Locally constrained homomorphisms on graphs of bounded degree and bounded treewidth*

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Abstract. A homomorphism from a graph G to a graph H is locally bijective, injective, or surjective if its restriction to the neighborhood of every vertex of G is bijective, injective, or surjective, respectively. We prove that the problems of testing whether a given graph G allows a homomorphism to a given graph H that is locally bijective, injective or surjective, respectively, are NP-complete even when G has bounded pathwidth or when both G and H are of bounded maximum degree. We complement these hardness results by showing that the three problems are polynomial-time solvable if G has bounded treewidth and in addition G or H has bounded maximum degree.

1 Introduction

A *graph homomorphism* from a graph $G = (V_G, E_G)$ to a graph $H = (V_H, E_H)$ is a mapping $\varphi : V_G \to V_H$ that maps adjacent vertices of G to adjacent vertices of H, i.e., $\varphi(u)\varphi(v) \in E_H$ whenever $uv \in E_G$. The notion of a graph homomorphism is well studied within the literature due to its many practical and theoretical applications; we refer to the textbook of Hell and Nešetřil [21] for a survey.

We write $G \to H$ to indicate the existence of a homomorphism from G to H. We call G the *guest graph* and H the *host graph*. We denote the vertices of H by $1, \ldots, |H|$ and call them *colors*. The reason for doing this is that graph homomorphisms generalize graph colorings: there exists a homomorphism from a graph G to a complete graph on k vertices if and only if G is k-colorable. The problem of testing whether $G \to H$ for two given graphs G and H is called the Hom problem. If only the guest graph is part of the input and the host graph is *fixed*, i.e., not part of the input, then this problem is denoted as H-Hom. The

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classical result in this area is the Hell-Nešetřil dichotomy theorem which states that H-Hom is solvable in polynomial time if H is bipartite, and NP-complete otherwise [20].

We consider so-called *locally constrained* homomorphisms. The *neighborhood* of a vertex u in a graph G is denoted $N_G(u) = \{v \in V_G \mid uv \in E_G\}$. If for every $u \in V_G$ the restriction of φ to the neighborhood of u, i.e., the mapping $\varphi_u : N_G(u) \to N_H(\varphi(u))$, is injective, bijective, or surjective, then φ is said to be *locally injective*, *locally bijective*, or *locally surjective*, respectively. Locally bijective homomorphisms are also called *graph coverings*. They originate from topological graph theory [3,27] and have applications in distributed computing [1,2,5] and in constructing highly transitive regular graphs [4]. Locally injective homomorphisms are also called *partial graph coverings*. They have applications in models of telecommunication [12] and in distance constrained labeling [13]. Moreover, they are used as indicators of the existence of homomorphisms of derivative graphs [28]. Locally surjective homomorphisms are also called *color dominations* [26]. In addition they are known as *role assignments* due to their applications in social science [9,29,30]. Just like locally bijective homomorphisms they also have applications in distributed computing [7].

If there exists a homomorphism from a graph G to a graph H that is locally bijective, locally injective, or locally surjective, respectively, then we write $G \xrightarrow{B} H$, $G \xrightarrow{I} H$, and $G \xrightarrow{S} H$, respectively. We denote the decision problems that are to test whether $G \xrightarrow{B} H$, $G \xrightarrow{I} H$, or $G \xrightarrow{S} H$ for two given graphs G and H by LBHom, LIHom and LSHom, respectively. Here, we always assume that guest graphs are *simple*, i.e., without multiple edges and self-loops. Host graphs do not contain multiple edges either. However, in order to describe situations in which vertices with the same color may be adjacent, host graphs can contain self-loops.

All three problems are known to be NP-complete when both guest and host graphs are given as input (see below for details), and attempts have been made to classify their computational complexity when only the guest graph belongs to the input and the host graph is fixed. The corresponding problems are denoted by H-LBHom, H-LIHom, and H-LSHom, respectively. Roberts and Sheng [30] classified the complexity of H-LSHom for all fixed pattern graphs H on at most two vertices. Their work was completed by Fiala and Paulusma [15], who proved the following dichotomy. The H-LSHom problem is polynomial-time solvable if one of the following three cases holds: either H has no edge, or one of its components consists of a single vertex incident with a self-loop, or H is simple and bipartite and has at least one component isomorphic to an edge. In all other cases the H-LSHom problem is NP-complete, even for the class of bipartite graphs [15]. The complexity classification of H-LBHom and H-LIHom is still open, although many partial results are known; we refer to the papers [12,25] and to the survey by Fiala and Kratochvíl [11] for both NP-complete and polynomially solvable cases.

Instead of fixing the host graph, another natural restriction is to only take guest graphs from a special graph class. Heggernes et al. [22] proved that LBHom is Graph Isomomorphism-complete when the guest graph is chordal, and polynomial-time solvable when the guest graph is interval. In contrast, LSHom is NP-complete when the guest graph is chordal and polynomial-time solvable when the guest graph is proper interval, whereas LIHom is NP-complete even for guest graphs that are proper interval [22]. It is also known that the problems LBHom and LSHom are polynomial-time solvable when the guest graph is a tree [16].

In this paper we focus on the following line of research. A graph F is called a *core* if there exists no homomorphism from F to any proper subgraph of F. Dalmau et al. [8] proved that the Hom problem is polynomial-time solvable when the guest graph belongs to any fixed graph class, the cores of which have bounded treewidth. In particular, this result implies that Hom is polynomial-time solvable when the guest graph has bounded treewidth. Grohe [18] strengthened the result of Dalmau et al. [8] by proving that under a certain complexity assumption (namely $\mathsf{FPT} \neq \mathsf{W}[1]$) the Hom problem can be solved in polynomial time if and only if this condition holds.

