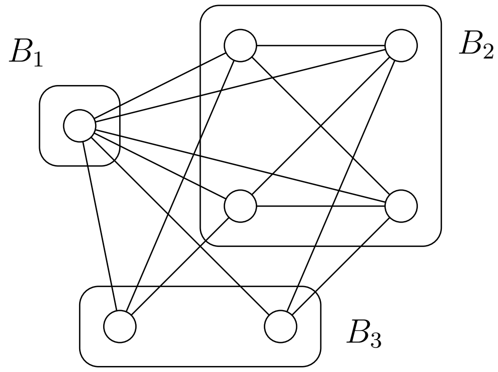
3 Structural aspects

3.1 Degree matrices

Other structural properties can be captured by equitable partitions and degree matrices mentioned already in the previous section.

Similarly as graph isomorphisms, any locally bijective homomorphism must maintain the degree of a vertex as well as the degrees of its neighbors, the degrees of neighbors of neighbors, etc. Hence, it is natural to look for an equivalence relation on the vertex set of a given graph such that the vertices in the same class cannot be distinguished by their degrees, the degrees of their neighbors, etc.

Definition 7. A partition of the vertex set of a graph G into disjoint classes is called an *equitable partition* if vertices in the same class have the same numbers of neighbors in all classes of the partition.



$$\mathbf{M} = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

Figure 6: An example of a minimal equitable partition and the corresponding degree refinement matrix.

Any equitable partition is characterized by the associated *degree matrix* whose rows and columns are indexed by the blocks of the partition, and the entry in the i-th row and j-th column describes how many neighbors a vertex from the i-th block has in the j-th block.

Every finite graph admits a unique minimal equitable partition. In this case a canonical ordering can be imposed on the blocks, so the corresponding degree matrix, called the *degree refinement matrix*, is also defined uniquely. See Fig. 6 for an example of a degree refinement matrix obtained from the minimal equitable partition of a graph.

Corneil introduced the notion of equitable partition in his PhD. thesis in 1968 [12, 11] as a heuristic for the graph isomorphism problem. It is worth mentioning that it was independently discovered by McKay [49] in 1976 in his master's thesis but with giving credits to Hopcroft's paper [36] from 1971 for the routine for minimizing the number of states of a finite automaton.

The notion of equitable partitions has soon become a folklore, and so it frequently appears without any reference [2, 43, 38, 29]. It was later implemented by McKay as a subroutine of a graph isomorphism software called Nauty [50, 48].

It can be decided efficiently whether a given matrix $\mathbf{M}$ is a degree matrix by solving a set of linear equations with variables describing the block sizes. If some entry satisfies $m_{i,j} \neq 0$ then also $m_{j,i} \neq 0$ and the fraction $\frac{m_{i,j}}{m_{j,i}}$ determines the ratio of the sizes of the i-th and the j-th blocks. It is enough to verify whether these conditions allow a nontrivial solution of block sizes. In the affirmative case a suitable graph can be constructed explicitly.

It is not hard to observe that a single graph may allow several equitable partitions yielding distinct degree matrices. In particular, any partition of

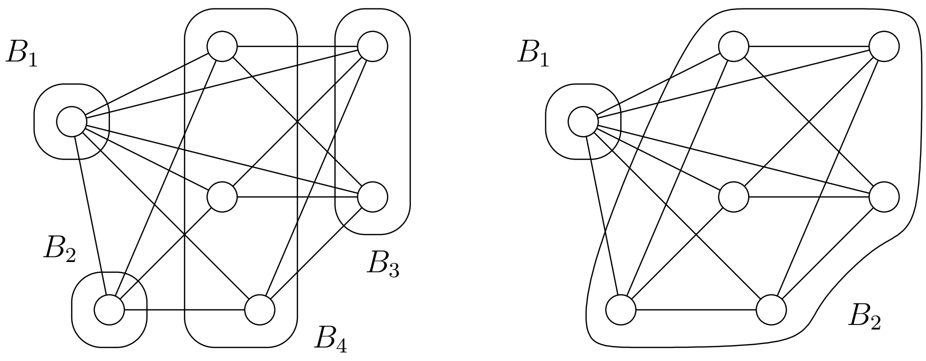


Figure 7: Two equitable partitions of the same graph, the left one is finer than the right one.

the complete graph is an equitable partition, hence complete graphs allow exponentially many degree matrices.

An equitable partition is said to be *finer* than another one when every class of the first partition is a subset of some class of the latter one. E.g., the partition into singletons is finer than any other equitable partition of the same graph.

According to our knowledge, the proof of the fact that the relation "being finer" defines a lattice on the set of all equitable partitions of a fixed graph was first published by Everett and Borgatti [16] in 1991. On the other hand, this has been already mentioned as 'well known' by McKay [50] in 1981. The proof follows a classical approach of forming intersection and union of algebraic structures.

The relation of being finer can be directly translated onto the class of degree matrices as follows:

Definition 8. For two degree matrices $\mathbf{M}$ of order k and $\mathbf{N}$ of order l we write $\mathbf{M} \to \mathbf{N}$, and say that $\mathbf{M}$ is *above* $\mathbf{N}$, if there exists a partition $R_1, \ldots, R_l$ of the index set $\{1, 2, \ldots, k\}$ such that

$$\sum_{j \in R_s} m_{i,j} = n_{r,s}$$

for every $i \in R_r$ and for every $s = 1, \ldots, l$.

The meaning is straightforward — vertices in blocks with indices in R_r cannot be distinguished in the new partition with fewer blocks.

