All those Ramsey classes

(Ramsey classes with closures and forbidden homomorphisms)

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Abstract

We prove the Ramsey property of classes of ordered structures with closures and given local properties. This generalises earlier results: the Nešetřil-Rödl Theorem, the Ramsey property of partial orders and metric spaces as well as the author's Ramsey lift of bowtie-free graphs. We use this framework to give new examples of Ramsey classes. Among others, we show (and characterise) the Ramsey property of convexly ordered S-metric spaces and prove the Ramsey Theorem for Finite Models (i.e. structures with both functions and relations) thus providing the ultimate generalisation of Structural the Ramsey Theorem. We also show the Ramsey Theorem for structures with linear ordering on relations ("totally ordered structures"). All of these results are natural, easy to state, yet proofs involve most of the theory developed here.

We characterise classes of structures defined by forbidden homomorphisms having a Ramsey lift and extend this to special cases of classes with closures. We apply this to prove the Ramsey property of many Cherlin-Shelah-Shi classes.

In several instances our results are the best possible and confirm the meta-conjecture that Ramsey classes are "close" to lifted universal classes.

Contents

1 Introduction

Extending classical early results, structural Ramsey theory originated at the beginning of 70's in a series of papers [28, 27, 54, 30, 1, 49], see [66] for references. Proper foundation was given by introducing the notions of Ramsey class, an **A**-Ramsey property and an ordering property [47, 54]. However the list of Ramsey classes, which may be seen as top of the line of Ramsey properties, was somewhat limited and this was also encouraged by the connection to ultrahomogeneous structures [50]: all Ramsey classes of undirected graphs were known earlier [55] (and this has been also verified recently for oriented graphs [40]). This connection led to the classification programme for Ramsey classes [52] and, in an important new twist, to the connection to the topological dynamics and ergodic theory [41].

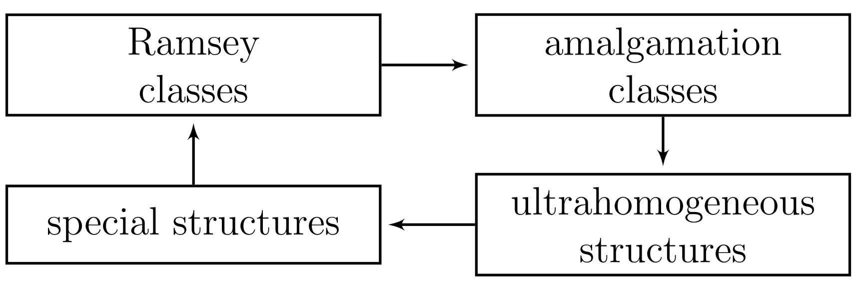
This development also led to rethinking of some of fundamentals of Ramsey theory. This paper is a contribution to this development. We present in this paper the far reaching generalisations which started from authors solution of the bowtie-free problem [35].

Let us start with the key definition of this paper. Let $\mathcal{K}$ be a class of structures endowed with embeddings between its members. For objects $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all sub-objects of $\mathbf{B}$, which are isomorphic to $\mathbf{A}$. (By a sub-object we mean that the inclusion is an embedding.) Using this notation the definition of a Ramsey class gets the following form:

A class $\mathcal{C}$ is a *Ramsey class* if for every two objects $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}$ and for every positive integer k there exists an object $\mathbf{C}$ in $\mathcal{C}$ such that the following holds: For every partition $\binom{\mathbf{C}}{\mathbf{A}}$ into k classes there exists an $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ belongs to one class of the partition. It is usual to shorten the last part of the definition to $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$.

Which classes are Ramsey? In other words: which structures allow such an ultimate generalisation of the Ramsey theorem?

These questions may be less illusive than it seems on the first glance as we can use the above-mentioned connection of Ramsey classes and ultrahomogeneous structures. The Ramsey Classification Programme was symbolised in [52] by the following diagram:



Here is the Ramsey Classification Programme in words: Under mild assumptions every Ramsey class leads to an amalgamation class (by [50], and in full generality [52, 41]) and amalgamation classes in turn lead to (infinite) ultrahomogeneous structures (Fraïssé limits). This is the source of (Lachlan,Cherlin) Classification Programme of Ultrahomogeneous Structures [46, 6, 4]. Not every ultrahomogeneous structure leads to a Ramsey class. We need to make the structure even more uniform often adding some additional information (like ordering). For such special ultrahomogeneous structure we can then hope to prove the last implication.

Recently this programme took a more concrete form [3, 70, 48] asking whether every ω-categorical ultrahomogeneous structure $\mathbf{A}$ has a finite (or precompact) expansion (called here lift) so that the corresponding class of all finite substructures (i.e. its age) is Ramsey. (Such a lift is obtained by a homogenising procedure and we treat it in Section 3 in full detail.)

If such (more concrete) approach would be true then the lack of symmetry (expressed by ultrahomogeneity) and lack of rigidity (expressed by special lifts) would be only obstacles for Ramseyness and the Ramsey Classification Programme. However, recently Evans [22] found examples of ultrahomogeneous structures (of Hrushovski type) which have no precompact Ramsey lift. His result relates to the most important case when the lifted class is defined by finitely many additional relations of every arity. This indicates that the answer to the Classification Programme may be more complicated then originally thought. (See [23] for refinement of [22] using the main result of this paper.)

Yet amalgamation is a central necessary condition for Ramsey classes. The main result (Theorem 2.2) gives a necessary structural condition on the class of ordered structures which implies the Ramsey property. The condition can be seen as a variant of a well established notion of the Fraïssé's amalgamation classes (and we call it an $(\mathcal{R},\mathcal{U})$-multiamalgamation class) with an explicit closure description $\mathcal{U}$ and additional assumptions about the local finiteness of the completions relative to a given Ramsey class $\mathcal{R}$. Theorem 2.2 is inspired by our recent result for bowtie-free graphs.

However we also isolate the more easily formulated Theorem 2.1 which, somewhat surprisingly, gives sufficient conditions for a Ramsey subclass of a Ramsey class: local finiteness and strong amalgamation are enough. The combination of these two theorems makes this approach flexible and easy to apply.

The structural Ramsey theory uses the Partite Construction as its main proof technique. It was developed by Nešetřil-Rödl in a series of papers [54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 53]. We use the Partite Construction in the form with unary closures as stated in our earlier paper [35] extending it to non-unary closures and further generalising the Iterated Partition Construction [53] to a local amalgamation argument. In a way our paper is further evidence for the surprising effectivity of the Partite Construction in the structural Ramsey Theory. This paper presents the most general formulation of the Partite Construction. Of course one could formulate this in the categorical terms (as opposed to the finite model-theoretic language as presented here) but it remains to be seen if such translation would produce any new interesting Ramsey classes. Nevertheless our general Theorems 2.1 and 2.2 present an unified approach to many ad-hoc applications of the Partite Construction.

Many examples of amalgamation classes are multiamalgamation classes. This allows us to give in Section 4 multiple examples of applications of the main result. Our starting point is the Nešetřil-Rödl Theorem [54]. Our examples of Ramsey classes include known examples such as (finite) acyclic graphs and partially ordered sets with linear extension [60], ordered metric spaces [53] or convexly ordered $\mathcal{H}$-colourable graphs. Many new examples follow. Particularly we fully characterise the Ramsey property of metric spaces with a given set of distances (in Section 4.3.3, Theorem 4.29) thus solving one problem in [68] and contribute to problem in [41]. We also consider classes with function symbols and give the first examples of classes defining partial orders not only on vertices, but also on n-tuples and neighbourhoods. These examples as well provide better understanding of the nature of Theorems 2.1 and 2.2. As a consequence we are able to prove the Ramsey Theorem for Finite Models (Theorem 4.26) and the Ramsey Theorem for Totally Ordered Structures (Theorem 4.32). These results may be viewed as new structural generalisations of the Ramsey Theorem. Both of these results are easy to state yet combine most of the techniques developed in this paper.

In Section 3 we are interested in classes of structures with a (possibly infinite) set of forbidden homomorphisms (or, more precisely, homomorphism-embeddings). Such classes were studied earlier (e.g. in [43, 42, 11]). In order to reach the level of the description needed for Ramsey constructions (particularly for the Partite Construction) we have to describe our classes more explicitely. This leads to notions of pieces and witnesses defined in our earlier papers [37, 38]. These notions are elaborated here in a greater detail (and generality) to follow Ramsey constructions. The whole process can be described as homogenisation (the term coined in [18]) and it amounts to describe the class by special rooted subgraphs.

As a particular case we develop a way to give an explicit Ramsey lift for classes defined by forbidden homomorphism-embeddings (a more restricted notion of homomorphism which is an embedding on irreducible structures) together with a restricted form of (irreducibly rooted) closure. Generalising the Ramsey lift of classes defined by forbidden homomorphisms from a finite set [65] and bowtie-free graphs [35] we give both satisfactory condition for Ramsey properties of classes defined by forbidden homomorphisms from an infinite set. This leads to a complete characterisation of Ramsey classes defined by means of forbidden homomorphisms (Theorem 3.7). A bit surprisingly classes with Ramsey lifts are exactly those which ω-categorical universal structures (see Corollary 3.8 for details). For classes defined by forbidden monomorphisms the situation is much more complex (even on the side of universality where algorithm undecidability is conjectured) and we essentially prove that the Ramsey property in many instances does not present any new restriction, see Theorem 4.37.

As it is well known from the beginning of the structural Ramsey theory, the orderings of the structures play a special rôle. In fact, Ramsey classes always fix a linear order [41, 2]. We can not escape this here. In Section 2 our results take the form of implications and we do not have to speak about ordering at all. It is implicit and will be mentioned in examples illustrating general results. In Section 3 we incorporate the ordering in the language. In Section 4 we relate this to the more traditional approach of structures with additional ordering of vertices by considering Ramsey lifts which adds the order.

Some of the results of this paper were outlined in our conference paper [36].

All in all Ramsey classes are not isolated examples. The rich spectrum of our examples should perhaps convince the interested reader about this.

1.1 Preliminaries

Most of our examples are relational structures. In fact later (in Section 2) we find it convenient to treat finite models (including functions) as relational structures (this is specified Section 4.3.2). We follow standard notations.