Our Contribution. We investigate whether the aforementioned results of Dalmau et al. [8] and Grohe [18] remain true when we consider locally constrained homomorphisms instead of general homomorphisms. In Section 4 we provide a negative answer to this question by showing that all three problems LBHom, LIHom and LSHom are already NP-complete when the guest graph has bounded pathwidth, and therefore bounded treewidth. We also show that the three problems are NP-complete if we bound the maximum degrees of both the guest graph and the host graph. On the positive side, in Section 3, we show that all three problems can be solved in polynomial time if we bound the treewidth of the guest graph and at the same time bound the maximum degree of the guest graph or the host graph. Because a graph class of bounded maximum degree has bounded treewidth if and only if it has bounded clique-width [19], all three problems are also polynomial-time solvable when we bound the clique-width and the maximum degree of the guest graph.

2 Preliminaries

Let G be a graph. The *degree* of a vertex v in G is denoted by $d_G(v) = |N_G(v)|$, and $\Delta(G) = \max_{v \in V_G} d_G(v)$ denotes the maximum degree of G. Let φ be a homomorphism from G to a graph H. Moreover, let G' be an induced subgraph of G, and let φ' be a homomorphism from G' to H. We say that φ *extends* (or, equivalently, is an *extension* of) φ' if $\varphi(v) = \varphi'(v)$ for every $v \in V_{G'}$.

A *tree decomposition* of G is a tree $T = (V_T, E_T)$, where the elements of V_T, called the *nodes* of T, are subsets of V_G such that the following three conditions are satisfied:

1. for each vertex $v \in V_G$, there is a node $X \in V_T$ with $v \in X$,
2. for each edge $uv \in E_G$, there is a node $X \in V_T$ with $\{u, v\} \subseteq X$,
3. for each vertex $v \in V_G$, the set of nodes $\{X \mid v \in X\}$ induces a connected subtree of T.

The *width* of a tree decomposition T is the size of a largest node X minus one. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G. A *path decomposition* of G is a tree decomposition T of G where T is a path. The *pathwidth* of G is the minimum width over all possible path decompositions of G. By definition, the pathwidth of G is at least as high as its treewidth. A tree decomposition T is *nice* [23] if T is a binary tree, rooted in a root R such that the nodes of T belong to one of the following four types:

1. a *leaf node* X is a leaf of T,
2. an *introduce node* X has one child Y and $X = Y \cup \{v\}$ for some vertex $v \in V_G \setminus Y$,
3. a *forget node* X has one child Y and $X = Y \setminus \{v\}$ for some vertex $v \in Y$,
4. a *join node* X has two children Y, Z satisfying $X = Y = Z$.

An *equitable partition* of a connected graph G is a partition of its vertex set in blocks $B_1, \ldots, B_k$ such that any vertex in B_i has the same number $m_{i,j}$ of neighbors in B_j. We call the matrix $M = (m_{i,j})$ corresponding to the coarsest equitable partition of G (in which the blocks are ordered in some canonical way; cf. [1]) the *degree refinement matrix* of G, denoted as $\mathrm{drm}(G)$.

We need the following lemma. A proof of the first statement in this lemma can be found in the paper of Fiala and Kratochvíl [12], whereas the second statement is due to Kristiansen and Telle [26].

Lemma 1. *Let G and H be two graphs. Then the following two statements hold:*

(i) if $G \xrightarrow{I} H$ and $\mathrm{drm}(G) = \mathrm{drm}(H)$, then $G \xrightarrow{B} H$
(ii) if $G \xrightarrow{S} H$ and $\mathrm{drm}(G) = \mathrm{drm}(H)$, then $G \xrightarrow{B} H$.

3 Polynomial-Time Results

In the case of locally injective and locally bijective homomorphisms, our polynomial-time result follows from general results on constraint satisfaction problems. In order to explain this, we need some additional terminology.

A relational structure $(A, R_1, \ldots, R_k)$ is a finite set A, called the *base set*, together with a collection of relations $R_1, \ldots, R_k$. The arities of these relations determine the vocabulary of the structure. A homomorphism between two relational structures of the same vocabulary is a mapping between the base sets such that all the relations are preserved.

Fiala and Kratochvíl [10] observed that locally injective and locally bijective homomorphisms between graphs can be expressed as homomorphisms between relational structures as follows. A locally injective homomorphism $f : G \to H$ can be expressed as a homomorphism between relational structures (V_G, E_G, E'_G) and (V_H, E_H, E'_H), where the new binary relation E' consists of pairs of distinct vertices that have at least one common neighbour. Since f maps distinct neighbors of a vertex v to distinct neighbors of $f(v)$, we get that f is a homomorphism of the associated relational structures. On the other hand, if (V_G, E_G, E'_G) and (V_H, E_H, E'_H) are constructed from G and H as described above, and if f is a homomorphism between them, then the relations E'_G and E'_H guarantee that no two vertices with a common neighbor in G are mapped to the same target in H. In other words, f is a locally injective homomorphism between the graphs G and H. An analogous construction works for locally bijective homomorphisms. Here, we need to express G using two binary relations E_G and E'_G as above, together with $\Delta(G) + 1$ unary relations. A unary relation can be viewed as a set: here, the i-th set will consists of all vertices of degree $i - 1$. These unary relations guarantee that degrees are preserved, and consequently that the associated graph homomorphisms are locally bijective.

The *Gaifman graph* $\mathcal{G}_{\mathcal{A}}$ of a relational structure $\mathcal{A} = (A, R_1, \ldots, R_k)$ is the graph with vertex set A, where distinct u and v are joined by an edge if they are bound by some relation. Formally $u, v \in E_{\mathcal{G}_{\mathcal{A}}}$ if and only if for some relation R_i of arity r and $(a_1, \ldots, a_r) \in R_i$ it holds that $\{u, v\} \subseteq \{a_1, \ldots .a_r\}$.

As a direct consequence of a result of Dalmau et al. [8], the existence of a homomorphism between two relational structures $\mathcal{A}$ and $\mathcal{B}$ can be decided in polynomial time if the treewidth of $\mathcal{G}_{\mathcal{A}}$ is bounded by a constant. This leads to Theorem 1 below. Since $G \xrightarrow{I} H$ implies that $\Delta(H) \geq \Delta(G)$, this theorem also holds if we bound the maximum degree of H instead of G.