For example, the degree matrices of the partitions from Fig. 7 can be compared as follows:

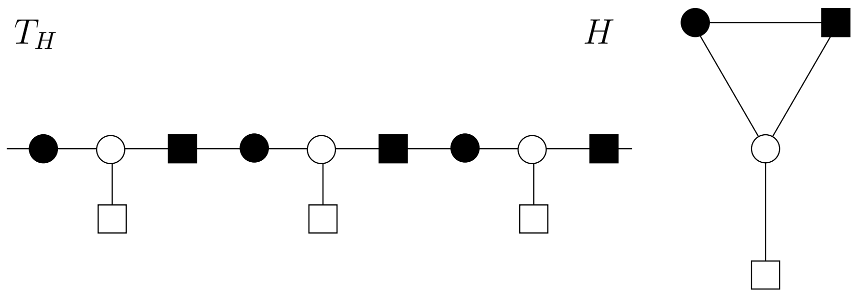


Figure 8: An example of the universal cover and a locally bijective homomorphism $T_H \xrightarrow{B} H$.

$$\mathbf{M} = \left(\begin{array}{c|ccc} 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow[\{1\},\{2,3,4\}]{R_1,R_2=} \begin{pmatrix} 0 & 6 \\ 1 & 3 \end{pmatrix} = \mathbf{N}$$

Besides its natural definition, this matrix order has several interesting properties closely tied to the existence of locally bijective homomorphisms:

- Every connected component of the matrix order $\to$ has exactly one minimum element, which is a degree refinement matrix.
- The lower ideal consisting of the matrices lying below the adjacency matrix of a graph H contains all degree matrices of H.
- Only adjacency matrices may occur above the adjacency matrix of a graph H.
- These matrices are the adjacency matrices of graphs G that allow locally bijective homomorphisms $G \xrightarrow{B} H$. In other words this *matrix order* contains as a suborder the *graph order* defined by existence of locally bijective homomorphisms.

3.2 Universal cover

The notion of *universal cover* is well established in topology of continuous spaces. In the discrete case, the universal cover of a graph H is the only (possibly infinite) tree T_H that allows a locally bijective homomorphism to H: $T_H \xrightarrow{B} H$. See Fig. 8 for an example.

The universal cover T_H can be constructed explicitly as follows: First, choose an arbitrary vertex $u_1 \in V_H$. For the vertex set of T_H take the set of all finite walks starting from u_1 that do not traverse the same edge in two consecutive steps. Two such walks form an edge in T_H of one is an extension of the other by a single edge. Observe that any different choice of the initial vertex u_1 provides an isomorphic tree, hence the universal cover is unique upto an isomorphism. Note further that the universal cover of a graph can be straightforwardly constructed from any of its degree matrices.

Obviously, the mapping that assigns every vertex of T_H the last vertex of the corresponding path in H, is a locally bijective homomorphism $T_H \xrightarrow{B} H$.

More can be seen when such homomorphisms act between different graphs. When a graph G admits a locally constrained homomorphism f into a connected graph H, we can lift this homomorphism to a homomorphism f' acting between the associated universal covers T_G and T_H: Simply the image of a walk is the walk formed by the images of the vertices of the original walk:

$$f'(u_1, \dots, u_k) := (f(u_1), \dots, f(u_k)).$$

Moreover, the local constraint of f is maintained by f' as well. As locally constrained homomorphisms between trees provide their inclusion, we get in particular that:

- if f is locally injective, then the tree T_G is a subtree of T_H;
- if f is locally surjective, then T_H is a subtree of T_G;
- if f is locally bijective, then T_G and T_H are isomorphic.

3.3 Cantor-Bernstein type theorem

With the help of universal covers one can derive interesting structural properties of locally constrained homomorphisms. For example, Fiala and Maxová proved in 2006 an analogue of the classical theorem from the set theory due to Cantor and Bernstein:

Theorem 9 (Fiala, Maxová [25]). *If a graph G admits a locally injective homomorphism f to a finite and connected graph H as well as a locally surjective homomorphism g to H, then all locally constrained homomorphisms between G and H are locally bijective.*

Note that this result is not just a direct consequence of Cantor-Bernstein theorem, since the two assumed locally constrained homomorphisms may act (injectively/surjectively) between a neighborhood in G but different neighborhoods in H.

Idea of the proof of Theorem 9. Let $\mathrm{diam}(H)$ be the diameter of the graph H, i.e., the maximum distance in H. Assume that the universal cover T_H is initiated in a vertex x chosen such that the first $\mathrm{diam}(H)+1$ levels of T_H contain as many vertices as possible. Denote by A the set of vertices of the first $\mathrm{diam}(H)+1$ levels of T_H. In the graph G, pick an arbitrary vertex u such that $g(u)=x$ and denote by B the vertices of the first $\mathrm{diam}(H)+1$ levels of T_G initiated in the vertex u. Finally let C contain the vertices of the first $\mathrm{diam}(H)+1$ levels of T_H initiated in $f(u)$.

We get that

$$|A| \leq |B| \leq |C| \leq |A|.$$

The first inequality follows from the fact that the derived mapping g' is surjective, the second one holds because f' is injective, and the last one by the choice of A.

Hence both f' and g' are bijections and consequently the original mappings f and g must be locally bijective. □

The statement can be weakened in the way that either f or g might be assumed to be locally bijective. The proof can be slightly adjusted so that it suffices to assume that the graphs G and H have the same universal cover or the same degree (refinement) matrix.

In particular, as a special case we get the following two theorems that play an important role in the classification of the computational complexity of locally constrained homomorphisms.

Theorem 10 (Fiala, Kratochvíl [21])**.** *If two graphs G and H share the same degree matrix, then every locally injective homomorphism is locally bijective.*

Theorem 11 (Kristiansen, Telle [41])**.** *If two graphs G and H share the same degree matrix, then every locally surjective homomorphism is locally bijective.*

3.4 Common covers

We have already mentioned the following theorem of Leighton:

Theorem 12 (Leighton [43])**.** *For connected finite graphs H and H', the following conditions are equivalent:*

1. *H and H' share a common finite cover,*
2. *H and H' have the same universal cover,*
3. *H and H' share a common (possibly infinite) cover,*
4. *H and H' have the same degree refinement matrix.*

Idea of the proof. The first three implications are straightforwardly obtained from the definitions:

$1 \Rightarrow 2$: If $G \xrightarrow{B} H$, then G and H have the same universal cover.

$2 \Rightarrow 3$: The universal cover is a particular choice of a cover.

$3 \Rightarrow 4$: If $G \xrightarrow{B} H$, then G and H have the same degree refinement matrix.

The only difficult implication is $4 \Rightarrow 1$ and it is, indeed, rather technical for the general case. For illustration we include here a proof for the case when the graphs H and H' are k-regular, i.e., both share the same degree refinement matrix $\mathbf{M}_G = \mathbf{M}_{G'} = (k)$ of order one.