A language L is a set of relational symbols $R \in L$, each associated with natural number $a(R)$ called *arity*. A *(relational) L-structure* $\mathbf{A}$ is a pair $(A, (R_{\mathbf{A}}; R \in L))$ where $R_{\mathbf{A}} \subseteq A^{a(R)}$ (i.e. $R_{\mathbf{A}}$ is a $a(R)$-ary relation on A). The set A is called the *vertex set* or the *domain* of $\mathbf{A}$ and elements of A are *vertices*. The language is usually fixed and understood from the context (and it is in most cases denoted by L). If set A is finite we call $\mathbf{A}$ *finite structure*. We consider only structures with countably many vertices. The class of all (countable) relational L-structures will be denoted by $\mathrm{Rel}(L)$.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R_{\mathbf{B}}; R \in L))$ is a mapping $f : A \to B$ satisfying for every $R \in L$ the implication $(x_1, x_2, \dots, x_{a(R)}) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \dots, f(x_{a(R)})) \in R_{\mathbf{B}}$. (For a subset $A' \subseteq A$ we denote by $f(A')$

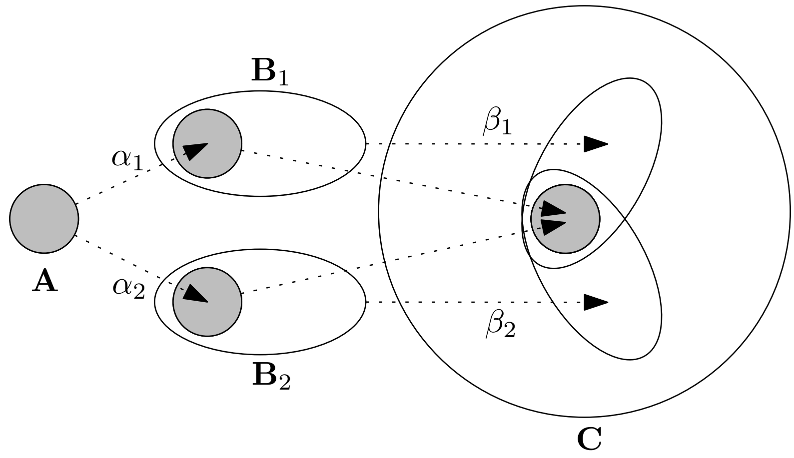


Figure 1: An amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

the set $\{f(x); x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a structure.) If f is injective, then f is called a *monomorphism*. A monomorphism is called *embedding* if the above implication is equivalence, i.e. if for every $R \in L$ we have $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$. If f is an embedding which is an inclusion then $\mathbf{A}$ is a *substructure* (or *subobject*) of $\mathbf{B}$. For an embedding $f : \mathbf{A} \to \mathbf{B}$ we say that $\mathbf{A}$ is *isomorphic* to $f(\mathbf{A})$ and $f(\mathbf{A})$ is also called a *copy* of $\mathbf{A}$ in $\mathbf{B}$. Thus $\binom{\mathbf{B}}{\mathbf{A}}$ is defined as the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

We now review some more standard model-theoretic notions (see e.g. [33]).

Let $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ be relational structures and α_1 an embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 an embedding of $\mathbf{A}$ into $\mathbf{B}_2$, then every structure $\mathbf{C}$ with embeddings $\beta_1 : \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2. See Figure 1. We will call $\mathbf{C}$ simply an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ (as in the most cases α_1 and α_2 can be chosen to be inclusion embeddings).

We say that an amalgamation is *strong* when $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$. Less formally, a strong amalgamation glues together $\mathbf{B}_1$ and $\mathbf{B}_2$ with an overlap no greater than the copy of $\mathbf{A}$ itself. A strong amalgamation is *free* if there are no tuples in any relations of $\mathbf{C}$ spanning both vertices of $\beta_1(B_1 \setminus \alpha_1(A))$ and $\beta_2(B_2 \setminus \alpha_2(A))$.

An *amalgamation class* is a class $\mathcal{K}$ of finite structures satisfying the following three conditions:

1. **Hereditary property:** For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;

2. **Joint embedding property:** For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C}$ contains both $\mathbf{A}$ and $\mathbf{B}$ as substructures;

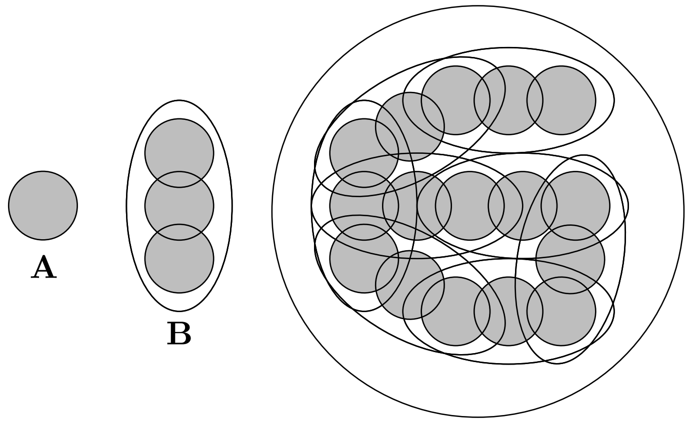


Figure 2: Construction of a Ramsey object by multiamalgamation.

3. **Amalgamation property:** For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

We will refine amalgamation classes in Definition 2.8. The full role of amalgamation classes will be discussed in Section 3.

2 Construction of Ramsey classes

The main results of this paper will be introduced here. Several old and new concepts have to be recalled and introduced in this section.

2.1 Statement of the results

First, we develop a generalised notion of amalgamation which will serve as useful tool for the construction of Ramsey objects. As schematically depicted in Figure 2, Ramsey objects are a result of amalgamation of multiple copies of a given structure which are all performed at once. In a non-trivial class this leads to many problems. Instead of working with complicated amalgamation diagrams we split the amalgamation into two steps — the construction of (up to isomorphism unique) free amalgamation (which yields an incomplete or "partial" structure) followed then by a completion. Formally this will be done as follows:

Definition 2.1. An L-structure $\mathbf{A}$ is *irreducible* if for every pair of distinct vertices u, v there is tuple $\vec{t} \in R_{\mathbf{A}}$ (of some relation $R \in L$) such that $\vec{t}$ contains both u and v.

Thus the irreducibility is meant with respect to the free amalgamation. The irreducible structures are our building blocks. Moreover in structural Ramsey theory we are fortunate that most structures are (or may be interpreted as) irreducible (for example thanks to a linear ordering).

We introduce the following stronger notion of homomorphism.

Definition 2.2. A homomorphism $f : \mathbf{A} \to \mathbf{B}$ is *homomorphism-embedding* if f restricted to any irreducible substructure of $\mathbf{A}$ is an embedding to $\mathbf{B}$.

While for (undirected) graphs the homomorphism and homomorphism-embedding coincide, for relational structures they differ.

Definition 2.3. Let $\mathbf{C}$ be a structure. An irreducible structure $\mathbf{C}'$ is a *completion* of $\mathbf{C}$ if there exists homomorphism-embedding $\mathbf{C} \to \mathbf{C}'$. If there is a homomorphism-embedding $\mathbf{C} \to \mathbf{C}'$ which is one-to-one, we call $\mathbf{C}'$ a *strong completion*.

In particular interest will be whether there exists a completion in a given class $\mathcal{K}$ of structures. In this case we speak about $\mathcal{K}$*-completion*.

Remark (on completion and holes)**.** Completion may be seen as a generalised form of amalgamation and strong completion as a generalised form of strong amalgamation. To see that let $\mathcal{K}$ be a class of irreducible structures. The (strong) amalgamation property of $\mathcal{K}$ can be equivalently formulated as follows: For $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is a (strong) completion of the free amalgamation (which itself is not necessarily in $\mathcal{K}$) of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

Free amalgamation may result in a reducible structure. The pairs of vertices where one vertex belong to $\mathbf{B}_1 \setminus \alpha_1(\mathbf{A})$ and the other to $\mathbf{B}_2 \setminus \alpha_2(\mathbf{A})$ are never both contained in a single tuple of any relation. Such pairs can be thought of as *holes* and a completion is then a process of filling in the holes to obtain irreducible structures while preserving all embeddings of irreducible structures.

The following is the key definition. It defines main property for obtaining the Ramsey classes.

Definition 2.4. Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is *locally finite subclass of* $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ there is finite integer $n = n(\mathbf{C}_0)$ such that every structure $\mathbf{C}$ has strong $\mathcal{K}$-completion (i.e. there exists $\mathbf{C}' \in \mathcal{K}$ that is a strong completion of $\mathbf{C}$) providing that it satisfies the following:

1. there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$ (in other words, $\mathbf{C}_0$ is a, not necessarily strong, $\mathcal{R}$-completion of $\mathbf{C}$), and,
2. every substructure of $\mathbf{C}$ with at most n vertices has a strong $\mathcal{K}$-completion.

True meaning of Definition 2.4 will be manifested in the examples bellow and in Section 4.2.

Example. Our running examples will be provided by metric spaces. Consider a language L containing binary relations R^1, R^2, R^3, and R^4 which we interpret as distances. Let $\mathcal{R}$ be the class of all irreducible finite structures where all four relations are symmetric, irreflexive and every pair of distinct vertices is in precisely one of relations R^1, R^2, R^3, or R^4 ($\mathcal{R}$ may be viewed a class of 4-edge-coloured complete graphs). Let $\mathcal{K}$ be a subclass of $\mathcal{R}$ of those structures satisfying the triangle inequality ($\mathcal{K}$ is the class of finite metric spaces with distances 1, 2, 3, and 4). Every structure $\mathbf{C}$ which has a completion to some $\mathbf{C}_0 \in \mathcal{R}$ (and thus all relations are symmetric and irreflexive and every pair of distinct vertices is in at most one relation) can be completed to a metric space if and only if it contains no non-metric triangles (i.e. a triangle with distances 1–1–3, 1–1–4 or 1–2–4) and no 4-cycle with distances 1–1–1–4. It follows that $\mathcal{K}$ is a locally finite subclass of $\mathcal{R}$ and for every $\mathbf{C}_0 \in \mathcal{R}$ we can put $n(\mathbf{C}_0) = 4$.

Now consider the subclass $\mathcal{K}_{1,3}$ of all metric spaces which use only distances one and three. It is easy to see that $\mathcal{K}_{1,3}$ is not a locally finite subclass of $\mathcal{R}$. Denote by $\mathbf{T} \in \mathcal{R}$ the triangle with distances 1–1–3. Now consider cycle of length n with one edge of distance three and the others of distance one. Such cycle has completion $\mathbf{T}$ however there is no strong $\mathcal{K}_{1,3}$-completion, while every proper substructure (path consisting of at most one edge with distance three and other with distance one) does have a $\mathcal{K}_{1,3}$-strong completion. It follows that there is no $n(\mathbf{T})$ and thus $\mathcal{K}_{1,3}$ is not locally finite subclass of $\mathcal{R}$. We will further discuss metric spaces in Sections 4.2.2 and 4.3.3.

Our first result gives a surprisingly compact sufficient condition for Ramsey classes:

Theorem 2.1. *Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with strong amalgamation. Then $\mathcal{K}$ is Ramsey.*

Explicitly: For every pair of structures $\mathbf{A}, \mathbf{B}$ in $\mathcal{K}$ there exists structure $\mathbf{C} \in \mathcal{K}$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Note that this theorem has the form of an implication: If $\mathcal{R}$ is a Ramsey class then also (a more special subclass) $\mathcal{K}$ is Ramsey.

