STATISTICS OF ORDERINGS

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ABSTRACT. In this paper we study the relationship between ordered graphs (i.e. graphs with given ordering of either vertices or its edges) and graphs which are unordered.

We establish the existence of sparse graphs where the probabilities for ordered graphs can be computed with arbitrary precision from unordered frequencies.

This establishes the strong quantitative ordering property of graphs and it extends the key result of Angel, Lyons, and Kechris. We extend this to orderings of pairs. This is more difficult and the dependency is less precise as one has to rely on the canonical ordering lemma.

1. INTRODUCTION

Let $H = (V, E)$ be an undirectred graph, $G = (V', E')$ its supgraph. Suppose that $(V, \leq)$, $(V', \leq')$ are linear orderings of V and V'. Is then a subgraph of G when the orderings are taken into account? More precisely, does there exist a 1–1 mapping $f : V' \to V$ which satisfies

$$\{u, v\} \in E' \iff \{f(u), f(v)\} \in E$$

(i.e. f is an embedding of G' into G)

$$u \leq' v \iff f(u) \leq f(v).$$

(i.e. f is monotone).

Such mapping is called monotone embedding of $(G, \leq') \to (H, \leq')$.

The answer to this question is positive in the sense that for given G' a graph G always exists.

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Lemma 1 (Ordering Lemma [7]). *For any G there exists H such that for any choices of orderings $(V(G), \leq)$, $(V(H), \leq')$ there exists a monotone embedding $(G, \leq) \to (H, \leq')$.*

The lemma was isolated in [7] in the context of structural Ramsey theory where it plays a key role in the context of study od Ramsey classes, see [10]. It is then important that Lemma 1 holds in much stronger form when we restrict G and H to a particular classes of graphs (such as triangle free graphs):

Lemma 2 (restricted ordering lemma [9]). *Let $\mathcal{F}$ be a finite set of 2-connected graphs. By $Frob(\mathcal{F})$ we denote all those graphs which do not contain any $F \in \mathcal{F}$ as a subgraph. Then in Lemma 1 $G \in Forb(\mathcal{F})$ then we may choose also $H \in Forb(\mathcal{F})$.*

In the other words the class $Forb(\mathcal{F})$ has ordering property [9, 5].

The proof of Lemma 2 given in [9] is probabilistic and is called random placement construction [9]. In [2] Angel, Kechris, and Lyons observed (in the context of topological dynamics) that the same construction proves the following quantitative version:

Lemma 3 (quantitative restricted ordering lemma [2]). *Let $\mathcal{F}$ be as in Lemma 2. Then for every $\varepsilon > 0$ and every $G \in Forb(\mathcal{F})$ with k vertices there exists $H \in Forb(\mathcal{F})$ such that*

$$\left| \frac{emb((G, \leq), (H, \leq))}{emb(G, H)} - \frac{1}{k!} \right| < \varepsilon$$

If in every ordering $\leq$ of H all the orderings of G are present approximately equally likely.

The proof of Lemma 3 follows readily from the random placement construction of [9] applying Chernoff inequality.

Here we generalize these results in two directions.

First we prove a localized version of Lemma 3. This will allow us to compute frequencies of order types of G from the local statistics of ordering of H. This is stated as Theorem 1. For the uniform distribution this implies Lemma 3 and has further consequences such as Sparsification Lemma (stated as Corollary 3).

In another direction we deal with ordering of subgraphs. In this paper we concentrate on the ordering of edges. An unpublished result of Leeb describes the canonical orderings of 2-element subsets of large sets. This is stated and proved in Section 3, Theorem 2.

We then prove an analogue of Lemma 3 for orderings of edges. This is stated as Theorem 3. This involves a carefull analysis of canonical orderings (which were studied e.g. in [6]). For the sake of completeness, we include a proof of unpublished result of K. Leeb (cf. [6]).

2. Ordering of vertices

In this section we prove Theorem 1 and its three corollaries one of them beeing above Lemma 3. Instead of dealing with orderings of vertices it is convenient to code orderings by permutations:

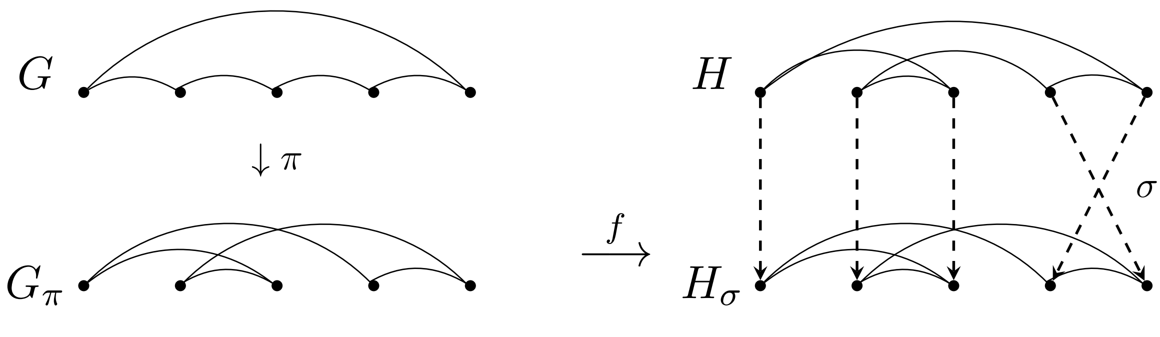
For graph $G = (V, E)$ let $\{v_1, v_2, \ldots, v_n\} = V$ be a fixed (standard) enumeration of vertices of G. For each graph we fix a standard enumeration. Then any ordering $(V, \leq)$ is coded by a permutation σ of $\{1, \ldots, n\}$ defined by $\sigma(i) \leq \sigma(j)$ iff $v_i \leq v_j$. This situation will be denoted by (G, σ).