Theorem 1. *The problems* LBHOM *and* LIHOM *can be solved in polynomial time on input pairs (G, H) where G has bounded treewidth and bounded degree.*

Proof. Observe that in both cases the Gaifman graph $\mathcal{G}_{\mathcal{A}} = G^2$, i.e., the graph arising from G by adding an edge between any two vertices at distance 2. It suffices to observe that $\mathrm{tw}(G^2) \leq \Delta(G)(\mathrm{tw}(G) + 1) - 1$, as we can transform any tree decomposition T of G of width $\mathrm{tw}(G)$ into a desired tree decomposition of G^2 by adding to each node X of T all the neighbors of every vertex from X. □

In Section 4 we show that if either one of the two conditions in Theorem 1 is dropped, i.e., if we allow either the treewidth or the maximum degree of G to be unbounded, then both problems, as well as LSHOM, become intractable.

To our knowledge, locally surjective homomorphisms have not yet been expressed as homomorphisms between relational structures. Hence, in the proof of Theorem 2 below, we present a polynomial-time algorithm for LSHOM when G has bounded treewidth and bounded degree. We first introduce some additional terminology.

Let φ be a locally surjective homomorphism from G to H. Let $v \in V_G$ and $p \in V_H$. If $\varphi(v) = p$, i.e., if φ maps vertex v to color p, then we say that p is *assigned* to v. By definition, for every vertex $v \in V_G$, the set of colors that are assigned to the neighbors of v in G is exactly the neighborhood of $\varphi(v)$ in H. Now suppose we are given a homomorphism φ' from an induced subgraph G' of G to H. For any vertex $v \in V_{G'}$, we say that v *misses* a color $p \in V_H$ if $p \in N_H(\varphi'(v)) \setminus \varphi(N_{G'}(v))$, i.e., if φ' does not assign p to any neighbor of v in G', but any locally surjective homomorphism φ from G to H that extends φ' assigns p to some neighbor of v.

Let T be a nice tree decomposition of G rooted in R. For every node $X \in V_T$, we define G_X to be the subgraph of G induced by the vertices of X together with the vertices of all the nodes that are descendants of X. In particular, we have $G_R = G$.

Definition 1. *Let $X \in V_T$, and let $c : X \to V_H$ and $\mu : X \to 2^{V_H}$ be two mappings. The pair (c, μ) is* feasible *for G_X if there exists a homomorphism φ from G_X to H satisfying the following three conditions:*

(i) $c(v) = \varphi(v)$ *for every* $v \in X$;
(ii) $\mu(v) = N_H(\varphi(v)) \setminus \varphi(N_{G_X}(v))$ *for every* $v \in X$;
(iii) $\varphi(N_G(v)) = N_H(\varphi(v))$ *for every* $v \in V_{G_X} \setminus X$.

In other words, a pair (c, μ) consists of a coloring c of the vertices of X, together with a collection of sets $\mu(v)$, one for each $v \in X$, consisting of exactly those colors that v misses. Informally speaking, a pair (c, μ) is feasible for G_X if there is a homomorphism $\varphi : G_X \to H$ such that φ "agrees" with the coloring c on the set X, and such that none of the vertices in $V_{G_X} \setminus X$ misses any color. The idea is that if a pair (c, μ) is feasible, then such a homomorphism φ might have an extension φ^* that is a locally surjective homomorphism from G to H. After all, for any vertex $v \in X$ that misses a color when considering φ, this color might be assigned by φ^* to a neighbor of v in the set $V_G \setminus V_{G_X}$.

We now prove a result for LSHom similar to Theorem 1. which provides a polynomial time bound when G has bounded treewidth and H has bounded maximum degree. Since $G \xrightarrow{S} H$ implies that $\Delta(G) \geq \Delta(H)$, our polynomial-time result below applies also if we bound the maximum degree of G instead of H.

Theorem 2. *The problem* LSHom *can be solved in polynomial time on input pairs* (G, H) *where* G *has bounded treewidth and* H *has bounded degree.*

Proof. Let G be the guest graph and H the host graph. We may assume without loss of generality that both G and H are connected, as otherwise we just consider all pairs (G_i, H_j) separately, where G_i is a connected component of G and H_j is a connected component of H. Because G has bounded treewidth, we can compute a tree decomposition of G of width $\text{tw}(G)$ in linear time using Bodlaender's algorithm [6]. We transform this tree decomposition into a nice tree decomposition T of G with width $\text{tw}(G)$ with at most $4|V_G|$ nodes using the linear-time algorithm of Kloks [23]. Let R be the root of T and let $k = \text{tw}(G) + 1$.

For each node $X \in V_T$, let F_X be the set of all feasible pairs (c, μ) for G_X. For every feasible pair $(c, \mu) \in F_X$ and every $v \in X$, it holds that $\mu(v)$ is a subset of $N_H(c(v))$. Since $|X| \leq k$ and $|N_H(c(v))| \leq \Delta(H)k$ for every $v \in X$ and every mapping $c : X \to V_H$, this implies that $|F_X| \leq |V_H|^k 2^{\Delta(H)k}$ for each $X \in V_T$. As we assumed that both k and $\Delta(H)$ are bounded by a constant, the set F_X is of polynomial size with respect to $|V_H|$.

The algorithm considers the nodes of T in a bottom-up manner, starting with the leaves of T and processing a node $X \in V_T$ only after its children have been processed. For every node X, the algorithm computes the set F_X in the way described below. We distinguish between four different cases. The correctness of each of the cases easily follows from the definition of a locally surjective homomorphism and Definition 1.

1. *X is a leaf node of T.* We consider all mappings $c : X \to V_H$. For each mapping c, we check whether c is a homomorphism from G_X to H. If not, then we discard c, as it can not belong to a feasible pair due to condition (i) in Definition 1. For each mapping c that is not discarded, we compute the unique mapping μ satisfying $\mu(v) = N_H(c(v)) \setminus c(N_{G_X}(v))$ for each $v \in X$, and we add the pair (c, μ) to F_X. It follows from condition (ii) that the obtained set F_X indeed contains all feasible pairs for G_X. As there is no vertex in $V_{G_X} \setminus X$, every pair (c, μ) trivially satisfies condition (iii). The computation of F_X can be done in $O(|V_H|^k k(\Delta(H) + k))$ time in this case.
2. *X is a forget node.* Let Y be the child of X in T, and let $\{u\} = Y \setminus X$. Observe that $(c, \mu) \in F_X$ if and only if there exists a feasible pair $(c', \mu') \in F_Y$ such

that $c(v) = c'(v)$ and $\mu(v) = \mu'(v)$ for every $v \in X$, and $\mu'(u) = \emptyset$. Hence we examine each $(c', \mu') \in F_Y$ and check whether $\mu'(u) = \emptyset$ is satisfied. If so, we first restrict (c', μ') on X to get (c, μ) and then we insert the obtained feasible pair into F_X. This procedure needs $O(|F_Y| k \Delta(H))$ time in total.