Remark (on irreducibility)**.** The condition on $\mathcal{R}$ to be a class of irreducible structures may seem too weak. It is however trivially satisfied in all applications we discuss. Why? The irreducibility is usually guaranteed by orderings. It is a non-trivial fact that for every Ramsey class (which is an age of homogeneous structure) fixes a linear order on vertices [41, 2]. In such cases (see remarks on interpretations in Section 4.2.3) we can assume that every structure in every Ramsey class has a binary relation representing the order. This order makes the structure irreducible in the sense of Definition 2.1.

Numerous concrete examples of Ramsey classes are implied by Theorem 2.1, particularly those related to classes of relational structures defined by means of forbidden homomorphisms and forbidden homomorphism embeddings. We summarise these applications in Section 4.

However Theorem 2.1 is not the end of the story. In this paper we are able to deal with classes of structures with both relations and functions (or operations) — i.e. with finite *models*. We find it convenient to deal with functions implicitly by not making them part of our language. Instead we control "functionality" of relations by a degree condition. Because our constructions are based on strong amalgamation classes we will at several places take an advantage of the fact that we can interpret given symbol as both a function and a relation. Note that the closure description will be denoted by $\mathcal{U}$. (By the lack of other letters $\mathcal{U}$, may stand for "uzávěr"—Czech word for closure. Also a special rôle will be played by unary closures. So you may think about it in this way.)

Definition 2.5. A *closure description* $\mathcal{U}$ is a (possibly infinite) set of pairs $(R^U, \mathbf{R})$ where R^U is a relational symbol of arity n and $\mathbf{R}$ is a non-empty irreducible structure on vertices $\{1, 2, \ldots, m\}$, $m \leq n$. We will refer to relations R^U as to *closure relations* to tuples in relations R^U as the *closure tuples* and to structures $\mathbf{R}$ as the *roots of the closures*.

Given a structure $\mathbf{A}$ the closure description can be understood as follows. Every pair $(R^U, \mathbf{R}) \in \mathcal{U}$ declares that the relation $R^U_{\mathbf{A}}$ of arity n is a function that assigns to every embedding of $\mathbf{R} \to \mathbf{A}$ an $(n - |R|)$-tuple of vertices of $\mathbf{A}$. We always assume that for every $\vec{t} \in R^U_{\mathbf{A}}$ the first $|R|$ vertices denote the copy of $\mathbf{R}$ and the remaining $(n - |R|)$ the vertices assigned to the copy of $\mathbf{R}$. This interpretation leads to:

Definition 2.6. Given a structure $\mathbf{A}$ with relation $R_{\mathbf{A}}$ of arity n, the $R_{\mathbf{A}}$-*out-degree* of a k-tuple $(v_1, v_2, \ldots, v_k)$ is the number of $(n-k)$-tuples $(v_{k+1}, v_{k+2}, \ldots, v_n)$ such that $(v_1, v_2, \ldots, v_n) \in R_{\mathbf{A}}$.

Let $\mathcal{U}$ be a closure description. We say that a structure $\mathbf{A}$ is *$\mathcal{U}$-closed* if for every pair $(R^U, \mathbf{R}) \in \mathcal{U}$ it holds that the $R^U_{\mathbf{A}}$-out-degree of an $|R|$-tuple $\vec{t}$ (of vertices of $\mathbf{A}$) is one if and only if the tuple $\vec{t}$ represents an embedding of $\mathbf{R}$ to $\mathbf{A}$ and zero otherwise.

Let $\mathbf{A}$ be an $\mathcal{U}$-closed structure and $B \subseteq A$. The *$\mathcal{U}$-closure of B in $\mathbf{A}$*, denote by $\mathrm{Cl}^{\mathcal{U}}_{\mathbf{A}}(B)$, is the minimal $\mathcal{U}$-closed substructure of $\mathbf{A}$ containing B.

Observe that because roots are non-empty, the empty structure is always $\mathcal{U}$-closed. Special cases that are important to us deserve special names: If every root has only one vertex, we speak about the *unary closure*. For a unary closure descriptions the closure of a subset is always a union of closures of individual vertices.

Remark. For amalgamation classes of irreducible ordered structures our definition of closure is equivalent with the model-theoretic definition of the algebraic closure (see Definition 4.11) considered in the Fraïssé limit of the class. This follows from the fact that the closure relations are definable in the structure and thus their equivalent need to be already present in the language. The advantage of our definition here is that the degree condition here is easier to control than the abstract closure one.

Remark. Notice that the roots of closures are irreducible substructures. For $\mathcal{U}$-closed structures containing holes, the completion to an $\mathcal{U}$-closed structure will turn reducible substructures irreducible and may thus involve a need to introduce new vertices and new relations to add closures. Note also that here closures do not satisfy further properties. For classes of structures with closures which in addition satisfy some axioms completion may be a difficult task. Iterated and simultaneous amalgamations may produce many holes and completing this (to a structure in $\mathcal{K}$) is the key problem.

To make the verification of the existence of a completion easier, we further refine it to the following (which suffices for our Ramsey applications):

Definition 2.7. Let $\mathbf{C}$ be a structure and let $\mathbf{B}$ be irreducible substructure of $\mathbf{C}$. We say that irreducible structure $\mathbf{C}'$ is a *completion of* $\mathbf{C}$ *with respect to copies of* $\mathbf{B}$ if there exists function $f : C \to C'$ such that for every $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ function f restricted to $\widetilde{B}$ is an embedding of $\widetilde{\mathbf{B}}$ to $\mathbf{C}'$.

If $\mathbf{C}'$ belong to a given class $\mathcal{K}$, then $\mathbf{C}'$ is called *$\mathcal{K}$-completion of* $\mathbf{C}$ *with respect to copies of* $\mathbf{B}$.

This is the weakest notion of completion which preserve the Ramsey property for a given structures $\mathbf{A}$ and $\mathbf{B}$. Note that f does not need to be homomorphism-embedding (and even homomorphism).

We now state all necessary conditions for our second main result:

Definition 2.8. Let L be a language, $\mathcal{R}$ be a Ramsey class of finite irreducible L-structures and $\mathcal{U}$ be a closure description (in the language L). We say that a subclass $\mathcal{K}$ of $\mathcal{R}$ is an $(\mathcal{R},\mathcal{U})$-*multiamalgamation class* if the following conditions are satisfied:

1. **$\mathcal{U}$-closed structures:** $\mathcal{K}$ consists of finite $\mathcal{U}$-closed L-structures.

2. **Hereditary property for $\mathcal{U}$-closed substructures:** For every $\mathbf{A} \in \mathcal{K}$ and an $\mathcal{U}$-closed substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$.

3. **Strong amalgamation property:** For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which contains a strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2 as a substructure.

4. **Locally finite completion property:** Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{C}_0 \in \mathcal{R}$. Then there exists $n = n(\mathbf{B}, \mathbf{C}_0)$ such that if $\mathcal{U}$-closed L-structure $\mathbf{C}$ satisfies the following:

   (a) there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$ (in other words, $\mathbf{C}_0$ is a completion of $\mathbf{C}$), and,

   (b) every substructure of $\mathbf{C}$ with at most n vertices has a $\mathcal{K}$-completion.

   Then there exists $\mathbf{C}' \in \mathcal{K}$ that is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$.

Remark. We shall see that this seemingly elaborated definition is in fact very flexible and easy to apply. For an amalgamation class $\mathcal{K}$ of irreducible structures it is up to interpretation always possible to construct a closure description $\mathcal{U}$ such that $\mathcal{K}$ satisfies the first three conditions in Definition 2.8. (Only exception are amalgamation classes which give a Fraïssé limit containing a closure of empty set. Those can be always corrected by appropriate interpretation.) Also as in our definition the empty set is always $\mathcal{U}$-closed we get strong joint embedding property: For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C}$ contains both $\mathbf{A}$ and $\mathbf{B}$ as (vertex) disjoint substructures.

It is the locally finite completion property which is the crucial condition for $\mathcal{K}$ to be a Ramsey class. Notice the difference between Definitions 2.8 and 2.4. In the case of strong amalgamation classes we use strong completions (in Definition 2.4) while in Definition 2.8 we use just completions. In the second case the bound on number of vertices is significantly less useful given the fact that the completions may identify vertices and reduce size of a substructure. In many applications it is however possible to show the existence of strong completions.

We can now state our main result as:

Theorem 2.2. *Every* $(\mathcal{R},\mathcal{U})$*-multiamalgamation class* $\mathcal{K}$ *is Ramsey.*

In Section 4 we give many examples illustrating usefulness of this result. Presently it covers most of the examples of Ramsey classes of structures. The strong amalgamation assumption will be achieved by a convenient lift (expansion). This is the subject of Section 4.

2.2 Proof structure of Theorems 2.1 and 2.2

The overall structure of proof of Theorems 2.1 and 2.2 is depicted in Figure 2.1. We give an explicit construction of Ramsey objects. For given structures $\mathbf{A}$ and $\mathbf{B}$ we first apply the Nešetřil-Rödl Theorem to obtain Ramsey $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ and subsequently we use three variants of the Partite Construction to obtain Ramsey structure $\mathbf{C}$ with desired properties.

In Sections 2.3 and 2.4 we give a new Partite Construction for classes with closures (generalising our techniques introduced in [35] and strengthening them to non-unary closures). In Section 2.5 we introduce the Iterated Partite Construction for strong amalgamation classes (extending results of [53, 65]) and finally we combine both to obtain our main results in Section 2.6.

To construct $\mathcal{U}$-closed structures (see Definition 2.6) we proceed in several steps. The following notions capture two "weaker" notions of closed structures and substructures which will be used in our constructions.

Definition 2.9. Let $\mathcal{U}$ be a closure description and $\mathbf{A}$ a substructure of $\mathbf{B}$. We say that $\mathbf{A}$ is $\mathcal{U}$*-substructure* of $\mathbf{B}$ if for every pair $(R^U, \mathbf{R}) \in \mathcal{U}$ and every tuple $\vec{t} \in R^B_{\mathbf{U}}$ such that all root vertices are in A it follows that all vertices of $\vec{t}$ are in A.

In other words there is no vertex $v \in B \setminus A$ with pair $(R^U, \mathbf{R}) \in \mathcal{U}$ and tuple $\vec{s} \in R^B_{\mathbf{U}}$ containing v such that first $|R|$ elements of $\vec{s}$ are in A.

The main property of $\mathcal{U}$-substructure is captured by the following easy lemma.

