Midsummer Combinatorial Workshop 2013

Dušan Knop (ed.)

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Preface

The 19th Prague Midsummer Combinatorial Workshop was held from July 29th to August 2nd 2013 in our beautiful building Malostranské náměstí 25. This of course contributed to the comfort of the participants as all the activities (including the lunches) could be taken on the same site. Besides, as it was expressed by several participants, the renovated faculty building surely belongs to the most beautiful math and computer science departments in the world! The workshop was organized by the Department of Applied Mathematics (KAM) of Charles University jointly with DIMATIA, CE-ITI and Computer Science Institute (IUUK) of Charles University. Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Peter J. Cameron among the participants. The list of speakers is included in this booklet.

As it already became a tradition, the workshop benefited from participation of young researchers and PhD students. For example four undergraduate students from the USA and five undergraduate students from Charles University, together with their mentors Glen Wilson from US side and Martin Balko from Prague side took part in the workshop, within the DIMATIA-DIMACS program International REU.

The workshop followed an informal daily routine with morning and early afternoon discussions and presentations. This report reflects some of the presentations during the workshop. Perhaps you can digest some of the atmosphere at the workshop from these proceedings, and you can also see that the fruitful exchange of ideas led directly to some new results and papers.

This volume was edited by Dušan Knop. Most of the contributions were supplied by the authors in an electronic form. In a few cases, slight typographical changes were necessary. We apologize for any possible inaccuracies which might have occurred in the editing process.

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We hope to meet again in 2014 the same midsummer week!

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Perfect digraphs: Answers and Questions

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Joint work with Winfried Hochstättler.

We consider some problems that occur when we generalize the notion of perfect graphs to digraphs using the notion of dichromatic number [6]. The *dichromatic number* $\vec{\chi}(D)$ of a digraph D is the smallest cardinality of a color set for a vertex coloring of D without monochromatic directed cycles. Undirected graphs are considered as symmetric digraphs. The *clique number* $\vec{\omega}(D)$ of D is the size of the largest symmetric clique in D. D is called *perfect* if, for any induced subdigraph H of D, $\vec{\chi}(H) = \vec{\omega}(H)$.

The *symmetric part* $S(D)$ of D is the graph of all symmetric arcs of D. A digraph D is a *superorientation* of a graph G if G is the underlying graph of D. A superorientation D of G is *clique-acyclic* if it does not contain complements of directed cycles. A *filled odd hole/antihole* is a digraph D, so that $S(D)$ is an odd hole/antihole.

We obtain the following characterization of perfect digraphs.

Theorem 1.1. *A digraph $D = (V, A)$ is perfect if and only if $S(D)$ is perfect and D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph.*

This result can be combined with several strong results to obtain some corollaries. Using the SPGT [3] we obtain a strong perfect digraph theorem.

Corollary 1.2. *A digraph $D = (V, A)$ is perfect if and only if it does neither contain a filled odd hole/antihole nor a directed cycle $\vec{C}_n$, $n \geq 3$, as induced subdigraph.*

Using the results of Grötschel, Lovász, and Schrijver [4] we obtain the following two complexity results.

Corollary 1.3. *There is a polynomial time algorithm to determine an induced acyclic subdigraph of maximum cardinality of a perfect digraph D.*

Corollary 1.4. *k-coloring of perfect digraphs is in $\mathcal{P}$ for any $k \geq 1$.*

Question 1.5. *Are there other interesting $\mathcal{NP}$-hard problems on digraphs that are polynomially solvable for perfect digraphs?*

Note that a digraph may be perfect but its complement may be not perfect. However, using the Weak Perfect Graph Theorem [5] we obtain

Corollary 1.6. *A digraph D is perfect if and only if its loopless complement $\overline{D}$ is a clique-acyclic superorientation of a perfect graph.*

Using the theorem on kernel solvability of perfect graphs [2] we obtain

Corollary 1.7. *For any perfect digraph D, its complement $\overline{D}$ has a kernel.*

The preceding result is in contrast to the following theorem.

Theorem 1.8. *It is $\mathcal{NP}$-complete to decide whether a perfect digraph has a kernel.*

Question 1.9. *Are there other problems that are $\mathcal{NP}$-complete or co-$\mathcal{NP}$-complete for digraphs in general as well as for perfect digraphs?*

The proof of the following intractibility result uses a similar reduction as for the $\mathcal{NP}$-completeness proof of recognizing digraphs containing a $\vec{C}_n$, $n \geq 3$, in [1].

Theorem 1.10. *The recognition of perfect digraphs is co-$\mathcal{NP}$-complete.*

Corollary 1.11. *The recognition of clique-acyclic superorientations of perfect graphs is co-$\mathcal{NP}$-complete.*

Question 1.12. *What is the complexity of recognizing superorientations of perfect graphs that have a kernel?*

Question 1.13. *Are there other interesting solvable problems on perfect graphs that have generalizations to perfect digraphs which are (co-)$\mathcal{NP}$-hard?*

We note two open questions that date back to the work of Neumann-Lara.

Question 1.14. *Determine the maximum dichromatic number of a tournament of order n.*

Conjecture 1.15. *Orientations of planar graphs are 2-colorable.*

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List Hadwiger conjecture and extremal K_5-minor-free graphs with fixed girth

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Joint work with David R. Wood.

We survey the results connected to the so called List Hadwiger conjecture that every K_t-minor-free graph is t-choosable. It is true for $t \leq 5$. Since there was no progress for larger values of t the following weak version was popularised:

There exists a constant c such that every K_t-minor-free graph is ct-choosable.

Kawarabayashi and Mohar explicitly stated that c might be $3/2$, Wood conjectured $c = 1$.

Barát, Joret and Wood proved that the List Hadwiger conjecture is false for $t \geq 8$ and $c \geq 4/3$ in the weak version.

We recall that there are K_6-minor-free graphs that are not 5-choosable. Mader proved that every K_6-minor-free graph is 7-degenerate and therefore 8-choosable by the greedy algorithm. We ask whether this can be improved and every K_6-minor-free graph is 6-degenerate and therefore 7-choosable. We also conjecture that possibly they are even 6-choosable.

There are numerous refinements of the same question with a constraint on the girth. We recall Mader's idea that large girth and minimum degree implies a large complete minor. In particular, every K_5-minor-free graph of girth at least 11 must have a vertex of degree 2. It is easy to improve this to girth 6. We ask the question whether girth 5 suffices here. Every such graph is 3-choosable by Thomassen's planar result and Wagner's theorem, which is an affirmative indication. However, the dodecahedron is a counterexample. We modify the question and ask whether every K_5-minor-free graph of girth 5 is either planar or have a vertex of degree 2. We show a construction refuting this modified conjecture.

Similar constructions are used to show the extremal number of this class of graphs; K_5-minor-free and girth 5.

Two of my favorite problems

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1 Minimizing monochromatic progressions

Problem 1.1. *Find the coloring of* $1, 2, \ldots, n$ *using* r *colors which has the fewest number of monochromatic* k*-term arithmetic progressions (i.e., sets of* $a, a+d, a+2d, \ldots, a+(k-1)d$ *all having the same color).*

As an example for $n = 28$, $k = 3$ and $r = 2$ the (unique!) best known pattern is `rrBBrrrBBBBBBrrrrrrBBBrrBBB`.

Note that van der Waerden's theorem shows that there must be at least one monochromatic k-term arithmetic progression for n sufficiently large. Frankl, Graham and Rödl extended this to show that in face there must be $xn^2 + o(n^2)$ monochromatic patterns (i.e., a positive fraction must be monochromatic).

Theorem 1.2 (Parrilo-Robertson-Saracino; Butler-Costello-Graham)**.** *Expanding the following coloring gives*

$$\frac{117}{2192}n^2 + O(n) = \frac{117}{137} \cdot \frac{1}{16}n^2 + O(n)$$

monochromatic 3*-APs:*

$$\underbrace{\mathtt{r}\cdots\mathtt{r}}_{28}\underbrace{\mathtt{B}\cdots\mathtt{B}}_{6}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{28}\underbrace{\mathtt{B}\cdots\mathtt{B}}_{37}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{59}\underbrace{\mathtt{B}\cdots\mathtt{B}}_{116}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{116}\underbrace{\mathtt{B}\cdots\mathtt{B}}_{59}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{37}\underbrace{\mathtt{B}\cdots\mathtt{B}}_{28}\underbrace{\mathtt{r}\cdots\mathtt{r}}_{6}\underbrace{\mathtt{B}\cdots\mathtt{B}}_{28}$$

It is conjectured that this is best possible. One way to approach this is to turn it from a discrete problem to a continuous problem; this was done in the paper of Butler, Costello and Graham to show that the above is locally optimal.

For $r = 2$ and $k = 4$ a very different type of coloring was found by Lu and Peng that has $\frac{1}{72}n^2 + O(n)$ monochromatic 4-APs. Namely, given $\ell = \sum b_i \cdot 11^i$ and j is the smallest index so that $b_j \neq 0$, then

$$\text{color } \ell \quad \begin{cases} \text{red} & \text{if } b_j = 1,\ 3,\ 4,\ 5, \text{ or } 9; \\ \text{blue} & \text{if } b_j = 2,\ 6,\ 7,\ 8, \text{ or } 10. \end{cases}$$

More generally, the following is believed true and has been verified for small cases; a general approach is not currently known.