3. *X is an introduce node.* Let Y be the child of X in T, and let $\{u\} = X \setminus Y$. Observe that $(c, \mu) \in F_X$ if and only if there exists a feasible pair $(c', \mu') \in F_Y$ such that, for every $v \in Y$, it holds that $c(v) = c'(v)$, $\mu(v) = \mu'(v) \setminus c(u)$ if $uv \in E_G$, and $\mu(v) = \mu'(v)$ if $uv \notin E_G$. Hence, for each $(c', \mu') \in F_Y$, we consider all $|V_H|$ mappings $c : X \to V_H$ that extend c'. For each such extension c, we test whether c is a homomorphism from G_X to H by checking the adjacencies of $c(u)$ in H. If not, then we may safely discard c due to condition (i) in Definition 1. Otherwise, we compute the unique mapping $\mu : X \to 2^{V_H}$ satisfying

$$\mu(v) = \begin{cases} N_H(c(u)) \setminus c(N_{G_X}(u)) & \text{if } v = u \\ \mu'(v) \setminus c(u) & \text{if } v \neq u \text{ and } uv \in E_G \\ \mu'(v) & \text{if } v \neq u \text{ and } uv \notin E_G \ , \end{cases}$$

and we add the pair (c, μ) to F_X; due to condition (ii), this pair (c, μ) is the unique feasible pair containing c. Computing the set F_X takes at most $O(|F_Y||V_H| k \Delta(H))$ time in total.

4. *X is a join node.* Let Y and Z be the two children of X in T. Observe that $(c, \mu) \in F_X$ if and only if there exist feasible pairs $(c_1, \mu_1) \in F_Y$ and $(c_2, \mu_2) \in F_Z$ such that, for every $v \in X$, $c(v) = c_1(v) = c_2(v)$ and $\mu(v) = \mu_1(v) \cap \mu_2(v)$. Hence the algorithm considers every combination of $(c_1, \mu_1) \in F_Y$ with $(c_2, \mu_2) \in F_Z$ and if they agree on the first component c, the other component μ is determined uniquely by taking the intersection of $\mu_1(v)$ and $\mu_2(v)$ for every $v \in X$. This procedure computes the set F_X in $O(|F_Y||F_Z| k \Delta(H))$ time in total.

Finally, observe that a locally surjective homomorphism from G to H exists if and only if there exists a feasible pair (c, μ) for G_R such that $\mu(v) = \emptyset$ for all $v \in R$. Since T has at most $4|V_G|$ nodes, we obtain a total running time of $O(|V_G|(|V_H|^k 2^{\Delta(H)k})^2 k \Delta(H))$. As we assumed that both $k = \mathrm{tw}(G) + 1$ and $\Delta(H)$ are bounded by a constant, our algorithm runs in polynomial time. □

Note that Theorem 1 can be derived by solving LIHom using a dynamic programming approach that strongly resembles the one for LSHom described in the proof of Theorem 2, together with the fact that (G, H) is a yes-instance of LBHom if and only if it is a yes-instance for both LIHom and LSHom. In a dynamic programming algorithm for solving LIHom, instead of keeping track of sets $\mu(v)$ of colors that a vertex $v \in X$ is missing, we keep track of sets $\alpha(v)$ of colors that have already been assigned to the neighbors of a vertex $v \in X$. This is because in a locally injective homomorphism from G to H, no color may be assigned to more than one neighbor of any vertex. Using this observation, it is straightforward to adjust Definition 1 in such a way that it works for locally injective instead of locally surjective homomorphisms. We omit further details, but we expect that a dynamic programming algorithm of this kind will have smaller hidden constants in the running time estimate than the more general method of Dalmau et al. [8].

4 NP-Completeness Results

For the NP-hardness results in Theorem 3 below we use a reduction from the 3-Partition problem. This problem takes as input a multiset A of $3m$ integers, denoted in the sequel by $\{a_1, a_2, \ldots, a_{3m}\}$, and a positive integer b, such that $\frac{b}{4} < a_i < \frac{b}{2}$ for all $i \in \{1, \ldots, 3m\}$ and $\sum_{1 \le i \le 3m} a_i = mb$. The task is to determine whether A can be partitioned into m disjoint sets $A_1, \ldots, A_m$ such that $\sum_{a \in A_i} a = b$ for all $i \in \{1, \ldots, m\}$. Note that the restrictions on the size of each element in A implies that each set A_i in the desired partition must contain exactly three elements, which is why such a partition $A_1, \ldots, A_m$ is called a 3-*partition* of A. The 3-Partition problem is strongly NP-complete [17], i.e., it remains NP-complete even if the problem is encoded in unary.

Theorem 3. (★)[1] *The following three statements hold:*

(i) LBHom *is* NP-*complete on input pairs* (G, H) *where* G *has pathwidth* 5;
(ii) LSHom *is* NP-*complete on input pairs* (G, H) *where* G *has pathwidth* 4;
(iii) LIHom *is* NP-*complete on input pairs* (G, H) *where* G *has pathwidth* 2.

Proof. First note that all three problems are in NP. We prove each statement separately. In fact we will strengthen these three statements by showing that they hold even when in addition the pathwidth of H is 3, 3 and 2, respectively.