For graphs G, H we denote by $\mathrm{emb}(G, H)$ the number of all embeddings $G \to H$. (Recall, that an injective mapping $f : V(G) \to V(H)$ is an embedding providing $\{x, y\} \in E(G)$ iff $\{f(x), f(y)\} \in E(H)$). Similarly, by $\mathrm{emb}((G, \pi), (G, \sigma))$ we denote the number if all embeddings $f : G \to H$ which satisfy $\pi(i) < \pi(j)$ iff $\sigma(f(i)) < \sigma(f(j))$ (with respect to standard enumerations).

The purpose of this section is to prove the following:

For $\pi : [k] \to [k]$ and $\sigma : [n] \to [n]$, we will denote by $\mathrm{emb}((G, \pi), (H, \sigma))$ the number of all embeddings $f : G \to H$ with $\pi(i) < \pi(j)$ if and only if $\sigma(f(i)) < \sigma(f(j))$.

The figure below shows such f for $[k] = [n] = [5]$ and $G \cong H \cong C_5$, $\pi = (1\ 2\ 5\ 4\ 3)$, and $\sigma = (4\ 5)$



In the Theorem below (which is an asymptotic statement for $n \to \infty$) we will use the following notation.

- Set $[k] = \{1, 2, \ldots, k\}$ and let $(\pi_1, \pi_2, \ldots, \pi_{k!})$ be a fixed labeling of all permutations $\pi : [k] \to [k]$. For $n \geq k$, let $\sigma : [n] \to [n]$ be a permutation, and for $X = \{x_1 < x_2 < \cdots < x_k\} \subset [n]$ let σ_X be a permutation on $[k]$ induced by σ and X defined by $\sigma_X(i) < \sigma_X(j)$ if and only if $\sigma(x_i) < \sigma(x_j)$. We define a *$k$-statistic of σ* as a $k!$-tuple $(s_1^\sigma, s_2^\sigma, \ldots, s_{k!}^\sigma)$, where $s_\ell^\sigma = |\{X \colon \sigma_X = \pi_\ell\}| / \binom{n}{k}$ for $\ell = 1, 2, \ldots, k!$.
- G and H are graphs with vertex sets $[k]$ and $[n]$ respectively.
- $\overrightarrow{a} = (a_1, \ldots, a_{k!})$ is a stochastic vector i.e., $\sum a_\ell = 1$ and $a_\ell \geq 0$ for all $\ell = 1, 2, \ldots, k!$.
- $\sigma : [n] \to [n]$ is a permutation with k-statistic $(s_1, s_2, \ldots, s_{k!})$.

Theorem 1. *For every 2-connected G and $\overrightarrow{a} = (a_1, \ldots, a_{k!})$, there exists H such that for any $\sigma : [n] \to [n]$ the following holds for every $\ell = 1, 2, \ldots, k!$*

$$emb((G, \pi_\ell), (H, \sigma)) = (b_\ell + o(1))emb(G, H) \tag{1}$$

where

$$b_\ell = \sum_{i,j} \{a_i s_j \colon \pi_j \circ \pi_i = \pi_\ell\} \tag{2}$$

and $o(1) \to 0$ as $n \to \infty$. Moreover, H can be chosen in such a way that girth(H) = girth(G).

We mention two special cases of Theorem 1.

(i) Taking $\overrightarrow{a} = (0, 0, \ldots, 1, 0, \ldots, 0)$, where the entry one corresponds to the identity permutation $\pi : [k] \to [k]$. The formula (2) yields $b_\ell = s_\ell$, for every $\ell = 1, 2, \ldots, k!$, and hence by (1) there exists H so that

$$\mathrm{emb}((G, \pi_\ell), (H, \sigma)) = (s_\ell + o(1))\mathrm{emb}(G, H)$$

holds for every $\ell = 1, 2, \ldots, k!$ and $\sigma : [n] \to [k]$.

(ii) Taking $\overrightarrow{a} = (1/k!, 1/k!, \ldots, 1/k!)$, we similarly infer that $b_\ell = 1/k!$ for all $\ell = 1, 2, \ldots, k!$ and hence there exists H so that

$$\mathrm{emb}((G, \pi_\ell), (H, \sigma)) = (1/k!)\mathrm{emb}(G, H)$$

holds for all $\ell = 1, 2, \ldots, k!$ and $\sigma : [n] \to [n]$.

This statement has several interesting corollaries (which provided motivation for it):

Corollary 1 (local global frequencies)**.** *For every 2-connected G, $|G| = k$, there exists H such that: For every (H, σ), σ with k-statistics $(s_1, \ldots, s_{k!})$*

$$emb((G, \pi_\ell), (H, \sigma)) = (s_\ell + o(1))(emb(G, H).$$

Moreover girth G = girth H.

Corollary 2 (uniformity of orders)**.** *For every 2-connected G, $|G| = k$, there exists H such that: For every (G, π), (H, σ) we have:*

$$emb((G, \pi), (H, \sigma)) = (\frac{1}{k!} + o(1))emb(G, H).$$

Moreover H may be chosen with the same girth as G.

Both corollaries are immediate consequences of Theorem 1. For Corollary 1 it suffices to consider stochastic vector $\vec{(a)} = (1, 0, 0, \ldots, 0)$ where 1 denotes the frequence of the identity. It then follows that $b_\ell = s_\ell$ for every ℓ.

For Corollary 2 is suffices to consider vector $\vec{(a)} = (1/k!, \ldots 1/k!)$. In this case $b_\ell = 1/k!$ for every l which yields the statement.