Lemma 2.3. *For every closure description* $\mathcal{U}$ *the following holds:*

1. *Let* $\mathbf{A}$ *be a substructure of an* $\mathcal{U}$*-closed structure* $\mathbf{B}$*. Then* $\mathbf{A}$ *is an* $\mathcal{U}$*-substructure if and only if* $\mathbf{A}$ *is* $\mathcal{U}$*-closed.*

2. *Let* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *be* $\mathcal{U}$*-closed structures and* $\mathbf{A}$ *an* $\mathcal{U}$*-closed substructure of both* $\mathbf{B}_1$ *and* $\mathbf{B}_2$*. Then the free amalgamation of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *over* $\mathbf{A}$ *is an* $\mathcal{U}$*-closed structure.*

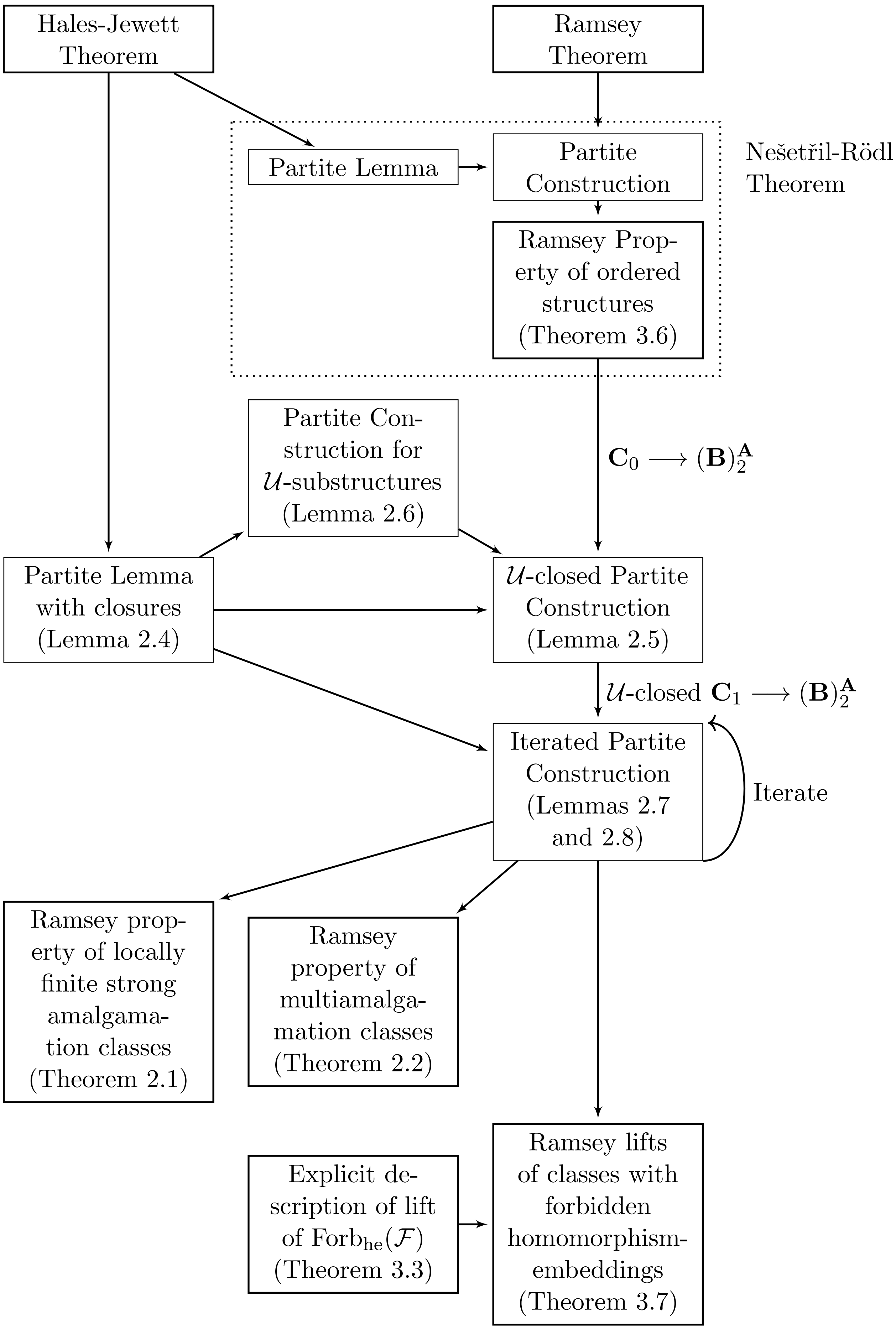


Figure 3: The structure of proofs of the main results.

Proof. Proof is easy. In 2. we use the fact that roots in $\mathcal{U}$ are irreducible structures. $\square$

While constructing $\mathcal{U}$-closed structures it is also useful to consider the following partial notion:

Definition 2.10. Let $\mathcal{U}$ be a closure description. We say that $\mathbf{A}$ is $\mathcal{U}$*-semi-closed* if for every pair $(R^U, \mathbf{R}) \in \mathcal{U}$ it holds that the $R^U_{\mathbf{A}}$-out-degree of a $|R|$-tuple $\vec{t}$ of vertices of $\mathbf{A}$ is at most one if there is an embedding from $\mathbf{R}$ to to $\vec{t}$, and, zero otherwise.

The following concept of size will be the basic parameter for our induction in the Iterated Partite Construction:

Definition 2.11. The $\mathcal{U}$*-size* of structure $\mathbf{B}$ is the number of vertices of the smallest substructure $\mathbf{A}$ of $\mathbf{B}$ such that the $\mathcal{U}$-closure of $\mathbf{A}$ in $\mathbf{B}$ is $\mathbf{B}$.

Observe that for every substructure $\mathbf{B}_0$ of $\mathcal{U}$-closed structure $\mathbf{B}$ the $\mathcal{U}$-size of $\mathbf{B}_0$ is the same as the $\mathcal{U}$-size of the $\mathcal{U}$-closure of $\mathbf{B}_0$ in $\mathbf{B}$.

2.3 Partite Lemma with closures

The basic part of our construction of Ramsey objects with a given closure is the closure refinement of the Partite Lemma [62] which deals with the following objects.

Definition 2.12 ($\mathbf{A}$-partite system)**.** Let L be a language and $\mathbf{A}$ be a L-structure. Assume $A = \{1, 2, \ldots, a\}$. An $\mathbf{A}$*-partite* L*-system* is a tuple $(\mathbf{A}, \mathcal{X}_{\mathbf{B}}, \mathbf{B})$ where $\mathbf{B}$ is an L-structure and $\mathcal{X}_{\mathbf{B}} = \{X^1_{\mathbf{B}}, X^2_{\mathbf{B}}, \ldots, X^a_{\mathbf{B}}\}$ is a partition of the vertex set of $\mathbf{B}$ into a classes ($X^i_{\mathbf{B}}$ are called *parts* of $\mathbf{B}$) such that

1. mapping π which maps every $x \in X^i_{\mathbf{B}}$ to i, $i = 1, 2, \ldots, a$, is a homomorphism-embedding $\mathbf{B} \to \mathbf{A}$ (π is called the *projection*);

2. every tuple in every relation of $\mathbf{B}$ meets every class $X^i_{\mathbf{B}}$ in at most one element (i.e. these tuples are called *transversal* with respect to the partition).

Remark. Our definition differs from the definition used in [62]. We do not treat the linear order explicitely and also assume the existence of a homomorphism-embedding $\mathbf{B} \to \mathbf{A}$ (which yields to a simpler construction in the proof of the Partite Lemma). However this formulation of the partite system does not lead directly to proof of the Nešetřil-Rödl Theorem itself. We aim for simplicity here.

The *isomorphisms* and *embeddings* of $\mathbf{A}$-partite systems, say of $\mathbf{B}_1$ into $\mathbf{B}_2$, are defined as the isomorphisms and embeddings of structures together with the condition that all parts are being preserved (the part $X^i_{\mathbf{B}_1}$ is mapped to $X^i_{\mathbf{B}_2}$ for every $i = 1, 2, \ldots, a$). In other words the following diagram commutes:

$$\begin{tikzcd} \mathbf{B}_1 \arrow[r] \arrow[dr, "\pi_1"] & \mathbf{B}_2 \arrow[d, "\pi_2"] \\ & \mathbf{A} \end{tikzcd}$$

Lemma 2.4 (Partite Lemma with closures)**.** *Let L be a language, $\mathcal{U}$ be a closure description in the language L, $\mathbf{A}$ be a finite $\mathcal{U}$-closed L-structure, and, $\mathbf{B}$ be a finite $\mathcal{U}$-semi-closed $\mathbf{A}$-partite L-system. Then there exists a finite $\mathcal{U}$-semi-closed $\mathbf{A}$-partite L-system $\mathbf{C}$ such that*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Moreover there exists a family $\mathcal{B}$ of copies of $\mathbf{B}$ in $\mathbf{C}$ such that:

1. *For every 2-colouring of all substructures of $\mathbf{C}$ which are isomorphic to $\mathbf{A}$ there exists $\widetilde{\mathbf{B}} \in \mathcal{B}$ such at all the substructures of $\widetilde{\mathbf{B}}$ which are isomorphic to $\mathbf{A}$ are monochromatic (thus $\mathcal{B}$ is a Ramsey system of copies of $\mathbf{B}$ in $\mathbf{C}$).*

2. *Every $\widetilde{\mathbf{B}} \in \mathcal{B}$ is an $\mathcal{U}$-substructure of $\mathbf{C}$.*

Finally if $\mathbf{B}$ is $\mathcal{U}$-closed then $\mathbf{C}$ is $\mathcal{U}$-closed, too.

Remark. Our proof is based on the proof of the Partite Lemma in [62] by the application of the Hales-Jewett theorem [29, 51]. We give an easy description of $\mathbf{C}$ as a product. This simplification follows from the assumption that $\mathbf{B}$ is an $\mathbf{A}$-partite-system and have a homomorphism-embedding projection to $\mathbf{A}$. This easier description of $\mathbf{C}$ allows us to verify the additional properties of $\mathbf{C}$ needed to carry our proof: the existence of a homomorphism-embedding, $\mathcal{U}$-closedness and the additional requirement on the Ramsey system of copies $\mathcal{B}$ being all $\mathcal{U}$-substructures. The key observation of our earlier paper [35] is that the unary closures (i.e. closures of vertices) can be preserved by the Partite Construction. We show this in a full generality (by a different technique which use nested Partite Construction instead of free amalgamation) in Section 2.4.

For completeness, we briefly recall the Hales-Jewett Theorem [29]: Consider a family of functions $f : \{1, 2, \ldots, N\} \to \Sigma$ for some finite alphabet Σ.