Conjecture 1.3. *For fixed k (=AP-length) and r (=colors) there is a coloring of $1, \ldots, n$ which can* beat *random coloring.*

2 Induced universal graphs

Problem 2.1. *Given a family $\mathcal{F}$ of graphs, construct a* small *graph F which contains each graph in $\mathcal{F}$ as an induced subgraph.*

This has been done for several families. Notably Moon did it for the family of all graphs and showed that the smallest such universal graph (in terms of number of vertices N) has $2^{(n-1)/2} < N < 2n2^{(n-1)/2}$. Chung made major impact by looking at trees, planar graphs, and graphs with bounded arboricity. In particular she established the following.

Theorem 2.2. *Let F be an induced universal graph for $\mathcal{F}$. If every graph in $\mathcal{H}$ can be edge-partitioned into k graphs in $\mathcal{F}$, then there is an induced universal graph H where*

$$|V(H)| \leq |V(F)|^k \text{ and } |E(H)| \leq k|E(F)||V(F)|^{2k-2}.$$

This theorem can be easily extended to multigraphs, directed graphs, hypergraphs, etc. In particular once with a good decomposition then we can construct good universal graphs. For example, Butler used decomposition of regular graphs of even degree k into cycles to construct universal graphs for graphs with bounded degree using $Cn^{\lceil k/2 \rceil}$ vertices, which for k even is within a constant of optimal.

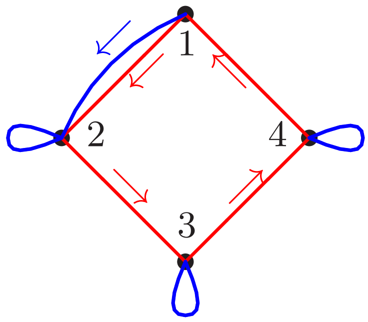
Almost no work has been done in this area for hypergraphs. A starting point would be to find a decomposition theorem into smaller, more manageable hypergraphs.

Synchronization

Peter J. Cameron

A (finite deterministic) *automaton* has a finite set of *states* and a finite set of *transitions*, each a map on the set of states. Think of a black box with coloured buttons on front, which changes its state in a certain way each time a button is pressed.

The automaton is *synchronizing* if there is a sequence of transitions which brings it into the same state from any starting state; such a sequence is called a *reset word*. Here is an example.



It can be checked easily that BRRRBRRRB is a reset word of length 9. In fact, this is the shortest reset word.

The *Černý Conjecture* asserts that if an n-state automaton is synchronizing, then it has a reset word of length at most $(n-1)^2$. The above example, with the square replaced by an n-gon, shows that this would be best possible. The problem has been open for about 45 years. The best known bound is cubic.

It is known that testing whether an automaton is synchronizing is in P, but finding the length of the shortest reset word is NP-hard.

Since we can compose transitions, it makes sense to study the *transformation semigroup* generated by the transitions; the automaton is synchronizing if and only if this semigroup contains an element of rank 1, in which case we say that the semigroup is *synchronizing*.

There is just one kind of obstruction to synchronization. A *graph* here has no loops, multiple edges, or directed edges.

Theorem 1.1. *A transformation semigroup S is non-synchronizing if and only if there is a non-null graph Γ on the set of states such that*

- *S is contained in the semigroup of endomorphisms of Γ;*
- *the clique number and chromatic number of Γ are equal.*

Recently João Araújo and I (and others) have been studying the case where all but one of the transitions are permutations: that is, S is generated by a permutation group G and a single non-permutation f. I will say that G *synchronizes* f if $\langle G, f \rangle$ is synchronizing.

Rystsov showed that a permutation group G is *primitive* (that is, preserves no non-trivial equivalence relation on the domain) if and only if it synchronizes every map of rank $n - 1$. This theorem has a particularly simple proof using the above theorem about graph endomorphisms.

It is not true that a primitive group synchronizes every non-permutation. Indeed, the class of such "synchronizing" groups is contained between the classes of primitive and doubly transitive groups. Both of these classes have recognition algorithms in P, but this is not known for the class of synchronizing groups. (The best algorithm we have does the following: given the primitive group G, find all the G-invariant graphs, and check each of them to see whether its clique number and chromatic number are equal. If we find one, the group is non-synchronizing.)

However, Araújo made the following bold conjecture:

Conjecture If the permutation group G is primitive, then it synchronizes every *non-uniform* map (one for which the inverse images of points in the image do not all have the same size).

I concluded the talk with a brief discussion of the (small amount of) progress we have made on this conjecture.

The Erdos-Hajnal Conjecture, product tournaments and the strong EH property

Krzysztof Choromanski

The celebrated and still unresolved Erdos-Hajnal Conjecture states that for every undirected graph H there exists constant $\epsilon(H) > 0$ such that every n-vertex undirected graph G that does not have H as an induced subgraph contains a clique or a stable set of size at least $n^{\epsilon(H)}$.

Some time ago its directed version was formulated. The directed version is equivalent to the undirected one and states that for every tournament H there exists $\epsilon(H) > 0$ such that every n-vertex H-free tournament T contains a transitive subtournament of size at least $n^{\epsilon(H)}$.

In this talk we present some very recent results concerning the directed version of the Conjecture.

In particular we introduce a very useful definition of the strong EH-property and using it prove the Conjecture for new classes of tournaments containing infinitely many prime tournaments. Furthermore, we introduce new procedure for combining two tournaments satisfying the conjecture to get the bigger one that also satisfies it. This method is different that the so-called substitution procedure (the only method of that flavour known before) because using it one can produce infinitely many prime tournaments satisfying the Conjecture. Since prime tournaments are crucial in the research on the conjecture, this new method may potentially help to prove the Conjecture for many families of tournaments for which it is still open.

It has been already used by us to prove that all but at most one 6-vertex tournaments satisfy the Conjecture.

Restricted degree sequences

Péter L. Erdős

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With the current burst of network theory research (especially in connection with social and biological networks) there is a renewed interest on realizations of given degree sequences and uniform sampling of those realizations. In this paper we propose a new degree sequence problem: we want to find graphical realizations of a given degree sequence on labeled vertices, where certain would-be edges are *forbidden*. Then we want to sample uniformly all possible realizations.

More precisely let's fix a labeled underlying vertex set V of n elements. The *degree sequence* $\mathbf{d}(G)$ of a *simple* graph $G = (V, E)$ is the sequence of its vertex degrees: $\mathbf{d}(G)_i = d(v_i)$. Call $\mathcal{F} \subset \binom{V}{2}$ the set of **forbidden** edges. The **restricted degree sequence** problem (or RDS for short) $\mathbf{d}^{\mathcal{F}}$ is to find a graphical realization G of $\mathbf{d}$ which completely avoids the elements of $\mathcal{F}$.

It is easy to see that the restricted degree sequence problem is very closely related to Tutte's f-factor theorem. However, while Tutte's approach provides a polynomial time algorithm to decide wether a given degree sequence satisfies a $\mathbf{d}^{\mathcal{F}}$ problem, it does not provide all possible realizations.

If $\mathcal{F}$ is empty then the RDS simplifies to the original degree sequence problem, which can be solved efficiently with Havel's greedy algorithm ([4]). A slightly modified algorithm can provide all possible realizations ([6]).

When G_1 and G_2 are two realizations of $\mathbf{d}$ one can prove that G_1 can be transformed into G_2 by the means of **swap** operation through a series of valid graphic realizations. (It was proved already in [7]). A similar statement applies for all RDS problem $\mathbf{d}^{\mathcal{F}}$ through a suitable generalization of the swap operation, called **$\mathcal{F}$-swap** (see [1]).

In the paper [2], as a first step, we solved the RDS problem if the forbidden edges form the union of a (not necessarily maximal) 1-factor and a (possible empty) star in a bipartite graph. (Here $\mathcal{F}$ consists of the forbidden edges among the vertex classes only.)

As it turns out the RDS problem with a **factor + star** forbidden set can be solved by the means of a slightly modified Havel-type swap operation. This new operation makes the space of all possible realizations connected, and - using standard technics - one can show the fast mixing nature of the natural Markov chain Monte Carlo sampling method in case of **half-regular** bipartite graphs. (In a half-regular bipartite graph in one class the vertices are uniform.)

This result is a common generalization of the well-known theorem of Kannan, Tetali and Vempala (on sampling regular bipartite graphs, [5]); a recent result of Greenhill (on sampling regular directed graphs, [3]), and others, providing new proofs for them ([2]). More importantly this new generalization is a **self-reducible** problem which ensures that all the mentioned Markov chain methods can efficiently count the approximate number of the corresponding realizations.

Open problem: Find other forbidden edge set $\mathcal{F}$ for which the restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$ can be solved with a Havel-type greedy algorithm.

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Boundary value problems on a weighted path

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Joint work with A.M. Encinas and A. Carmona.

We determine explicit expressions for the Green matrix associated with any regular boundary value problem on a weighted path, via orthogonal polynomials. The weights are determined by the coefficients of the three terms recurrence relation defining the polynomials. We use similar techniques to the ones for solving boundary value problems associated with ordinary differential equations.

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A red-blue intersection problem

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Joint work with C. Cortés, M.A. Garrido, C.I. Grima, A. Márquez, A. Moreno-Gonzĺez, J. Valenzuela, and M.T. Villar.

A geometric intersection problem that has been intensively studied is the *bichromatic segment intersection problem*: given two sets of segments in the plane, say red and blue, whose total size is n, report all the intersections between red segments and blue segments. These red-blue intersections are called *bichromatic intersections*. In the case where monochromatic intersections exist (i.e., intersections between segments having the same color) which is considered to be the difficult case, Agarwal [1] and Chazelle [3] showed that the k bichromatic intersections can be reported in $O(k+n^{4/3}\log^{O(1)} n)$ time using a partitioning technique called *cuttings*. See also [2, 4, 5] for more information about this topic.