A particular step in the proof of Theorem 1 implies the following which is perhaps of independant interest:

Corollary 3 (sparsification lemma). *For every $l, k \geq 2$, $\varepsilon > 0$ there exists n and $\mathcal{M} \subseteq \binom{[n]}{k}$ such that*

(1) $([n], \mathcal{M})$ has no cycles of length $< l$
(2) For every permutation σ on $[n]$ it holds that $|s_i^\sigma - s_i^\sigma(\mathcal{M})| < \varepsilon$ where $s_i^\sigma(\mathcal{M}) = |\{M \in \mathcal{M}; \sigma_M = \pi_i\}|/|\mathcal{M}|$.

Before we give a proof of Theorem 1, we will introduce some further concepts and notation:

- For a set $E = \{x_1 < \cdots < x_k\}$ and permutation $\pi : [k] \to [k]$, let $\overrightarrow{E_\pi} = (x_{\pi(1)}, \ldots, x_{\pi(k)})$ be an orientation of E (i.e. ordered k-tuple). On the other hand, if $\overrightarrow{E} = (y_1, \ldots, y_k)$ is an ordered k-tuple, we denote the set $\{y_1, \ldots, y_k\}$ by E.
- An ordered k-graph with vertex set $[n]$ is a system $\overrightarrow{\mathcal{E}} \subset [n]^k$ of ordered k-tuples $(y_1, \ldots, y_k)$ such that $y_i \neq y_j$ for all $1 \leq i < j \leq k$ and such that, for any k-set $E \in \binom{[n]}{k}$ there is at most one orientation $\overrightarrow{E} \in \overrightarrow{\mathcal{E}}$
- For an oriented k-graph $\mathcal{E}$ we denote by $\mathcal{E} = \{E \colon \overrightarrow{E} \in \overrightarrow{\mathcal{E}}\}$
- A cycle of length h in a k-graph $\mathcal{E}$ is a sequence of h distinct vertices and h distinct edges $x_1E_1x_2E_2, \ldots, x_{h-1}E_{h-1}x_hE_h$ which satisfies $x_i \in E_i$, $x_{i+1} \in E_i$ for $i = 1, 2, \ldots, h-1$ and $x_h \in E_h$ and $x_1 \in E_h$.

  One can observe that $\mathcal{E}$ contains no cycle of length $< g$ if $\left|\bigcup \mathcal{E}'\right| \geq (k-1)|\mathcal{E}'|+1$ for every $\mathcal{E}' \subset \mathcal{E}$, $|\mathcal{E}'| < g$. (see [3]). A cycle of length H in an oriented k-graph $\overrightarrow{\mathcal{E}}$ is a sequence $x_1\overrightarrow{E_1}x_2 \ldots x_{h-1}\overrightarrow{E}_{h-1}x_h\overrightarrow{E}_1$ such that $x_1E_1 \ldots E_{h-1}x_hE_1$ is a cycle of length h in $\mathcal{E}$. (In other words, cycles in $\overrightarrow{\mathcal{E}}$ and $\mathcal{E}$ are in one-to-one correspondence, and we do not require cycles in $\overrightarrow{\mathcal{E}}$ to be "directed".)
- The girth of $\mathcal{E}$ ($\overrightarrow{\mathcal{E}}$ resp.) is the length of a shortest cycle in $\mathcal{E}$.
- Naturally we can view each oriented k-graph $\overrightarrow{\mathcal{E}}$ with vertex set $[n]$ as a union $\overrightarrow{\mathcal{E}} = \bigcup\{\overrightarrow{\mathcal{E}}_\lambda \colon \lambda : [k] \to [k]\}$ where $\overrightarrow{E}_\lambda = \{(x_{\lambda(1)}, \ldots, x_{\lambda(k)}), x_1 < x_2 < \cdots < x_k\}$.
- For $\sigma : [n] \to [n]$, $\pi : [k] \to [k]$ and

$$\overrightarrow{E} = (y_1, y_2, \ldots, y_k) \in \overrightarrow{\mathcal{E}}, \tag{3}$$

  we will set $\sigma_{\overrightarrow{E}} = \pi$ if

$$\sigma(y_i) < \sigma(y_j) \Leftrightarrow \pi(i) < \pi(j).$$

  Consequently for each $\ell = 1, 2, \ldots, k!$ we may define

$$S_\ell^\sigma(\overrightarrow{\mathcal{E}}) = \{\overrightarrow{E} \in \overrightarrow{\mathcal{E}} : \sigma_{\overrightarrow{E}} = \pi_\ell\} \text{and} s_\ell^\sigma(\overrightarrow{\mathcal{E}}) = |S_\ell^\sigma(\overrightarrow{\mathcal{E}})|$$

- Note that in (3) we may view $(y_1, \ldots, y_k) \in \overrightarrow{\mathcal{E}}$ $y_i = x_{\lambda(i)}$, for some $\lambda : [k] \to [k]$, where $x_1 < x_2 < \cdots < x_k$.

Lemma 4. *For every $k, g, \varepsilon, 0 < \varepsilon < 1/(g-2)$ and stochastic vector $\overrightarrow{a} = (a_1, \dots, a_{k!})$, there exists an oriented k-graph $\overrightarrow{\mathcal{E}} \subset [n]^k$, $|\overrightarrow{\mathcal{E}}| = \Theta(n^{1+\varepsilon}$ and girth $\overrightarrow{\mathcal{E}} = g$ such that for any $\sigma : [n] \to [n]$ and any $\ell = 1, 2, \dots, k!$, $s_\ell^\sigma(\overrightarrow{\mathcal{E}}) = (b_\ell + o(1))|\overrightarrow{\mathcal{E}}|$, where b_ℓ is given by the formula* (2).

