Fractional coloring of triangle-free planar graphs*

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Abstract

We prove that every planar triangle-free graph on n vertices has fractional chromatic number at most $3 - \frac{1}{n+1/3}$.

1 Introduction

Coloring of triangle-free planar graphs is an attractive topic. It started with Grötzsch's theorem [7], stating that such graphs are 3-colorable. Since then, several simpler proofs have been given, e.g., by Thomassen [13, 14]. Algorithmic questions have also been addressed: while most proofs readily yield quadratic algorithms to 3-color such graphs, it takes considerably more effort to obtain asymptotically faster algorithms. Kowalik [11] proposed an algorithm running in time $O(n \log n)$, which relies on the design of an advanced data structure. More recently, Dvořák *et al.* [2] managed to obtain a linear-time algorithm, yielding at the same time a yet simpler proof of Grötzsch's theorem.

The fact that all triangle-free planar graphs admit a 3-coloring implies that all such graphs have an independent set containing at least one third of the vertices. Albertson *et al.* [1] had conjectured that there is always a larger independent set, which was confirmed by Steinberg and Tovey [12] even in a stronger sense: all triangle-free planar n-vertex graphs admit a 3-coloring

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where not all color classes have the same size, and thus at least one of them forms an independent set of size at least $\frac{n+1}{3}$. This bound turns out to be tight for infinitely many triangle-free graphs, as Jones [9] showed. As an aside, let us mention that the graphs built by Jones have maximum degree 4: this is no coincidence as Heckman and Thomas later established that all triangle-free planar n-vertex graphs with maximum degree at most 3 have an independent set of order at least $\frac{3n}{8}$, which again is a tight bound—actually attained by planar graphs of girth 5.

All these considerations naturally lead to investigate the fractional chromatic number of triangle-free planar graphs. Indeed, as we shall later see, this invariant actually corresponds to a weighted version of the independence ratio. In addition, since $\chi_f(G) \leqslant \chi(G)$ for every graph G, Grötzsch's theorem implies that $\chi_f(G) \leqslant 3$ whenever G is triangle-free and planar. On the other hand, Jones's construction shows the existence of triangle-free planar graphs with fractional chromatic number arbitrarily close to 3. Thus one wonders whether there exists a triangle-free planar graph with fractional chromatic number exactly 3. Let us note that this happens for the circular chromatic number χ_c, which is a different relaxation of ordinary chromatic number such that $\chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G)$ for every graph G.

The purpose of this work is to answer this question. We do so by establishing the following upper bound on the fractional chromatic number of triangle-free planar n-vertex graphs, which depends on n.

Theorem 1. *Every planar triangle-free graph on n vertices has fractional chromatic number at most $3 - \frac{1}{n+1/3}$.*

Consequently, no (finite) triangle-free planar graph has fractional chromatic number equal to 3. We also note that the bound provided by Theorem 1 is tight up to the multiplicative factor. Indeed, the aforementioned construction of Jones [9] yields, for each $n \geqslant 2$ such that $n \equiv 2 \pmod 3$, a triangle-free planar graph G_n with $\alpha(G_n) = \frac{n+1}{3}$. Consequently, $\chi_f(G_n) \geqslant \frac{3n}{n+1} = 3 - \frac{3}{n+1}$.

Our result can be improved for triangle-free planar graphs with maximum degree at most four, giving an exact bound for such graphs.

Theorem 2. *Every planar triangle-free n-vertex graph of maximum degree at most four has fractional chromatic number at most $\frac{3n}{n+1}$.*

Furthermore, the graphs of Jones's construction contain a large number of separating 4-cycles (actually, all their faces have length five). We show that planar triangle-free graphs of *maximum degree* 4 and *without separating 4-cycles* cannot have fractional number arbitrarily close to 3.