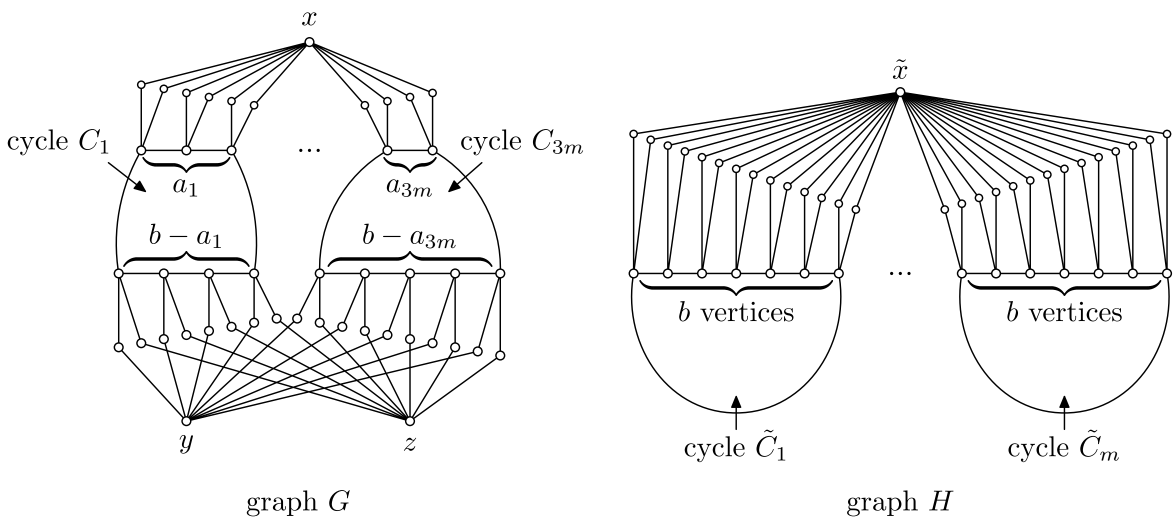

Fig. 1. A schematic illustration of the graphs G and H that are constructed from a given instance (A, b) of 3-Partition in the proof of statement (i) in Theorem 3. See also Figure 2 for a more detailed illustration of the "leftmost" part of G and the "rightmost" part of H, including more labels.

We only prove statement (i) here; the proofs of statements (ii) and (iii) can be found in the appendix. Given an instance (A, b) of 3-Partition we construct two graphs G and H as follows; see Figures 1 and 2 for some helpful illustrations.

[1] Proofs of results marked with an asterisk have been moved, in whole or in part, to the appendix due to page restrictions.

The construction of G starts by taking $3m$ disjoint cycles $C_1, \ldots, C_{3m}$ of length b, one for each element of A. For each $i \in \{1, \ldots, 3m\}$, the vertices of C_i are labeled $u_1^i, \ldots, u_b^i$ and we add, for each $j \in \{1, \ldots, b\}$, two new vertices p_j^i and q_j^i as well as two new edges $u_j^i p_j^i$ and $u_j^i q_j^i$. We then add three new vertices x, y and z. Vertex x is made adjacent to vertices $p_1^i, p_2^i \ldots, p_{a_i}^i$ and $q_1^i, q_2^i \ldots, q_{a_i}^i$ for every $i \in \{1, \ldots, 3m\}$. Finally, the vertex y is made adjacent to every vertex p_j^i that is not adjacent to x, and the vertex z is made adjacent to every vertex q_j^i that is not adjacent to x. This finishes the construction of G.

To construct H, we take m disjoint cycles $\tilde{C}_1, \ldots, \tilde{C}_m$ of length b, where the vertices of each cycle $\tilde{C}_i$ are labeled $\tilde{u}_1^i, \ldots, \tilde{u}_b^i$. For each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, b\}$, we add two vertices $\tilde{p}_j^i$ and $\tilde{q}_j^i$ and make both of them adjacent to $\tilde{u}_j^i$. Finally, we add a vertex $\tilde{x}$ and make it adjacent to each of the vertices $\tilde{p}_j^i$ and $\tilde{q}_j^i$. This finishes the construction of H.

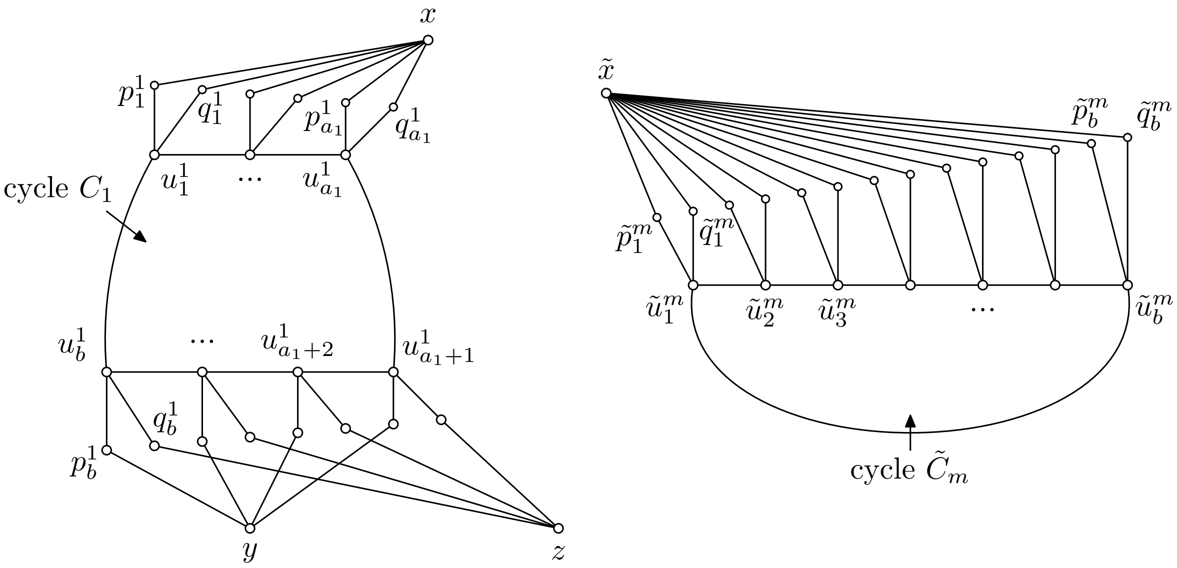

Fig. 2. More detailed illustration of parts of the graphs G and H in Figure 1.

