On First-Order Definable Colorings

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Abstract

We address the problem of characterizing H-coloring problems that are first-order definable on a fixed class of relational structures. In this context, we give several characterizations of a homomorphism dualities arising in a class of structure.

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1. Introduction

Recall that *classical model theory* studies properties of abstract mathematical structures (finite or not) expressible in first-order logic [19], and *finite model theory* is the study of first-order logic (and its various extensions) on finite structures [10], [26].

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Constraint Satisfaction Problems (CSPs), and more specifically H-coloring problems, are standard examples of problems which can be expressed in monadic second order logic but usually not in first-order logic. Of course, expressing a H-coloring problem in first-order logic would be highly appreciable, as it would allow fast checking (in at most polynomial time) although problems expressed in monadic second order logic are usually NP-complete. In this direction, it has been proved by Hell and Nešetřil [17] that in the context of finite undirected graphs the H-coloring problem is NP-complete unless H is bipartite, in which case the H-coloring problem belongs to P. This, and a similar dichotomy result of Schaefer [39] led Feder and Vardi [13, 14] to formulate the celebrated Dichotomy Conjecture which asserts that, for every constraint language over an arbitrary finite domain, the corresponding constraint satisfaction problems are either solvable in polynomial time, or are NP-complete. It was soon noticed that this conjecture is equivalent to the existence of a dichotomy for (general) H-coloring problems, and in fact it suffices to prove it for oriented graphs (see [14] and [18]).

Alternately, the class P can be described as the class of problems expressible (on ordered structures) in first-order logic with a least fixed point operator [42, 20]. On the other hand the class NP may be characterized (up to polynomial equivalence) as the class of all problems which have a lift determined by forbidden homomorphisms from a finite set [24]. Hence, we are led naturally to the question of descriptive complexity of classes of structures corresponding to H-coloring problems. Particular case is the question whether a H-coloring problem may be expressed in first-order logic or not.

In this paper, we will consider the relativized version of the problem of first-order definability of H-coloring problems to graphs (or structures) belonging to a fixed class $\mathcal{C}$:

Problem 1. *Given a fixed class $\mathcal{C}$ of graphs (directed graphs, relational structures), determine which H-coloring problems are first-order definable in $\mathcal{C}$, that is for which graphs (directed graphs, relational structures) H there exists a first-order sentence ϕ_H such that*

$$\forall G \in \mathcal{C} : \qquad (G \models \phi_H) \iff (G \to H).$$

The case where $\mathcal{C}$ is the whole class of all finite graphs (all finite directed graphs, all finite relational structures with given finite signature) is well understood. Atserias [1, 2] and Rossman [37] proved that in this case first-order definable H-colorings correspond exactly to *finite homomorphism dualities*, and these dualities have been fully characterized (for undirected graphs, by Nešetřil and Pultr [34]; for directed graphs, by Komárek [21]; for general finite structures, by Nešetřil and Tarif [35]):

Theorem 1 ([35]). *For any signature σ and any finite set $\mathcal{F}$ of σ-structures the following two statements are equivalent:*

1. *There exists D such that $\mathcal{F}$ and D form a finite duality, that is:*

$$\forall \text{ finite } G: \qquad (\forall F \in \mathcal{F},\ F \nrightarrow G) \quad \iff \quad (G \to D)$$

2. *$\mathcal{F}$ is homomorphically equivalent to a set of finite (relational) trees.*

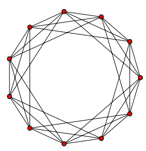
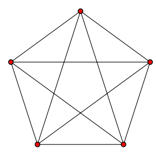
An example of such a duality for finite directed graphs is the Gallai-Hasse-Roy-Vitaver theorem [15, 16, 38, 43], which states that for every directed graph $\vec{G}$ it holds:

$$\vec{P}_{k+1} \nrightarrow \vec{G} \qquad \iff \qquad \vec{G} \to \vec{T}_k.$$

For general classes of graphs the answer is more complicated . For instance, let $\mathcal{C}$ be the class of toroidal graphs and let ϕ be the sentence

$$\forall x_0 \forall x_1 \ldots \forall x_{10} \quad \bigvee_{i=0}^{10} \neg(x_i \sim x_{i+1}) \vee \neg(x_i \sim x_{i+2}) \vee \neg(x_i \sim x_{i+3}),$$

where additions are considered modulo 11 and where $u \sim v$ denotes that u and v are adjacent. Then, it follows from [41] that a graph $G \in \mathcal{C}$ satisfies ϕ if and only if it is 5-colourable. This property can be alternatively be expressed by the following *restricted duality*:

$\forall G \in \mathcal{C}$:  $\not\to \quad G \quad \iff \quad G \quad \longrightarrow$ 

The situation, for general classes of graphs or structures, is quite complex. The problem can be split into two sub-problems. Namely, with respect to a fixed class $\mathcal{C}$:

1. Are first-order definable H-colorings the same as finite restricted dualities?
2. How to characterize finite restricted dualities?

Although characterizing those graphs H such that H-coloring is first-order definable in the class $\mathcal{C}$ is undoubtedly a complex and difficult goal, characterizing those graphs (directed graphs, structures) F such that forbidding homomorphisms from F is equivalent to some coloring problem (that is: templates of restricted singleton dualities) may be easier.

Furthermore, as such a generality might well still be out of reach, we may also restrict ourselves and consider which classes are "nice" in the sense that they contain all the first-order coloring we should intuitively expect, that is all those coloring problems, which arise from forbidding a connected template. In this direction, we proved in [29] (and we sketch a proof below in Section 3) that *classes with bounded expansion* (a concept introduced in [28]) have all such restricted dualities. In this paper we prove here that the converse holds: Under some mild assumptions, a class has all restricted dualities if and only if it is a class with bounded expansion. The characterization of those classes that have all restricted dualities is the subject of Section 3 to 6.

Then we address in Section 7 the problem of determining whether first-order definable H-colorings are the same as finite restricted dualities in a class $\mathcal{C}$. This is related to a classical model theoretical topic: homomorphism preservation theorems. Finally, we relate the problem of characterizing first-order definable colorings of hereditary addable topologically closed class of graphs to classical conjectures of Thomassen [40] and Erdős–Hajnal [12].