Theorem 3. *There exists $\delta > 0$ such that every planar triangle-free graph of maximum degree at most four and without separating 4-cycles has fractional chromatic number at most $3 - \delta$.*

Dvořák and Mnich [5] proved that there exists $\beta > 0$ such that all planar triangle-free n-vertex graphs without separating 4-cycles contain an independent set of size at least $n/(3-\beta)$. This gives an evidence that the restriction on the maximum degree in Theorem 3 might not be necessary.

Conjecture 1. *There exists $\delta > 0$ such that every planar triangle-free graph without separating 4-cycles has fractional chromatic number at most $3 - \delta$.*

Faces of length four are usually easy to deal with in the proofs by collapsing; thus the following seemingly simpler variant of Conjecture 1 is likely to be equivalent to it.

Conjecture 2 (Dvořák and Mnich [5])**.** *There exists $\delta > 0$ such that every planar graph of girth at least five has fractional chromatic number at most $3 - \delta$.*

2 Notation and auxiliary results

Let μ be the Lebesgue measure on real numbers. Let G be a graph. If a function φ assigns to each vertex of G a measurable subset of $[0,1]$ and $\varphi(u) \cap \varphi(v) = \emptyset$ for all edges uv of G, we say that φ is a *fractional coloring* of G. Let $f\colon V(G) \to Q \cap [0,1]$ be a function with rational values. If the fractional coloring ϕ satisfies $\mu(\varphi(v)) \geqslant f(v)$ for every $v \in V(G)$, then we say that φ is an *f-coloring* of G. If $\mu(\varphi(v)) = f(v)$ for every $v \in V(G)$, then we say that φ is a *tight f-coloring*. Note that if G has an f-coloring, then it also has a tight one. For $x \in Q \cap [0,1]$, let c_x denote the constant function assigning the value x to each vertex of G. The *fractional chromatic number* of G is defined as

$$\chi_f(G) = \frac{1}{\sup \{x \in Q \cap [0,1] \;:\; G \text{ has a } c_x\text{-coloring}\}}.$$

Let $w\colon V(G) \to \mathbf{R}^+$ be an arbitrary function. For a set $X \subseteq V(G)$, by $w(X)$ we mean $\sum_{v \in X} w(v)$. Let $w(f) = \sum_{v \in V(G)} f(v)w(v)$. An integer $N \geqslant 1$ is a *common denominator* of f if $Nf(v)$ is an integer for every $v \in V(G)$. Setting $[N] = \{1, \ldots, N\}$, a function $\psi\colon V(G) \to \mathcal{P}([N])$ is an *(f,N)-coloring* of G if $\psi(u) \cap \psi(v) = \emptyset$ for every $uv \in E(G)$ and $|\psi(v)| \geqslant Nf(v)$ for every $v \in V(G)$. The (f,N)-coloring is *tight* if $|\psi(v)| = Nf(v)$ for every $v \in V(G)$.

The fractional chromatic number of a graph can be expressed in various equivalent ways, based on its well known linear programming formulation and duality. The proof of the following lemma can be found e.g. in Dvořák *et al.* [6, Theorem 2.1].

Lemma 4. *Let G be a graph and $f\colon V(G) \to Q \cap [0,1]$ a function. The following statements are equivalent.*

- *The graph G has an f-coloring.*
- *There exists a common denominator N of f such that G has an (f,N)-coloring.*
- *For every $w\colon V(G) \to \mathbf{R}^+$, there exists an independent set $X \subseteq V(G)$ with $w(X) \geqslant w(f)$.*

We need several results related to Grötzsch's theorem. The following lemma was proved for vertices of degree at most three by Steinberg and Tovey [12]. The proof for vertices of degree four follows from the results of Dvořák and Lidický [4], as observed by Dvořák *et al.* [3].

Lemma 5. *If G is a triangle-free planar graph and v is a vertex of G of degree at most four, then there exists a 3-coloring of G such that all neighbors of v have the same color.*

In fact, Dvořák *et al.* [3] proved the following stronger statement.