We now show that there exists a locally bijective homomorphism from G to H if and only if (A, b) is a yes-instance of 3-Partition. Let us first assume that there exists a locally bijective homomorphism φ from G to H. Since φ is a degree-preserving mapping, we must have $\varphi(x) = \tilde{x}$. Moreover, since φ is locally bijective, the restriction of φ to $N_G(x)$ is a bijection from $N_G(x)$ to $N_H(\tilde{x})$. Again using the definition of a locally bijective mapping, this time considering the neighborhoods of the vertices in $N_H(\tilde{x})$, we deduce that there is a bijection from the set $N_G^2(x) := \{u_j^i \mid 1 \leq i \leq 3m, 1 \leq j \leq a_i\}$, i.e., from the set of vertices in G at distance 2 from x, to the set $N_H^2(\tilde{x}) := \{\tilde{u}_j^k \mid 1 \leq k \leq m, 1 \leq j \leq b\}$ of vertices that are at distance 2 from $\tilde{x}$ in H. For every $k \in \{1, \ldots, m\}$, we define a set $A_k \subseteq A$ such that A_k contains element $a_i \in A$ if and only if $\varphi(u_1^i) \in \{\tilde{u}_1^k, \ldots, \tilde{u}_b^k\}$. Since φ is a bijection from $N_G^2(x)$ to $N_H^2(\tilde{x})$, the sets $A_1, \ldots, A_m$ are disjoint; moreover each element $a_i \in A$ is contained in exactly one of them. Observe that the subgraph of G induced by $N_G^2(x)$ is a disjoint union of $3m$ paths of lengths $a_1, a_2, \ldots, a_{3m}$, respectively, while the subgraph of H induced by $N_H^2(\tilde{x})$ is a disjoint union of m

cycles of length b each. The fact that φ is a homomorphism and therefore never maps adjacent vertices of G to non-adjacent vertices in H implies that $\sum_{a \in A_i} a = b$ for all $i \in \{1, \ldots, m\}$. Hence $A_1, \ldots, A_m$ is a 3-partition of A.

For the reverse direction, suppose there exists a 3-partition $A_1, \ldots, A_m$ of A. We define a mapping φ as follows. We first set $\varphi(x) = \varphi(y) = \varphi(z) = \tilde{x}$. Let $A_i = \{a_r, a_s, a_t\}$ be any set of the 3-partition. We map the vertices of the cycles C_r, C_s, C_t that are at distance 2 from x to the vertices of the cycle $\tilde{C}_i$ in the following way: $\varphi(u^r_j) = \tilde{u}^i_j$ for each $j \in \{1, \ldots, a_r\}$, $\varphi(u^s_j) = \tilde{u}^i_{a_r+j}$ for each $j \in \{1, \ldots, a_s\}$, and $\varphi(u^t_j) = \tilde{u}^i_{a_r+a_s+j}$ for each $j \in \{1, \ldots, a_t\}$. The vertices of C_r, C_s and C_t that are at distance more than 2 from x in G are mapped to vertices of $\tilde{C}_i$ such that the vertices of C_r, C_s and C_t appear in the same order as their images on $\tilde{C}_i$. In particular, we set $\varphi(u^r_j) = \tilde{u}^i_j$ for each $j \in \{a_r + 1, \ldots, b\}$; the vertices of the cycles C_s and C_t that are at distance more than 2 from x are mapped to vertices of $\tilde{C}_i$ analogously. After the vertices of the cycles $C_1, \ldots, C_{3m}$ have been mapped in the way described above, it remains to map the vertices p^i_j and q^i_j for each $i \in \{1, \ldots, 3m\}$ and $j \in \{1, \ldots, b\}$.

Let p^i_j, q^i_j be a pair of vertices in G that are adjacent to x, and let u^i_j be the second common neighbor of p^i_j and q^i_j. Suppose $\tilde{u}^k_\ell$ is the image of u^i_j, i.e., suppose that $\varphi(u^i_j) = \tilde{u}^k_\ell$. Then we map p^i_j and q^i_j to $\tilde{p}^k_\ell$ and $\tilde{q}^k_\ell$, respectively. We now consider the neighbors of y and z in G. By construction, the neighborhood of y consists of the $2mb$ vertices in the set $\{p^i_j \mid a_{i+1} \leq j \leq b\}$, while $N_G(z) = \{q^i_j \mid a_{i+1} \leq j \leq b\}$.

Observe that $\tilde{x}$, the image of y and z, is adjacent to two sets of mb vertices: one of the form $\tilde{p}^k_\ell$, the other of the form $\tilde{q}^k_\ell$. Hence, we need to map half the neighbors of y to vertices of the form $\tilde{p}^k_\ell$ and half the neighbors of y to vertices of the form $\tilde{q}^k_\ell$ in order to make φ a locally bijective homomorphism. The same should be done with the neighbors of z. For every vertex $\tilde{u}^k_\ell$ in H, we do as follows. By construction, exactly three vertices of G are mapped to $\tilde{u}^k_\ell$, and exactly two of those vertices, say u^i_j and u^g_h, are at distance 2 from y in G. We set $\varphi(p^i_j) = \tilde{p}^k_\ell$ and $\varphi(p^g_h) = \tilde{q}^k_\ell$. We also set $\varphi(q^i_j) = \tilde{q}^k_\ell$ and $\varphi(q^g_h) = \tilde{p}^k_\ell$. This completes the definition of the mapping φ.