In this work, we introduce a variation of the bichromatic segment intersection problem in the case where monochromatic intersections exist. Instead of reporting all bichromatic intersections between two sets of colored segments, we are given two sets of colored points defining two sets of colored segments, and study the problem of reporting the set of segments of each color intersected by segments of the other color.

Let R and B be two disjoint sets of n_r red points and n_b blue points in the plane, respectively, such that no three points of $R \cup B$ lie on the same line. Let $n = n_r + n_b$. A line segment defined by two red points is a *red segment*, and that defined by two blue points is a *blue segment*. Let S_b be the set of blue segments that intersect at least one red segment, and let S_r be the set of red segments crossed by at least one blue segment.

Theorem 1.1. *The sets S_b and S_r can be computed in $O(n^2)$ time and space.*

Theorem 1.2. *Computing $|S_b|$ and $|S_r|$ is 3-Sum hard.*

Corollary 1.3. *Computing S_b and S_r is 3-Sum hard.*

[1] Supported by projects 2009/FQM-164 and 2010/FQM-164.

As a natural extension of our study, it would be interesting to consider the problem in 3D, i.e., given two sets of points in 3D, consider the same problem but using monochromatic triangles instead of segments. The goal is to improve the trivial brute force algorithm for the problem.

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Cops and robbers on infinite graphs

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The original cop-and-robber game is played by two players on an undirected reflexive graph (connected, no multiple edges) in *rounds*, each of which consists of two *moves*, first one by the cop, then one by the robber. At round 0 the cop chooses a vertex, then the robber chooses one. At round $i > 0$, first the cop moves to a vertex adjacent to her current vertex, then the robber moves to a neighbour of his current vertex. The object of the game is for the cop to occupy the same vertex as the robber, and for the robber to escape such a situation. For finite graphs, the graphs on which the cop has a winning strategy (called *cop-win*) were characterised in [8] and, independently, in [9, 10, 11]. The characterization depends strongly on the existence of retractions implied by the cop's winning strategy. It follows from the characterisation that chordal and bridged graphs are cop win. A second characterization is given in [8] that also applies to infinite graphs and leads to a simple pseudo-polynomial algorithm that can check in time in $O(n^{5(k+\ell)})$ whether k cops can catch ℓ robbers on a general oriented graph[2] with n vertices and give their optimal strategies. Is not obvious to check the condition for infinite graphs, even for one cop and one robber.

Anstee and Farber asked if (countably) infinite bridged graphs are cop-win. Obviously not - trees are chordal, hence bridged, and a ray (one-way infinite path) is clearly not cop-win. But there is a stronger statement to be made, [6]. While Theorem 0.1 and its generalisation to arbitrary cardinalities in Theorem 0.3 can be proven directly by construction, it follows easily by compactness from Theorem 0.2, whose proof is more instructive, if also long.

Theorem 0.1. *There is a countably infinite chordal graph of diameter two that is not cop-win.*

Theorem 0.2. *For every $k \in \mathbb{N}$ there is a chordal graph of diameter two on which the cop needs at k rounds to catch the robber.*

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[2]A *general oriented graph* is a directed graph obtained from a reflexive undirected graph by replacing each edge by a pair of symmetric arcs and each loop by an arc, and then removing some of the resulting arcs.

Theorem 0.3. *For every infinite cardinal κ there is a chordal graph diameter two on κ vertices that is not cop-win.*

It is, therefore, interesting to search for infinite cop-win graphs. Such graphs must have vertices of infinite degree since, as Gavenčiak observed (private communication), infinite locally finite graphs cannot be cop-win (think breadth-first-search and König's lemma). Finding non-trivial infinite cop-win graphs is surprisingly difficult. One recent example was constructed by Anna Lubiw and Hamideh Vosoughpour based on visibility in polygons.

One obvious candidate to consider is the random graph R. It has the property that for any finite disjoint subsets A, B of vertices there is a vertex $u_{A,B}$ adjacent to all the vertices of A and no vertices of B. This clearly allows a robber to escape from a finite number of cops, so the graph is not cop-win and requires $\aleph_0$ cops to catch a robber. A reasonable question can then be asked: how dense (sic) do the cops have to be to win? It turns out, seem [3], that if the density (*cop-density*) is defined as $\lim_{i<\omega} \frac{c(G_i)}{|V(G_i)}|$ with G_i an induced subgraph of G_{i+1}, $R = (\cup_{i<\omega} V(G_i), \cup_{i<\omega} E(G_i))$ and $c(G_i)$ the number of cops necessary and sufficient to catch a robber on G_i, then any real number in $[0, 1]$ can be realised as cop-density even if we insist, reasonably, that each G_i be connected.

An interesting class of examples can be obtained by taking powers of cop-win graphs, be they finite or infinite. For a graph G and an infinite cardinal $\kappa \geq |V(G)|$, the κ-power ${}^{\kappa}G$ of a graph G has as its vertices the functions from $\kappa \times V(G)$ into $V(G)$ that differ in only finitely many coordinates from the canonical base function (projection) $b_G : \kappa \times V(G) \longrightarrow V(G)$ defined by $b_G(\alpha, v) = v$. This is based on a construction of Imrich from the 1980's. Perhaps surprisingly, a κ-power of G is vertex-transitive (see [2] whose proof for *canonical* powers extends) and (work in progress) is are cop-win if and only if G is. This contrasts with (an easy exercise) the fact that a finite vertex-transitive graph is never cop-win.

The κ-powers of a graph have other interesting properties. For example, ${}^{\kappa}P_3$ (a path on three vertices) already contains all countable graphs and no uncountable star (even if κ is uncountable). Yet it is not homogeneous since then it would be the Rado graph which we have seen is not cop-win.

This writing is in the hope of generating some interest in infinite graph theory in general and cop-and-robbers on infinite graphs in particular. It must, therefore, be concluded by some open questions that either are directly related to cop-and-robber games, or are off-shoots of research into them.

1. Find more classes of (non-trivial) infinite cop-win graphs.

2. For each $k < \omega$, find classes of infinite graphs G with $c(G) = k$, if possible. Perhaps an infinite graphs requires either one or infinitely many cops to catch a single robber.

3. Can (finite or infinite) cop-win graphs on which the cop wins in at most (exactly?) $d > 1$ rounds be characterised?

4. Let P be a property of graphs and let G be a countable graph with P. Find examples of P such that every finite subset of $V(G)$ is contained in a finite induced subgraph of G that also has P. One simple example is chordality: every finite subset of an infinite chordal graph is itself chordal. It is also true, but more difficult to prove, for bridgeness, see [5]. The origin of this probleme is [6].

5. Characterise properties P described above.

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Polynomial graph invariants from homomorphism numbers

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The number of homomorphisms $\hom(G, K_k)$ from a graph G to the complete graph K_k is the value of the chromatic polynomial of G at a positive integer k. This motivates:

Definition 1.1 ([2])**.** A sequence of graphs $(H_{\mathbf{k}})$, $\mathbf{k} = (k_1, \ldots, k_h) \in \mathbb{N}^h$, is *strongly polynomial* if for every graph G there is a polynomial $p(G; x_1, \ldots, x_h)$ such that $\hom(G, H_{\mathbf{k}}) = p(G; k_1, \ldots, k_h)$ for every $\mathbf{k} \in \mathbb{N}^h$.

Many important graph polynomials $p(G)$ are determined by strongly polynomial sequences of graphs $(H_{\mathbf{k}})$: e.g. the Tutte polynomial, the Averbouch–Godlin–Makowsky polynomial [1] (which includes the matching polynomial) and the Tittmann–Averbouch–Godlin polynomial [4] (which includes the independence polynomial).

We are interested here in how to construct strongly polynomial sequences from basic building blocks, rather than verifying whether or not a given sequence of graphs – such as that of hypercubes (Q_k) – is strongly polynomial.

Proposition 1.2 ([2] and [3])**.** *If (H_k) is strongly polynomial, and each H_k simple, then the complements $(\overline{H_k})$ and line graphs $(L(H_k))$ are strongly polynomial. Also, (ℓH_k) is strongly polynomial in k, ℓ.*

For graph H, we let H^ℓ denote the same graph but with ℓ loops attached to each vertex (if H already has loops, then these are first deleted before adding the ℓ loops).

Proposition 1.3 ([3])**.** *If (H_k) is strongly polynomial, with at most one loop on each vertex of H_k, then (H_k^0) and (H_k^1) are strongly polynomial. More generally, (H_k^ℓ) is strongly polynomial in k, ℓ.*

Proposition 1.4 ([2] and [3])**.** *If (F_j), (H_k) are strongly polynomial, then the disjoint unions $(F_j \cup H_k)$, joins $(F_j + H_k)$, direct products $(F_j \times H_k)$, and lexicographic products $(F_j[H_k])$ are all strongly polynomial in j, k.*

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Example 1.5. *Beginning with the trivial strongly polynomial sequence* (K_1)*, the following are strongly polynomial:*

(i) multiple: $(kK_1) = (\overline{K_k})$

(ii) complement: (K_k) *(chromatic polynomial)*

(iii) loop-addition*:* (K_k^ℓ) *(Tutte polynomial)*

(iv) join: $(K_{k-j}^1 + K_j^\ell)$ *(Averbouch–Godlin–Makowsky polynomial)*

The main contribution of our paper [3] is to give a new method of constructing strongly polynomial sequences of graphs. (This method can be used to produce the graph sequence determining the Tittmann–Averbouch–Makowsky polynomial.)