Proof. We procede by random construction. Consider an oriented random k-graph $\overrightarrow{\mathcal{E}} = \bigcup \overrightarrow{\mathcal{E}}_\lambda$, where for each $\lambda : [k] \to [k]$ the k-tuples $(x_{\lambda(1)}, x_{\lambda(2)}, \dots, x_{\lambda(k)})$ with $x_1 < x_2 < \cdots < x_k$ are selected into $\overrightarrow{\mathcal{E}}_\lambda$ independently, each with probability $a_\lambda p$, where $p = n^{1+\varepsilon-k}$. The proof will follow from the two facts below:

Claim 1. *For each λ and π, permutations on $[k]$, and $\sigma : [n] \to [n]$, the following holds with probability $1 - o(1/n!)$*

(*i*) $|\overrightarrow{\mathcal{E}}_\lambda| = (a_\lambda + o(1))p\binom{n}{k}$, *and*
(*ii*) *For $\pi_\ell = \pi\lambda^{-1}$ if $s_l \gg \log n/n^\varepsilon$ then the subset $\overrightarrow{\mathcal{E}}_{\lambda,\pi_\ell,\sigma} \subset \overrightarrow{\mathcal{E}}_\lambda$ consisting of the vectors $(x_{\lambda(1)}, x_{\lambda(2)}, \dots, x_{\lambda(k)})$, $x_1 < x_2 < \cdots < x_k$ with the property*

$$\sigma(x_{\lambda(i)}) < \sigma(x_{\lambda(j)}) \Leftrightarrow \pi(i) < \pi(j)$$

satisfies

$$|\overrightarrow{\mathcal{E}}_{\lambda,\pi_\ell,\sigma}| = (1 + o(1))a_\lambda s_\ell p\binom{n}{k}$$

For $h = 2, 3, \dots, g-1$ and hypergraph $\mathcal{E}$ let $c_h(\mathcal{E})$ be the number of edges contained in cycles of length h. Then the following holds:

Claim 2. *Expected number of cycles of length $< g$ in a random hypergraph $\mathcal{E} = \cup_\lambda\{\mathcal{E}_\lambda, \lambda : [k] \to [j]\}$ is at most*

$$\mathbb{E}(\sum_{h=2}^{g-1} c_h(\mathcal{E})) = o(n^{1+\varepsilon})$$

Proof of Claim 1. The proof of both (i) and (ii) will follow from the Chernoff inequality, which states that for a binomial random variable $\mathrm{Bi}(N, q)$,

$$\mathrm{Prob}(|\mathrm{Bi}(N,q) - Nq| > \delta Nq) < \exp\Big(-\delta^2\frac{Nq}{3}\Big) \tag{4}$$

First observe that if $a_\lambda = 0$ then $\overrightarrow{\mathcal{E}}_\lambda = \emptyset$ by definition and hence both (i) and (ii) are straightforward.

Assume therefore that $a_\lambda \neq 0$. Since $|\overrightarrow{\mathcal{E}}_\lambda|$ is a binomial random variable with $N = \binom{n}{k}$, $q = a_\lambda p$ and expected value $Nq = \Omega(n^{1+\varepsilon})$, we infer from (4) that

$$|\overrightarrow{\mathcal{E}}_\lambda| = (1 + o(1))Nq = (1 + o(1))a_\lambda p\binom{n}{k}$$

fails to hold with probability $\ll 1/n!$ if we set δ to satisfy

$$\sqrt{\log n / n^{\varepsilon}} \ll \delta \ll o(1).$$

Hence (i) is verified.

In order to prove (ii), we will find useful to refer to the following diagram.

$$\begin{array}{ccc}
x_1 < x_2 < \cdots < x_k & & 1, 2, \ldots, k \\
\big\downarrow \lambda & & \big\downarrow \lambda \\
x_{\lambda(1)}, x_{\lambda(2)}, \ldots, x_{\lambda(k)} & & \lambda(1), \lambda(2), \ldots, \lambda(k) \\
\big\downarrow \sigma & & \big\downarrow \pi_\ell = \pi\lambda^{-1}
\end{array}$$

$$(\sigma(x_{\lambda(1)}), \ldots, \sigma(x_{\lambda(k)})) \quad \sim \quad (\pi(1), \pi(2), \ldots \pi(k)) \tag{5}$$

where the "squiggle" $\sim$ denotes the fact that

$$\sigma(x_{\lambda(i)}) < \sigma(x_{\lambda(j)}) \Leftrightarrow \pi(i) < \pi(j).$$

Since $|\overrightarrow{\mathcal{E}}_{\lambda,\pi_\ell,\sigma}|$ counts those k-tuples $\overrightarrow{\mathcal{E}}_\lambda$ which satisfy (5) it follows from the diagram that

$$\begin{aligned}
\overrightarrow{\mathcal{E}}_{\lambda,\pi_\ell,\sigma} &= \overrightarrow{\mathcal{E}}_\lambda \cap \mathcal{S}_\ell, \text{ where} \\
\mathcal{S}_\ell &= \left\{ X = \{x_1 < x_2 < \cdots < x_k\} \in \binom{[n]}{k} : \sigma_X = \pi_\ell \right\}
\end{aligned}$$

Similarly as before, we use the fact that $|\overrightarrow{\mathcal{E}}_{\lambda,\pi_\ell,\sigma}|$ is the binomial random variable $\mathrm{Bi}(N, q)$ with $N = |\mathcal{S}_\ell| = s_\ell \binom{n}{k}$ and $q = a_\lambda p$. Using our assumption on s_ℓ, we infer that $Nq \gg n \log n$, and hence by the Chernoff inequality,

$$\begin{aligned}
Prob(|\vec{\mathcal{E}}_{\lambda,\pi_\ell,\sigma}|) = (1 \pm \delta) a_\lambda s_h p \binom{n}{k} \geq 1 - \exp(-\delta^2 a_\lambda s_\ell p \binom{n}{k}) \geq \\
\geq 1 - \exp(-100 n \log n) = 1 - o(1/n!)
\end{aligned}$$

as claimed. $\square$

Since there are only $O(n!)$ choices to select a tuple of permtations π_ℓ, λ and σ we infer that with probability $1 - o(1)$ $|\vec{\mathcal{E}}_{\lambda,\pi_ell,\sigma}| = (1 + o(1)) a_\lambda s_\ell p \binom{n}{k}$ holds for all choices of π_ℓ, λ and σ.