2. Preliminaries

2.1. Taxonomy of Classes of Graphs

In the following, we denote by $\mathcal{G}raph$ the class of all finite graphs. A class of graphs $\mathcal{C}$ is *monotone* (resp. *hereditary*, *topologically closed*) if every subgraph (resp. every induced subgraph, every subdivision) of a graph in $\mathcal{C}$ also belongs to $\mathcal{C}$. Notice that if a class $\mathcal{C}$ is both hereditary and topologically closed it is also monotone: If H is a subgraph of a graph $G \in \mathcal{C}$, the graph H is an induced subgraph of the graph $G' \in \mathcal{C}$ obtained from G by subdividing every edge not in H, hence $H \in \mathcal{C}$. For a graph G, we denote by $\omega(G)$ its clique number, by $\chi(G)$ its chromatic number, and by $\overline{\mathrm{d}}(G)$ the average degree of its vertices. By extension, for a class of graphs $\mathcal{C}$ we define

$$\begin{aligned}\omega(\mathcal{C}) &= \sup\{\omega(G),\ G \in \mathcal{C}\}\\ \chi(\mathcal{C}) &= \sup\{\chi(G),\ G \in \mathcal{C}\}\\ \overline{\mathrm{d}}(\mathcal{C}) &= \sup\{\overline{\mathrm{d}}(G),\ G \in \mathcal{C}\}\end{aligned}$$

We proposed in [30, 32, 28] a general classification scheme for graph classes which is based on the density of shallow (topological) minors (we refer the interested reader to the monography [33]). This classification can be defined in several very different ways and we give here one of the simplest definitions:

Let p be a half-integer. A graph H is a *shallow topological minor* of a graph G *at depth* p if some $\leq 2p$-subdivision of H is a subgraph of G; the set of all shallow topological minors of G at depth p is denoted by $G \mathbin{\widetilde{\nabla}} p$ and, more generally, $\mathcal{C} \mathbin{\widetilde{\nabla}} p$ denotes the class of all shallow topological minors at depth p of graphs in $\mathcal{C}$.

A class of undirected graphs $\mathcal{C}$ is *somewhere dense* if there exists an integer p such that the p-th subdivision of every finite graph H may be found as a subgraph of some graph graph in $\mathcal{C}$; it is *nowhere dense* otherwise.

In other words, the class $\mathcal{C}$ is nowhere dense if

$$\forall p \in \mathbb{N}, \qquad \omega(\mathcal{C} \mathbin{\widetilde{\nabla}} p) < \infty.$$

A particular type of nowhere dense classes will be of particular importance in this paper: A class $\mathcal{C}$ has *bounded expansion* [28] if

$$\forall p \in \mathbb{N}, \qquad \overline{\mathrm{d}}(\mathcal{C} \mathbin{\nabla} p) < \infty.$$

Among the numerous equivalent characterizations that can be given for the property of having bounding expansion, we will make use of a characterization based on the chromatic numbers of the shallow topological minors of the graphs in the class. This characterization can be deduced from the following result of Dvořák [8, 9] (see also [33]):

Lemma 2. *Let $c \geq 4$ be an integer and let G be a graph with minimum degree $d > 56(c-1)^2 \frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.* □

Hence the following characterization of classes having bounded expansion:

Lemma 3. *A class $\mathcal{C}$ has bounded expansion if and only if it holds*

$$\forall p \in \mathbb{N}, \qquad \chi(\mathcal{C} \mathbin{\widetilde{\nabla}} p) < \infty. \tag{1}$$

Proof. According to Lemma 2, for every graph G and every integer p there exists an integer C such that:

$$\overline{\mathrm{d}}(G \mathbin{\widetilde{\nabla}} p) \leq C\,\chi(G \mathbin{\widetilde{\nabla}} (2p + 1/2))^4.$$

Moreover, as every graph G is $(\lfloor \overline{\mathrm{d}}(G \mathbin{\widetilde{\nabla}} 0) \rfloor + 1)$-colorable, every graph in $G \mathbin{\widetilde{\nabla}} p$ is $(\lfloor \overline{\mathrm{d}}(G \mathbin{\widetilde{\nabla}} p) \rfloor + 1)$-colorable, that is:

$$\overline{\mathrm{d}}(G \mathbin{\widetilde{\nabla}} p) \geq \chi(G \mathbin{\widetilde{\nabla}} p) - 1.$$

The result follows from these two inequalities. □

Thus we see that all three parameters $\overline{\mathrm{d}}$, ω, and χ can be used to define bounded expansion classes.

2.2. Relational Structures

We recall some basic definitions, notations and result of model theory. Our terminology is standard, cf [10, 25]:

A *signature* σ is a finite set of relation symbols, each with a specified arity. A *σ-structure* $\mathbf{A}$ consists of a *universe* A, or *domain*, and an *interpretation* which associates to each relation symbol $R \in \sigma$ of some arity r, a relation $R^{\mathbf{A}} \subseteq A^r$.

A σ-structure $\mathbf{B}$ is a *substructure* of $\mathbf{A}$ if $B \subseteq A$ and $R^{\mathbf{B}} \subseteq R^{\mathbf{A}}$ for every $R \in \sigma$. It is an *induced substructure* if $R^{\mathbf{B}} = R^{\mathbf{A}} \cap B^r$ for every $R \in \sigma$ of arity r. Notice the analogy with the graph-theoretical concept of subgraph and induced subgraph. A substructure $\mathbf{B}$ of $\mathbf{A}$ is *proper* if $\mathbf{A} \neq \mathbf{B}$. If $\mathbf{A}$ is an induced substructure of $\mathbf{B}$, we say that $\mathbf{B}$ is an *extension* of $\mathbf{A}$. If $\mathbf{A}$ is a proper induced substructure, then $\mathbf{B}$ is a *proper extension.* If $\mathbf{B}$ is the disjoint union of $\mathbf{A}$ with another σ-structure, we say that $\mathbf{B}$ is a *disjoint extension* of $\mathbf{A}$. If $S \subseteq A$ is a subset of the universe of $\mathbf{A}$, then $\mathbf{A} \cap S$ denotes the *induced substructure generated by* S; in other words, the universe of $\mathbf{A} \cap S$ is S, and the interpretation in $\mathbf{A} \cap S$ of the r-ary relation symbol R is $R^{\mathbf{A}} \cap S^r$.

The *Gaifman graph* $\mathrm{Gaifman}(\mathbf{A})$ of a σ-structure $\mathbf{A}$ is the graph with vertex set A in which two vertices $x \not\simeq y$ are adjacent if and only if there exists a relation R of arity $k \geq 2$ in σ and $v_1, \ldots, v_k \in A$ such that $\{x, y\} \subseteq \{v_1, \ldots, v_k\}$ and $(v_1, \ldots, v_k) \in R^{\mathbf{A}}$.

A *block* of a σ-structure $\mathbf{A}$ is a tuple $(R, x_1, \ldots, x_k)$ such that $R \in \sigma$ has arity k and $(x_1, \ldots, x_k) \in R^{\mathbf{A}}$. The *incidence graph* $\mathrm{Inc}(\mathbf{A})$ is the bipartite graph (A, B, E) where A is the universe of $\mathbf{A}$, B is the set of all *blocks* of $\mathbf{A}$, and E is the set of the pairs $\{(R, x_1, \ldots, x_k), y\} \subseteq B \times A$ such that $y \in \{x_1, \ldots, x_k\}$. Thus for us $\mathrm{Inc}(\mathbf{A})$ is a simple graph. No multiple edges are needed for our purposes.

For $k \geq 2$, a *circuit* in a relational structure $\mathbf{A}$ is a cycle $(x_1, y_1, \ldots, x_k, y_k)$ of $\mathrm{Inc}(\mathbf{A})$ where for every $1 \leq a \leq k$ if $y_a = (R_a, z_1^a, \ldots, z_k^a)$ then there exist $1 \leq i < j \leq r_a$ (where r_a is the arity of R_a) such that $x_a = z_i^a$ and $x_{a+1} = z_j^a$ (where we define $x_{k+1} = x_1$); the integer k is the *length* of the circuit. A structure without a circuit is *acyclic.* In the case where $\mathbf{A}$ is an undirected structure, a circuit is called a *cycle.*

A *homomorphism* $\mathbf{A} \to \mathbf{B}$ between two σ-structure is defined as a mapping $f : A \to B$ which satisfies for every relational symbol $R \in \sigma$ the following:

$$(x_1, \ldots, x_k) \in R^{\mathbf{A}} \quad \Longrightarrow \quad (f(x_1), \ldots, f(x_k)) \in R^{\mathbf{B}}.$$

The class of all σ-structures is denoted by $\mathrm{Rel}(\sigma)$.