A *combinatorial line* $\mathcal{L}$ is a pair (ω, h) where $\emptyset \neq \omega \subseteq \{1, 2, \ldots, N\}$ and h is a function from $\{1, 2, \ldots, N\} \setminus \omega$ to Σ. The combinatorial line $\mathcal{L}$ describes a family of all those functions $f : \{1, 2, \ldots, N\} \to \Sigma$ that are constant on ω and $f(i) = h(i)$ otherwise. The Hales-Jewett theorem guarantees, for sufficiently large N, that for every 2-colouring of functions $f : \{1, 2, \ldots, N\} \to \Sigma$ there exists a monochromatic combinatorial line.

Proof of Lemma 2.4. Assume without loss of generality $A = \{1, 2, \ldots, a\}$ and denote by $\mathcal{X}_{\mathbf{B}} = \{X^1_{\mathbf{B}}, X^2_{\mathbf{B}}, \ldots, X^a_{\mathbf{B}}\}$ the parts of $\mathbf{B}$. We take N sufficiently large (that will be specified later) and construct an $\mathbf{A}$-partite L-system $\mathbf{C}$ with parts $\mathcal{X}_{\mathbf{C}} = \{X^1_{\mathbf{C}}, X^2_{\mathbf{C}}, \ldots, X^a_{\mathbf{C}}\}$ as follows:

1. For every $1 \leq i \leq a$ let $X^i_{\mathbf{C}}$ be the set of all functions
$$f : \{1, 2, \ldots, N\} \to X^i_{\mathbf{B}}.$$
2. For every relation $R \in L$, put
$$(f_1, f_2, \ldots, f_{a(R)}) \in R_{\mathbf{C}}$$
if and only if for every $1 \leq i \leq N$ it holds that
$$(f_1(i), f_2(i), \ldots, f_{a(R)}(i)) \in R_{\mathbf{B}}.$$

This completes the construction of $\mathbf{C}$.

We shall check that indeed $\mathbf{C}$ is an $\mathcal{U}$-semi-closed $\mathbf{A}$-partite L-system with parts $\mathcal{X}_{\mathbf{C}} = \{X^1_{\mathbf{C}}, X^2_{\mathbf{C}}, \ldots, X^a_{\mathbf{C}}\}$. Most of this follows immediately from the definition. We only verify that $\mathbf{C}$ is $\mathcal{U}$-semi-closed. For the contrary assume the existence of pair $(R^U, \mathbf{R}) \in \mathcal{U}$, an embedding $f : \mathbf{R} \to \mathbf{C}$, $|R|$-tuple of $(r_1, r_2, \ldots, r_{|R|})$ of vertices of $f(\mathbf{R})$ such that the $R^U_{\mathbf{C}}$-out-degree of $(r_1, r_2, \ldots, r_{|R|})$ is more than one. Denote by m the number of vertices of $\mathbf{R}$ and n the arity of R^U. Because the $R^U_{\mathbf{A}}$-out-degree of is more than one we have $(n-m)$-tuples $(f_{m+1}, f_{m+2}, \ldots, f_n) \neq (f'_{m+1}, f'_{m+2}, \ldots, f'_n)$ such that:
$$(r_1, r_2, \ldots, r_m, f_{m+1}, f_{m+2}, \ldots, f_n) \in R^U_{\mathbf{C}}, \text{ and,}$$
$$(r_1, r_2, \ldots, r_m, f'_{m+1}, f'_{m+2}, \ldots, f'_n) \in R^U_{\mathbf{C}}.$$
By the construction of $\mathbf{C}$ we know that for every $1 \leq j \leq N$:
$$(r_1(j), r_2(j), \ldots, r_m(j), f_{m+1}(j), f_{m+2}(j), \ldots, f_n(j)) \in R^U_{\mathbf{B}}, \text{ and,}$$
$$(r_1(j), r_2(j), \ldots, r_m(j), f'_{m+1}(j), f'_{m+2}(j), \ldots, f'_n(j)) \in R^U_{\mathbf{B}}.$$

Since $R^U_{\mathbf{A}}$-out-degree is at most one in $\mathbf{B}$ we know that $f_k(j) = f'_k(j)$ for every $m < k \leq n$ and $1 \leq j \leq N$, a contradiction. The second part of definition of $\mathcal{U}$-semi-closed structure is trivially satisfied by the existence of the projection.

By a similar argument it follows that if $\mathbf{B}$ is $\mathcal{U}$-closed then also $\mathbf{C}$ is $\mathcal{U}$-closed.

Now we describe the Ramsey family $\mathcal{B}$ of copies of $\mathbf{B}$. Let $\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_t$ be the enumeration of all substructures of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Put $\Sigma = \{1, 2, \ldots, t\}$ which we consider as an alphabet. Each combinatorial line $\mathcal{L} = (\omega, h)$ in Σ^N corresponds to an embedding $e_{\mathcal{L}} : \mathbf{B} \to \mathbf{C}$ which assigns to every vertex $v \in X^p_{\mathbf{B}}$ a function $e_{\mathcal{L}}(v) : \{1, 2, \ldots, N\} \to X^p_{\mathbf{B}}$ (i.e. a vertex of $X^p_{\mathbf{C}}$) such that:

$$e_{\mathcal{L}}(v)(i) = \begin{cases} v \text{ for } i \in \omega, \text{ and,} \\ \text{the unique vertex in } \widetilde{A}_{h(i)} \cap X^p_{\mathbf{B}} \text{ otherwise.} \end{cases}$$

It follows from the construction of $\mathbf{C}$ and from the fact that $\mathbf{B}$ has a projection $\mathbf{A}$ that the $e_{\mathcal{L}}$ is an embedding.

Let the family $\mathcal{B}$ consist from all copies $e_{\mathcal{L}}(\mathbf{B})$ for some combinatorial line $\mathcal{L}$. We first check that every copy in $\mathcal{B}$ is $\mathcal{U}$-substructure of $\mathbf{C}$ (condition 2 above). Assume, to the contrary, that there is $\widetilde{\mathbf{B}} \in \mathbf{B}$ which corresponds to a combinatorial line $\mathcal{L} = (\omega, h)$, pair $(R^U, \mathbf{R}) \in \mathcal{U}$ and $\vec{t} = (f_1, f_2, \ldots, f_{a(R^U)}) \in R^U_{\mathbf{C}}$ such that $\{f_1, f_2, \ldots, f_{|R|}\} \subseteq \widetilde{B}$ and there is a closure vertex f in $\vec{t}$ such that $f \in C \setminus \widetilde{B}$. Because $\mathbf{C}$ is $\mathcal{U}$-semi-closed, $(f_1, f_2, \ldots, f_{|R|})$ can not be a root of a closure tuple within $\widetilde{\mathbf{B}}$ (because $\mathbf{C}$ is $\mathcal{U}$-semi-closed this would imply that $f \in \widetilde{B}$). By construction of $\mathbf{C}$ it follows that $(f_1(i), f_2(i), \ldots, f_{|R_i|}(i))$ do not form a root of a closure tuple for some $i \in \omega$. On the other hand, by construction of $\mathbf{C}$ and because $\vec{t} \in R^U_{\mathbf{C}}$, it follows $(f_1(i), f_2(i), \ldots, f_{|R_i|}(i))$ must form a root of closure tuple for every $1 \leq i \leq N$, a contradiction.

It remains to check the property 1. (i.e. that $\mathcal{B}$ is a Ramsey system of copies of $\mathbf{B}$). Let N be the Hales-Jewett number guaranteeing a monochromatic line in any 2-colouring of the N-dimensional cube over an alphabet Σ. Now assume that $\mathcal{A}_1, \mathcal{A}_2$ is a 2-colouring of all copies of $\mathbf{A}$ in $\mathbf{C}$. Using the construction of $\mathbf{C}$ we see that among copies of $\mathbf{A}$ are copies induced by an N-tuple $(\widetilde{\mathbf{A}}_{u(1)}, \widetilde{\mathbf{A}}_{u(2)}, \ldots, \widetilde{\mathbf{A}}_{u(N)})$ of copies of $\mathbf{A}$ for every function $u : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, t\}$. However such copies are coded by the elements of the cube $\{1, 2, \ldots, t\}^N$ and thus there is a monochromatic combinatorial line $\mathcal{L}$. The monochromatic copy of $\mathbf{B}$ is then $e_{\mathcal{L}}(\mathbf{B})$ which belongs to $\mathcal{B}$.

$\square$

2.4 Partite Construction with closures

The main result of this section is the following Lemma which reflects the title of this section.

Lemma 2.5. *Let L be a language, $\mathcal{U}$ be a closure description,* $\mathbf{A}$, $\mathbf{B}$ *be a finite $\mathcal{U}$-closed L-structures and* $\mathbf{C}_0$ *a finite L-structure such that:*

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Then there exists finite $\mathcal{U}$-closed L-structure $\mathbf{C}$ *with a homomorphism-embedding* $\mathbf{C} \to \mathbf{C}_0$ *such that:*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

We first prove a weaker variant of Lemma 2.5 (the weakening consists in an additional assumption on $\mathbf{C}_0$):

Lemma 2.6. *Let L be a language, $\mathcal{U}$ be a closure description,* $\mathbf{A}$, $\mathbf{B}$ *be a finite $\mathcal{U}$-closed L-structures and* $\mathbf{C}_0$ *a finite L-structure such that:*

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Further assume that every copy of $\mathbf{A}$ *in* $\mathbf{C}_0$ *is $\mathcal{U}$-substructure of* $\mathbf{C}_0$*. Then there exists finite $\mathcal{U}$-closed L-structure* $\mathbf{C}$ *with a homomorphism-embedding* $\mathbf{C} \to \mathbf{C}_0$ *such that:*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Proof (an adaptation of [62]). Without loss of generality we can assume that $C_0 = \{1, 2, \ldots, c\}$. Enumerate all copies of $\mathbf{A}$ in $\mathbf{C}_0$ as $\{\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_b\}$. We shall define $\mathbf{C}_0$-partite $\mathcal{U}$-closed structures $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_b$ with the property that for every 2-colouring of copies of $\mathbf{A}$ in $\mathbf{P}_k$ there is a copy of $\mathbf{P}_{k-1}$ in $\mathbf{P}_k$ such that all copies of $\mathbf{A}$ with projection to $\widetilde{\mathbf{A}}_k$ are monochromatic. As usual in Partite Construction the systems $\mathbf{P}_k$ are called *pictures*. Put explicitly $\mathcal{X}_{\mathbf{P}_k} = \{X^1_k, X^2_k, \ldots, X^c_k\}$. Pictures will be constructed by induction on k.