Proof of Claim 2. It follows from (i) of Claim 1 that

$$|\overrightarrow{\mathcal{E}}| = \sum_\lambda |\overrightarrow{\mathcal{E}}_\lambda| = \sum_\lambda (a_\lambda + o(1))p\binom{n}{k} = p\binom{n}{k} = \Omega(n^{1+\varepsilon}).$$

Consequently, all we need to show is that the number of edges contained in cycles of length $< g$ is $o\big(p\binom{n}{k}\big)$. Since any cycle of length h has $(k-1)h$ vertices inducing h edges, we can bound the expected number of edges in question by ($c(k,h)$ is a constant depending on k and h only)

$$\sum_{h=2}^{g-1} n^{(k-1)h} p^h c(k,h) = O\left(\sum_{h=2}^{g-1} n^{\varepsilon h}\right) = O(n^{\varepsilon(g-1)}) = o(n^{1+\varepsilon})$$

which holds as long as $\varepsilon < 1/(g-2)$. □

By Claim 1 and Claim 2, we get a k-graph $\mathcal{E} = \bigcup_\lambda \mathcal{E}_\lambda$ satisfying the properties of Claim 1 for every σ and λ, which has $o\big(|\overrightarrow{\mathcal{E}}|\big)$ edges of length $< g$.

Proof of Lemma 2:

By Claims 3 and 4 we infer the existence of a k-graph with most of the edges not contained in cycles of length $< g$. Deleting the edges which are in short cycles we obtain a hypergraph which we denote (again) by $\mathcal{E} = \cup_\lambda \mathcal{E}_lambda$ with girth g and with the properties i) and ii) of Claim 3 still maintained.

By property i) we have

$$|\vec{\mathcal{E}}| = \sum_\lambda \vec{\mathcal{E}_\lambda} = \sum_\lambda (a_\lambda + o(1))p\binom{n}{k} = (1+o(1))p\binom{n}{k}$$

By property (ii) on the other hand

$$|\mathcal{E}_{\lambda,\vec{\pi_h},\sigma}| = (1+o(1))a_\lambda s_h p\binom{n}{k}, \text{for every} h = 1,2,\ldots,k! \text{and} \sigma.$$

Consequently $S_\ell^\sigma = |\{\vec{E} \in \vec{\mathcal{E}}, \sigma_{\vec{E}} = \pi_\ell\}| = \sum_\lambda |\vec{\mathcal{E}}_{\lambda,\pi_\ell,\sigma}| = (1+o(1))\sum_\lambda a_\lambda s_h p\binom{n}{k}$, where the summation is taken over all pairs of permutatuins $\pi_\ell, \lambda : [k] \to [k]$ with the property that $\pi_\ell = \pi_h \cdot \lambda$. Hence by (2) $\sum_\lambda a_\lambda s_h = b_\ell$ and the Lemma 2 follows. □

Proof of Theorem 1:

Theorem 1 is an easy consequence of Lemma 2. Consider a system of oriented k-tuples $\vec{\mathcal{E}} = \cup_\lambda \vec{\mathcal{E}_\lambda}$, with girth ≥ 3 and insert in ewach k-tuple $(x_\lambda(1), x_\lambda(2), \ldots, x_\lambda(k)) \in \vec{\mathcal{E}_\lambda}$ a copy λ image of G to obtain a graph H. This is possible without the conflict if $\mathcal{E}$ has no 2-cycles. It follows now from Lemma 2 that the resulting graph H satisfies properties of Theorem 1.

3. Ordering of subgraphs

Ordering lemma, canonical theorems and whole Ramsey theory belongs to study of extremal problems (in combinatorics and elsewhere). In such situatuins the higher "arity" of the problem the more difficult is the situation. Ordering theorems are not exception. One of the reasons for this is that we are dealing not with an arbitrary ordering but with orderings which are called canonical. In this paper we illustrate this on the simplest case – ordering of pairs.

We start with the description of canonical orderings. This was considered (and not published) by K. Leeb (see e.g. [6]) and we give full details here.

Let X be a set linearly ordered by $<_1$. We say that an ordering $<_2$ of pairs of X $\big($i.e. $\big(\binom{X}{2}, <_2\big)\big)$ is canonical w.r.t. $<_1$, if one of the possibilities is true:

$$(a, b,) <_2 (c, d) \text{ if}$$

(i) $a <_1 c$ or $a = c$ & $b <_1 d$
(ii) $a <_1 c$ or $a = c$ & $b \;{}_1\!> d$
(iii) $a \;{}_1\!> c$ or $a = c$ & $b <_1 d$
(iv) $a \;{}_1\!> c$ or $a = c$ & $b \;{}_1\!> d$
(v) $b \;{}_1\!> d$ or $b = d$ & $a \;{}_1\!> c$
(vi) $b \;{}_1\!> d$ or $b = d$ & $a <_1 c$
(vii) $b <_1 d$ or $b = d$ & $a \;{}_1\!> c$
(viii) $b <_1 d$ or $b = d$ & $a <_1 c$.