The definition of bounded expansion extends to classes of relational structures: a class $\mathcal{C}$ of relational structures has *bounded expansion* if the class of the Gaifman graphs of the structures in $\mathcal{C}$ has bounded expansion. It is immediate that two relational structures have the same Gaifman graph if they have the same incidence graph, but that the converse does not hold in general. For a class of relational structures $\mathcal{C}$, denote by $\mathrm{Inc}(\mathcal{C})$ the class of all the incidence graphs $\mathrm{Inc}(\mathbf{A})$ of the relational structures $\mathbf{A} \in \mathcal{C}$.

Proposition 4 ([33]). *Assume that the arities of the relational symbols in σ are bounded, and let $\mathcal{C}$ be an infinite class of σ-structures. Then the class $\mathcal{C}$ has bounded expansion if and only if the class $\mathrm{Inc}(\mathcal{C})$ has bounded expansion.*

3. Classes with All Restricted Dualities

A class of σ-structures $\mathbf{A}$ has *all restricted dualities* if every non-empty connected σ-structure has a restricted dual for $\mathcal{C}$, that is: for every non-empty connected σ-structure $\mathbf{F}$ there exists a σ-structure $\mathbf{D}$ such that $\mathbf{F} \nrightarrow \mathbf{D}$ and

$$\forall \mathbf{A} \in \mathcal{C}: \qquad (\mathbf{F} \to \mathbf{A}) \quad \iff \quad (\mathbf{A} \nrightarrow \mathbf{D}).$$

Note that this definition implies that also for any finite set $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_t$ of connected σ-structures there exists a σ-structure $\mathbf{D}$ such that $\mathbf{F}_i \nrightarrow \mathbf{D}$ (for $1 \le i \le t$) and

$$\forall \mathbf{A} \in \mathcal{C}: \qquad (\exists i \le t: \ \mathbf{F}_i \to \mathbf{A}) \quad \iff \quad (\mathbf{A} \nrightarrow \mathbf{D}).$$

We proved in [29] that bounded expansion classes have all restricted dualities:

Theorem 5. *Let $\mathcal{C}$ be a class with bounded expansion. Then for every connected graph F there exists a graph D such that (F, D) is a restricted homomorphism duality for $\mathcal{C}$:*

$$\forall G \in \mathcal{C} \qquad (F \to G) \quad \iff \quad (G \nrightarrow H). \tag{2}$$

Theorem 5 naturally extends to relational structures by considering Gaifman graphs:

Theorem 6. *let $\mathcal{K}$ be a class of relational structures. If the class of the Gaifman graphs of the structures in $\mathcal{K}$ has bounded expansion then the class $\mathcal{K}$ has all restricted dualities.*

How far is Theorem 6 from a characterization of classes with all restricted dualities? Can one characterize classes with all restricted dualities?

Below we give several characterization theorems which provide answers to these questions. These characterizations use various tools and various restrictions on classes, and perhaps indicate the rich context of this phenomenon: In Section 4 we present a characterization by means of metric and approximations. in Section 5 we present characterization of reorientation-closed classes having all restricted dualities; in Section 6 we present characterization of topologically closed classes having all restricted dualities.

While an application of the first characterization leads to an alternative proof of Theorem 6 (using [27, 28]), which we present in Section 4, the second and third characterizations are on the way of a characterization of first-order definable coloring, either by considering lifts (reorientations) or interpretations (subdivisions). This leads us to a study of relativized homomorphisms preservation theorems, which we review in Section 7. Combining this with the characterization of topological classes having all restricted dualities, this leads to interesting problems in the spirit of classical combinatorial problems of Thomassen [40] and Erdős–Hajnal [12].

Remark 7. The metric characterization is interesting in itself and fits to broader context which can be outlined as follows: The finite dualities are equivalently characterized by properties of the homomorphism quasi-order $(\mathrm{Rel}(\sigma), \leq)$ (where we put $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A} \longrightarrow \mathbf{B}$, cf. [18]): The dual pairs $\mathbf{F}, \mathbf{D}$ are in $1{-}1$ correspondence with connected gaps i.e. with pairs $(\mathbf{A}, \mathbf{B})$ where $\mathbf{B}$ is connected, $\mathbf{A} < \mathbf{B}$ and there is no $\mathbf{C}$ with $\mathbf{A} < \mathbf{C} < \mathbf{B}$. For restricted dualities we do not have such a nice correspondence. However the role of the homomorphism order is taken by the completion of the homomorphism order and by the approximation as explained in Section 4. However for the completion of the homomorphism order we have full characterization of dualities (see [31, 33]). Details will appear elsewhere.

4. Characterization by Approximations

For a structure $\mathbf{A}$ and an integer t, define $\Theta^t(\mathbf{A})$ as the minimum order of a structure $\mathbf{B}$ such that

- $\mathbf{A} \to \mathbf{B}$,
- every substructure $\mathbf{F}$ of $\mathbf{B}$ with order $|F| < t$ has a homomorphism to $\mathbf{A}$.

Intuitively, such a structure $\mathbf{B}$ can be seen as approximate core of $\mathbf{A}$: For $t \geq |B|$, $\mathbf{A}$ and $\mathbf{B}$ are homomorphism-equivalent and $\mathbf{B}$ is the core of $\mathbf{A}$ (alternately, $\mathbf{B}$ is the minimal retract of $\mathbf{A}$). A structure $\mathbf{B}$ with the above properties and order $\Theta^t(\mathbf{A})$ is called a *t-approximation* of (the homomorphism equivalence class of) $\mathbf{A}$.

Theorem 8. *Let $\mathcal{C}$ be a class of σ-structures. Then $\mathcal{C}$ is bounded and has all restricted dualities if and only if $\mathcal{C}$ for every integer t we have $\sup_{\mathbf{A}\in\mathcal{C}} \Theta^t(\mathbf{A}) < \infty$.*