We start with a simple graph H encoded by a coloured rooted tree T. For each vector of edge variables $\mathbf{k} = (k_s : s \in E(T)) \in \mathbb{N}^{|E(T)|}$ corresponds another coloured rooted tree $T^{\mathbf{k}}$, which is recursively constructed from T using the following operation of "branching" until no more edge variables remain.

Definition 1.6 (k-branching)**.** Let $s = uv$ be an edge of a coloured rooted tree with endpoint u nearest the root. Suppose s has been assigned edge variable k. To *branch* at s, remove the edge variable k from s and create k isomorphic copies of the subtree T_v of T rooted at v, each pendant from u by a copy of edge s. (Isomorphism here includes colours on vertices in $V(T_v)$ and any remaining variables on edges in $E(T_v)$.)

For the statement of our main theorem we restrict attention to the following ways of representating a graph H by a coloured rooted tree T:

(i) H as a subgraph of the closure of T: colour $v \in V(T) = V(H)$ with a subset of $\{0, 1, \dots, \text{height}(T)\}$ (consisting of levels in T of vertices u on the path from v to the root of T such that $uv \in E(H)$);

(ii) ornamented version of (i): strongly polynomial sequence $(F_{v;j_v})$ for each vertex $v \in V(H)$, colour as in (i) paired with $F_{v;j_v}$;

(iii) cotree T encoding cograph H: colour non-leaf of T from $\{\cup, +\}$, leaves of T comprise $V(H)$.

Theorem 1.7 ([3]). *Let T be a coloured rooted tree representing a graph H under one of the schemes (i), (ii) or (iii) above, and let $\mathbf{k} = (k_s : s \in E(T))$ be branching variables on the edges of T. Let $H_{\mathbf{k}}$ be the graph represented by the coloured rooted tree $T^{\mathbf{k}}$ obtained from T after recursively k_s-branching on edge s for each $s \in E(T)$.*

Then $(H_{\mathbf{k}})$ is strongly polynomial in $\mathbf{k}$.

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Graph Homomorphisms Through Graph Powers

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Abstract

In this talk, we introduce two kinds of power for graphs [2, 3]. First, for a given graph G, we consider $G^{\frac{r}{s}}$, i.e., the rth power of the sth subdivision of G, and we present some basic properties of this power.

In the sequel, we introduce the graph power $G^{\widetilde{\frac{2s+1}{2r+1}}}$. We show that these powers can be considered as the dual of each other. Precisely, we show that

$$G^{\frac{2r+1}{2s+1}} \longrightarrow H \Longleftrightarrow G \longrightarrow H^{\widetilde{\frac{2s+1}{2r+1}}}.$$

Next, we review some coloring properties of graph powers [4]. In this regard, we show that if $\frac{2r+1}{2s+1} \leq \frac{\chi_c(G)}{3(\chi_c(G)-2)}$, then $\chi_c(G^{(2r+1)/(2s+1)}) = \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1}$. Also, we present an upper bound for the fractional chromatic number of subdivision graphs. Precisely, we show that $\chi_f(G^{1/(2s+1)}) \leq \frac{(2s+1)\chi_f(G)}{s\chi_f(G)+1}$.

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Reconstructing Rooted Phylogenetic Networks

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Let X be a finite set. A *(rooted) phylogenetic tree* on X is a rooted tree with no indegree-1 outdegree-1 vertices whose leaves are bijectively labelled by the elements of X. Phylogenetic trees are especially popular in biology, where the set X represents a collection of species and the tree describes a hypothesis of how these species are related. To model more complex genealogical relationships, the following more general model has been introduced. A *(rooted) phylogenetic network* on X is a rooted directed acyclic graph with no indegree-1 outdegree-1 vertices whose leaves are bijectively labelled by the elements of X. To quantify the amount of reticulation (i.e. non-treelike evolutionary events), the *reticulation number* of a phylogenetic network N is defined as

$$\sum_{v \in V(N) \setminus \{\text{root}\}} d^-(v) - 1.$$

A "tree-based" approach for constructing phylogenetic networks is to first construct a phylogenetic tree for each gene, and then to find a phylogenetic network that "contains" each of these trees. To formalize this, we say that a phylogenetic tree T is *displayed* by a phylogenetic network N if T can be obtained from a subgraph of N by contracting edges. Given two rooted phylogenetic trees T_1 and T_2, the problem Minimum Reticulation is to find a rooted phylogenetic network with minimum reticulation number that displays T_1 and T_2.

Recently, we showed that there exists a constant factor approximation algorithm for Minimum Reticulation if and only if there exists such an algorithm for Directed Feedback Vertex Set [2]. However, whether such an algorithm exists is still an open question.

Much less is known about the generalization of this problem to more than two phylogenetic trees. In particular, there is no algorithm with running time $O(c^{|X|})$ known, for any constant c, even for instances consisting of three binary phylogenetic trees. In addition, it is not known whether the problem is fixed parameter tractable when the number of phylogenetic trees and their degrees are unrestricted (see [3]).

A "character based" approach for constructing phylogenetic networks aims to construct them directly from character data, optimizing e.g. the "parsimony score". A p-state *character* on a set S is a function $\alpha : S \to \{1, \ldots, p\}$. Given a phylogenetic network N and a p-state character τ on $V(N)$, the *change* $c_\tau(e)$ on edge $e = (u, v)$ of N w.r.t. τ is 0 if $\tau(u) = \tau(v)$ and 1 otherwise. We distinguish between two variants of the parsimony score of phylogenetic networks, both of which generalize the well-known parsimony score of phylogenetic trees (which can be computed in polynomial time).

The *hardwired parsimony score* of a phylogenetic network N on X and p-state character α on X is given by

$$PS_{\text{hw}}(N, \alpha) = \min_{\tau} \sum_{e \in E(N)} c_\tau(e),$$

where the minimum is taken over all p-state characters τ on $V(N)$ that extend α.

The *softwired parsimony score* of a phylogenetic network N and p-state character α is given by

$$PS_{\text{sw}}(N, \alpha) = \min_{T \in \mathcal{T}(N)} PS_{\text{hw}}(T, \alpha),$$

where $\mathcal{T}(N)$ is the set of phylogenetic trees on X displayed by N.

Computing the hardwired parsimony score is closely related to the problem Minimum Multiterminal Cut. Consequently, it is polynomial-time solvable for binary characters and, for general p-state characters, 1.3438-approximable and fixed-parameter tractable (FPT) in the parsimony score.

Computing the, biologically more relevant, softwired parsimony score of a phylogenetic network is, unfortunately, much harder. There is no polynomial-time approximation algorithm that approximates $PS_{sw}(N, \alpha)$ to a factor $|X|^{1-\epsilon}$, for a rooted phylogenetic network N and a binary character α, for any constant $\epsilon > 0$, unless P = NP [1].

Some interesting open questions remain. First of all, are there any restricted classes of networks for which computing the (hardwired or softwired) parsimony score is easier? In particular, is there a polynomial-time algorithm for networks with bounded treewidth (of the underlying undirected graph)? In addition, is there an approximation algorithm for computing the softwired parsimony score of a binary phylogenetic network that is *tree-child*, i.e. in which each non-leaf vertex has at least one child with indegree 1? Furthermore, can we go beyond computing the parsimony score of a given network, and search for a network with optimal parsimony score (e.g. for a fixed reticulation number)?

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Splittability of Permutation Classes

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Joint work with Pavel Valtr.

We say that a permutation π is *merged* from permutations ρ and τ, if we can color the elements of π red and blue so that the red elements are order-isomorphic to ρ and the blue ones to τ. Claesson, Jelínek and Steingrímsson [4] have shown that every permutation π that avoids 1324 can be merged from a permutation avoiding 132 and a permutation avoiding 213. From this, it follows that there are at most 16^n 1324-avoiding permutations of order n. This argument can be extended to more general patterns, showing in particular that if σ is a layered pattern of size k, then there are at most $(2k)^{2n}$ σ-avoiding permutations of size n. This bound is again based on an argument showing that a permutation avoiding a certain pattern can be merged from permutations avoiding smaller patterns.

Motivated by these results, we introduce the concept of splittability of permutation classes; we say that a hereditary permutation class C is *splittable* if it has two proper hereditary subclasses A and B such that every element of C can be obtained by merging an element of A with an element of B. We address the general problem of identifying which permutation classes are splittable. We mostly focus on *principal classes*, i.e., classes defined by avoidance of a single forbidden pattern, although some of our results are applicable to general hereditary classes as well.

On the negative side, we show that every permutation class closed under inflations is unsplittable. This implies, in particular, that if σ is a simple permutation, then the class $\mathrm{Av}(\sigma)$ of all σ-avoiding permutations is unsplittable. We also find examples of unsplittable classes that are not closed under inflations, e.g., the class of layered permutations or the class of 132-avoiding permutations.

On the positive side, we show that if σ is a direct sum of two nonempty permutations and has size at least four, then $\mathrm{Av}(\sigma)$ is splittable. This extends the results of Claesson et al. [4], who address the situation when σ is a direct sum of three permutations, with an extra assumption on one of the three summands.

Apart from these results, we will also show that splittability is closely related to other previously studied structural properties of classes of relational structures. In particular, splittability is related to the notions of Ramseyness and amalgamation [2, 3, 7].