Note that, if an order $<_2$ is canonical w.r.t. $<_1$, then it is canonical w.r.t. reverse order $\prec_1$ defined by

$$x \prec_1 y \text{ iff } y <_1 x$$

Moreover, $<_2$ is not canonical w.r.t any other ordering of the set X.

Theorem 2. *For all k, there exists $n = n(k)$ such that any ordering $<_2$ of $\binom{[n]}{2}$ yields a k-element subset $X = \{x_1 < x_2 < \cdots < x_k\} \subset [n]$ so that $\big(\binom{X}{2}, <_2\big)$ is canonical w.r.t. the natural order on X.*

Below, all hypergraphs will have linearly ordered vertex sets. Without loss of generality, we will therefore assume that these vertex sets are sets of integers with their natural order.

Let G and H be two graphs (r-uniform hypergraphs resp.) with $V(G) = [k]$ and $V(H) = [n], k \leq n$. An embedding $f\colon G \to H$ is an order preserving isomorphism onto an induced subgraph of H. We define by $\mathrm{emb}(G, H)$ the number of all embeddings $f\colon G \to H$.

Let $\mathcal{G}$ and $\mathcal{H}$ be 3-graphs with vertex sets $[k]$ and $[N]$, respectively. Let moreover $\lessdot_2$ be a canonical order of $\binom{[k]}{2}$ and $<_2$ an arbitrary order of $\binom{[N]}{2}$.

Notation 1. Let $\binom{\mathcal{H}}{\mathcal{G}}$ be a system of all induced copies of $\mathcal{G}$ in $\mathcal{H}$ with isomorphism between $\mathcal{G}$ and $\mathcal{G}'$ being not necessarily vertex order preserving map between $V(\mathcal{G})$ and $V(\mathcal{G}')$.

For $(\mathcal{G}, \lessdot_2)$, with canonical order on its pairs, we say that $\mathcal{G}' \in \binom{\mathcal{H}}{\mathcal{G}}$ is a *canonical copy* of $\mathcal{G}$ of the (vertex) order preserving map between $V(\mathcal{G}) = [k]$ and $V(\mathcal{G}')$ is also preserving order on pairs. This map, however, may be different from the isomorphism between $\mathcal{G}$ and $\mathcal{G}'$. Let $\binom{\mathcal{H}}{\mathcal{G}}_{\text{can}}$ be a system of all canonical copies of $\mathcal{G}$ in $\mathcal{H}$. To make this more precise, these are all copies $\mathcal{G}$ for which $<_2$ restricted to $\binom{V(\mathcal{G}')}{2}$ is a canonical order of pairs. In particular, since there are eight types of canonical orders of $[k]$, the restriction of $<_2$ to $\binom{V(\mathcal{G}')}{2}$ is not for a canonical copy of $\mathcal{G}'$ uniquely defined.

Finally let $\binom{\mathcal{H}}{\mathcal{G}}^{\text{op}}_{\text{can}}$ be the set of those copies $\mathcal{G}' \in \binom{\mathcal{H}}{\mathcal{G}}_{\text{can}}$ with isomorphism $\varphi : V(\mathcal{G}) \to V(\mathcal{G}')$ being also order preserving map of vertices and pairs.

If $\mathcal{H} = (V, E)$ is a k-uniform hypergraph, we denote by $\mathcal{H}^{(2)} = (V^{(2)}, E^{(2)})$ a $\binom{k}{2}$-uniform hypergraph with vertex set $V^{(2)} = \binom{V}{2}$ and edge set $E^{(2)} = \left\{\binom{H}{2} : H \in E\right\}$.

Finally recall that a cycle of length g in an r-uniform hypergraph is a sequence of distinct edges $H_1, H_2, \ldots, H_g \in \mathcal{H}$ for which there are distinct vertices $v_1, v_2, \ldots, v_g$, $v_i \in H_i \cap H_{i+1}$ (mod g). Observe that $\mathcal{H}^{(2)}$ contains no cycle of length $\leq g$ if for any $\tilde{g}$, $2 \leq \tilde{g} \leq g$ and any collection of edges $H_1, H_2, \ldots, H_{\tilde{g}} \in E$,

$$\left|\bigcup_{i=1}^{\tilde{g}} H_i\right| \geq (k-2)\tilde{g} + 1$$

holds.

For simplicty of the notation we first formulate the main result of this section for 3-uniform hypergraphs.

Theorem 3. *Let $\mathcal{G}$, $|V(\mathcal{G})| = k$ be a 3-uniform hypergraph. Then for every $\varepsilon > 0$, there exists a 3-uniform $\mathcal{H} = ([N], E)$ and a system $\mathcal{E} \subset \binom{\mathcal{H}}{\mathcal{G}}$ such that for any order $<_2$ of $\binom{[N]}{2}$*

$$\frac{\left|\binom{\mathcal{H}}{\mathcal{G}}^{op}_{can} \cap \mathcal{E}\right|}{\left|\binom{\mathcal{H}}{\mathcal{G}}_{can} \cap \mathcal{E}\right|} = \binom{Aut\ \mathcal{G}}{k!} \pm \varepsilon.$$

Moreover, for all $\varepsilon > 0$ and integer $g \geq 3$, $\mathcal{H}$ and $\mathcal{E}$ can be chosen so that $\mathcal{E}^{(2)} = \left\{\binom{V(\mathcal{G}')}{2} : \mathcal{G}' \in \mathcal{E}\right\}$ has no cycles of length $3, 4, \ldots, g$.

First we establish the following:

Lemma 5. *For all k, g, δ, $0 < \delta < 2/(g-1)$, there exists $\mathcal{F} \subset \binom{[N]}{k}$ (N large) such that*

(1) $|\mathcal{F}| \geq \Omega(N^{2+\delta})$

(2) girth $\mathcal{F}^{(2)} \geq g$ *(and thus in particular* $|F \cap F'| \leq 2$ *whenever* $F \neq F'$, $F, F' \in \mathcal{F}$.