Without loss of generality we may assume that both graphs are k-edge colorable, with the color set $\{1, 2, \ldots, k\}$. Otherwise we take the categorical product with a single edge which results in a k-regular bipartite graph $H \times K_2$ which is k-edge colorable. The projection to the first coordinate guarantees that $H \times K_2 \xrightarrow{B} H$.

For the vertex set of the common cover G we take the Cartesian product $V_H \times V_{H'}$ and for the edges we choose the pairs $((u, u'), (v, v'))$ such that the edges $(u, v) \in E_H$ and $(u', v') \in E_{H'}$ are of the same color. In other words, the edge set of G is the union of k matchings, where the i-th matching is obtained as the categorical product of the matchings of H and H' induced by the edges of color i.

The projection to the first coordinate witnesses that $G \xrightarrow{B} H$, while the projection to the second one implies $G \xrightarrow{B} H'$.

For graphs with more complex degree refinement matrices, an analogous construction for edges between pairs of blocks is presented in [43]. It requires some more effort to show that these parts can be combined together. □

3.5 Colored directed multigraphs

The notion of locally constrained homomorphisms can be extended to similar structures: directed graphs, multigraphs or even to hypergraphs. For the classification of locally bijective homomorphisms, the extension to colored directed multigraphs plays an important role. A *multigraph* is a graph which is allowed to have multiple edges and loops. In a directed one, some edges are directed, while some may be undirected. The 'colored' attribute means that vertices and edges come with an extra information about their color. Multigraphs still correspond to simplicial complexes, but for their covering projections we need an extra information about mapping of the edges. We formally define locally bijective homomorphisms as vertex mappings that preserve the colored degrees. Obviously, every covering projection gives rise to a locally bijective homomorphism. And it is a neat consequence of old and celebrated Petersen theorems that the converse is also true — every locally bijective homomorphism can be extended to a covering projection.

To be able to present the arguments in more details, we need to be more precise with the definitions.

Definition 13. A *colored directed multigraph* is a triple $G = (V_G, E_G \cup L_G \cup D_G), \phi_G, c_G$, where V_G is the set of vertices, E_G is the set of proper (undirected) edges, L_G is the set of (undirected) loops, D_G is the set of directed edges, $\phi_G : E_G \cup L_G \cup D_G \longrightarrow \binom{V_G}{2} \cup V_G \cup V_G \times V_G$ is the indicator function (such that $\phi_G(E_G) \subseteq \binom{V_G}{2}$, $\phi_G(L_G) \subseteq V_G$, and $\phi_G(D_G) \subseteq V_G \times V_G$), and $c_G : V_G \cup E_G \cup L_G \cup D_G \longrightarrow C$ is a coloring of vertices and edges, C being the set of colors.

The *colored degree* of a vertex u is

$$\deg_G^{b-}(u) = |\{e \in D_G : c_G(e) = b, \phi_G(e) = (u, \_\,)\}| \text{ and}$$

$$\deg_G^{b+}(u) = |\{e \in D_G : c_G(e) = b, \phi_G(e) = (\_\,, u)\}|$$

if b is a color of directed edges, and

$$\begin{aligned}\deg_G^{b}(u) = \; & |\{e \in E_G : c_G(e) = b, u \in \phi_G(e)\}| \\ & +2|\{e \in L_G : c_G(e) = b, \phi_G(e) = u\}|\end{aligned}$$

when b is a color of undirected edges and loops.

Note that directed loops are simply counted as directed edges with the same starting and ending vertices. However, undirected loops are distinguished, because they contribute by a different amount to the degrees of their vertices, but also because of different behavior with respect to covering projections (as defined below).

Definition 14. (Covering projection) Let G and H be colored directed multigraphs with the same set C of colors. A mapping $g : V_G \cup E_G \cup L_G \cup D_G \longrightarrow V_H \cup E_H \cup L_H \cup D_H$ is a *cdm-covering projection* if $g(V_G) \subseteq V_H$, $g(E_G) \subseteq E_H \cup L_H$, $g(L_G) \subseteq L_H$, $g(D_G) \subseteq D_H$, and the following conditions are fulfilled

- $c_G(x) = c_H(g(x))$ for all $x \in V_G \cup E_G \cup L_G \cup D_G$,
- $\phi_H(g(e)) = g(\phi_G(e))$ for all $e \in E_G \cup L_G \cup D_G$,
- the preimage of every proper edge $e \in E_H$ such that $\phi_H(e) = \{x, y\}$ is a matching containing all vertices of G that g maps to x or y,
- the preimage of every undirected loop $e \in L_H$ such that $\phi_H(e) = x$ is a disjoint union of cycles containing all vertices of G that g maps to x,
- the preimage of every directed non-loop edge $e \in D_H$ such that $\phi_H(e) = (x, y)$ is a matching containing all vertices of G that g maps to x or y, and
- the preimage of every directed loop $e \in D_H$ such that $\phi_H(e) = (x, x)$ is a disjoint union of directed cycles containing all vertices of G that g maps to x.

Definition 15. (Locally bijective homomorphism) Let G and H be colored directed multigraphs with the same set C of colors. A mapping $f : V_G \longrightarrow V_H$ is a *locally bijective homomorphism* if

- $c_G(u) = c_H(f(u))$ for all $u \in V_G$,
- $|\{e \in E_G : c_G(e) = b, \phi_G(e) = \{u, x\} \text{ with } f(x) = v\}| = |\{e \in E_H : c_H(e) = b, \phi_H(e) = \{f(u), v\}|$ for every $u \in V_G$, every $b \in C$ and all $v \in V_H, v \neq u$,
- $|\{e \in E_G : c_G(e) = b, \phi_G(e) = \{u, x\} \text{ with } f(x) = f(u)\}| + 2|\{e \in L_G : c_G(e) = b, \phi_G(e) = u\}| = 2|\{e \in L_H : c_H(e) = b, \phi_H(e) = f(u)|$ for every $u \in V_G$ and every $b \in C$,
- $|\{e \in D_G : c_G(e) = b, \phi_G(e) = (u, x) \text{ with } f(x) = v\}| = |\{e \in D_H : c_H(e) = b, \phi_H(e) = (f(u), v)|$ for every $u \in V_G$, every $b \in C$ and all $v \in V_H$, and

- $|\{e \in D_G : c_G(e) = b, \phi_G(e) = (x,u) \text{ with } f(x) = v\}| = |\{e \in D_H : c_H(e) = b, \phi_H(e) = (v, f(u))|$ for every $u \in V_G$, every $b \in C$ and all $v \in V_H$.