Since the mapping φ preserves adjacencies, it clearly is a homomorphism. In order to show that φ is locally bijective, we first observe that the degree of every vertex in G is equal to the degree of its image in H; in particular, $d_G(x) = d_G(y) = d_G(z) = d_H(\tilde{x}) = mb$. From the above description of φ we get a bijection between the vertices of $N_H(\tilde{x})$ and the vertices of $N_G(v)$ for each $v \in \{x, y, z\}$. For every vertex p^i_j that is adjacent to x and u^i_j in G, its image $\tilde{p}^k_\ell$ is adjacent to the images $\tilde{x}$ of x and $\tilde{u}^k_\ell$ of u^i_j. For every vertex p^i_j that is adjacent to y (respectively z) and u^i_j in G, its image $\tilde{p}^k_\ell$ or $\tilde{q}^k_\ell$ is adjacent to $\tilde{x}$ of y (respectively z) and $\tilde{u}^k_\ell$ of u^i_j. Hence the restriction of φ to $N_G(p^i_j)$ is bijective for every $i \in \{1, \ldots, 3m\}$ and $j \in \{1, \ldots, b\}$, and the same clearly holds for the restriction of φ to $N_G(q^i_j)$. The vertices of each cycle C_i are mapped to the vertices of some cycle $\tilde{C}_k$ in such a way that the vertices and their images appear in the same order on the cycles. This, together with the fact that the image $\tilde{u}^k_\ell$ of every vertex u^i_j is adjacent to the images $\tilde{p}^k_\ell$ and $\tilde{q}^k_\ell$ of the neighbors p^i_j and q^i_j of u^i_j, shows that the restriction of φ

to $N_G(u_j^i)$ is bijective for every $i \in \{1, \ldots, 3m\}$ and $j \in \{1, \ldots, b\}$. We conclude that φ is a locally bijective homomorphism from G to H.

In order to show that G has pathwidth 5, let us first consider the subgraph of G depicted on the left-hand side of Figure 2; we denote this subgraph by L_1, and we say that the cycle C_1 *defines* the subgraph L_1. The graph L_1' that is obtained from L_1 by deleting vertices x, y, z and edge $u_1^1 u_b^1$ is a caterpillar, i.e., a tree in which there is a path containing all vertices of degree more than 1. Since caterpillars are well-known to have pathwidth 1, graph L_1' has a path decomposition P_1' of width 1. Starting with P_1', we can now obtain a path decomposition of the graph L_1 by simply adding vertices x, y, z and u_1^1 to each node of P_1'; this path decomposition has width 5. Every cycle C_i in G defines a subgraph L_i of G in the same way C_1 defines the subgraph L_1. Suppose we have constructed a path decomposition P_i of width 5 of the subgraph L_i for each $i \in \{1, \ldots, 3m\}$ in the way described above. Since any two subgraphs L_i and L_j with $i \neq j$ have only the vertices x, y, z in common, and these three vertices appear in all nodes of each of the path decompositions P_i, we can arrange the $3m$ path decompositions $P_1, \ldots, P_{3m}$ in such a way that we obtain a path decomposition P of G of width 5. Similar but easier arguments can be used to show that H has pathwidth 3. □

We now consider the case where only the maximum degree of G is bounded.

Theorem 4. (★) *The problems* LBHom*,* LIHom *and* LSHom *are* NP*-complete on input pairs* (G, K_4) *where* G *has maximum degree* 3*.*

5 Conclusion

We leave it as an open problem whether the bounds on the pathwidth and the treewidth of the graphs constructed in our hardness proofs in Theorem 3 can be reduced further. Here, we only mention the following result, which is known already for the problems LBHom and LSHom [16].

Theorem 5. (★) *The* LIHom *problem is polynomial-time solvable on input pairs* (G, H) *where* G *is a tree.*

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Appendix

This appendix contains all the proofs that were omitted from the main body of the paper due to page restrictions.

Proof of Theorem 3 (continued). In the main body of the paper, we proved that LBHom is NP-hard on input pairs (G, H) where G has pathwidth 5. This NP-hardness reduction for the locally bijective case can also be used to prove that LIHom and LSHom are NP-hard for input pairs (G, H) where G has pathwidth 5. This follows from the claim that $G \xrightarrow{B} H$ if and only if $G \xrightarrow{S} H$ if and only if $G \xrightarrow{I} H$ for the gadget graphs G and H displayed in Figure 1. This claim can be seen as follows. First suppose that $G \xrightarrow{B} H$. Then, by definition, $G \xrightarrow{S} H$ and $G \xrightarrow{I} H$. Now suppose that $G \xrightarrow{I} H$ or $G \xrightarrow{S} H$. Since it can easily be verified that

$$\mathrm{drm}(G) = \mathrm{drm}(H) = \begin{pmatrix} 0 & 0 & 2mb \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix},$$

we can use Lemma 1 (i) or (ii), respectively, to deduce that $G \xrightarrow{B} H$. However, we can strengthen the hardness results for the locally injective and surjective cases by reducing the pathwidth of the guest graph to 4 and 2, respectively, as claimed in statements (ii) and (iii) of Theorem 3. In order to do so, we give the following alternative constructions below.

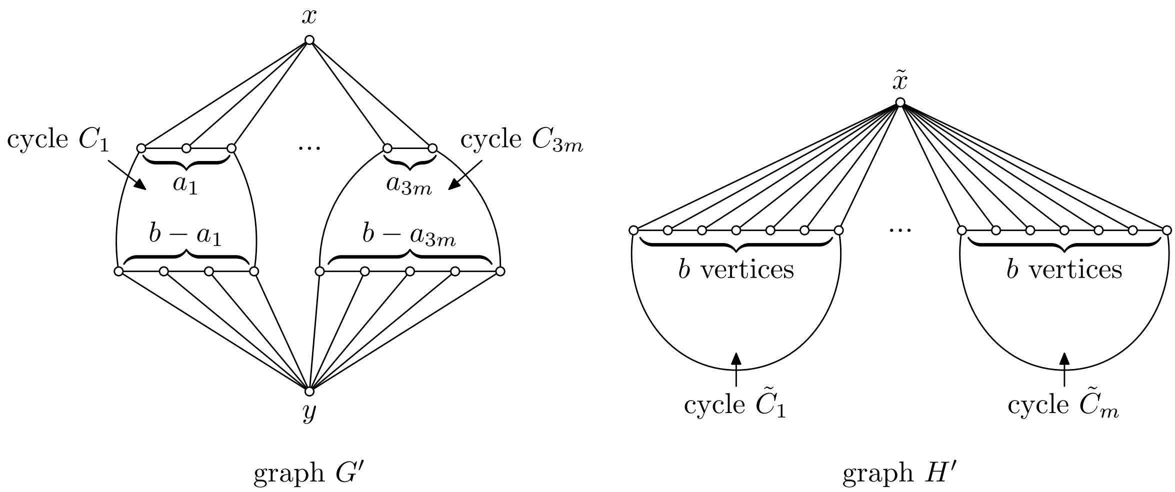

Fig. 3. A schematic illustration of the graphs G' and H' that are constructed from a given instance (A, b) of 3-Partition in the proof of statement (ii) in Theorem 3.