Lemma 6. *There exists an integer $D \geqslant 4$ with the following property. Let G be a triangle-free planar graph without separating 4-cycles and let X be a set of vertices of G of degree at most four. If the distance between every two vertices in X is at least D, then there exists a 3-coloring of G such that all neighbors of vertices of X have the same color.*

Let G be a triangle-free plane graph. A 5-face $f = v_1v_2v_3v_4v_5$ of G is *safe* if v_1, v_2, v_3 and v_4 have degree exactly three, their neighbors $x_1, \ldots, x_4$ (respectively) not incident with f are pairwise distinct and non-adjacent, and

- the distance between x_2 and v_5 in $G - \{v_1, v_2, v_3, v_4\}$ is at least four, and
- $G - \{v_1, v_2, v_3, v_4\}$ contains no path of length exactly three between x_3 and x_4.

Lemma 7 (Dvořák *et al.* [2, Lemma 2.2])**.** *If G is a plane triangle-free graph of minimum degree at least three and all faces of G have length five, then G has a safe face.*

Finally, let us recall the folding lemma, which is frequently used in the coloring theory of planar graphs.

Lemma 8 (Klostermeyer and Zhang [10])**.** *Let G be a planar graph with odd-girth $g > 3$. If $C = v_0v_1 \dots v_{r-1}$ is a facial circuit of G with $r \neq g$, then there is an integer $i \in \{0, \dots, r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (where indices are taken modulo r) is also of odd-girth g.*

3 Proofs

First, let us show a lemma based on the idea of Hilton *et. al.* [8].

Lemma 9. *Let G be a planar triangle-free graph and let $w : V(G) \to R^+$ be an arbitrary function. If $v \in V(G)$ has degree at most 4, then G contains an independent set X such that $w(X) \geqslant \frac{w(V(G))+w(v)}{3}$.*

Proof. Lemma 5 implies that there exists a 3-coloring of G such that all neighbors of v have the same color. Consequently, G has an f_v-coloring for the function f_v such that $f_v(z) = 1/3$ for $z \in V(G) \setminus \{v\}$ and $f_v(v) = 2/3$. By Lemma 4, there exists an independent set $X \subseteq V(G)$ such that $w(X) \geqslant w(f_v) = \frac{w(V(G))+w(v)}{3}$. □

Theorem 2 now readily follows.

Proof of Theorem 2. Let G be a planar triangle-free n-vertex graph of maximum degree at most four. Consider any function $w\colon V(G) \to \mathbf{R}^+$, and let v be the vertex to which w assigns the maximum value. We have $w(v) \geqslant w(V(G))/n$. By Lemma 9, there exists an independent set X such that $w(X) \geqslant \frac{w(V(G))+w(v)}{3} \geqslant \frac{n+1}{3n} w(V(G))$. Therefore, for every $w\colon V(G) \to \mathbf{R}^+$, there exists an independent set X with $w(X) \geqslant w(c_{(n+1)/(3n)})$. By Lemma 4, it follows that the fractional chromatic number of G is at most $\frac{3n}{n+1}$. □

Similarly, Lemma 6 implies Theorem 3.

Proof of Theorem 3. Let D be the constant of Lemma 6, let $\delta_0 = \frac{1}{3 \cdot 4^D}$ and $\delta = \frac{9\delta_0}{3\delta_0+1} = \frac{3}{4^D+1}$. Let G be a planar triangle-free graph of maximum degree at most four and without separating 4-cycles. Clearly, it suffices to prove that G has a $c_{1/3+\delta_0}$-coloring.

Let G' be the graph obtained from G by adding edges between all pairs of vertices at distance at most $D-1$. The maximum degree of G' is less than 4^D, and thus G' has a coloring by at most 4^D colors. Let C_1, ..., C_{4^D} be the color classes of this coloring. For $i \in \{1, \ldots, 4^D\}$, let f_i be the function defined by $f_i(v) = 2/3$ for $v \in C_i$ and $f_i(v) = 1/3$ for $v \in V(G) \setminus C_i$. Lemma 6 ensures that G has an f_i-coloring.