We also establish a less direct, but perhaps more useful, connection between splittability and coloring of circle graphs. Let σ_k be the permutation $k(k-1)\cdots 32(k+1)$ of size $k+1$. From our general results, it follows that all σ_k-avoiding permutations can be merged from a bounded number, say $f(k)$, of 213-avoiding permutations. Moreover, we prove that the smallest such $f(k)$ is equal to the smallest number of colors needed to properly color every circle graph with no clique of size k. We may therefore exploit previous results on colorings of circle graphs [1, 6, 5] to deduce results on splittability of permutation classes.

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Arrangements of pseudocircles and circles

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An arrangement of pseudocircles is a finite collection of Jordan curves in the plane with the additional properties that **(i)** every two curves meet in at most two points; and **(ii)** if two curves meet in a point p, then they cross at p.

We say that two arrangements $\mathcal{C} = (c_1, \ldots, c_n)$ and $\mathcal{D} = (d_1, \ldots, d_n)$ are equivalent if there is a homeomorphism φ of the plane onto itself such that $\varphi[c_i] = d_i$ for all $i \in \{1, \ldots, n\}$. Linhart and Ortner (2005) gave an example of an arrangement of five pseudocircles that is not equivalent to an arrangement of circles, and they conjectured that every arrangement of at most four pseudocircles is equivalent to an arrangement of circles. Here we prove their conjecture.

We consider two related recognition problems. The first is the problem of deciding, given a (combinatorial description of a) pseudocircle arrangement, whether it is equivalent to an arrangement of circles. The second is deciding whether it is equivalent to an arrangement of convex pseudocircles. We prove that both problems are NP-hard, answering questions of Bultena, Grünbaum and Ruskey (1998) and of Linhart and Ortner (2008).

We also give an example of an arrangement of convex pseudocircles with the property that its intersection graph (i.e. the graph with one vertex for each pseudocircle and an edge between two vertices if and only if the corresponding pseudocircles intersect) cannot be realised as the intersection graph of a family of circles. This disproves a folklore conjecture communicated to us by Pyatkin.

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Open questions on matroids and list colouring

Tamás Király

The list edge-colouring conjecture states that the list edge-chromatic number $\chi'_l(G)$ equals the edge-chromatic number $\chi'(G)$ for every loopless graph G. In other words, if every edge of G has a list of $\chi'(G)$ possible colours, then it is possible to choose a proper edge-colouring from the lists. The bipartite case was solved by Galvin in 1995 [2], but the general conjecture is still open.

In this talk we discuss possible generalizations of Galvin's theorem to matroids, and some related open questions on matchings. Our starting point is the following result of Seymour [5].

Theorem 1.1. *Let $M = (S, \mathcal{I})$ be a matroid where S can be covered by k independent sets, and let L_s be a list of k colours for every $s \in S$. It is possible to choose a colour from each list such that every monochromatic set is independent.*

It is tempting to formulate a common generalization of this result and Galvin's theorem by extending it to matroid intersection. Given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on a common ground set S, we can define their chromatic number $\chi(M_1, M_2)$ as the minimum number of common independent sets that can cover S. The list chromatic number $\chi_l(M_1, M_2)$ is the smallest number k such that given arbitrary lists L_s of k colours ($s \in S$), it is possible to choose a colour from each list such that every monochromatic set is a common independent set.

Question 1.2. *For which matroid pairs (M_1, M_2) is it true that $\chi_l(M_1, M_2) = \chi(M_1, M_2)$?*

We are not aware of any matroid pair where equality does not hold; on the other hand, there are very few classes of matroids where the answer is known. Galvin's theorem means that equality always holds for two partition matroids; some other simple cases are discussed in [3], where it is observed that Galvin's theorem implies $\chi_l(M_1, M_2) = \chi(M_1, M_2)$ for any two transversal matroids. The following are some interesting special cases where the question is still open:

- two strongly base orderable matroids,

- a partition matroid and a graphic matroid,
- a special case of the previous one: colouring of a digraph where each colour class must be a branching,
- $\chi(M_1, M_2) = 2$ (that is, lists of size 2),
- if the total number of colours is $\chi(M_1, M_2) + 1$, i.e. each element has a single forbidden colour.

Let us mention a special case of this last problem. Let M_1 be a matroid of rank $k+1$ that can be partitioned into k disjoint bases $B_1, \dots, B_k$, and let $S_1, \dots, S_{k+1}$ be disjoint transversals of the family $B_1, \dots, B_k$. If M_2 is the partition matroid defined by classes $S_1, \dots, S_{k+1}$, and the forbidden colour of elements in S_i is the i-th colour, then we obtain the following conjecture.

Conjecture 1.3. *In the above setting, there exist disjoint independent sets $I_1, \dots, I_{k+1}$ of M_1 such that*

$$|I_i \cap S_j| = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

Notice that such sets exist if the family $B_1, \dots, B_k$ has a transversal that is a common independent set of M_1 and M_2. Indeed, let I be the transversal; then the sets $I, B_1 \setminus I, \dots, B_k \setminus I$, taken in an appropriate order, satisfy the conditions of the conjectureecture. In fact, the existence of such a transversal is conjectured by Aharoni et al. [1] for any pair of matroids.

Conjecture 1.4 ([1])**.** *Let M_1 and M_2 be two matroids of rank $k+1$ on ground set S, and suppose that S can be partitioned into common bases $B_1, \dots, B_k$. Then the family $B_1, \dots, B_k$ has a transversal that is independent in both matroids.*

It should be noted that contrary to the list colouring conjecture, this is open even for matchings in bipartite graphs:

Conjecture 1.5 (Aharoni, Berger)**.** *If a bipartite graph is the disjoint union of k matchings $M_1, \dots, M_k$ of size $k+1$, then it has a matching that contains exactly one edge from each of $M_1, \dots, M_k$.*

Kotlar and Ziv [4] proved the approximate result that such a matching exists if $M_1, \dots, M_k$ all have size $\lfloor \frac{5}{3}k \rfloor$. Finally, let us conclude with another possible generalization of Galvin's theorem that is related to the above conjectures on matroids, but does not seem to be implied by them.

Conjecture 1.6. *Let G be a bipartite graph, and k a positive integer. If every edge has a list of k possible colours, then we can choose edge colours so that every node v is incident to at least $\min\{d_G(v), k\}$ different colours.*

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Domination Game

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The *domination game* is played on a graph G by two players, named Dominator and Staller. They alternatively select vertices of G such that each chosen vertex enlarges the set of vertices dominated before the move on it. Dominator's goal is that the game is finished as soon as possible, while Staller wants the game to last as long as possible. It is assumed that both play optimally. Game 1 and Game 2 are variants of the game in which Dominator and Staller has the first move, respectively. The *game domination number* $\gamma_g(G)$, and the *Staller-start game domination number* $\gamma_g'(G)$, is the number of vertices chosen in Game 1 and Game 2, respectively. The game was introduced in [1] and studied afterwards in several papers. In this talk we will present main results obtained so far and point out to intrinsic differences between the game domination number and usual domination number.

A fundamental result, one of which was proved [2] and the other half in [5], asserts that for any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$, holds. The theorem will be discussed and tools for its proof presented. In view of the theorem, a pair (r, s) of integers is called *realizable* if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma_g'(G) = s$. By the theorem, only possible realizable pairs are: (r, r), $(r, r+1)$, $(r, r-1)$. As proved in [6], pairs $(r, r), r \geq 2$, $(r, r+1), r \geq 1$, and $(2k, 2k-1), k \geq 2$, can actually be realized by 2-connected graphs, while pairs $(2k+1, 2k), k \geq 2$ are realizable by connected graphs [6]. On the other hand, for any integer $\ell \geq 1$, there exists a graph G and its spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$ [2]. IN the same paper it was moreover proved that for any $m \geq 3$ there exists a 3-connected graph G_m and its 2-connected spanning subgraph H_m such that $\gamma_g(G_m) \geq 2m - 2$ and $\gamma_g(H_m) = m$.

In the second part of the talk some open problems will be discussed. Two 3/5-conjectures from [5] will be presented. The first asserts that for an n-vertex forest T without isolated vertices, $\gamma_g(T) \leq \frac{3n}{5}$ and $\gamma_g'(T) \leq \frac{3n+2}{5}$, while the second asserts that the same conclusions hold for any connected graph. The first conjecture has been very recently (cf. [4]) proved for forests in which no two leaves are at distance 4. Extremal families of graphs for these 3/5-conjectures constructed in [3] will be presented. The talk will be concluded with the following algorithmic problems: (1) What is the computational complexity of the game domination number? (2) What is the computational complexity of the game domination number on trees? (3) Can we say **anything** about the computational complexity of the domination game?

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Finitely forcible graphons

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Joint work with Roman Glebov, Tereza Klimošová and Jan Volec.

Graphons are analytic objects associated with convergent sequences of dense graphs. The theory of dense graph limits has recently been built in a series of papers authored by several authors, in particular, by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi. Finitely forcible graphons, i.e., those determined by densities of finitely many subgraphs, play an important role because of their relation to extremal graph theory. Lovász and Szegedy intensively studied finitely forcible graphons and they conjectured that the topological space of typical vertices of a finitely forcible graphon must be compact and its dimension is always finite. We disprove both conjectures.

We construct a finitely forcible graphon such that the associated space is not compact and it even fails to be locally compact. We also provide another construction of a finitely forcible graphon such that the associated space of typical vertices has a subspace homeomorphic to $[0,1]^\infty$, i.e., its dimension is infinite for all standard notions of dimension.

The Spectrum of Modularity matrix of Graphs with Pure Anticommunity Structure

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Joint work with Gilles Caporossi.