1. The picture $\mathbf{P}_0$ is constructed as a disjoint union of copies of $\mathbf{B}$: for every copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}_0$ we consider a new isomorphic and disjoint copy $\widetilde{\mathbf{B}}'$ in $\mathbf{P}_0$ which intersects the part X^l_0 if and only if $\widetilde{\mathbf{B}}$ intersects and that the projection of $\widetilde{\mathbf{B}}'$ is $\widetilde{\mathbf{B}}$ (see Figure 4). This is indeed $\mathcal{U}$-closed as no tuples in any relations between copies are added.

2. Let the picture $\mathbf{P}_k$ be already constructed. Let $\mathbf{B}_k$ be the substructure of $\mathbf{P}_k$ induced by $\mathbf{P}_k$ on vertices which projects to $\widetilde{\mathbf{A}}_{k+1}$. By the assumption that $\widetilde{\mathbf{A}}_{k+1}$ is an $\mathcal{U}$-substructure of $\mathbf{C}_0$ we also know that $\mathbf{B}_k$ is $\mathcal{U}$-substructure of $\mathbf{P}_k$.

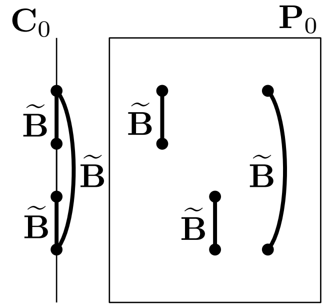


Figure 4: The construction of $\mathbf{P}_0$.

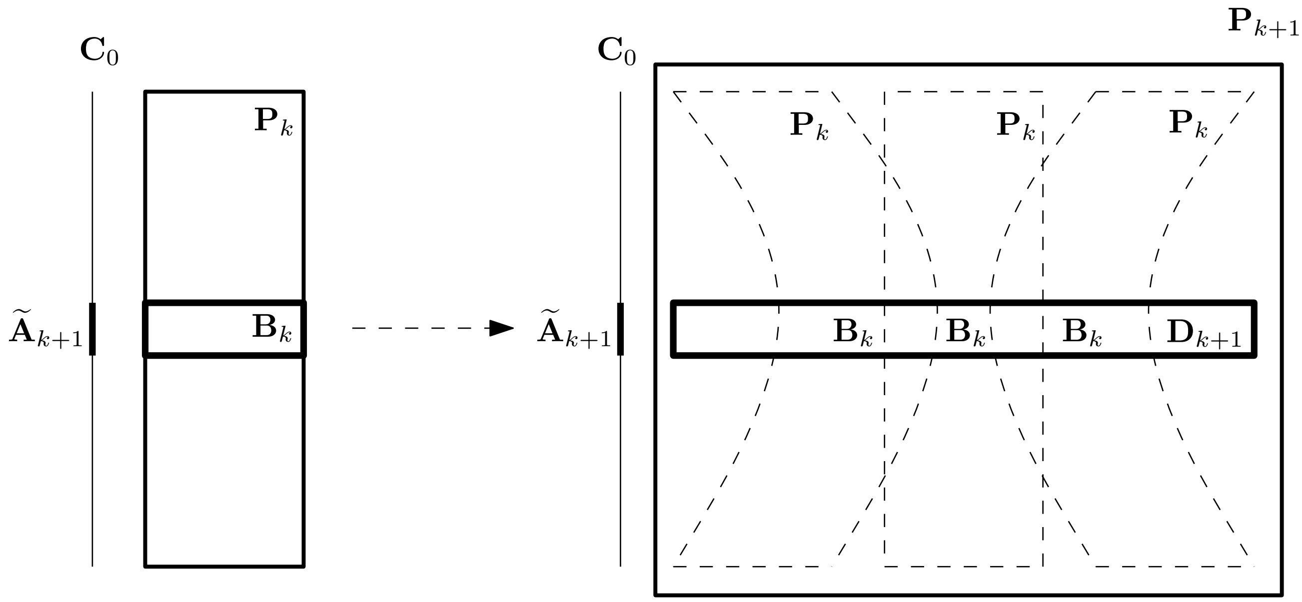


Figure 5: The construction of Picture $\mathbf{P}_{k+1}$ from Picture $\mathbf{P}_k$.

In this situation we use Partite Lemma 2.4 to obtain an $\mathcal{U}$-closed $\widetilde{\mathbf{A}}_{k+1}$-partite system $\mathbf{D}_{k+1}$ and a Ramsey system $\mathcal{B}_{k+1}$ of copies of $\mathbf{B}_k$ which are $\mathcal{U}$-substructures of $\mathbf{D}_{k+1}$. Now consider all copies in $\mathcal{B}_{k+1}$ and extend each of these structures to a copy of $\mathbf{P}_k$ by a free amalgamation. These copies are disjoint outside $\mathbf{D}_{k+1}$ and preserve the parts of all the copies. The result of this multiple amalgamation is $\mathbf{P}_{k+1}$. The construction is depicted in Figure 5. By repeated application of Lemma 2.3 we know that $\mathbf{P}_{k+1}$ is $\mathcal{U}$-closed because it is a result of a sequence of free amalgamations of $\mathcal{U}$-closed structures over $\mathcal{U}$-substructures.

Put $\mathbf{C} = \mathbf{P}_b$. It follows easily that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: by a backward induction on k one proves that in any 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists a copy $\widetilde{\mathbf{P}}_0$ of $\mathbf{P}_0$ such that the colour of a copy of $\mathbf{A}$ in $\mathbf{P}_0$ depends only on its projection. As this in turn induces colouring of copies of $\mathbf{A}$ in $\mathbf{C}$, we obtain a monochromatic copy of $\mathbf{B}$ in $\widetilde{\mathbf{P}}_0$. $\square$

Proof of Lemma 2.5. We apply again the Partite Construction as in the proof of Lemma 2.6. However we repeatedly use Lemma 2.6 as the crucial step in the "picture induction".

Assume that $C_0 = \{1, 2, \dots, c\}$. Enumerate all copies of $\mathbf{A}$ in $\mathbf{C}_0$ as $\{\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \dots, \widetilde{\mathbf{A}}_b\}$. We shall define $\mathbf{C}_0$-partite $\mathcal{U}$-closed structures (pictures) $\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_b$ with the property that for every 2-colouring of copies of $\mathbf{A}$ in $\mathbf{P}_k$ there is a copy of $\mathbf{P}_{k-1}$ in $\mathbf{P}_k$ such that all copies of $\mathbf{A}$ with projection to $\widetilde{\mathbf{A}}_k$ are monochromatic. Again we proceed by the induction on k.

1. The picture $\mathbf{P}_0$ is again constructed as a disjoint union of copies of $\mathbf{B}$: for every copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}_0$ we consider a new isomorphic and disjoint copy $\widetilde{\mathbf{B}}'$ in $\mathbf{P}_0$ which intersects the part X_0^l if and only if $\widetilde{\mathbf{B}}$ intersects and that the projection of $\widetilde{\mathbf{B}}'$ is $\widetilde{\mathbf{B}}$. Clearly $\mathbf{P}_0$ is $\mathcal{U}$-closed.

2. Let the picture $\mathbf{P}_k$ be already constructed. Let $\mathbf{B}_{k+1}$ be the $\mathcal{U}$-semi-closed substructure of $\mathbf{P}_k$ induced by $\mathbf{P}_k$ on vertices which projects to $\widetilde{\mathbf{A}}_{k+1}$. Observe that in this setting $\mathbf{B}_{k+1}$ is not necessarily $\mathcal{U}$-closed because $\widetilde{\mathbf{A}}_{k+1}$ may not be an $\mathcal{U}$-substructure of $\mathbf{C}_0$.

   In this situation we use Partite Lemma 2.4 to obtain an $\mathcal{U}$-semi-closed $\widetilde{\mathbf{A}}_{k+1}$-partite system $\mathbf{D}_{k+1}$ and a Ramsey system $\mathcal{B}_{k+1}$ of copies of $\mathbf{B}_k$ which are all $\mathcal{U}$-substructures of $\mathbf{D}_{k+1}$. Now consider all copies in $\mathcal{B}_{k+1}$ and extend each of these structures to a copy of $\mathbf{P}_k$. These copies are disjoint outside $\mathbf{D}_{k+1}$ and preserve the parts of all the copies. The result of this multiple amalgamation is denoted by $\mathbf{O}_{k+1}$. ($\mathbf{O}$ stands for Czech "obrázek" — a little picture. At this moment we further refine the Partite Construction. In the construction of Picture $\mathbf{P}_{k+1}$ from $\mathbf{P}_k$ we sendwich $\mathbf{O}_{k+1}$ which itself is a result of the Partite Construction.) Note that because $\mathbf{B}_k$ is not necessarily an $\mathcal{U}$-substructure of $\mathbf{P}_k$ also $\mathbf{O}_{k+1}$ is not necessarily $\mathbf{U}$-semi-closed.

   Denote by $\mathcal{A}_{k+1}$ the set of all copies of $\mathbf{A}$ in $\mathbf{O}_{k+1}$ with projection to $\widetilde{\mathbf{A}}_{k+1}$. We show that for every pair $(R^U, \mathbf{R}) \in \mathcal{U}$ and $|R|$-tuple $\vec{t}$ of vertices of $\mathbf{O}_{k+1}$ such that $R^U_{\mathbf{O}_{k+1}}$-out-degree of $\vec{t}$ is more than one it holds that $\vec{t}$ is never contained in a copy of $\mathbf{A}$ in $\mathcal{A}_{k+1}$. This follows from the fact that the higher degrees can only be created by means of free amalgamations used to construct $\mathbf{O}_{k+1}$. All copies of $\mathbf{B}_{k+1}$ in $\mathcal{B}_{k+1}$ are $\mathcal{U}$-substructures of $\mathbf{D}_{k+1}$ and both $\mathbf{D}_{k+1}$ and $\mathbf{P}_k$ are $\mathcal{U}$-closed. The amalgamation thus never introduce closure tuple out of the copy of $\mathbf{A}$ in $\mathcal{A}_{k+1}$.

   To apply Lemma 2.6 we turn the $\mathbf{C}_0$-partite system $\mathbf{O}_{k+1}$ to a relational structure $\mathbf{O}^+_{k+1}$ in an extended language L^+ to represent parts by means of unary relations. Explicitely, we put $L^+ = L \cup \{R^{X_i}; i \in C_0\}$ and the arity of all new relations is one. The L^+-structure $\mathbf{O}^+_{k+1}$ is constructed as follows:

(a) $O^+_{k+1} = O_{k+1}$ (i.e. $\mathbf{O}^+_{k+1}$ has same vertices as $\mathbf{O}_{k+1}$),

(b) for every relation $R \in L$ put $R_{\mathbf{O}^+_{k+1}} = R_{\mathbf{O}_{k+1}}$ (i.e. $\mathbf{O}^+_{k+1}$ has same original relations as $\mathbf{O}_{k+1}$),

(c) $(v) \in R^{X_i}_{\mathbf{O}^+_{k+1}}$ if and only if $i = \pi(v)$.