Proof. Assume $\mathcal{C}$ is bounded and has all restricted dualities and let $t \in \mathbb{N}$ be an integer. Let $\mathbf{Z}$ be a strict bound of $\mathcal{C}$, that is a structure such that for every $\mathbf{A} \in \mathcal{C}$ it holds $\mathbf{A} \to \mathbf{Z}$ but $\mathbf{Z} \nrightarrow \mathbf{A}$. As the sequence $(\Theta^t(\mathbf{A}))_{t\in\mathbb{N}}$ is obviously non-decreasing, we may assume without loss of generality that $t \geq |Z|$. For a structure $\mathbf{A} \in \mathcal{C}$, let $\mathcal{F}_t(\mathbf{A})$ be the set of all connected cores $\mathbf{T}$ of order at most t such that $\mathbf{T} \nrightarrow \mathbf{A}$. This set is not empty as it contains the core of $\mathbf{Z}$. For $\mathbf{T} \in \mathcal{F}_t(\mathbf{A})$, let $\mathbf{D_T}$ be the dual of $\mathbf{T}$ relative to $\mathcal{C}$ and let $\mathbf{A}'$ be the product of all the $\mathbf{D_T}$ for $\mathbf{T} \in \mathcal{F}_t(\mathbf{A})$. First notice that for every $\mathbf{T} \in \mathcal{F}_t(\mathbf{A})$ we have $\mathbf{T} \nrightarrow \mathbf{A}$ hence $\mathbf{A} \to \mathbf{D_T}$. It follows that $\mathbf{A} \to \mathbf{A}'$. Let $\mathbf{T}'$ be a connected substructure of order at most t of $\mathbf{A}'$. Assume for contradiction that $\mathbf{T}' \nrightarrow \mathbf{A}$. Then $\mathrm{Core}(\mathbf{T}') \in \mathcal{F}_t(\mathbf{A})$ hence $\mathbf{A}' \to \mathbf{D_{T'}}$ thus $\mathbf{T}' \nrightarrow \mathbf{A}'$ (as for otherwise $\mathbf{T}' \to \mathbf{D_{T'}}$), a contradiction. Thus $\mathbf{T}' \to \mathbf{A}$. It follows that $\Theta^t(\mathbf{A}) \leq |A'| \leq C(t)$ for some suitable finite constant $C(t)$ independent of $\mathbf{A}$ (for instance, one can choose $C(t)$ to be the product of the orders of all the duals relative to $\mathcal{C}$ of connected cores of order at most t).

Conversely, assume that we have $\sup_{\mathbf{A}\in\mathcal{C}} \Theta^t(\mathbf{A}) < \infty$ for every $t \in \mathbb{N}$. The class $\mathcal{C}$ is obviously bounded by the disjoint union of all non-isomorphic minimal order 1-approximations of the structures in $\mathcal{C}$. Let $\mathbf{F}$ be a connected σ-structure, let $t \geq |F|$, and let $\mathcal{D}$ be a set of t-approximations of all the structures $\mathbf{A} \in \mathcal{C}$ such that $\mathbf{F} \nrightarrow \mathbf{A}$. As all the $\Theta^t(\mathbf{A})$ are bounded by some constant $C(t)$, the set $\mathcal{D}$ is finite. If $\mathcal{D}$, let $D_t(\mathbf{F})$ be the empty substructure. Otherwise, let $D_t(\mathbf{F})$ be the disjoint union of all the graphs in $\mathcal{D}$. First notice that $\mathbf{F} \nrightarrow D_t(\mathbf{F})$ as for otherwise $\mathbf{F}$ would have a homomorphism to some structure in $\mathcal{D}$ (as $\mathbf{F}$ is connected), that is to some t-approximation $\mathbf{B}'$ of a structure $\mathbf{B}$ such that $\mathbf{F} \nrightarrow \mathbf{B}$ (this would contradict $\mathbf{F} \to \mathbf{B}'$). Also, if $\mathbf{F} \to \mathbf{A}$ then $\mathbf{A} \nrightarrow D_t(\mathbf{F})$ (for otherwise $\mathbf{F} \to D_t(\mathbf{F})$) and if $\mathbf{F} \nrightarrow \mathbf{A}$ then $\mathcal{D}$ contains a t-approximation $\mathbf{A}'$ of $\mathbf{A}$ thus $\mathbf{A} \to D_t(\mathbf{F})$. Altogether, $D_t(\mathbf{F})$ is a dual of $\mathbf{F}$ relative to $\mathcal{C}$. □

As an application of this characterization, we can outline a proof of Theorem 6.

Proof of Theorem 6 (sketch). Let $\mathcal{K}$ be a class of relational structures. Assume the class of the Gaifman graphs of the structures in $\mathcal{K}$ has bounded expansion. Let $\mathbf{A} \in \mathcal{K}$, and let $t \in \mathbb{N}$ be at least as large as the maximum arity of a relation in the signature of $\mathbf{A}$.

The *tree-depth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest whose closure includes G as a subgraph. A property of tree-depth is that there exists a function $\digamma : \mathbb{N} \to \mathbb{N}$ with the property that if the Gaifman graph of a structure $\mathbf{B}$ has tree-depth at most t then there exists a homomorphism $f : \mathbf{B} \to \mathbf{B}$ such that $|f(\mathbf{B})| \leq \digamma(t)$ [27]. For integer t, we defined in [27] the graph invariant χ_t as follows: for a graph G, $\chi_t(G)$ is the minimum number of colors need in a coloring of G such that the union of every subset of $k \leq t$ color classes induces a subgraph with tree-depth at most k. It has been proved in [28] that a class of graphs $\mathcal{C}$ has bounded expansion if and only if for every integer t it holds $\sup\{\chi_t(G) : \ G \in \mathcal{C}\} < \infty$ (this is related to Lemma 3 above).

We consider a coloring c of the Gaifman graph of $\mathbf{A}$ by $N = \chi_t(\mathrm{Gaifman}(\mathbf{A}))$ colors, which is such that the union of every subset of $k \leq t$ color classes induces a subgraph with tree-depth at most k. It follows that I for each $I \in \binom{[N]}{t}$ there exists a homomorphism $f_I : \mathbf{A}_I \to \mathbf{A}_I$ such that $|f_I(\mathbf{A}_I)| \leq \digamma(t)$, where $\mathbf{A}_I$ denotes the substructure of $\mathbf{A}$ induced by elements with color in I. Define the equivalence relation $\sim$ on the domain of $\mathbf{A}$ by

$$x \sim y \quad \iff \quad c(x) = c(y) \text{ and } \forall I \in \binom{[N]}{t}\ f_I(x) = f_I(y).$$

Define the structure $\hat{\mathbf{A}}$ (with same signature as $\mathbf{A}$) whose domain is the set of the equivalence classes $[x] \in A/\sim$, and relations are defined by

$$([x_1], \ldots, [x_{k_i}]) \in R_i^{\hat{\mathbf{A}}} \quad \iff \quad \forall I \in \binom{[N]}{t}\ (f_I(x_1), \ldots, f_I(x_{k_i})) \in R_i^{\mathbf{A}}.$$

We also define a N-coloration of $\hat{\mathbf{A}}$ by $\hat{c}([x]) = c(x)$. One checks easily that $\hat{\mathbf{A}}$ and $\hat{c}$ are well defined. By construction, $x \mapsto [x]$ is a homomorphism $\mathbf{A} \to \hat{\mathbf{A}}$. Moreover, for every $I \in \binom{[N]}{t}$ the mapping $[x] \mapsto f_I(x)$ is a homomorphism $\hat{\mathbf{A}}_I \to \mathbf{A}_I$ (where $\hat{\mathbf{A}}_I$ is the substructure of $\hat{\mathbf{A}}$ induced by colors in I). It follows that

$$|\Theta^t(\mathbf{A})| \leq |\hat{A}| \leq \digamma(t)^{N^t} \leq \digamma(t)^{\chi_t(\mathrm{Gaifman}(\mathcal{K}))^t}.$$

According to Theorem 8, this implies that the class $\mathcal{K}$ has all restricted dualities. □

For an alternate proof, we refer the reader to [29, 33].