Consider any function $w : V(G) \to R^+$. There exists $i \in \{1, \ldots, 4^D\}$ such that $w(C_i) \geqslant w(V(G))/4^D$. By Lemma 4 applied for f_i, we conclude that G contains an independent set X such that $w(X) \geqslant w(f_i) = \frac{w(V(G))+w(C_i)}{3} \geqslant (1/3+\delta_0)w(V(G)) = w(c_{1/3+\delta_0})$. Since the choice of w was arbitrary, Lemma 4 implies that G has a $c_{1/3+\delta_0}$-coloring. □

The proof of Theorem 1 is somewhat more involved. Let $\varepsilon = 1/9$ and for $n \geqslant 1$, let $b(n) = 1/3 + \varepsilon/n$. Let G be a plane triangle-free graph. We say that G is a *counterexample* if G does not have a $c_{b(|V(G)|)}$-coloring. We say that G is a *minimal counterexample* if G is a counterexample and no plane triangle-free graph with fewer than $|V(G)|$ vertices is a counterexample. Since b is a decreasing function, every minimal counterexample is connected.

Lemma 10. *If G is a minimal counterexample, then G is 2-connected. Consequently, the minimum degree of G is at least two.*

Proof. Since $b(n) \leqslant 1/2$, every counterexample has at least three vertices; hence, it suffices to prove that G is 2-connected, and the bound on the minimum degree will follow. Let n be the number of vertices of G.

Suppose that G is not 2-connected, and let G_1 and G_2 be subgraphs of G such that $G = G_1 \cup G_2$, the graph G_1 intersects G_2 in exactly one vertex v, and both $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$ are greater than 1. Since $n = n_1 + n_2 - 1$, we have $n_1, n_2 < n$, and thus neither G_1 nor G_2 is a counterexample. Consequently, G_i has a $c_{b(n_i)}$-coloring for $i \in \{1, 2\}$. Since b is a decreasing function, we deduce that G_i has a $c_{b(n)}$-coloring and hence, by Lemma 4, there exists $N \geqslant 1$ such that G_i has a $(c_{b(n)}, N)$-coloring φ_i. By permuting the colors if necessary, we can assume that $\varphi_1(v) = \varphi_2(v)$, and thus $\varphi_1 \cup \varphi_2$ is a $(c_{b(n)}, N)$-coloring of G. This contradicts the assumption that G is a counterexample. □

Lemma 11. *If G is a minimal counterexample, then every face of G has length exactly 5.*

Proof. Let n be the number of vertices of G. Suppose that G has a face f of length other than 5. Since G is triangle-free, it has odd girth at least five, and by Lemma 8, there exists a path $v_1v_2v_3$ in the boundary of f such that the graph G' obtained by identifying v_1 with v_3 to a single vertex z has odd girth at least five as well. It follows that G' is triangle-free. Since G is a minimal counterexample, G' has a $c_{b(n-1)}$-coloring, and by giving both v_1 and v_3 the color of z, we obtain a $c_{b(n-1)}$-coloring of G. Since $b(n) < b(n-1)$, this contradicts the assumption that G is a counterexample. □

Given a counterexample G on n vertices, a function $w\colon V(G) \to \mathbf{R}^+$ is a *witness* if G has no independent set X satisfying $w(X) \geqslant w(c_{b(n)})$. By Lemma 4, every counterexample has a witness. Let us now state a useful special case of Lemma 9.

Lemma 12. *If G is a counterexample on n vertices, w is a witness and $v \in V(G)$ has degree at most three, then $w(v) < 3\varepsilon w(V(G))/n$.*

Proof. Let n be the number of vertices of G. By Lemma 9, there exists an independent set $X \subseteq V(G)$ with $w(X) \geqslant \frac{w(V(G))+w(v)}{3}$. On the other hand, since w is a witness, we have $w(X) < w(c_{b(n)}) = \frac{w(V(G))}{3} + \frac{\varepsilon}{n}w(V(G))$. The claim of this lemma follows. □

Lemma 13. *If G is a minimal counterexample, then G has minimum degree at least three.*

Proof. Let n be the number of vertices of G and let $w\colon V(G) \to \mathbf{R}^+$ be a witness for G. By Lemma 10, the graph G has minimum degree at least two. Suppose that $v \in V(G)$ has degree two. By Lemma 12, we have $w(v) < 3\varepsilon w(V(G))/n$.