(This can be seen as a special case of lift which will be in greater generality discussed in Section 3.) Analogously we turn $\mathbf{C}_0$-partite L-system $\mathbf{P}_k$ to L^+-structure $\mathbf{P}^+_k$. Next we turn the L-structure $\widetilde{\mathbf{A}}_{k+1}$ to an L^+-structure $\mathbf{A}^+$ by putting $A^+ = \widetilde{A}_{k+1}$, $R_{\mathbf{A}^+} = R_{\widetilde{\mathbf{A}}_{k+1}}$ for every $R \in L$, and $(v) \in R^{Xv}_{\mathbf{A}^+}$ for every $v \in A^+$. Finally construct an closure description (in language L^+) $\mathcal{U}^+$ consiting of all pairs $(R^U, \mathbf{S}^+)$ where $R^U \in L$, $\mathbf{S}^+$ is L^+-structure such that there exists an $(R^U, \mathbf{S}) \in \mathcal{U}$ and $S = S^+$, $R_{\mathbf{S}} = R_{\mathbf{S}^+}$, for every $R \in L$ (that is $\mathcal{U}^+$ extends every root of $\mathcal{U}$ by the unary relations in every possible way and thus represent the same closures regardless the new unary relations).

We verify premises of Lemma 2.6 for these L^+-structures. Because the projection is explicitely represented by the unary relations in L^+, it follows that $\mathbf{O}^+_{k+1} \longrightarrow (\mathbf{P}^+_k)^{\mathbf{A}^+}_2$. This holds as all copies of $\mathbf{A}^+$ in $\mathbf{O}^+_{k+1}$ corresponds to copies of $\mathbf{A}_{k+1}$ in $\mathcal{A}_{k+1}$ and the Ramsey property for those copies is given by Lemma 2.4. We also verified that all such copies are $\mathcal{U}$-substructures of $\mathbf{O}_{k+1}$ and consequently all copies of $\mathbf{A}^+$ in $\mathbf{O}^+_{k+1}$ are $\mathcal{U}^+$-substructures. It follows, by the application of Lemma 2.6, that there exists $\mathcal{U}^+$-closed L^+-structure $\mathbf{P}^+_{k+1}$ such that $\mathbf{P}^+_{k+1} \longrightarrow (\mathbf{P}^+_k)^{\mathbf{A}^+}_2$ with a homomorphism-embedding to $\mathbf{O}^+_{k+1}$.

The $\mathcal{U}$-closed $\mathbf{C}_0$-partite L-system $\mathbf{P}_{k+1}$ is then constructed by reinterpreting $\mathbf{P}^+_{k+1}$ as partite system: vertices of $\mathbf{P}_{k+1}$ are same as vertices of $\mathbf{P}^+_{k+1}$, all tuples in all relations in the language L are also the same. The parts are determined by unary relations R^{X^i} (for every $i \in C_0$ and $v \in P^+_{k+1}$ it holds that $v \in X^i_{k+1}$ if and only if $(v) \in R^{X^i}_{\mathbf{P}_{k+1}}$).

Put $\mathbf{C} = \mathbf{P}_b$. Again, analogously to proof of Lemma 2.6, by the backward induction, it follows that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. $\square$

2.5 Iterated Partite Construction

Next, in the course of developing the proof of Theorem 2.1 and 2.2 we generalise the Iterated Partite Construction introduced in [53]. This is the main tool for obtaining Ramsey objects where substructures of a given size have completion in a given strong amalgamation class.

Lemma 2.7 (j completion imlies $j+1$ completion)**.** *Let L be a language, $\mathcal{U}$ be a closure description in the language L, $\mathcal{K}$ a class of finite irreducible $\mathcal{U}$-closed L-structures which is hereditary for $\mathcal{U}$-closed substructures and which has strong amalgamation and $j \geq 0$. Let $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and $\mathbf{C}_0$ be a finite $\mathcal{U}$-closed L-structure such that*

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Further assume that either $j = 0$ or $j > 0$ and every substructure of $\mathbf{C}_0$ with $\mathcal{U}$-size at most j has a $\mathcal{K}$-completion. Then there exists an $\mathcal{U}$-closed L-structure $\mathbf{C}$ with a homomorphism-embedding $\mathbf{C} \to \mathbf{C}_0$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$$

and every substructure of $\mathbf{C}$ of $\mathcal{U}$-size at most $j+1$ has $\mathcal{K}$-completion.

Proof. For the fourth (and last) time we apply the Partite Construction. We proceed analogously to the proofs of Lemmas 2.5 and 2.6. Again, we enumerate all copies of $\mathbf{A}$ in $\mathbf{C}_0$ as $\{\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_b\}$. We then define $\mathcal{U}$-closed $\mathbf{C}_0$-partite systems $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_b$ such that:

(i) every substructure of $\mathbf{P}_k$, $0 \leq k \leq b$, of $\mathcal{U}$-size at most $j+1$ has a $\mathcal{K}$-completion, and,

(ii) in any 2-colouring of $\binom{\mathbf{P}_k}{\mathbf{A}}$, $1 \leq k \leq b$, there exists a copy $\widetilde{\mathbf{P}}_{k-1}$ such that all copies of $\mathbf{A}$ with a projection to $\widetilde{\mathbf{A}}_k$ are monochromatic.

As before we know that putting $\mathbf{C} = \mathbf{P}_b$ we have the desired Ramsey property $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. It remains to prove (i) and (ii).

Put explicitly $\mathcal{X}_{\mathbf{P}_k} = \{X^1_k, X^2_k, \ldots, X^c_k\}$. We proceed by an induction on k.

1. The Picture $\mathbf{P}_0$ is constructed the same way as in the proof of Lemma 2.6 as a disjoint union of copies of $\mathbf{B}$: for every copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}_0$ we consider a new isomorphic and disjoint copy $\widetilde{\mathbf{B}}'$ in $\mathbf{P}_0$ which intersects the part X^l_0 if and only if $\widetilde{\mathbf{B}}$ intersects (so the projection of $\widetilde{\mathbf{B}}'$ is $\widetilde{\mathbf{B}}$). Clearly $\mathbf{P}_0$ has a $\mathcal{K}$-completion that can be constructed by a series of strong amalgamations over empty set giving property (i).

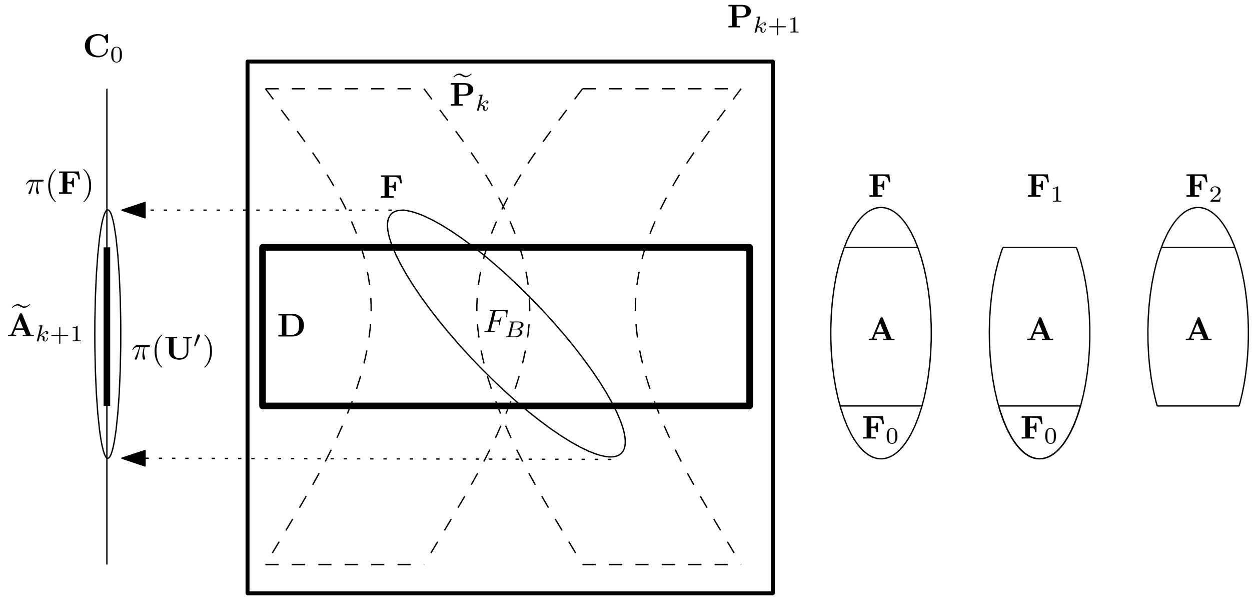


Figure 6: The completion of $\mathbf{F}$ by a strong amalgamation over an $\mathcal{U}$-closed irreducible substructure.

2. Let the Picture $\mathbf{P}_k$ be already constructed. Let $\mathbf{B}_k$ be the $\mathcal{U}$-substructure of $\mathbf{P}_k$ induced by $\mathbf{P}_k$ on vertices which projects to $\widetilde{\mathbf{A}}_{k+1}$. $\mathbf{P}_{k+1}$ is constructed the same way as in the proof of Lemma 2.6: We use Partite Lemma 2.4 to obtain an $\mathcal{U}$-closed $\widetilde{\mathbf{A}}_{k+1}$-partite system $\mathbf{D}_{k+1}$ and the Ramsey system $\mathcal{B}_{k+1}$. Now consider all copies in $\mathcal{B}_{k+1}$ and extend each of these structures to a copy of $\mathbf{P}_k$ (using free amalgamation). These copies are disjoint outside $\mathbf{D}_{k+1}$. In this extension we preserve the parts of all the copies. The result of this multiple amalgamation is $\mathbf{P}_{k+1}$. Because $\mathbf{D}_{k+1} \longrightarrow (\mathbf{B}_k)_2^{\widetilde{\mathbf{A}}_{k+1}}$ we know that $\mathbf{P}_{k+1}$ satisfies (ii).

   Because $\mathbf{P}_{k+1}$ is created by a series of free amalgamations of $\mathcal{U}$-closed structures over $\mathcal{U}$-substructures it follows that $\mathbf{P}_k$ is $\mathcal{U}$-closed.

   We show (i) for $\mathbf{P}_{k+1}$. Assume the contrary and denote by $\mathbf{F}$ substructure of $\mathcal{U}$-size at most $j+1$ with no $\mathcal{K}$-completion and choose $\mathbf{F}$ with smallest $\mathcal{U}$-size (see schematic Figure 6). Because $\mathcal{K}$ is has the strong amalgamation over empty set this implies that $\mathbf{F}$ is connected.