5. Classification by Orientations

Let $\mathbf{A}, \mathbf{B}$ be two structures. The structure $\mathbf{B}$ is a *weak reorientation* of $\mathbf{A}$ is

- for all $(x_1, \ldots, x_{r_i}) \in R_i^{\mathbf{A}}$, there exists a permutation π of $\{1, \ldots, r_i\}$ such that $(x_{\pi(1)}, \ldots, x_{\pi(r_i)}) \in R_i^{\mathbf{B}}$;

- for all $(y_1, \ldots, y_{r_i}) \in R_i^{\mathbf{B}}$, there exists a permutation ρ of $\{1, \ldots, r_i\}$ such that $(y_{\rho(1)}, \ldots, y_{\rho(r_i)}) \in R_i^{\mathbf{A}}$.

Notice that this obviously defines an equivalence relation, and that $R_i^{\mathbf{A}}$ and $R_i^{\mathbf{B}}$ can have different cardinality. For instance, the symmetric orientation and any simple orientation of an undirected graph G are weak-reorientation of each other.

Let A be the universe of $\mathbf{A}$ and let $<$ be a linear order on A. Then $\mathbf{B}$ is the *linear $<$-reorientation* of $\mathbf{A}$ if $\mathbf{B}$ is a weak reorientation of $\mathbf{A}$ satisfying

$$\forall (y_1, \ldots, y_{r_i}) \in R_i^{\mathbf{B}} \quad y_1 < y_2 < \cdots < y_{r_i}.$$

(Notice that $\mathbf{A}$ has a unique linear $<$-reorientation for each linear order $<$.) Extending the well known topological sorting of graphs, it is easily checked that a structure $\mathbf{A}$ is acyclic (see definition in Section 2.2) if and only if there exists a linear order $<$ on A such that $\mathbf{A}$ is its own linear $<$-reorientation.

For a class $\mathcal{C}$ we define

- the class $\mathcal{C}_{\mathrm{wor}}$ has the class of all weak reorientations of structures in $\mathcal{C}$;

- the class $\mathcal{C}_{\mathrm{acyc}}$ has the class of all acyclic weak reorientations of structures in $\mathcal{C}$.

Theorem 9. *Let $\mathcal{C}$ be a class. The following properties are equivalent:*

1. *the class $\mathcal{C}$ has bounded expansion;*
2. *the class $\mathcal{C}_{\mathrm{wor}}$ has all restricted dualities;*
3. *for every integer p, there is $\mathbf{D}_p$ with no circuits of length at most p such that*

$$\forall \mathbf{A} \in \mathcal{C}_{\mathrm{acyc}}, \quad \mathbf{A} \to \mathbf{D}_p.$$

Proof. We prove the equivalence by means of three implications:

(1) $\Rightarrow$ (2) is a direct consequence of Theorem 6.

(2) $\Rightarrow$ (3) is straightforward (consider the product of the duals of all the minimal structures with a circuit of length at most p).

(3) ⇒ (1) is proved by contradiction: Assume that (3) holds and that $\mathcal{C}$ does not have bounded expansion. According to Proposition 4 the class $\mathrm{Inc}(\mathcal{C})$ does not have bounded expansion. According to Lemma 3 there exists an integer p such that $\mathrm{Inc}(\mathcal{C}) \,\widetilde{\nabla}\, p$ has unbounded chromatic number. Let N be the order of $\mathbf{D}_{p+1}$. There exists in $\mathrm{Inc}(\mathcal{C})$ a graph G which contains the $\leq p$-subdivision S of a graph H having chromatic number strictly greater than N. We may further assume that the minimum degree of H is strictly greater than the maximum arity of a relational symbol in σ. Let $\mathbf{A} \in \mathcal{C}$ be such that G is isomorphic to the incidence graph of $\mathbf{A}$. By the assumptions on the minimal degree of H, the branching vertices of G correspond to vertices of $\mathbf{A}$. Consider a linear order $<$ on the universe A of $\mathbf{A}$ such that every branch of S will correspond to a monotone sequence. Consider the linear $<$-reorientation $\mathbf{B}$ of $\mathbf{A}$. According to (3), there exists a homomorphism $f : \mathbf{B} \to \mathbf{D}_{p+1}$. Moreover, the two endpoints of a branch of S cannot have the same image by f as then a circuit of length at most p would exist in $\mathbf{D}_{p+1}$. It follows that any two adjacent vertices in H are mapped by f to distinct vertices of $\mathbf{D}_{p+1}$ hence $\chi(H) \leq |D_{p+1}|$, a contradiction.

□

6. Characterization by Subdivisions

Theorem 10. *Let $\mathcal{C}$ be a topologically closed class of undirected graphs. Then the following properties are equivalent:*

1. *the class $\mathcal{C}$ has bounded expansion;*
2. *the class $\mathcal{C}$ has all restricted dualities;*
3. *for every odd integer g there exists a non-bipartite graph H_g with odd-girth at least g such that every graph $G \in \mathcal{C}$ with odd-girth at least g has a homomorphism to H_g*

Proof. The proof follows from the next three implications:

- (1) ⇒ (2) is a direct consequence of Theorem 5.
- (2) ⇒ (3) is straightforward (consider for H_g a dual of C_g for $\mathcal{C}$).
- (3) ⇒ (1) is proved by contradiction: assume that (3) holds and that $\mathcal{C}$ does not have bounded expansion. According to Lemma 3 there exists an integer p such that $\mathcal{C} \,\widetilde{\nabla}\, p$ has unbounded chromatic number. As $\mathcal{C}$ is topologically closed there exists an odd integer $g \geq p$ and a graph $G_0 \in \mathcal{C}$ such that G_0 is the $(g-1)$-subdivision of a graph H_0 with chromatic number $\chi(H_0) > |H_g|$. According to (3), there exists a homomorphism $f : G_0 \to H_g$. As $C_g \nrightarrow H_g$,

the ends of a path of length g cannot have the same image by f. It follows that any two adjacent vertices in H_0 correspond to branching vertices of G_0 which are mapped by f to distinct vertices of H_g. It follows that $\chi(H_0) \leq |H_g|$, a contradiction.

□

7. Homomorphism Preservation Theorems

Suppose that an H-coloring problem is first-order definable. By this we mean that there is a first-order sentence Φ such that

$$G \to H \qquad \Longleftrightarrow \qquad G \models \Phi.$$

It immediately follows that $\neg\Phi$ is preserved by homomorphisms:

$$G \models \neg\Phi \quad \text{and} \quad G \to G' \qquad \Longrightarrow \qquad G' \models \neg\Phi$$

(for otherwise $G \to G' \to H$ hence $G \models \Phi$, a contradiction).

Such a property suggests that such a formula Φ could be equivalent to a formula with a specific syntactic form. Indeed the classical *Homomorphism Preservation Theorem* asserts that a first-order formula is preserved under homomorphisms on all structures if, and only if, it is logically equivalent to an existential-positive formula. The terms "all structures", which means finite and infinite structures, is crucial in the statement of these theorems.