Obviously the restriction of a cdm-covering projection to the vertex sets of the multigraphs is a locally bijective homomorphism in the sense of the previous definition. This obvious necessary condition is also sufficient, as it is shown by the following theorem.

Theorem 16 (Kratochvíl, Proskurowski, Telle [37])**.** *Let G and H be colored directed multigraphs with the same set C of colors. Every locally bijective homomorphism $f : V_G \longrightarrow V_H$ can be extended to a cdm-covering projection of G to H.*

Proof. Suppose $f : V_G \longrightarrow V_H$ is a locally bijective homomorphism. We define $g(x) = f(x)$ for all vertices $x \in V_G$. We need to extend this definition to edges of G.

Let $e \in E_G$ be a proper edge of G such that $\phi_G(e) = \{u, x\}$ and $c_G(e) = b$. Let k be the number of edges of color b that connect $f(u)$ to $f(x)$ in H, and assume $f(u) \neq f(x)$. Consider the subgraph of G induced by the vertices that map onto $f(u)$ or $f(x)$, and by the edges of color b that connect them. By the assumption, this subgraph is k-regular and bipartite. It follows from König-Hall theorem (and was proved earlier by Petersen [55] in 1891) that the edges of this bipartite (multi)-graph can be properly colored by k colors, i.e., partitioned into k perfect matchings. Now we map all edges of one matching onto one of the k edges connecting $f(u)$ and $f(x)$ in H.

If $f(u) = f(x)$ (this covers also the case of $e \in L_G$ being a loop), the subgraph of G induced by the vertices that map onto $f(u)$, and by the edges of color b that connect them is not necessarily bipartite, but it is surely $2k$-regular. It follows from Petersen theorem [55] that this subgraph can be partitioned into 2-factors, i.e., into k collections of disjoint cycles, each collection covering all of its vertices. Again, map the edges of each collection to one loop of color b around the vertex $f(u)$ in H.

The arguments for directed edges are similar, even simpler. □

Let us now return to simple graphs. Observe that whenever $G \xrightarrow{B} H$, then every leaf (i.e., a vertex of degree one) has to be mapped only on a leaf in H. Hence, it is enough to encode in both graphs the numbers of leaf-neighbors of the vertices of higher degree and prune all leaves. This process can be iterated until there is no leaf in either of the graphs. At the end we get at each vertex u of G and H an extra information that encodes the structure of the tree that was attached to the rest of the graph in u and that was removed during the pruning procedure. We view this code as the color of the vertex. Clearly, only vertices with the same color can be mapped onto one another by a locally bijective homomorphism (cf. Theorem 2).

Obviously, locally bijective homomorphisms between the original graphs are in one-to-one correspondence with the color respecting locally bijective homomorphisms of the resulting colored graphs (upto automorphisms of the removed trees).

We can extend this reduction rule to vertices of degree two as follows: Any path of length k between vertices of degree at least three which contains only vertices of degree two must be mapped onto a path of the same length and the same color pattern under any locally bijective homomorphism. Hence each such path can be replaced by a single edge, with an extra information about the color pattern. Analogously to the case of vertices, this can be viewed as a color of the edge. Since the pattern may not be symmetric, the direction in which the pattern was encoded provides also the direction of the edge. As some pairs of vertices might be connected by several paths, multiple edges or loops may appear by this construction. (See Fig. 9.)

Every locally bijective homomorphism between the original graphs translates to a covering projection between the resulting multigraphs, in the sense of the definition above, and vice versa. But as we have already seen, the existence of a covering projection is equivalent to existence of a locally bijective homomorphism between the resulting colored directed multigraphs. Hence we have shown:

Proposition 17 (Kratochvíl, Proskurowski, Telle [37]). *Let G and H be simple graphs and let G' and H' be colored directed multigraphs obtained from G and H by the above described procedure. Then $G \xrightarrow{B} H$ if and only if $G' \xrightarrow{B} H'$.*

What we gained by the transformation to colored directed multigraphs is the fact that both G' and H' have minimum total degree greater than two. It is hoped that this fact may help avoid simple polynomially solvable cases and could allow a more concise description of the borderline between polynomially solvable and NP-complete instances of the problem of testing existence of locally bijective homomorphisms to fixed parameter graphs.

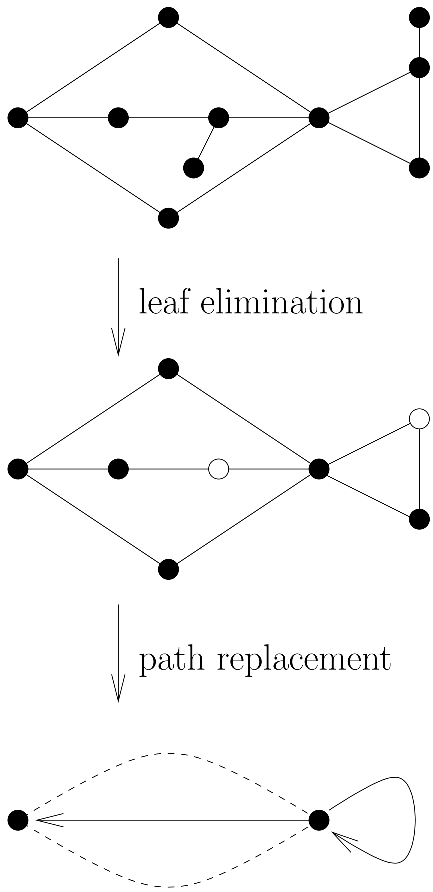


Figure 9: Conversion of a simple graph into a colored directed multigraph of minimum degree three.

It may seem that introduction of colors, multiedges, loops, and directed edges is a bit high price for a reduction to (multi)graphs of minimum degree at least three. However, the process can easily be reversed. For every colored directed multigraph H', one can construct a simple graph H so that the problems of deciding existence of locally bijective homomorphisms onto H (for simple graphs) and onto H' (for colored directed multigraphs) are polynomially equivalent.