Since G is a minimal counterexample, there exists $N \geqslant 1$ and a tight $(c_{b(n-1)}, N)$-coloring ψ of $G - v$. Let $f(x) = b(n-1)$ for $x \in V(G - v)$ and $f(v) = 1 - 2b(n-1)$. Clearly, ψ extends to an (f, N)-coloring of G. By

Lemma 4, there exists an independent set $X \subseteq V(G)$ such that

$$\begin{aligned} w(X) &\geqslant w(f) \\ &= b(n-1)w(V(G)) - (3b(n-1)-1)w(v) \\ &> b(n-1)w(V(G)) - \frac{3(3b(n-1)-1)\varepsilon}{n} w(V(G)) \\ &= \left[b(n-1) - \frac{9\varepsilon^2}{n(n-1)} \right] w(V(G)) \\ &= \left[b(n) + \frac{\varepsilon}{n(n-1)} - \frac{9\varepsilon^2}{n(n-1)} \right] w(V(G)) \\ &= b(n)w(V(G)) = w(c_{b(n)}). \end{aligned}$$

This contradicts that w is a witness for G. $\square$

Lemma 14. *No minimal counterexample contains a safe* 5*-face.*

Proof. Let G be a minimal counterexample containing a safe 5-face $f = v_1v_2v_3v_4v_5$, and let $x_1, \ldots, x_4$ be the neighbors of $v_1, \ldots, v_4$, respectively, that are not incident with f. Let n be the number of vertices of G and let $w\colon V(G) \to \mathbf{R}^+$ be a witness for G. By Lemma 12, we have $w(v_i) < 3\varepsilon w(V(G))/n$ for $1 \leqslant i \leqslant 4$.

Let G' be the graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by identifying x_2 with v_5 into a new vertex u_1, and x_3 with x_4 into a new vertex u_2. Since f is safe, G' is triangle-free. Since G is a minimal counterexample, there exists $N \geqslant 1$ and a tight $(c_{b(n-6)}, N)$-coloring ψ of G'. Let $f(x) = b(n-6)$ for $x \in V(G - \{v_1, v_2, v_3, v_4\})$ and $f(v_i) = 1 - 2b(n-6)$ for $1 \leqslant i \leqslant 4$. We use ψ to design an (f, N)-coloring of G.

Let $\psi(x_2) = \psi(v_5) = \psi(u_1)$ and $\psi(x_3) = \psi(x_4) = \psi(u_2)$. Let $\psi(v_1)$ be a subset of $[N] \setminus (\psi(x_1) \cup \psi(v_5))$ of size $f(v_1)N$, and let $\psi(v_2)$ be a subset of $[N] \setminus (\psi(x_2) \cup \psi(v_1))$ of size $f(v_2)N$. Let $M_3 = [N] \setminus (\psi(v_2) \cup \psi(x_3))$ and $M_4 = [N] \setminus (\psi(v_5) \cup \psi(x_4))$. Note that $|M_3| \geqslant f(v_3)N$ and $|M_4| \geqslant f(v_4)N$. Furthermore, since $\psi(x_3) = \psi(x_4)$ and $\psi(v_2) \cap \psi(v_5) = \emptyset$, we have $|M_3 \cup M_4| = 1 - |\psi(x_3)| = 1 - b(n-6) \geqslant f(v_3) + f(v_4)$. Therefore, we can choose disjoint sets $\psi(v_3) \subseteq M_3$ and $\psi(v_4) \subseteq M_4$ of size $f(v_3)N = f(v_4)N$. This gives an (f, N)-coloring of G.