It was not known until recently whether this theorem would hold when relativized to the finite. In fact other well known theorems relating preservation under some specified algebraic operation and certain syntactic forms, like Łoś-Tarski theorem or Lyndon's theorem, fail in the finite. However, the finite relativization of the homomorphism preservation has been proved to hold by B. Rossman [37]. In its "graph form" the result may be stated as follows:

Theorem 11 ([37]). *Let ϕ be a first order formula. Then,*

$$G \to H \text{ and } G \models \phi \quad \Longrightarrow \quad H \models \phi$$

holds for all finite graphs G and H if and only if for finite graphs ϕ is equivalent to an existential first-order formula.

It follows that for finite structures, the only H-coloring problems which are expressible in first-order logic are those for which there exists a finite family $\mathcal{F}$ of finite structures with the property that for every graph G the following finite homomorphism duality holds:

$$\exists F \in \mathcal{F} \quad F \to G \qquad \Longleftrightarrow \qquad G \nrightarrow H. \tag{3}$$

If we want to relativize Theorem 11, we should consider each relativization as a new problem. The Łoś-Tarski theorem, for instance, holds in general, yet fails when relativized to the finite, but holds when relativized to hereditary classes of structures with bounded degree which are closed under disjoint union [3]. These examples stress again that some properties of structures (in general) and graphs (in particular) need, at times, to be studied in the context of a fixed class, in order to state a relativized version of a general statement which could fail in general.

In this context Atserias, Dawar and Kolaitis defined classes of graphs called *wide*, *almost wide* and *quasi-wide* (cf. [6] for instance). It has been proved in [3] that the extension preservation theorem holds in any class $\mathcal{C}$ that is wide, hereditary (i.e. closed under taking substructures) and closed under disjoint unions, that is hereditary classes with bounded degree which are closed under disjoint unions. Also, it has been proved in [4] [5] that the homomorphism preservation theorem holds in any class $\mathcal{C}$ that is almost wide, hereditary and closed under disjoint unions. Almost wide classes of graphs include classes of graphs which exclude a minor [22].

In [7] Dawar proved that the homomorphism preservation theorem holds in any hereditary quasi-wide class that is closed under disjoint unions. This is a strengthening of the result proved in [4].

Theorem 12 ([7])**.** *Let $\mathcal{C}$ be a hereditary addable quasi-wide class of graphs. Then the homomorphism preservation theorem holds for $\mathcal{C}$.*

However, we have proved that hereditary quasi-wide classes of graphs are exactly hereditary nowhere dense classes [30]:

Theorem 13. *A hereditary class of graphs $\mathcal{C}$ is quasi-wide if and only if it is nowhere dense.*

Thus it follows from Theorems 12 and 13 that the relativization of the homomorphism preservation theorem holds for every hereditary addable nowhere dense class of graphs. But nowhere dense classes are not the only classes with relativized homomorphism preservation theorem. We now show that relativized homomorphism preservation theorems are preserved by particular interpretations, from which will deduce that relativized homomorphism preservation theorems hold for the classes $\mathrm{Sub}_q(\mathcal{G}raph)$ of all q-subdivisions of finite graphs. This is of particular interest as somewhere dense classes (i.e. classes which fail to be nowhere dense) are characterized by containment of classes $\mathrm{Sub}_q(\mathcal{G}raph)$ for some q.

In the framework of the model theoretical notion of *interpretation* (see, for instance [25, pp. 178-180]), we can construct the q-subdivision $\mathsf{I}(G)$ of a graph G by means of first-order formulas on the q-tuples of vertices of G:

- vertices of $\mathsf{I}(G)$ are the equivalence classes x of the $(q+1)$-tuples $(v_1, \dots, v_{q+1})$ with form

$$(\overbrace{u, \dots, u}^{j}, \overbrace{v, \dots, v}^{q+1-j})$$

where u and v are adjacent vertices in G (and $0 \leq j \leq q+1$), where tuples of the form

$$(\overbrace{u, \dots, u}^{j}, \overbrace{v, \dots, v}^{q+1-j}) \text{ and } (\overbrace{v, \dots, v}^{q+1-j}, \overbrace{u, \dots, u}^{j})$$

are identified;

- edges of $\mathsf{I}(G)$ are those pairs $\{x, y\}$ where x and y have representative of the form

$$(\overbrace{u, \dots, u}^{j}, \overbrace{v, \dots, v}^{q+1-j}) \text{ and } (\overbrace{u, \dots, u}^{j+1}, \overbrace{v, \dots, v}^{q-j})$$

(for some $u, v \in G$ and $0 \leq j \leq q+1$).

A main interest of such an logical construction (called interpretation) lies in the following property:

Proposition 14 (See, for instance [25], p. 180). *For every first-order formula $F[v_1, \dots, v_k]$ there exists a formula $\mathsf{I}(f)[\overline{w}_1, \dots, \overline{w}_k]$ with $k(q+1)$ free variables (each $\overline{w}_i$ represents a succession of $(q+1)$ free variables) such that for every graph G and every $(x_1, \dots, x_k) \in \mathsf{I}(G)^k$ the three following conditions are equivalent:*

1. $\mathsf{I}(G) \models F[x_1, \dots, x_k]$;
2. *there exist $\overline{b}_1 \in x_1, \dots, \overline{b}_k \in x_k$ such that $G \models \mathsf{I}(F)[\overline{b}_1, \dots, \overline{b}_k]$;*
3. *for all $\overline{b}_1 \in x_1, \dots, \overline{b}_k \in x_k$ it holds $G \models \mathsf{I}(F)[\overline{b}_1, \dots, \overline{b}_k]$.*

In particular, it holds:

Corollary 15. *For every closed first order formula Φ (in the language of graphs) there exists a closed first order formula Ψ such that for every graph G we have*

$$G \models \Psi \quad \iff \quad \mathrm{Sub}_{2p}(G) \models \Phi. \tag{4}$$

Lemma 16. *If the homomorphism preservation theorem holds for a hereditary class of graphs $\mathcal{C}$, it also holds for the class $\mathrm{Sub}_q(\mathcal{C})$ of all q-subdivisions of the graphs in $\mathcal{C}$.*

Proof. If q is odd then the property is obvious as $\mathcal{C}$ contains at most two homomorphism equivalence classes, the one of K_1 and the one of K_2. Hence we can assume q is even and we define $p = q/2$.