3.6 Locally constrained homomorphisms as CSP

The recently intensively studied Constraint Satisfaction Problem is, in its general formulation, the question of existence of a homomorphism between two relational structures of the same type. A famous dichotomy conjecture of Feder and Vardi [17] states that for every fixed target structure (called usually the *template*), the CSP problem is either polynomially solvable or NP-complete. Similar dichotomy is hoped for in the case of locally constrained homomorphisms, and (mostly partial) results are presented in the next chapter. In this section we show that two of our three homomorphism types can be straightforwardly reformulated as CSP.

For locally injective homomorphisms, we can reduce as follows. Given two graphs G and H, we introduce two binary relations — E and D. We set E_H to be the edges of H (more precisely, for every edge (x, y) of H, we put both ordered pairs $(x, y), (y, x)$ in the template relation E_H), and E_G will be the edges of G. For the other relation, we put in D_H all pairs of distinct vertices of H, and in D_G all pairs of vertices of G that share a common neighbor. A relational homomorphism of these structures then directly corresponds to a locally injective homomorphism of the graphs — the E relation controls that edges are mapped onto edges (i.e., that we have a homomorphism), while the D relation controls the required local injectivity.

For locally bijective homomorphisms, we add d unary relations $B^1, \ldots, B^d$, where d is the maximum degree in H. Then B^i_H will contain all vertices of degree i in H, and B^i_G will contain all vertices of degree i in G. Adding these constraints will guarantee that the resulting homomorphism is degree-preserving, and hence locally bijective.

It is interesting to note that though also locally surjective homomorphisms can be polynomially reduced to CSP (CSP is NP-complete and existence of locally surjective homomorphisms is clearly in NP), a simple direct reduction is not obvious and in fact not known.

It is, however, important to realize that the presented reductions to CSP do not guarantee dichotomy of the computational complexity. And they would not guarantee it even if the Feder-Vardi conjecture for CSP were proved. Only the polynomial cases of CSP would translate, but if the resulting CSP problem is NP-complete, we cannot deduce anything for the original locally constrained homomorphism one. This is because the CSP problem may be hard on inputs that do not arise from the locally constrained homomorphism problem. And indeed, this is the case. The D relation encodes graph coloring, and so the CSP formulation becomes NP-complete as soon as H has at least three vertices. However, many larger graphs H allow polynomial time algorithms (both for locally bijective and for locally injective homomorphisms), as we will see in the next chapter. Somewhat surprisingly it is the case of locally surjective homomorphisms where dichotomy is known.

4 Computational complexity

4.1 Locally bijective homomorphisms

Bodlaender [8] proved in 1989 that every cover G of a connected graph H is a uniform emulation, that means that a parallel algorithm designed for the processor network G can be emulated on H where each node of H corresponds to a constant number of nodes of G. The same paper provided the complete characterization of covers of the ring, the grid, the cube, the cube connected cycles, the tree and the complete graphs. Moreover it is shown there that the decision problem whether a graph G covers a graph H is at least as hard as the graph isomorphism problem, even if the ratio $\frac{|V_G|}{|V_H|}$ is fixed.

In the concluding remarks Bodlaender asked the computational complexity of the following decision problem:

H-Locally Bijective Homomorphism	H-LBiHom
Parameter: A graph H.	
Instance: A graph G.	
Question: Does $G \xrightarrow{B} H$ hold?	

Abello, Fellows and Stillwell [1] showed in 1991 that there are both polynomially solvable and NP-complete cases. Since then a considerable effort was devoted to the attempts to fully characterize the computational complexity of H-LBiHom, but the goal is still not at sight. Only partial results have been achieved so far, but we find at least some of them interesting both for relative generality of the results and for the proof techniques. We will provide an overview of them in this chapter. Unless stated otherwise, we assume that all graphs are simple, i.e., without loops or multiple edges.

Proposition 18. *The H-LBiHom problem is solvable in polynomial time if the only degree partition of H is the partition into singletons.*

Proof. According to Theorem 12, the graphs G and H must have the same degree refinement matrix in order to have a chance for the existence of a locally bijective homomorphism, as this is a necessary condition. Hence we start by checking this condition.

The degree refinement matrix of the graph G can be constructed in time $O(|V_G|^4)$ from the minimum equitable partition obtained by the following iterative algorithm:

Algorithm 1: The minimum equitable partition

Input: A graph G.
Output: The minimum equitable partition $\mathcal{B}$ of G.
Initialize $\mathcal{B}_1 := (B_{1,1}) = (V_G)$, $k_1 := 1$, $t := 1$.
repeat
  For every $u \in V_G$, compute its degree vector $\mathbf{d}(u)$ defined by $\mathbf{d}(u)_i := |N(u) \cap B_{i,t}|$ for $i \in [1, k_t]$.
  Increment $t := t + 1$.
  Sort the degree vectors lexicographically and define the partition $\mathcal{B}_t$ such that each class $B_{i,t}$ consists of the vertices whose degree vector is the i-th in the sorted order.
  Set k_t to be the number of classes of $\mathcal{B}_t$.
until $k_t = k_{t-1}$;
return $\mathcal{B}_t$

On the other hand, the equality of the degree refinement matrices is also a sufficient condition. Any locally bijective homomorphism $f : G \xrightarrow{B} H$ maps vertices from the i-th block of the partition of G onto the vertices from the i-th block of H. Hence, the mapping f is uniquely defined when each block of H consists of only one vertex. In addition, the images of neighbors of any vertex are determined, hence this mapping f is a locally bijective homomorphism. □

The above proposition can be extended to the case when the target graph has at most two vertices in each class of the minimum equitable partition [38]. The existence of a locally bijective homomorphism can be expressed by a Boolean formula with at most two literals in each clause (the 2-SAT problem) which is well known to be solvable in linear time [6]. The transformation goes as follows:

For every block of H with two vertices we regard one of them as the true value and the other as false. For every vertex u in G that should be mapped onto such pair, we introduce a Boolean variable x_u.

If a block of G contains two vertices u and v connected by an edge or if these two vertices share a common neighbor, we introduce clauses $(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v)$.