Community structure detection methods assume that the network of interest divides naturally into subgroups, so the job is to find these groups. The number and size of the groups is determined by the network itself. It is possible that no good division even exists.
A given group of vertices in a network is considered to be a *community* if the number of edges within the group is significantly more than we expect by chance.
A *modularity*, proposed by Newman and Girvan (2004), is a measure of the quality of a particular division of the network. It is proportional to the number of links falling within groups of vertices minus the expected number in an equivalent network with links placed at random.
For a graph G with n vertices and m edges, the modularity is defined as

$$Q = \frac{1}{2m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij} - \frac{d_i d_j}{2m}\right)\delta_{ij}$$

where a_{ij} is the element of the adjacency matrix of G, d_i is the degree of vertex i and δ_{ij} is a Kronecker delta function which equals one if and only if vertices i and j belongs to the same group.
The main goal is to maximize the modularity by choosing an appropriate division of the network. If the number of links within the group is no better than random, then the modularity is zero.
The modularity can be written as

$$Q = \frac{1}{4m}\mathbf{s}^{\tau} B\mathbf{s},$$

where $B = A - \frac{1}{2m}\mathbf{d}\mathbf{d}^{\tau}$ is the *modularity matrix* and $\mathbf{s}$ is the index vector with values ± 1.
The modularity matrix is symmetric real matrix with all row sums eqaul to zero. It is a special rank-one modification of the adjacency matrix.

The maximization of modularity relies on spectral partitioning method. It requires calculation of the components of the leading eigenvector i.e. the eigenvector that belongs to the largest eigenvalue of B. The modularity is maximized by dividing the vertices according to the signs of the elements of the leading eigenvector. The magnitudes of these elements measure how strong the corresponding vertex belongs to the community. If the components of the leading eigenvector are all equal to one, then the graph is indivisible.

For a graph division into more than two communities, this algorithm can be easily applied.

Although there are various numerical results concerning the largest eigenvalue of modularity matrix, not much is known about its spectral properties. I present the main results concerning the sufficient condition for indivisibility of a graph:

Theorem 1 For a complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ with $n = n_1 + \ldots + n_k$ vertices, let $n'_1 > n'_2 > \ldots > n'_{k'}$ be the sequence of all distinct eigenvalues among $n_1, \ldots, n_k$, and let s_i, $i = 1, \ldots, n_k$ be the number of occurences of n'_i among $n_1, \ldots, n_k$. The spectrum of modularity matrix of $K_{n_1,n_2,\ldots,n_k}$ consists of:

(i) an eigenvalue 0 of multiplicity $n - k + 1$

(ii) an eigenvalue $-n'_i$ of multiplicity $s_i - 1$ whenever $s_i \geq 2$ and

(iii) $k' - 1$ eigenvalues λ, one from each of the intervals $(-n'_i, -n'_{i-1})$, $i = 2, \ldots, k'$ satisfying

$$\sum_{p=1}^{k} \frac{n_p^2(n - n_p)}{2m(\lambda + n_p)} = 1.$$

Theorem 2 A connected graph has the largest eigenvalue of modularity matrix equal to zero if and only if it is a complete multipartite graph.

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The Arithmetic of the Random Poset

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For a countable relational structure $\mathcal{A}$, the class of all finite structures that embed into $\mathcal{A}$ is called the *age* of $\mathcal{A}$ and we denote it by $\mathrm{age}(\mathcal{A})$. A class $\mathbf{K}$ of finite structures is an *age* if there is countable structure $\mathcal{A}$ such that $\mathbf{K} = \mathrm{age}(\mathcal{A})$. It is easy to see that a class $\mathbf{K}$ of finite structures is an age if and only if $\mathbf{K}$ is an abstract class (that is, closed for isomorphisms), there are at most countably many pairwise nonisomorphic structures in $\mathbf{K}$, $\mathbf{K}$ has the hereditary property (HP), and $\mathbf{K}$ has the joint embedding property (JEP). An age $\mathbf{K}$ is a *Fraïssé age* (= Fraïssé class = amalgamation class) if $\mathbf{K}$ satisfies the amalgamation property (AP). For every Fraïssé age $\mathbf{K}$ there is a unique (up to isomorphism) countable homogeneous structure $\mathcal{A}$ such that $\mathbf{K} = \mathrm{age}(\mathcal{A})$. We say that $\mathcal{A}$ is the *Fraïssé limit* of $\mathbf{K}$.

For example, the class of all finite linear orders is a Fraïssé class whose Fraïssé limit is isomorphic to $(\mathbb{Q}, <)$; the class of all finite graphs is a Fraïssé class whose Fraïssé limit is called the *random graph*; the class of all finite posets (= partially ordered sets) is a Fraïssé class whose Fraïssé limit is called the *random poset* and denoted by $\mathcal{P}$.

Note that $(\mathbb{Q}, <)$ is not only a Fraïssé limit, but also a carrier of a more elaborate algebraic structure – that of an ordered field. In this talk we show that the random poset $\mathcal{P}$ also carries an algebraic structure compatible with the underlying partial order, albeit not as rich as the one usually endowing $\mathbb{Q}$.

Let us first recall the Hubička-Nešetřil presentation $\mathcal{P}_\in$ of the random poset [2]. Fix a model $\mathcal{M}^{@}_{\mathrm{fin}}$ of hereditarily finite set theory with a single atom @. For $m \in \mathcal{M}^{@}_{\mathrm{fin}}$ let $\mathcal{L}_m = \{x : @ \notin x \in m\}$ and $\mathcal{R}_m = \{x \setminus \{@\} : @ \in x \in m\}$. Clearly, $@ \notin \mathcal{L}_m$ and $@ \notin \mathcal{R}_m$. On the other hand, for $a, b \in \mathcal{M}^{@}_{\mathrm{fin}}$ such that $@ \notin a$ and $@ \notin b$ let $(a \mid b) = a \cup \{x \cup \{@\} : x \in b\}$. It is easy to see that $@ \notin (a \mid b)$ and that if $@ \notin m$ then $m = (\mathcal{L}_m \mid \mathcal{R}_m)$. For $a, b \in \mathcal{M}^{@}_{\mathrm{fin}}$ we write $a \preccurlyeq b$ if

$$(\{a\} \cup \mathcal{R}_a) \cap (\{b\} \cup \mathcal{L}_b) \neq \varnothing.$$

Definition ([2]) Let $\mathcal{P}_\in$ be the set of all $m \in \mathcal{M}^{@}_{\mathrm{fin}}$ such that:

1. [correctness] $@ \notin m$, $\mathcal{L}_m \cup \mathcal{R}_m \subseteq \mathcal{P}_\in$, $\mathcal{L}_m \cap \mathcal{R}_m = \varnothing$;

2. [ordering] $x \preccurlyeq y$ for all $x \in \mathcal{L}_m$, $y \in \mathcal{R}_m$;
3. [completeness] $\mathcal{L}_x \subseteq \mathcal{L}_m$ for all $x \in \mathcal{L}_m$, and $\mathcal{R}_x \subseteq \mathcal{R}_m$ for all $x \in \mathcal{R}_m$.

Theorem 1.1 ([2]). *The binary relation $\preccurlyeq$ is a partial order on $\mathcal{P}_\in$. Moreover, $(\mathcal{P}_\in, \preccurlyeq)$ is isomorphic to the random poset.*

Let $\mathbb{S}$ be the class of John Conway's surreal numbers (see [1, 3]). By a slight abuse of notation, we can assume that each surreal number $x \in \mathbb{S}$ takes the form $x = (\mathcal{L}_x \mid \mathcal{R}_x)$. Let $\sqsubseteq$ denote Conway's ordering on $\mathbb{S}$. One of the main observations on the ordering of surreal numbers is that $\sqsubseteq$ is a quasiorder (it is not antisymmetric). We, therefore, write

$$x \approx y \quad \text{if} \quad x \sqsubseteq y \text{ and } y \sqsubseteq x$$

and say that $x, y \in \mathbb{S}$ are *equal* if $x \approx y$. We say that x and y are *identical* if $x = y$. An important observation in [2] is that it is safe assume that $\mathcal{P}_\in \subseteq \mathbb{S}$ (justifying the abuse of notation, that is, using $(\cdot \mid \cdot)$ as the pairing operator in the Hubička-Nešetřil presentation of the random poset, and as the pairing operator in the construction of surreal numbers). Moreover,

Theorem 1.2 ([2]). *For all $a, b \in \mathcal{P}_\in$, if $a \prec b$ then $a \sqsubset b$. In other words, the restriction of $\sqsubset$ to $\mathcal{P}_\in$ is a linear extension of $\prec$.*

Our main observation in this talk is the following proposition:

Proposition 1.3. *Using the underlying algebraic structure of the surreal numbers we can define a binary operation $+$ and a unary operation $-$ on $\mathcal{P}_\in$ such that $(\mathcal{P}_\in, +, -, 0, \preccurlyeq)$ is an ordered commutative monoid with involution.*

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Extremal problems on K_6-minor free graphs

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Let $Forb(K_k)$ be the class of all graphs without a clique minor of order k, $k \geq 2$. The sharp upper bound for the minimal degree $\delta(G)$, for any $G \in Forb(K_k)$ is known only up to $k = 5$. Apparently, this problem is difficult. We consider the case G is triangle-free. This problem as well is far from settled. For $k = 2, 3, 4, 5$, it is easy to check that $k - 2$ is sharp. What about the case $k = 6$? Does there exist a triangle-free graph G with $\delta(G) = 5$, and without K_6 minor? It seems tempting to answer this in the negative. However, to date we do not know the answer. Hence, we weaken the requirement by forbidding more minors other than K_6. The Petersen family consists of seven graphs that are obtained from the Petersen graph by applying the operations known as Δ-Y or Y-Δ transformations. The complete graph K_6 is one of the seven graphs in the Petersen family. We have more structure in the class of graphs forbidding all of these seven graphs as a minor. It is known as the class of linklessly embeddable graphs. Even settling the case of graphs avoiding all of the Peterson family minors would be very interesting.