The alternative hardness construction for LSHom is similar to but easier than the construction for LBHom; see Figure 3. Let (A, b) be an instance of 3-Partition. We construct a graph G' by taking $3m$ disjoint cycles $C_1, \ldots, C_{3m}$ of length b, and labeling the vertices of each cycle C_i with labels $u_1^i, \ldots, u_B^i$ in the same way as we labeled the vertices of the cycles C_i in the construction for LBHom (see also Figure 2). We then add two vertices x and y. For every $i \in \{1, \ldots, 3m\}$, we make x adjacent to each of the vertices $u_1^i, u_2^i \ldots, u_{a_i}^i$, and y is made adjacent to each of the vertices $u^i_{a_i+1}, \ldots, u^i_B$. Graph H' is obtained from the disjoint union of m cycles $\tilde{C}_1, \ldots, \tilde{C}_m$ of length b by adding one universal vertex $\tilde{x}$. Using similar arguments as the ones used in the NP-hardness proof of LBHom, it can be shown that there exists a locally surjective homomorphism φ from G' to H' if and only if (A, b) is a yes-instance of 3-Partition. Such a homomorphism φ' maps x and y to $\tilde{x}$, and maps the vertices of cycles $C_1, \ldots, C_{3m}$ to the vertices of cycles $\tilde{C}_1, \ldots, \tilde{C}_m$ in exactly the same way as φ mapped these vertices in the NP-hardness proof of LBHom. It is a routine exercise to show that G' has pathwidth 4 and that H' has pathwidth 3.

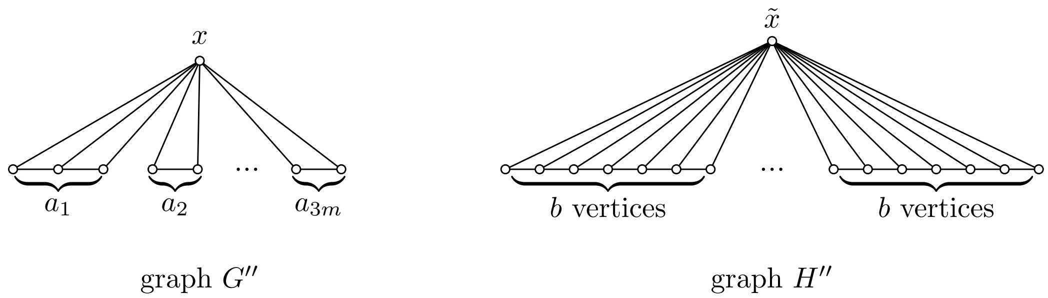

Fig. 4. A schematic illustration of the graphs G'' and H'' that are constructed from a given instance (A, b) of 3-Partition in the proof of statement (iii) in Theorem 3.

The reduction for LIHom is even easier; see Figure 4. Given an instance (A, b) of 3-Partition, we create a graph G'' by adding a universal vertex x to the disjoint union of $3m$ paths on $a_1, a_2, \ldots, a_{3m}$ vertices, respectively. Graph H'' is obtained from the disjoint union of m paths on b vertices by adding a universal vertex $\tilde{x}$. It is easy to verify that there exists a locally injective homomorphism φ'' from G'' to H'', mapping x to $\tilde{x}$ and all other vertices of G'' to the vertices of degree 2 or 3 in H'', if and only if (A, b) is a yes-instance of 3-Partition. The observation that both G'' and H'' have pathwidth 2 completes the proof of Theorem 3. □

Proof of Theorem 4. Kratochvíl and Křivánek [24] showed that K_4-LBHom is NP-complete, where K_4 denotes the complete graph on four vertices. Since a graph G allows a locally bijective homomorphism to K_4 if and only if G is 3-regular, K_4-LBHom is NP-complete on 3-regular graphs. Consequently, due to Lemma 1, K_4-LBHom is equivalent to K_4-LIHom and also to K_4-LSHom on 3-regular graphs. This completes the proof of Theorem 4. □

Proof of Theorem 5. The universal cover T_G of a connected graph G is the unique tree (which may have an infinite number of vertices) such that there is a locally bijective homomorphism from T_G to G. One way to define this mapping is as follows. Consider all finite walks in G that start from an arbitrary fixed vertex in G and that do not traverse the same edge in two consecutive steps. Each such walk will correspond to a vertex of T_G. We let two vertices of T_G be adjacent if and only if the two associated walks differ only in the presence of the last edge. Then the mapping f_G that maps every walk to its last vertex is a locally bijective homomorphism from T_G to G [1]. It is also known that $T_G = G$ if and only if

G is a tree [1]. Moreover, for any two graphs G and H, $G \xrightarrow{I} H$ implies that $T_G \xrightarrow{I} T_H$ [14].

Now let G be a tree and H be an arbitrary graph. We claim that $G \xrightarrow{I} H$ if and only if $T_G \xrightarrow{I} T_H$. The forward implication follows from above. To show the backward implication, suppose that $T_G \xrightarrow{I} T_H$. Then $G \xrightarrow{I} T_H$, because $T_G = G$. Let f be a locally injective homomorphism from G to T_H. Then, because $G \xrightarrow{I} T_H$ and $T_H \xrightarrow{B} H$, we have $G \xrightarrow{I} H$. To explain this, consider the mapping $f' : V_G \to V_H$ defined by $f'(u) = f_H(x)$ if and only if $f(u) = x$. Notice that f' is a locally injective homomorphism from G to H. The desired result follows from this claim combined with the fact that we can check in polynomial time whether $T_G \xrightarrow{I} T_H$ holds for two graphs G and H [16]. □