Whenever blocks of size two in H are connected by a matching we introduce for every edge $(u, v) \in E_G$ between the corresponding two blocks a collection of clauses according to the following rule:

- $(x_u \vee \neg x_v) \wedge (\neg x_u \vee x_v)$ if the matching in H connects vertices of the same value (true–true and false–false),
- $(x_u \vee x_v) \wedge (\neg x_u \vee \neg x_v)$ if vertices of opposite values (true–false and false–true) are joined in H.

Observe that satisfying evaluations of the formula are in one-to-one correspondence with homomorphisms that are locally bijective: The value of the variable x_u determines which of the two possible vertices shall be used as the image of u, and the clauses encode both the homomorphism requirements and the local constraints.

We focus for the moment on the colored directed multigraphs defined in the previous section. If the resulting multigraph H consists of only a single vertex, the H-LBiHom problem becomes easily solvable by the same arguments as were used to prove Proposition 18. The situation becomes more interesting when H has two vertices. In such a case the complete characterization is known:

Theorem 19 (Kratochvíl, Proskurowski, Telle [37])**.** *If H is a colored directed multigraph on two vertices, then the H-LBiHom problem is* NP-*complete if and only if the following conditions are satisfied:*

- *the two vertices of H form a class of an equitable partition,*
- *there is a color c such that the two vertices are connected by an edge of color c and both have a loop of the same color, and if the edges of this color c are directed, then each of the vertices is incident with at least three outgoing edges of color c.*

In all other cases the H-LBiHom problem for colored directed multigraphs H on two vertices is solvable in polynomial time.

Proof. If the minimum equitable partition can distinguish the two vertices, the locally bijective homomorphism is obtained by Proposition 18.

Denote by u and v the two vertices of H. In the remaining polynomial cases every color c induces either

- only a multiedge between u and v or
- only two multiloops on u and v of the same multiplicity.

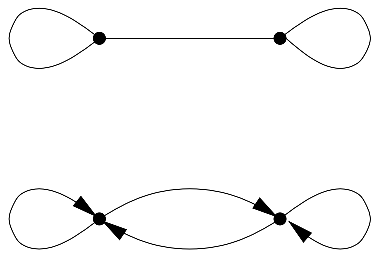


Figure 10: Undirected and directed multigraphs H that present the base cases in the characterization of Theorem 19. The top one is the smallest undirected multigraph defining an NP-complete instance, while the bottom one is largest connected monochromatic directed multigraph that defines a polynomially solvable variant of the H-LBiHom problem.

- a single directed loop on u and on v together with a pair of oppositely directed arcs between u and v (See Fig. 10).

Then the 2-SAT technique used above can be employed again. The two vertices of H encode the values true and false. Every edge of the first type will give rise to clauses that would ensure that the variables assigned to its vertices get different values, while every edge of the second type will give rise to clauses that would ensure the same value for the variables. For edges of the third type, each vertex of G has two incoming and two outgoing edges of this color. For the two in-neighbors, as well as for the two out-neighbors, we add clauses to ensure that the corresponding variables get different values.

For the NP-hardness part assume first that the color c induces in H unoriented edges and loops. Let k be the multiplicity of the edge between u and v and l the multiplicity of the two loops.

The idea is to reduce the following BW-(i, j) problem, which has been proven NP-complete by Kratochvíl, Proskurowski and Telle [39] for all pairs of positive integers i, j such that $i + j \geq 3$.

Black & White (i, j)-coloring	BW-i, j
Parameters: Integers i, j.	
Instance: An $(i + j)$-regular graph G.	
Question: Does G allow a coloring by two colors such that every vertex has i neighbors of the same color and j neighbors of the other color?	

Let G be an instance of the BW-$(2l, k)$ problem. We construct a colored directed multigraph $\widetilde{G}$ as follows: take two disjoint copies of G, call them G'

and G'', and give all their edges color c. In the sequel let x' and x'' denote the copies of the vertex $x \in V_G$ in G' and G'', respectively. Let $H^{\bar{c}}$ be the submultigraph of H arising by deleting all edges and loops of color c. Now for any $x \in V_G$ we also add into $\widetilde{G}$ a copy of $H^{\bar{c}}$ such that the two vertices of $H^{\bar{c}}$ are identified with x' and x''.

Assume first that a locally bijective homomorphism $f : \widetilde{G} \xrightarrow{B} H$ exists. Then the partition of $V_{G'}$ determined by $f^{-1}(u)$ and $f^{-1}(v)$ provides a valid black & white coloring of G as required by the BW-$(2l, k)$ problem.

In the other direction, fix a suitable black & white coloring of G. Define a mapping $f : \widetilde{G} \to H$ as follows: If $x \in V_G$ is white, set $f(x') = u$ and $f(x'') = v$. If x is black, set $f(x') = v$ and $f(x'') = u$. Straightforwardly, this f is a locally bijective homomorphism.

In the other case — when the edge color c is assigned to directed edges — the construction is analogous, but more technical.

First one proves that the H-LBiHomproblem is NP-complete for the following two graphs:

- H_1 — the graph consisting from one digon (two vertices joined by a pair of arcs directed in opposite ways) with two directed loops at each vertex, and
- H_2 — consisting from a double digon with a single directed loop at each vertex.

When H_i, $i = 1, 2$, is a proper subgraph of H, we take two copies of G_i — a graph that is difficult to decide the existence of a locally bijective homomorphism to H_i — and join every x' and x'' by $H \setminus H_1$ as in the undirected case. $\square$

A rich family of parameter graphs H that determine computationally difficult variants of the H-LBiHom problem is described by the following theorem:

Theorem 20 (Kratochvíl, Proskurowski, Telle [39], Fiala [19])**.** *The H-LBiHom problem is NP-complete for all simple k-regular graphs with $k \geq 3$.*

In fact this result disproved the conjecture of Abello et al. [1] stating that graphs H with trivial structure of the automorphism group (namely the *rigid* graphs, i.e., those who allow the identity to be the only automorphism) would provide polynomially solvable cases of the H-LBiHom problem.