On the general position subset selection problem[1]

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Joint work with David R. Wood[2].

Given $k < \ell$, Erdős [1] asked for the maximum integer $f(n, \ell, k)$ such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n, \ell, k)$ points with at most k collinear. For $k = 2$ this is known as the general position subset selection problem. The density version of the Hales–Jewett theorem implies that $f(n, \ell, k) \leq o(n)$. Füredi [2] showed that $f(n, 3, 2) \geq \Omega(\sqrt{n \ln n})$. Much more recently, Lefmann [4] proved that for ℓ fixed, $f(n, \ell, k) \geq \Omega(n^{(k-1)/k}(\ln n)^{1/k})$.

We consider the case when ℓ is not fixed, but varies as a function of n. Our general approach is to combine the Szemerédi–Trotter Theorem with various known results on independence numbers of uniform hypergraphs. For $k = 2$ we show that:

- If $\ell \leq O(\sqrt{n})$ then $f(n, \ell, 2) \geq \Omega(\sqrt{\frac{n}{\ln \ell}})$.
- If $\ell \leq O(n^{(1-\epsilon)/2})$ then $f(n, \ell, 2) \geq \Omega(\sqrt{n \log_\ell n})$.

For fixed $k \geq 3$ we show that:

- If $\ell \leq O(\sqrt{n})$ then $f(n, \ell, k) \geq \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}\right)$.
- If $\ell \leq O(n^{(1-\epsilon)/2})$ then $f(n, \ell, k) \geq \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}(\ln n)^{1/k}\right)$.

These results turn out to be useful in answering a symmetric version of the problem posed by Gowers on MathOverflow [3]. He asked for the minimum integer $\mathrm{GP}(q)$ such that every set of at least $\mathrm{GP}(q)$ points in the plane contains q collinear points or q points in general position. He noted that $\Omega(q^2) \leq \mathrm{GP}(q) \leq O(q^3)$. Our first result implies that $\mathrm{GP}(q) \leq O(q^2 \ln q)$.

[1] The full version of this paper is available at http://arxiv.org/abs/1208.5289 and will appear in SIAM J. Discrete Math.

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Our third result answers the natural generalisation of this problem, exchanging 'general position' for 'with at most k collinear', yielding $\mathrm{GP}_k(q) = \Theta(q^2)$.

We end our paper with a series of conjectures. The following conjecture would completely answer Gowers' question, showing that $\mathrm{GP}(q) = \Theta(q^2)$.

Conjecture 1.1. $f(n, \sqrt{n}, 2) \geq \Omega(\sqrt{n})$.

The following colouring conjecture would imply Conjecture 1.1.

Conjecture 1.2. *Every set P of n points in the plane with at most $\sqrt{n}$ collinear can be coloured with $O(\sqrt{n})$ colours such that each colour class is in general position.*

The problem of determining the correct asymptotics of $f(n, \ell, 2)$ (and $f(n, \ell, k)$) for fixed ℓ remains wide open.

Conjecture 1.3. *If ℓ is fixed, then $f(n, \ell, 2) \geq \Omega(n/\operatorname{polylog}(n))$.*

In the colouring setting, the following conjecture is actually equivalent to Conjecture 1.3.

Conjecture 1.4. *For all fixed $\ell \geq 3$, every set of n points in the plane with at most ℓ collinear can be coloured with $O(\operatorname{polylog}(n))$ colours such that each colour class is in general position.*

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Three open problems on arrangements of DP-ribbons

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A DP-ribbon is a cylinder with a distinguished core circle with a distinguished side and an arrangement of DP-ribbons is a finite family of at least two DP-ribbons pairwise attached as shown in Fig. 1.

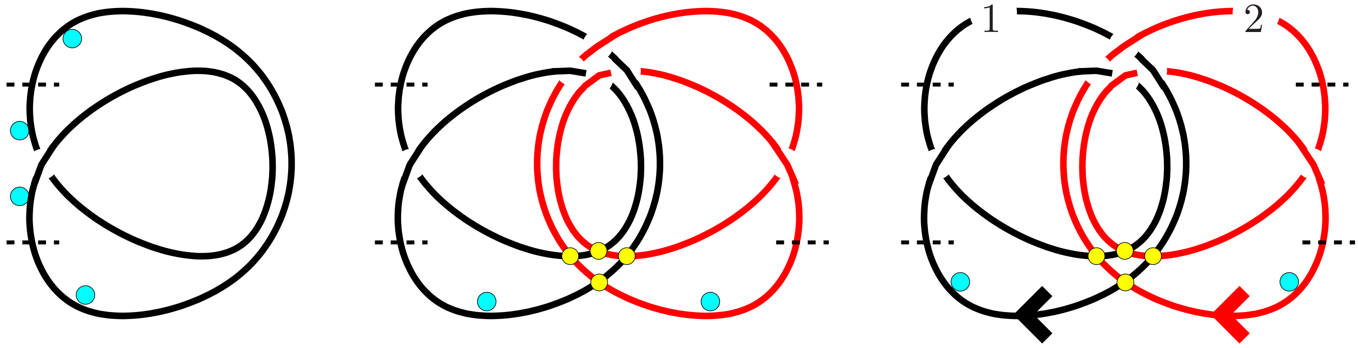

Figure 1: A DP-ribbon embedded in three-space (only the core circle is drawn, the distinguished side is indicated by small sky blue disks, and half-twists of the ribbon are indicated by horizontal dashed line segments), an arrangement of two DP-ribbons, and an indexed arrangement of two oriented DP-ribbons

The reader will easily check that the underlying surface of an arrangement of two DP-ribbons is a sphere with one crosscap and five boundaries. The genus of an arrangement of DP-ribbons is the genus of its underlying surface.

Theorem 0.1 ([2]). *The arrangements of DP-ribbons of genus* 1 *are exactly, modulo the adjunction of topological disks along their boundaries, the so-called arrangements of double pseudolines, i.e., the dual arrangements of finite families of pairwise disjoint convex bodies of (real two-dimensional) projective planes. Furthermore an arrangement of DP-ribbons is of genus 1 if and only if its subarrangements of size 3, 4 and 5 are of genus 1.*

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Theorem 0.2 ([2]). *There is a natural one-to-one and onto correspondence between indexed arrangements of n oriented DP-ribbons and the n-tuples of suffles of the $n-1$ circular sequences $\overline{jj}jj$, $j = 2, 3, \ldots, n$. In particular the number b_n of indexed arrangements of n oriented DP-ribbons is*

$$\left\{4^{n-2}\binom{4n-5}{3, 4, 4, \ldots, 4}\right\}^n$$

and the number a_n of arrangements of n DP-ribbons is bounded from below by

$$b_n/(2^n n!).$$

Problem 0.3. *Give asymptotic formulae (or, better, closed formulae) for the numbers $b_n(g)$ of indexed arrangements of n oriented DP-ribbons of genus g.*

Comment: Let $a_n(g)$ be the number of DP-ribbons of size n and genus g. The following values of the $a_3(g)$ and $b_3(g)$ have been obtained in collaboration with Carsten Lange using classical enumeration algorithms for multiset permutations, e.g., [5, 3].

g	1	2	3	4	5	6	7
$a_3(g)$	13	20	77	197	674	1127	2707
$b_3(g)$	216	636	2756	8292	29032	50848	123240

g	8	9	10	11	12	13	≥ 14
$a_3(g)$	5173	10073	11943	13633	9115	3290	0
$b_3(g)$	240196	475920	565016	653528	436496	157824	0

Problem 0.4. *Prove that an arrangement of five DP-ribbons is of genus 1 if and only if its subarrangements of size 3 and 4 are of genus 1.*

Comment : We ask for a non computer-assisted proof. So far we only know that an arrangement of five DP-ribbons whose subarrangements of size 4 are of genus 1 is of genus 1 or its subarrangements of size 4 belong to a well-defined family of few dozens of arrangements [2, Theorem 46]. A computer-assisted proof is therefore doable using modest computing ressources. Preliminary investigations in this direction, in collaboration with Carsten Lange, lead to the following values for the numbers $a_4^*(g)$ of arrangements of DP-ribbons of size 4 and genus g whose subarrangements of size 3 are of genus 1.

g	1	2	3	4	5	6	7	≥ 8
$a_4^*(g)$	6 570	0	455	0	18	0	1	0
$b_4^*(g)$	2 415 112	0	135 664	0	4 560	0	16	0
$\lceil b_4^*(g)/2^4 4! \rceil$	6290	0	354	0	12	0	1	0

The techniques developped in Ortner [4] might be relevant for a non computer-assisted proof.

Problem 0.5. *Devise a quadratic time algorithm to compute an arrangement of n double pseudolines presented by its subarrangements of size* 3*.*

Comment : In the case of crosscap arrangements, i.e., dual arrangements of finite families of pairwise disjoint convex bodies of affine planes, a strategy, based on the notion of pseudotriangulation, is proposed in [1].