By Lemma 4, there exists an independent set $X \subseteq V(G)$ such that

$$\begin{aligned} w(X) &\geqslant w(f) \\ &= b(n-6)w(V(G)) - (3b(n-6)-1)\sum_{i=1}^{4} w(v_i) \\ &> b(n-6)w(V(G)) - \frac{12(3b(n-6)-1)\varepsilon}{n} w(V(G)) \\ &= \left[b(n-6) - \frac{36\varepsilon^2}{n(n-6)} \right] w(V(G)) \\ &= \left[b(n) + \frac{6\varepsilon}{n(n-6)} - \frac{36\varepsilon^2}{n(n-6)} \right] w(V(G)) \\ &\geqslant b(n)w(V(G)) = w(c_{b(n)}). \end{aligned}$$

This contradicts that w is a witness for G. □

We can now establish Theorem 1.

Proof of Theorem 1. Note that $\frac{1}{3-\frac{1}{n+1/3}} = b(n)$. Suppose that there exists a planar triangle-free graph G on n vertices with fractional chromatic number greater than $3 - \frac{1}{n+1/3}$. Then G has no $c_{b(n)}$-coloring, and thus G is a counterexample. Therefore, there exists a minimal counterexample G_0. Lemmas 13, 11 and 7 imply that G_0 has a safe 5-face. However, that contradicts Lemma 14. □

References

[1] M. Albertson, B. Bollobás, and S. Tucker, *The independence ration and the maximum degree of a graph*, Congr. Numer., 17 (1976), pp. 43–50.

[2] Z. Dvořák, K. Kawarabayashi, and R. Thomas, *Three-coloring triangle-free planar graphs in linear time*, Trans. on Algorithms, 7 (2011), p. article no. 41.

[3] Z. Dvořák, D. Král', and R. Thomas, *Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies.* Manuscript.

[4] Z. Dvořák and B. Lidický, *3-coloring triangle-free planar graphs with a precolored 8-cycle*, ArXiv e-prints, 1305.2467 (2013).

[5] Z. Dvořák and M. Mnich, *Large Independent Sets in Triangle-Free Planar Graphs*, ArXiv e-prints, 1311.2749 (2013).

[6] Z. Dvořák, J.-S. Sereni, and J. Volec, *Subcubic triangle-free graphs have fractional chromatic number at most 14/5*, ArXiv e-prints, 1301.5296 (2013).

[7] H. Grötzsch, *Ein Dreifarbenzatz für Dreikreisfreie Netze auf der Kugel*, Math.-Natur. Reihe, 8 (1959), pp. 109–120.

[8] A. Hilton, R. Rado, and S. Scott, *A* (< 5)*-colour theorem for planar graphs*, Bull. London Math. Soc., 5 (1973), pp. 302–306.

[9] K. F. Jones, *Minimum independence graphs with maximum degree four*, in Graphs and applications (Boulder, Colo., 1982), Wiley-Intersci. Publ., Wiley, 1985, pp. 221–230.

[10] W. Klostermeyer and C. Q. Zhang, $(2+\epsilon)$*-coloring of planar graphs with large odd-girth*, J. Graph Theory, 33 (2000), pp. 109–119.

[11] L. Kowalik, *Fast 3-coloring triangle-free planar graphs*, in ESA, S. Albers and T. Radzik, eds., vol. 3221 of Lecture Notes in Computer Science, Springer, 2004, pp. 436–447.

[12] R. Steinberg and C. A. Tovey, *Planar Ramsey numbers*, J. Combin. Theory, Ser. B, 59 (1993), pp. 288–296.

[13] C. Thomassen, *Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane*, J. Combin. Theory, Ser. B, 62 (1994), pp. 268–279.

[14] ——, *A short list color proof of Grotzsch's theorem*, J. Combin. Theory, Ser. B, 88 (2003), pp. 189–192.