The proof of the theorem itself required as a tool a construction of a graph that allows many locally bijective homomorphisms to H. A *multicover* of a regular graph H is a graph G with a distinguished vertex u, such that for any vertex $v \in V_H$, every bijection $f' : N_G(u) \to N_H(v)$ between the neighborhoods of u and v can be extended to a locally bijective homomorphism $f : G \xrightarrow{B} H$ satisfying $f(u) = v$. The construction of this multicover G involves an algebraic method that generalizes the construction of common covers used by Angluin and Gardiner [3] and Leighton [43].

The first paper of Kratochvíl, Proskurowski and Telle [39] provided the construction of multicovers for k-edge colorable and $\lceil \frac{k+2}{2} \rceil$-edge connected k-regular graphs H and the general NP-hardness reduction from a hypergraph coloring problem to show that for such graphs the H-LBiHomproblem is NP-complete. Fiala [19] later observed that by using a double bipartite cover of H (which is always k-edge colorable) the NP-hardness result can be extended to all k-regular graphs.

The series of papers by Kratochvíl et al. [39, 37, 40, 38] from late 1990's exhibits these and also other approaches to establishing the most accurate boundary between the graphs for which the H-LBiHom problem is polynomially solvable or NP-complete. Besides the cases presented above, several nontrivial infinite classes of both polynomial and NP-complete instances were recognized. However, currently there is no plausible conjecture concerning the characterization of graphs H for which the H-LBiHom problem is polynomially solvable (assuming, of course, $\mathsf{P} \neq \mathsf{NP}$).

4.2 The other two local constraints

Analogously to the locally bijective homomorphisms, the following two decision problems arise naturally:

H-Locally Injective Homomorphism	H-LInHom
Parameter: A graph H.	
Instance: A graph G.	
Question: Does $G \xrightarrow{I} H$ hold?	

H-Locally Surjective Homomorphism	H-LSurHom
Parameter: A graph H.	
Instance: A graph G.	
Question: Does $G \xrightarrow{S} H$ hold?	

Theorems 10 and 11 have an immediate consequence:

Corollary 21. *If for some graph H the problem H-*LBiHom *is* NP*-complete, then both problems H-*LInHom *and H-*LSurHom *are* NP*-complete as well.*

In the case of locally surjective homomorphisms the full classification implying dichotomy on the H-LSurHomproblems is known:

Theorem 22 (Fiala, Paulusma [26]). *For a connected simple graph H, the H-*LSurHom *problem is solvable in polynomial time if and only if H has at most two vertices. In all other cases it is* NP*-complete.*

We note here that the dichotomy can be extended to possibly disconnected graphs with loops on some vertices, but the description of the polynomial cases is less trivial [26].

Corollary 21 has an interesting computational complexity consequence for the Frequency Assignment Problem. As it has been shown in Section 2.4, a graph has a circular $C_{2,1}$-labeling of span k if and only if it allows a locally injective homomorphism into $\overline{C_k}$, the complement of the cycle of length k. This complement is a $(k-3)$-regular graph, and so for every $k \geq 6$, $\overline{C_k}$-LBiHom and hence also $\overline{C_k}$-LInHom are NP-complete (cf. Theorem 20).

4.3 Locally injective homomorphisms

No such simple characterization, as for the H-LSurHom problem, is known for the H-LInHom problem. The current knowledge is far from a full characterization, but the known results undoubtedly show that the H-LInHom problem provides a much more colorful tapestry of polynomial and NP-complete instances.

According to Corollary 21 every NP-complete H-LBiHom problem translates the NP-hardness to the companion H-LInHom problem. The converse is, however, not true. Hence it is natural to study first those graphs H that are known to determine polynomially solvable instances of H-LBiHom. Let us illustrate this on the following two examples.

For positive integers a, b, c, let $\Theta(a, b, c)$ be the graph consisting of two vertices of degree three joined by three paths of lengths a, b and c, respectively. If we reduce the vertices of degree two we get a loopless colored (undirected) multigraph on two vertices, hence the $\Theta(a, b, c)$-LBiHom problem is solvable in polynomial time by Theorem 19.

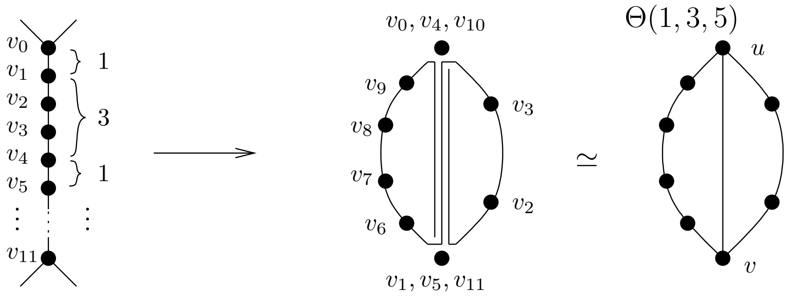


Figure 11: Example of a mapping of a path of length $m = 11$ into $\Theta(1,3,5)$ according to the pattern $1+3+1+5+1 = 11$.

On the other hand, the classification of the $\Theta(a,b,c)$-LInHom problems is surprisingly nontrivial. So far the dichotomy was obtained only in the case if at least two of the three parameters are the same:

Proposition 23 (Fiala, Kratochvíl [20]). *For integers a and b, $b \geq 2$, the $\Theta(a,b,b)$-*LInHom *problem is solvable in polynomial time if a and b are divisible by the same power of two. In all other cases the $\Theta(a,b,b)$-*LInHom *problem is* NP*-complete.*

Idea of the proof. We may assume that a and b are relatively prime, since the $\Theta(a,b,b)$-LInHom and $\Theta(ca,cb,cb)$-LInHom problems are polynomially equivalent for any positive integer c.

The first case is thus when both a and b are odd. Denote by u and v the two vertices of degree three of H.

Let G be an instance of the $\Theta(a,b,b)$-LInHom problem. Every vertex of degree three must be mapped either to u or to v, so this decision is one of the two key tasks to be resolved. Since $\Theta(a,b,b)$ is bipartite, so must be G and the answer for this task is (uniquely upto the automorphism of $\Theta(a,b,b)$) given by the bipartition of G.

Secondly, it is necessary to determine how the paths between the vertices of degree three in G will be mapped onto walks starting and ending in u or v. Let m be the length of such a path. If $m > ab$, than any mapping pattern starting or ending with a segment traversing along the a-path in $\Theta(a,b,b)$ or by a b-path is feasible, so the paths of such lengths provide no substantial constraints on the locally injective homomorphism.