Let Φ be a first order formula preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{C})$, where $\mathcal{C}$ is a hereditary class of graphs on which the homomorphism preservation theorem holds. Then we shall prove that there exists a finite family of $2p$-subdivided graphs $\mathcal{F}$, all of which satisfy Φ, and such that for any graph G it holds

$$\mathrm{Sub}_{2p}(G) \models \Phi \quad \iff \quad \exists F \in \mathcal{F} \quad \mathrm{Sub}_{2p}(F) \to \mathrm{Sub}_{2p}(G). \tag{5}$$

According to Corollary 15 there exists a first order formula Ψ such that for every graph G it holds

$$G \models \Psi \quad \iff \quad \mathrm{Sub}_{2p}(G) \models \Phi.$$

Assume that $G \models \Psi$ and $G \to H$, with $G, H \in \mathcal{C}$. Then $\mathrm{Sub}_{2p}(G) \models \Phi$ and $\mathrm{Sub}_{2p}(G) \to \mathrm{Sub}_{2p}(H)$. As Φ is preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{C})$ we get $\mathrm{Sub}_{2p}(H) \models \Phi$ hence $H \models \Psi$. Thus Ψ is preserved by homomorphisms on $\mathcal{C}$. As the homomorphism preservation theorem holds by assumption on $\mathcal{C}$, Ψ is equivalent on $\mathcal{C}$ with a positive first-order formula, that is: there exits a finite family $\mathcal{F}_0$ of finite graphs such that for every $G \in \mathcal{C}$ it holds:

$$G \models \Psi \quad \iff \quad \exists F \in \mathcal{F}_0 \quad F \to G.$$

Moreover, by considering the subgraphs induced by the homomorphic images of the graphs $F \in \mathcal{F}_0$ and as $\mathcal{C}$ is hereditary, we can assume $\mathcal{F}_0 \subseteq \mathcal{C}$. Thus every $F \in \mathcal{F}_0$ satisfies Ψ hence the $2p$-subdivision of the graphs in $\mathcal{F}_0$ satisfy Φ. Let $\mathcal{F}$ be the set of the $2p$-subdivisions of the graphs in $\mathcal{F}_0$. As Φ is preserved by homomorphisms on $\mathrm{Sub}_{2p}(\mathcal{C})$ it follows that for every graph $G \in \mathcal{C}$ if there exists $F \in \mathcal{F}$ such that $F \to \mathrm{Sub}_{2p}(G)$ then $\mathrm{Sub}_{2p}(G)$ satisfies Φ. Conversely, if $\mathrm{Sub}_{2p}(G)$ satisfies Φ for some $G \in \mathcal{C}$ then G satisfies Ψ, thus there exists $F \in \mathcal{F}_0$ such that $F \to G$ hence $\mathrm{Sub}_{2p}(F) \to \mathrm{Sub}_{2p}(G)$. □

We deduce this extension of Rossman's theorem to the class of p-subdivided graphs:

Corollary 17. *For every integer p, the homomorphism preservation theorem holds for* $\mathrm{Sub}_p(\mathcal{G}raph)$.

For a discussion on relativization of the homomorphism preservation theorem, we refer the reader to [33, Chapter 10].

8. On First-Order Definable H-colorings

Corollary 17 has the following negative consequence:

Lemma 18. *Let p be a positive integer, let $\mathcal{C}$ be a class of graphs which includes $\mathrm{Sub}_{2p}(\mathcal{G}raph)$, and let H be a non-bipartite graph (different from K_1) of odd-girth strictly greater than $2p+1$. Then there exists no first order formula Φ such that for every graph $G \in \mathcal{C}$ holds*

$$(G \models \Phi) \iff (G \to H).$$

Proof. Assume for contradiction that such a formula Φ exists. As $\neg\Phi$ is preserved by homomorphisms on $\mathcal{C}$ (hence on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$) it is equivalent on $\mathrm{Sub}_{2p}(\mathcal{G}raph)$ with an existential first-order formula, that is: there exists a finite family $\mathcal{F}$ such that for every graph G it holds:

$$\forall F \in \mathcal{F}\ F \nrightarrow \mathrm{Sub}_{2p}(G) \iff \mathrm{Sub}_{2p}(G) \to H.$$

Clearly, the graphs in $\mathcal{F}$ are non-bipartite. Let g be the maximum of girth of graphs in $\mathcal{F}$ and let G be a graph with chromatic number $\chi(G) > |H|$ and odd-girth $\mathrm{odd}-\mathrm{girth}(G) > g$. Then for every $F \in \mathcal{F}$ we have $F \nrightarrow \mathrm{Sub}_{2p}(G)$ hence $\mathrm{Sub}_{2p}(G) \to H$. However, has the odd-girth of H is strictly greater than $2p+1$ two branching vertices of $\mathrm{Sub}_{2p}(G)$ corresponding to adjacent vertices of G cannot be mapped to a same vertex. It follows that $|H| \geq \chi(G)$, a contradiction. □

Corollary 19. *Let $\mathcal{C}$ be a hereditary addable topologically closed class of graphs.*

Assume that for every integer p there is a non-bipartite graph H_p of odd-girth strictly greater than $2p+1$ and a first order formula Φ_p such that for every graph $G \in \mathcal{C}$ holds

$$(G \models \Phi_p) \iff (G \to H_p).$$

Then $\mathcal{C}$ is nowhere dense.

To the opposite, if $\mathcal{C}$ has bounded expansion, there exists for every integer p a non-bipartite graph H_p of odd-girth strictly greater than $2p+1$ and a first order formula Φ_p such that for every graph $G \in \mathcal{C}$ holds

$$(G \models \Phi_p) \iff (G \to H_p).$$

Indeed, consider for Φ_p the formula asserting that G contains an odd cycle of length at most $2p+1$, and for H_p the restricted dual of the cycle C_{2p+1} with respect to $\mathcal{C}$ (whose existence follows from Theorem 5.

Thus we are naturally led to the following conjecture.

Conjecture 1. Let $\mathcal{C}$ be a hereditary addable topologically closed class of graphs. The following properties are equivalent:

1. for every integer p there is a non-bipartite graph H_p of odd-girth strictly greater than $2p+1$ and a first order definable class $\mathcal{D}_p$ such that a graph $G \in \mathcal{C}$ is H_p-colorable if and only if $G \in \mathcal{D}_p$. Explicitly, there exists a formula Φ_p such that for every graph $G \in \mathcal{C}$ holds

$$(G \vDash \Phi_p) \quad \Longleftrightarrow \quad (G \to H_p);$$

2. the class $\mathcal{C}$ has bounded expansion.

Note that the hypothesis of the conjecture is only a bit weaker than the one of Theorem 10, which asserts that a topologically closed class of graphs has bounded expansion if and only if for every integer p there exists a non-bipartite graph H_p with odd-girth at least $2p+1$ such that every graph $G \in \mathcal{C}$ with odd-girth at least $2p+1$ has a homomorphism to H_p.

To prove Conjecture 1 we are missing a property allowing extract obstructions in nowhere dense classes that do not have bounded expansion. Existence of such obstructions is the core of the following conjecture, which (if true) would imply Conjecture 1. This (structural conjecture) is, we believe, very interesting on its own.

Conjecture 2. Let $\mathcal{C}$ be a monotone nowhere dense class that does not have bounded expansion. Then there exists an integer p such that $\mathcal{C}$ includes p-subdivisions of graphs with arbitrarily large chromatic number and girth.

In support to Conjecture 2, note that it would follow from a positive solution to any of the following two well known conjectures.

Conjecture 3 (Erdős and Hajnal [12]). For every integers g and n there exists an integer $N = f(g,n)$ such that every graph G with chromatic number at least N has a subgraph H with girth at least g and chromatic number at least n.

The case $g = 4$ of the conjecture was proved by Rödl [36], while the general case is still open. Remark that the existence of graphs of both arbitrarily high chromatic number and high girth is a well known result of Erdös [11].