(3) For every ordering $<_2$ *of* $\binom{N}{2}$*, there are at least* $\beta_k|\mathcal{F}|$ *k-tuples* $F \in \mathcal{F}$ *which are canonically ordered.*

Proof of Lemma 5. For given k, let $n = n(k)$ be the number from Theorem 1. Take $N \gg n$ sufficiently large and let $\mathcal{F}^{(k)}_{N,p}$ be a random subset of $\binom{[N]}{k}$ with k-tuples selected randomly and independently, each with probability $p = N^{2-k+\delta}$. We will show that in $\mathcal{F}^{(k)}_{N,p}$ with probability $1 - o(1)$ only a "few edges are in short cycles" and moreover (1) and (3).

Indeed, since the expected number of edges $\mathbf{E}(|\mathcal{F}^{(k)}_{N,p}|) = p\binom{N}{k}$, the Chernoff inequality immediately yields that $\text{Prob}\big(|\mathcal{F}^{(k)}_{N,p}| \geq \frac{1}{2k!}N^{2+\delta}\big) = 1 - o(1)$, establishing (1).

In order to address the property (3), consider $<_2$ ordering of pairs $\binom{[N]}{2}$. Due to Theorem 1, for every n-element subset S of $[N]$, there is a k-tuple $K \subset S$ so that $<_2$ restricted to $\binom{K}{2}$ is canonical. This in particular means that, fixing $<_2$, the set $\mathcal{K}^{<_2}$ of all canonical k-tuples has cardinality

$$|\mathcal{K}^{<_2}| \geq \frac{\binom{N}{k}}{\binom{n}{k}} \geq \left(\frac{N}{n}\right)^k.$$

Consequently, applying Chernoff inequality again (and now a bit more carefully), we infer that

$$\text{Prob}\Big[(|\mathcal{F}_{N,p} \cap \mathcal{K}^{<_2}| - \mathbf{E}|\mathcal{F}_{N,p} \cap \mathcal{K}^{<_2}|) > \frac{1}{2}\left(\frac{N}{n}\right)^k p\Big] < \exp(-\alpha_k N^k p).$$

(The last inequality here follows in view of the dependence of n on k, given by Theorem 1.)

Since $\alpha_k N^k p = \alpha_k N^{2+\delta}$ and since there are only $\binom{N}{2}! < \exp(2N^2 \log N)$ orderings of $\binom{N}{2}$ we infer that with probability

$$1 - \exp(-\alpha_k N^{2+\delta} + 2N^2 \log N) = 1 - o(1),$$

$$|\mathcal{F}_{N,p} \cap \mathcal{K}^{<_2}| \geq \frac{1}{2}\mathbf{E}|\mathcal{F}_{N,p} \cap \mathcal{K}^{<_2}| \geq \frac{1}{2}\frac{N^{2+\delta}}{n^k}(1 - o(1)) = \beta_k N^{2+\delta}$$

holds for all orderings $<_2$ of $\binom{N}{2}$.

Finally we shall address the short cycles in $\mathcal{F}_{N,p}$. First, observe that for any $\tilde{g} \leq g$, the expected number of $(k-2)\tilde{g}$ element subsets of $[N]$ which induce at least $\tilde{g}$ edges is at most $O(N^{(k-2)\tilde{g}}p^{\tilde{g}}) \leq N^{\delta g} = o(N^{2+\delta})$. Thus (say by the Markov inequality) with probability $1 - o(1)$ there are only $o(N^{2+\delta}) = o(|\mathcal{F}_{N,p}|)$ edges which are in short cycles of $\mathcal{F}^{(2)}$.

Deleting these edges yields a resulting graph since, due their small number, the properties (1) and (3) remain valid. □

Next we establish Theorem 3. The proof closely follows the idea from [9]. Given 3-uniform $\mathcal{G}$, $|V(\mathcal{G})| = k$ and an integer $g \geq 2$ consider a hypergraph $\mathcal{F}$ existence of which is ensured by Lemma 5.

Consider a space $\mathcal{F}(\mathcal{G})$ of all 3-uniform hypergraphs $\mathcal{H}$ obtained by inserting a copy of $\mathcal{G}$ into each $F \in \mathcal{F}$. Since there are $\frac{k!}{\text{Aut } \mathcal{G}}$ such insertions, we infer that

$$|\mathcal{F}(\mathcal{G})| = \Big(\frac{k!}{\text{Aut } \mathcal{G}}\Big)^{|\mathcal{F}|}.$$

Fix an arbitrary order $<_2$ of $\binom{[N]}{2}$ and let $F(<_2)$ be a family of all canonically ordered k-sets $F \in \mathcal{F}$. By (3) of Lemma 5, we infer that

$$|\mathcal{F}(<_2)| \geq \beta_k |\mathcal{F}| = \Omega(N^{2+\delta}).$$

On the other hand, permuting $[k]$ one obtains $r = \binom{k!}{\text{Aut } \mathcal{G}}$ distinct images $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_r$ of $\mathcal{G}$ on $[k]$ (with say $\mathcal{G} = \mathcal{G}_1$) and thus, due to the construction of $\mathcal{F}(\mathcal{G})$ (the insertions of images of $\mathcal{G}$ are chosen uniformly and independently for distinct $F \in \mathcal{F}(<_2) \subset \mathcal{F}$), we infer that for each $j = 1, 2, \ldots, r$, the expected number of F's inducing an ordered copy of $\mathcal{G}_j$ equals

$$\frac{\text{Aut } \mathcal{G}}{k!} |F(<_2)| = \Omega(N^{2+\delta}).$$

Thus if I_j denotes a random variable counting ordered copies of $\mathcal{G}_j$ in $\mathcal{F}(<_2)$, we infer by the Chernoff inequality that

$$\text{Prob}\Big[\Big|I_j - \frac{\text{Aut } \mathcal{G}}{k!}|\mathcal{F}(<_2)|\Big| > \frac{\varepsilon}{2}\frac{\text{Aut } \mathcal{G}}{k!}|\mathcal{F}(<_2)|\Big] < \exp(-\Omega(N^{2+\delta})).$$