However, all shorter paths have to be examined and for every length $m \leq ab$, it has to be decided whether there exists a locally injective homomorphism from the path to $\Theta(a,b,b)$ such that

- it starts and ends traversing along the a-path, or
- it starts along the a-path and ends along the b-path (then the opposite way is clearly possible as well), or
- it starts and ends along the b-paths.

Note that this is a finite problem as a and b are fixed.

Knowing this information we can reduce the task of path mapping to finding special kind of a factor graph. The factor will be a set of so called *flags*. Flags in this case are pairs $(w, e) \in V_{G'} \times E_{G'}$ where $w \in e$, and G' is the graph arising from G by contracting all vertices of degree two. The chosen flags represent those starting segments of paths in G that will be mapped onto the a-path in H.

There are additional requirements such as:

- Every vertex of degree three should be incident with one flag.
- The possibilities which flags of some edge could be chosen are determined by the mapping patterns derived from the length m of the path.

This problem of finding such flag factor translates to the question of finding a factor in a graph with degrees prescribed by intervals which was studied by Lovász [47, problem 7.19].

The NP-hardness of the $\Theta(a, b, b)$-LINHOMproblem when $\Theta(a, b, b)$ is not bipartite is obtained by a reduction from the BW(1,2) problem. Here, the preimages of the two vertices u and v in any locally injective homomorphism determine a feasible black & white coloring analogously to the arguments used in the proof of Theorem 19. $\square$

The next proposition summarizes the known NP-hard instances of the $\Theta(a, b, c)$-LINHOM problem.

Proposition 24 (Fiala, Kratochvíl, Pór [20, 24]). *For integers a, b and c, the $\Theta(a, b, c)$-LINHOM problem is NP-complete if one of the following conditions holds*

- $a = 1$, $b = 2$ *and* $c \geq 3$,
- $a = 1$, $b = 3$, $c \geq 4$ *and* c *is even,*
- $a + b$ *divides* c,

- *a, b and c are pairwise distinct and all three are odd.*

As an immediate consequence of the first case we get that the existence of an $L(2,1)$-labeling of span four is an NP-complete problem. It is enough to observe that $\overline{P_5} = \Theta(1,2,3)$ as is depicted in Fig. 5.

The NP-hardness in all cases is obtained by reductions from the BW(i,j) coloring problem. Edges of a graph whose coloring is demanded are replaced by paths of suitable length m which guarantees that every locally injective homomorphism from the derived graph to $\Theta(a,b,c)$ provides a valid coloring.

The suitable length m is described as a solution of a collection of linear equations with some uniqueness constraints, which is rather a number theoretic problem. In the last case of all parameters odd, the value of m was determined by further geometric arguments [24].

4.4 The list version of the H-LInHom problem

Finally let us focus on the list version of the H-LInHom problem. In this version, the instance of the problem consists of a graph G together with a set $L_u \subseteq V_H$ for each vertex $u \in V_G$. The sets L_u, called the *lists*, represent feasible images of the vertices. This is a natural setting from the Constraint Satisfaction Problem point of view since list represent unary relations on the domain V_G.

List H-Locally Injective Homomorphism $\mathcal{L}$-H-LInHom
Parameter: A graph H.
Instance: A graph G, lists $L_u \subseteq V_H$ for all $u \in V_G$.
Question: Does there exist $f : G \xrightarrow{I} H$ such that $f(u) \in L_u$ for every $u \in V_G$?

The $\mathcal{L}$-H-LInHom problem is, for every graph H, at least as difficult as the H-LInHom problem. It means that $\mathcal{L}$-H-LInHom is NP-complete for more graphs H. It is still a pleasant surprise that at least in this setting a full dichotomy was proved:

Theorem 25 (Fiala, Kratochvíl [22]). *The $\mathcal{L}$-H-LInHom problem is solvable in polynomial time if and only if the graph H has at most one cycle in each component of connectivity. In all other cases the $\mathcal{L}$-H-LInHom problem is NP-complete.*

It is of some interest that $\Theta(a, b, c)$ graphs were, together with other two graphs, the core cases of the NP-hardness proof. Also, the following interesting phenomenon plays an important role:

Lemma 26. *Let H be an induced subgraph of H'. If the $\mathcal{L}$-H-*LInHom *problem is* NP*-complete, then the $\mathcal{L}$-H'-*LInHom *problem is* NP*-complete as well.*

When H is an induced subgraph of H', it is easy to derive the result by assigning $L_u := V_H$ to every vertex $u \in V_G$.

It would be most desired to obtain an analogous result for the H-LBiHom and H-LInHom problems. In such setting it is necessary to restrict ourselves to those pairs of $H \subseteq H'$ where the coarsest equitable partition of H is finer or the same as the coarsest equitable partition of H'.

5 Conclusion

We hope that we have managed to convince the reader that locally constrained homomorphisms provide a realm of interesting interconnections and intriguing open questions. We would like to conclude by naming those open problems that we consider as most stimulating.

First of all, the conjecture of Negami on finite planar coverable graphs stands out as the main structural challenge. The connection to fundamental topological motivation as well as the names connected with the partial progress in its solution speak for themselves.

From the computational complexity point of view, the connection to Constraint Satisfaction Problem and the Dichotomy Conjecture of Feder and Vardi justify the hunt for the characterization of computational complexity of H-LInHom and H-LBiHom problems. The 'jungle' of complexity results on H-LInHom for Theta graphs is particularly annoying, including the fact that most known NP-completeness reductions are based on numerical characteristics of the path lengths, which – intuitively – should not play such an important role.

It came as a small suprise that the list version of H-LInHom allows such a simple characterization (and dichotomy). Again, one would like to see a similar result for the $\mathcal{L}$-H-LBiHom problem.

Last but not least, not enough attention was paid to planar variants of these problems. It is known that all generally NP-complete instances of $\mathcal{L}$-H-

LInHom remain NP-complete for planar inputs as well. It is also known that the K_4-LInHom problem is NP-complete for planar graphs. But these are isolated results and some unexpected surprises may show up if this direction of research is pursued.

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