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On the probability of planarity of a random graph near the critical point

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The random graph model $G(n, M)$ assigns uniform probability to graphs on n labelled vertices with M edges. A fundamental result of Erdős and Rényi [4] is that the random graph $G(n, M)$ undergoes an abrupt change when M is around $n/2$, the value for which the average vertex degree is equal to one. When $M = cn/2$ and $c < 1$, almost surely the connected components are all of order $O(\log n)$, and are either trees or unicyclic graphs. When $M = cn/2$ and $c > 1$, almost surely there is a unique giant component of size $\Theta(n)$. We direct to reader to the reference texts [3] and [8] for a detailed discussion of these facts.

We concentrate on the so-called critical window $M = \frac{n}{2}(1 + \lambda n^{-1/3})$, where λ is a real number, identified by the work of Bollobás [1, 2]. Let us recall that the excess of a connected graph is the number of edges minus the number of vertices. A connected graph is complex if it has positive excess. As $\lambda \to -\infty$, complex components disappear and only trees and unicyclic components survive, and as $\lambda \to +\infty$, components with unbounded excess appear. A thorough analysis of the random graph in the critical window can be found in [7] and [10], which constitute our basic references.

For each fixed λ, we denote the random graph $G\left(n, \frac{n}{2}(1 + \lambda n^{-1/3})\right)$ by $G(\lambda)$. The core $C(\lambda)$ of $G(\lambda)$ is obtained by repeatedly removing all vertices of degree one from $G(\lambda)$. The kernel $K(\lambda)$ is obtained from $C(\lambda)$ by replacing all maximal paths of vertices of degree two by single edges. The graph $G(\lambda)$ satisfies almost surely several fundamental properties, that were established in [10] by a subtle simultaneous analysis of the $G(n, M)$ and the $G(n, p)$ models.

1. The number of complex components is bounded.
2. Each complex component has size of order $n^{2/3}$, and the largest suspended tree in each complex component has size of order $n^{2/3}$.
3. $C(\lambda)$ has size of order $n^{1/3}$ and maximum degree three, and the distance between two vertices of degree three in $C(\lambda)$ is of order $n^{1/3}$.

4. $K(\lambda)$ is a cubic (3-regular) multigraph of bounded size.

The key property for us is the last one. It implies that almost surely the components of $G(\lambda)$ are trees, unicyclic graphs, and those obtained from a cubic multigraph K by attaching rooted trees to the vertices of K, and attaching ordered sequences of rooted trees to the edges of K. Some care is needed here, since the resulting graph may not be simple, but asymptotically this can be accounted for.

It is clear that $G(\lambda)$ is planar if and only if the kernel $K(\lambda)$ is planar. Then by counting planar cubic multigraphs it is possible to estimate the probability that $G(\lambda)$ is planar. To this end we use generating functions. The trees attached to $K(\lambda)$ are encoded by the generating function $T(z)$ of rooted trees, and complex analytic methods are used to estimated the coefficients of the corresponding series. This allows us to determine the exact probability

$$p(\lambda) = \lim_{n\to\infty} \Pr\left\{G\left(n, \tfrac{n}{2}(1+\lambda n^{-1/3})\right) \text{ is planar}\right\}.$$

In particular, we obtain $p(0) \approx 0.99780$.

This approach was initiated in the seminal paper by Flajolet, Knuth and Pittel [5], where the authors determined the threshold for the appearance of the first cycles in $G(n, M)$. A basic feature in [5] is to estimate coefficients of large powers of generating functions using Cauchy integrals and the saddle point method. This path was followed by Janson, Knuth, Łuczak and Pittel [7], obtaining a wealth of results on $G(\lambda)$. Of particular importance for us is the determination in [7] of the limiting probability that $G(\lambda)$ has given excess. The approach by Łuczak, Pittel and Wierman in [10] is more probabilistic and has as starting point the classical estimates by Wright [11] on the number of connected graphs with fixed excess. The range of these estimates was extended by Bollobás [1] and more recently the analysis was refined by Flajolet, Salvy and Schaeffer [6], by giving complete asymptotic expansions in terms of the Airy function.

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Existence of Friendship Hypergraphs

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Joint work with Leif Kjær Jørgensen.

A 3-*uniform friendship hypergraph* is a 3-uniform hypergraph in which, for all triples of vertices x, y, z there exists a unique vertex w, such that xyw, xzw and yzw are hyperedges in the hypergraph. Sós showed in [2] that such 3-uniform friendship hypergraphs on n vertices exist with a so called *universal friend* (a vertex which is in an hyperedge with all other pairs of vertices) if and only if a Steiner triple system, $S(2, 3, n-1)$ exists. Hartke and Vandenbussche used integer programming in [5] to search for 3-uniform friendship hypergraphs without a universal friend and found one on 8, three non-isomorphic on 16 and one on 32 vertices. So far, these five hypergraphs are the only known 3-uniform friendship hypergraphs. Li, van Rees, Seo and Singhi also used integer programming in [6] to show that the only 3-uniform friendship hypergraphs, with at most 12 vertices, are the ones found by Sós and Hartke and Vandenbussche.

A *cubeconstructed hypergraph* $\mathcal{H}$ is a hypergraph on 2^k vertices for $k \geq 2$ where the vertices are labelled with k-bit binary strings and xyz is a hyperedge in $\mathcal{H}$ if and only if $dist_H(x, y) + dist_H(x, z) + dist_H(y, z) = 2k$, where $dist_H(a, b)$ is the Hamming distance between a and b. In [3] we proved the following theorem.

Theorem 0.1. *The cubeconstructed hypergraphs are* 3*-uniform friendship hypergraphs on* $2^{k-1}(3^{k-1} - 1)$ *hyperedges.*

Furthermore we constructed 3-uniform friendship hypergraphs on 20 and 28 vertices using a computer, which inspired us to conjecture the following.

Conjecture 0.2. *For all n which is divisible by* 4 *and not divisible by* 3*, there exist a vertex-transitive* 3*-uniform friendship hypergraph on n vertices.*

We also we defined r*-uniform friendship hypergraphs* and stated that the existence of those with a universal friend, is dependent on the existence of a Steiner system, $S(r-1, r, n-1)$. As a result hereof, we know infinitely many 4-uniform friendship hypergraphs with a universal friend. Finally we showed how to construct a 4-uniform friendship hypergraph on 9 vertices and with no universal friend.

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Parameterized Complexity of Directed Steiner Tree and Domination Problems on Sparse Graphs

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Joint work with Mark Jones, Daniel Lokshtanov, M. S. Ramanujan, and Saket Saurabh.

We studied the parameterized complexity of the directed variant of the classical Steiner Tree problem on various classes of sparse graphs. While the parameterized complexity of Steiner Tree on undirected graphs parameterized by the number of terminals is well understood, not much is known about the parameterization by the number of non-terminals in the solution tree. All that is known for this parameterization is that both the directed and the undirected versions are W[2]-hard on general graphs, and hence unlikely to be fixed parameter tractable (FPT). The undirected Steiner Tree problem becomes FPT when restricted to sparse classes of graphs such as planar graphs, but the techniques used to show this result break down on directed planar graphs.

In this talk we precisely chart the tractability border for Directed Steiner Tree (DST) on sparse graphs parameterized by the number of non-terminals in the solution tree. Specifically, we show that the problem is fixed parameter tractable on graphs excluding a topological minor, but becomes W[2]-hard on graphs of degeneracy 2. On the other hand we show that if the subgraph induced by the terminals is required to be acyclic then the problem becomes FPT on graphs of bounded degeneracy.

Our algorithms for DST are based on a novel branching rule. To demonstrate the versatility of the new branching we use it to give improved parameterized algorithms for Dominating Set on graphs of bounded degeneracy and graphs excluding a topological minor. We further show that our algorithm achieves the best possible running time dependence on the solution size and degeneracy of the input graph, under standard complexity theoretic assumptions.

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Extension Complexity of Combinatorial Polytopes

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Let $P \subset \mathbb{R}^d, Q \subset \mathbb{R}^k$ be two polytopes. Q is called an extended formulation (EF) of P if P is the projection of Q. Defining the *size* of a polytope to be the minimum number of inequalities needed to describe it, we can define a measure of how complex a polytope is. To this end, we define the *extension complexity* of a polytope P to be the minimum over the sizes of all extended formulations of P.

It is known that the extension complexity of a polytope can be exponentially smaller than the number of inequalities needed to describe the polytope. For example, for regular n-gons the extension complexity is $\Theta(\log n)$.

Many combinatorial optimization problems have natural polytopes associated with them. For example, consider the traveling salesman problem. We can define the TSP polytope with parameter n to be the convex hull of the characteristic vectors of every traveling salesman tour of the complete graph K_n. If the extension complexity of this polytope is polynomial in n then one can solve the traveling salesman problem in polynomial time by optimizing a linear function over the extended formulation, thereby proving $P = NP$. Indeed, in the mid eighties claims of such polynomial extended formulations for the TSP polytope were made by Ted Swart. Yannakakis later proved that every extended formulation of the TSP polytope that satisfies certain symmetry must have large size. Swart's EF satisfied this symmetry requirement and was therefore proved to be wrong.

It remained open, however, whether another EF for TSP polytope existed that avoided the symmetry requirement and was of polynomial size. Recently an unconditional superpolynomial lower bound on the size of the TSP polytope was obtained by Fiorini et al, thereby answering this question. However a much more general question remains unanswered: Does every linear programming formulation of NP-hard problems have superpolynomial size?

To answer this question, one must handle the following problem first:

- How do we associate polytope with optimization problems in a canonical way?

- How do we translate lower bounds for extension complexity of a polytope associated with one problem to that of another problem?
- In particular, what subset of Turing reductions allow one to translate such lower bounds?

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