In particular,

$$\text{Prob}\Big[\Big|\frac{I_1}{\sum I_j} - \frac{\text{Aut } \mathcal{G}}{k!}\Big| > \varepsilon\Big] < \exp(-\Omega(N^{2+\delta})).$$

This is true for each of the $\binom{N}{2}! < \exp(2N \log N)$ possible orders $<_2$ of $\binom{N}{2}$ and thus in particular

$$\text{Prob}\Big[\Big|\frac{I_1}{\sum I_j} - \frac{\text{Aut } \mathcal{G}}{k!}\Big| > \varepsilon\Big] < \exp(-\Omega(N^{2+\delta}) + 2N \log N) = o(1)$$

holds for all orderings of $\binom{N}{2}$. Consequently, there is a 3-uniform hypergraph $\mathcal{H} \in \mathcal{F}(\mathcal{G})$ and a family

$$\mathcal{E} = \{\mathcal{G}' \colon V(\mathcal{G}') \in \mathcal{F}\} \subset \binom{\mathcal{H}}{\mathcal{G}}$$

with the properties of Theorem 3.

Proof of Theorem 2. Proof uses Ramsey theorem for coloring of triples and quadruples. Consecutively the bound for $n(k)$ is very large and we do not optimize this.

Let n be a large number and let $<_2$ be an ordering of $\binom{[n]}{2}$. Apart from $<_2$ we denote by $<_L$ the lexicographic order of $\binom{[n]}{2}$. For any $\{i,j,k,l\} \in \binom{[n]}{4}, i<j<k<l$, denote by $\sigma(i,j,k,l)$ the isomorphism type ordering $<_2$ restricted to $\binom{\{i,j,k,l\}}{2}$. By this we mean that if $i'<j'<k'<l'$ and the monotone bijection $\{i,j,k,l\} \to \{i',j',k',k'\}$ is also monotone with respect to $<_2$ then $\sigma(i,j,k,l) = \sigma(i',j',k',l')$. Now this induces a finite colouring of $\binom{[n]}{4}$ (by at most 6! colours). Applying Ramsey theorem we get a k-tuple $X = \{x_1 < x_2 < \cdots < k_k\}$ such that $\binom{X}{4}$ is homogeneous, i.e. $\sigma(M') = \sigma(M) = \sigma$ for all $M, M' \in \binom{X}{4}$. Now σ is a particular ordering of $\binom{\{1,2,3,4\}}{2}$. We have to distinguish few cases but each leads to one of the canonical orderings. □

From the fact that hypergraphs from a Ramsey class [1, 8] we get similarly also the following analogue (and in fact generalization) of ordering lemma:

Theorem 4. *Fix $p \geq 3$. Let $G = (V,E)$ be p-uniform ordered hypergraph (with ordering $<_G$). Then there exists a p-uniform ordered hypergraph $H = (W,F)$ (with ordering $<_H$) with the following property: for every ordering $<_2$ of set $\binom{W}{2}$ there exists a subhypergraph G' of H order isomorphic to G such that $<_2$ restricted to $\binom{V(G')}{2}$ is canonical with respect to the ordering of $V(G')$.*

This result is similar to line of research [6] where canonical orderings of Hales-Jewett cubes were considered.

Remark that here we have to assume $p \geq 3$ as for $p = 2$ we could order edges and non edges separately yielding non-canonical ordering.

Can we prove a quantitative version of this result? The answer is positive and this will be formulated as Theorem 5 if we recall and generalize formalism from triples to p-tuples.

After that we can formulate the analogue of Theorem 3 for p-uniform hypergraphs:

Theorem 5. *Let $\mathcal{G}$, $|V(\mathcal{G})| = K$ be a p-uniform hypergraph, $p > 2$. Then for every $\varepsilon > 0$, there exists a 3-uniform $\mathcal{H} = ([N], E)$ and a system $\mathcal{E} \subset \binom{\mathcal{H}}{\mathcal{G}}$ such that for any order $<_2$ of $\binom{[N]}{2}$*

$$\frac{\left|\binom{\mathcal{H}}{\mathcal{G}}^{op}_{can} \cap \mathcal{E}\right|}{\left|\binom{\mathcal{H}}{\mathcal{G}}_{can} \cap \mathcal{E}\right|} = \binom{Aut\ \mathcal{G}}{k!} \pm \varepsilon.$$

Moreover, for all $\varepsilon > 0$ and integer $g \geq 3$, $\mathcal{H}$ and $\mathcal{E}$ can be chosen so that $\mathcal{E}^{(2)} = \left\{\binom{V(\mathcal{G}')}{2} : \mathcal{G}' \in \mathcal{E}\right\}$ has no cycles of length $3, 4, \ldots, g$.

The proof follows the same lines as the proof of Theorem 3.

Concluding remarks

It is possible to prove the statement analogical to Theorems 2 and 3, and also Theorems 4 and 5, for canonical orderings of q-tuples in p-uniform hypergraphs (of course under assumption that $q < p$). This together with further generalizations should appear elsewhere. it remains to be seen whether the Theorem 1 (and its corollaries) have an interesting interpretation in the topological dynamics similar to [2].

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