Conjecture 4 (Thomassen [40]). For all integers c, g there exists an integer $f(c,g)$ such that every graph G of average degree at least $f(c,g)$ contains a subgraph of average degree at least c and girth at least g.

The case $g = 4$ of this conjecture is a direct consequence of the simple fact that every graph can be made bipartite by deleting at most half of its edges. The case $g = 6$ has been proved in [23].

That Conjecture 2 follows from Conjecture 4 is a direct consequence of Lemma 2.

References

[1] A. Atserias, *On digraph coloring problems and treewidth duality*, 20th IEEE Symposium on Logic in Computer Science (LICS), 2005, pp. 106–115.

[2] ———, *On digraph coloring problems and treewidth duality*, European J. Combin. **29** (2008), no. 4, 796–820.

[3] A. Atserias, A. Dawar, and M. Grohe, *Preservation under extensions on well-behaved finite structures*, 32nd International Colloquium on Automata, Languages and Programming (ICALP) (Springer-Verlag, ed.), Lecture Notes in Computer Science, vol. 3580, 2005, pp. 1437–1449.

[4] A. Atserias, A. Dawar, and P.G. Kolaitis, *On preservation under homomorphisms and unions of conjunctive queries*, Proceedings of the twenty-third ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, ACM Press, 2004, pp. 319 – 329.

[5] ———, *On preservation under homomorphisms and unions of conjunctive queries*, J. ACM **53** (2006), 208–237.

[6] A. Dawar, *Finite model theory on tame classes of structures*, Mathematical Foundations of Computer Science 2007 (L. Kučera and A. Kučera, eds.), Lecture Notes in Computer Science, vol. 4708, Springer, 2007, pp. 2–12.

[7] ———, *Homomorphism preservation on quasi-wide classes*, Journal of Computer and System Sciences **76** (2010), 324–332.

[8] Z. Dvořák, *Asymptotical structure of combinatorial objects*, Ph.D. thesis, Charles University, Faculty of Mathematics and Physics, 2007.

[9] ———, *On forbidden subdivision characterizations of graph classes*, European J. Combin. **29** (2008), no. 5, 1321–1332.

[10] H.-D. Ebbinghaus and J. Flum, *Finite model theory*, Springer-Verlag, 1996.

[11] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959), no. 1, 34–38.

[12] P. Erdős and A. Hajnal, *On chromatic number of graphs and set-systems*, Acta Math. Acad. Sci. Hungar. **17** (1966), 61–99.

[13] T. Feder and M.Y. Vardi, *Monotone monadic SNP and constraint satisfaction*, Proceedings of the 25rd Annual ACM Symposium on Theory of Computing (STOC), 1993, pp. 612–622.

[14] ———, *The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory.*, SIAM J. Comput. **28** (1999), no. 1, 57–104 (English).

[15] T. Gallai, *On directed graphs and circuits*, Theory of Graphs (Proc. Colloq. Tihany 1966), Academic Press, 1968, pp. 115–118.

[16] M. Hasse, *Zur algebraischen Begründung der Graphentheorie I*, Math. Nachrichten **28** (1964/5), 275–290.

[17] P. Hell and J. Nešetřil, *On the complexity of H-coloring*, J. Combin. Theory Ser. B (1990), no. 48, 92–110.

[18] ———, *Graphs and homomorphisms*, Oxford Lecture Series in Mathematics and its Applications, vol. 28, Oxford University Press, 2004.

[19] W. Hodges, *Model theory*, Cambridge University Press, 1993.

[20] N. Immerman, *Relational queries computable in polynomial time (extended abstract)*, Proceedings of the Fourteenth Annual ACM Symposium on Theory of Computing (New York, NY, USA), STOC '82, ACM, 1982, pp. 147–152.

[21] P. Komárek, *Good characterizations of graphs*, Ph.D. thesis, Charles University, Prague, 1988.

[22] M. Kreidler and D. Seese, *Monadic NP and graph minors*, Computer Science Logic, Lecture Notes in Computer Science, vol. 1584, Springer, 1999, pp. 126–141.

[23] D. Kühn and D. Osthus, *Every graph of sufficiently large average degree contains a C_4-free subgraph of large average degree*, Combinatorica **24** (2004), no. 1, 155–162.

[24] G. Kun and J. Nešetřil, *Forbidden lifts (NP and CSP for combinatorialists)*, European. J. Combin. **29** (2008), no. 4, 930–945.

[25] D. Lascar, *La théorie des modèles en peu de maux*, Cassini, 2009.

[26] L. Libkin, *Elements of finite model theory*, Springer-Verlag, 2004.

[27] J. Nešetřil and P. Ossona de Mendez, *Tree depth, subgraph coloring and homomorphism bounds*, European Journal of Combinatorics **27** (2006), no. 6, 1022–1041.

[28] ———, *Grad and classes with bounded expansion I. decompositions*, European Journal of Combinatorics **29** (2008), no. 3, 760–776.

[29] ———, *Grad and classes with bounded expansion III. restricted graph homomorphism dualities*, European Journal of Combinatorics **29** (2008), no. 4, 1012–1024.

[30] ———, *First order properties on nowhere dense structures*, The Journal of Symbolic Logic **75** (2010), no. 3, 868–887.

[31] ———, *From sparse graphs to nowhere dense structures: Decompositions, independence, dualities and limits*, European Congress of Mathematics, European Mathematical Society, 2010, pp. 135–165.

[32] ———, *On nowhere dense graphs*, European Journal of Combinatorics **32** (2011), no. 4, 600–617.

[33] ———, *Sparsity (graphs, structures, and algorithms)*, Algorithms and Combinatorics, vol. 28, Springer, 2012, 465 pages.

[34] J. Nešetřil and A. Pultr, *On classes of relations and graphs determined by subobjects and factorobjects*, Discrete Math. **22** (1978), 287–300.

[35] J. Nešetřil and C. Tardif, *Duality theorems for finite structures (characterizing gaps and good characterizations)*, Journal of Combinatorial Theory, Series B **80** (2000), 80–97.

[36] V. Rödl, *On the chromatic number of subgraphs of a given graph*, Proc. Amer. Math. Soc. **64** (1977), no. 2, 370–371.

[37] B. Rossman, *Homomorphism preservation theorems*, J. ACM **55** (2008), no. 3, 1–53.

[38] B. Roy, *Nombre chromatique et plus longs chemins d'un graphe*, Rev. Francaise Informat. Recherche Operationelle **1** (1967), 129–132.

[39] T.J. Schaefer, *The complexity of satisfiability problems*, STOC '78 Proceedings of the tenth annual ACM symposium on Theory of computing, 1978, pp. 216–226.

[40] C. Thomassen, *Girth in graphs*, Journal of Combinatorial Theory, Series B **35** (1983), 129–141.

[41] ———, *Five-coloring graphs on the torus*, Journal of Combinatorial Theory, Series B **62** (1994), no. 1, 11 – 33.

[42] M.Y. Vardi, *The complexity of relational query languages*, Proceedings of the 14th ACM Symposium on Theory of Computing, 1982, pp. 137–146.

[43] L.M. Vitaver, *Determination of minimal colouring of vertices of a graph by means of boolean powers of incidence matrix*, Doklady Akad. Nauk SSSR **147** (1962), 758–759.