   Consider the projection π from $\mathbf{F}$ to $\mathbf{C}_0$ (which is a homomorphism-embedding). Clearly the $\mathcal{U}$-size of $\mathbf{F}$ is greater or equal to $\mathcal{U}$-size of $\pi(\mathbf{F})$. First assume that $\mathcal{U}$-size of $\pi(\mathbf{F})$ is at most j. In this case, by the assumptions on $\mathbf{C}_0$, there exists a structure $\mathbf{F}' \in \mathcal{K}$ which is a completion of $\pi(\mathbf{F})$. It follows that $\mathbf{F}' \in \mathcal{K}$ is also a completion of $\mathbf{F}$. In the following we thus assume that $\mathcal{U}$-size of $\pi(\mathbf{F})$ and $\mathcal{U}$-size of $\mathbf{F}$ is $j+1$.

   Because in $\mathbf{D}_{k+1}$ is $\widetilde{\mathbf{A}}_{k+1}$-partite system and thus it has a projection to

$\widetilde{\mathbf{A}}_{k+1}$ we also know that $\mathbf{A} \in \mathcal{K}$ is a completion of $\mathbf{D}_{k+1}$. It follows that $\mathbf{F}$ contains some vertices not in $\mathbf{D}_{k+1}$. All other vertices were added by the means of free amalgamation of copies of $\mathbf{P}_k$. By (i) it follows that that $\mathbf{F}$ contains vertices belonging to more than one copy of $\mathbf{P}_k$ and also that $j \geq 1$. Denote by $\widetilde{\mathbf{P}}_k$ a copy of $\mathbf{P}_k$ which contain a vertex of $\mathbf{F}$ with no projection to $\widetilde{\mathbf{A}}_{k+1}$. Denote by F_B the set $\widetilde{P}_k \cap F \cap D_{k+1}$. Because $\mathbf{F}$ is connected, we know that the vertices of F_B forms a vertex cut of $\mathbf{F}$. The case that F_B is not vertex cut correspond to the case where F_B contains a relation of $\mathbf{P}_{k+1}$ which fails to be a relation of $\widetilde{P}_k$ which is impossible as we consider embeddings only.

Denote by $\mathbf{F}_0$ a connected component of $\mathbf{F}$ with cut F_B. Denote by $\mathbf{F}_1$ the structure induced by $\mathbf{F}$ on vertices $F_0 \cup F_B$ and $\mathbf{F}_2$ the structure induced by $\mathbf{F}$ on vertices $F \setminus F_0$. Clearly $\mathbf{F}$ is a free amalgamation of $\mathbf{F}_1$ and $\mathbf{F}_2$ over F_B. Denote by $\mathbf{F}'_1$ the $\mathcal{U}$-closure of $\pi(\mathbf{F}_1)$ in $\mathbf{C}_0$ and by $\mathbf{F}'_2$ the $\mathcal{U}$-closure of $\pi(\mathbf{F}_2)$ in $\mathbf{C}_0$. Because the structure induced by $\mathbf{F}$ on F_B is $\mathcal{U}$-closed in $\mathbf{F}$, the $\mathcal{U}$-size of $\mathbf{F}_1$ and $\mathbf{F}_2$ is at most j. By the assumptions on completions in $\mathbf{C}_0$ it follows that there is $\mathbf{F}''_1 \in \mathcal{K}$ which is a completion of $\mathbf{F}'_1$ and $\mathbf{F}''_2 \in \mathcal{K}$ which is a completion of $\mathbf{F}'_2$. The strong amalgamation of $\mathbf{F}''_1$ and $\mathbf{F}''_2$ over the $\mathcal{U}$-closure of $\pi(F_B)$ in $\mathbf{C}_0$ (which is a substructure of $\widetilde{\mathbf{A}}_{k+1}$ and because $\widetilde{\mathbf{A}}_{k+1}$ is irreducible, it must remain unchanged in both $\mathbf{F}''_1$ and $\mathbf{F}''_2$) is then the completion of $\mathbf{F}$ in $\mathcal{K}$. A contradiction with $\mathbf{F}$ having no $\mathcal{K}$-completion. This finishes the proof of (i).

□

Lemma 2.8. *Let L be a language, $\mathcal{U}$ be a closure description in the language L, $\mathcal{K}$ be a class of finite irreducible $\mathcal{U}$-closed L-structures which is hereditary for $\mathcal{U}$-closed substructures and which has strong amalgamation. Let $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, $n \geq 1$, and $\mathbf{C}_0$ be a finite $\mathcal{U}$-closed L-structure such that*

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Then there exists a finite $\mathcal{U}$-closed L-structure $\mathbf{C}$ with a homomorphism-embedding $\mathbf{C} \to \mathbf{C}_0$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$$

and moreover every substructure of $\mathbf{C}$ with at most n vertices has a $\mathcal{K}$-completion.

Proof. By the repeated application of Lemma 2.7 we construct a sequence of $\mathcal{U}$-closed L-structures $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \ldots, \mathbf{C}_n$ such that:

(i) $\mathbf{C}_j \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ for every $1 \leq j \leq n$,

(ii) there is a homomorphism-embedding $\mathbf{C}_j \to \mathbf{C}_{j-1}$ for every $1 < j \leq n$, and,

(iii) every substructure of $\mathbf{C}_j$ (for every $1 \leq j \leq n$) of $\mathcal{U}$-size at most j has a $\mathcal{K}$-completion.

It remains to verify that $\mathbf{C}_1$ satisfies (i), (ii) and (iii): Without loss of generality we can assume that every vertex (and thus also every closure of a vertex) in $\mathbf{C}_1$ is part of a copy of $\mathbf{B}$. It follows that every substructure of $\mathbf{C}_1$ of $\mathcal{U}$-size at most one has a $\mathcal{K}$-completion.

The statement of Lemma 2.8 then follows by putting $\mathbf{C} = \mathbf{C}_n$. □

2.6 Conclussion of the proofs

We start with a simple (interesting, possibly folkloristic) model-theoretic lemma which justifies the reason why Theorem 2.1 works with strong completions as opposed to Theorem 2.2.

Let $\mathcal{K}$ be a class of structures (in a given language L). Define the class $\mathcal{K}_0$ of *structures with strong $\mathcal{K}$-completion* as the class of all finite L-structures $\mathbf{A}$ such that there is $\mathbf{A}' \in \mathcal{K}$ which is a strong completion of $\mathbf{A}$. The class $\overline{\mathcal{K}}_0$ is the complementary class of all finite L-structures which have no strong $\mathcal{K}$-completion.

Lemma 2.9. *Let $\mathcal{K}$ be a hereditary class with strong amalgamation, $\mathcal{K}_0$ the class of structures with strong $\mathcal{K}$-completion and $\overline{\mathcal{K}}_0$ the class of structures with no strong $\mathcal{K}$-completion. Then $\mathcal{K}_0$ is the class of all finite structures in* $\mathrm{Forb}_{\mathrm{he}}(\overline{\mathcal{K}}_0)$.

Consequently if structure $\mathbf{A}$ has $\mathcal{K}$-completion if and only if it has a strong $\mathcal{K}$-completion.

Proof. Assume to the contrary that there is $\mathbf{F} \in \overline{\mathcal{K}}_0$, $\mathbf{A} \in \mathcal{K}_0$ and a homomorphism-embedding $h : \mathbf{F} \to \mathbf{A}$. Among all such examples take $\mathbf{F}$ with the minimal number of vertices. Without loss of generality we can assume that h identifies only two vertices u and v.

Let $\mathbf{A}' \in \mathcal{K}$ be the completion of $\mathbf{A}$. Now use the hereditarity and the strong amalgamation property $\mathcal{K}$ to produce an amalgamation $\mathbf{C} \in \mathcal{K}$ of $\mathbf{A}'$ and $\mathbf{A}'$ over $A' \setminus f(u)$. It follows that $\mathbf{C}$ is the completion of $\mathbf{F}$ a contradiction with $\mathbf{A}' \in \mathcal{K}_0$. □

After all the prepareations we are ready to complete the proofs of Theorems 2.1 and 2.2. Combining the results of previous sectoins this takes the following easy form.

Proof of Theorem 2.1. Given $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we use the Ramsey property of $\mathcal{R}$ ($\mathcal{K} \subseteq \mathcal{R}$) to obtain $\mathbf{C}_0 \in \mathcal{R}$ such that:

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Now we apply Lemma 2.8 (putting $\mathcal{U} = \emptyset$, $\mathbf{C}'_0 = \mathbf{C}_0$ and $n = n(\mathbf{C}_0)$) to obtain $\mathbf{C}$ satisfying

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$$

and having a strong $\mathcal{K}$-completion. $\square$

Proof of Theorem 2.2. Given $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we use the Ramsey property of $\mathcal{R}$ ($\mathcal{K} \subseteq \mathcal{R}$) to obtain $\mathbf{C}_0 \in \mathcal{R}$ such that:

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Now obtain $n = n(\mathbf{C}_0, \mathbf{B})$. Next apply Lemma 2.5 to obtain $\mathcal{U}$-closed $\mathbf{C}_1$,

$$\mathbf{C}_1 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2,$$

with a homomorphism-embedding to $\mathbf{C}_0$. Finally apply Lemma 2.8 to obtain $\mathbf{C}$,

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2,$$

with a homomorphism-embedding to $\mathbf{C}_1$ where every substructure of $\mathbf{C}$ on at most n vertices has a $\mathcal{K}$-completion. We have verified the assumptions of the local completetion property (Definition 2.8) for $\mathbf{C}$. It follows that there is $\mathbf{C}' \in \mathcal{K}$ which is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$. We obtained $\mathbf{C}'$ such that:

$$\mathbf{C}' \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

$\square$

3 Construction of Ramsey lifts

At least at the first glance Ramsey classes seem to be very special. In this section we focus on techniques of turning a class into a class with strong amalgamation where we can apply Theorem 2.1 by means of lifted language.

3.1 Ramsey classes and ultrahomogeneous structures

Let $\mathcal{K}$ be a class of structures. We say that structure $\mathbf{U}$ is *embedding-universal* (or shortly *universal*) for $\mathcal{K}$ if for every structure in $\mathcal{K}$ there is an embedding to $\mathbf{U}$. We say that a *class $\mathcal{K}$ contains an universal structure* if there exists structure $\mathbf{U} \in \mathcal{K}$ which is universal for $\mathcal{K}$. It is well known that universal objects may be constructed by an iterated amalgamation (Fraïssé limit) of finite objects. This leads to a stronger notion of a *$\mathcal{K}$-generic object*: For a class $\mathcal{K}$ we say that an object $\mathbf{H}$ is $\mathcal{K}$-generic if it is both universal for $\mathcal{K}$ and it is *ultrahomogeneous*. The later notion means the following: Every isomorphism φ between two finite substructures $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{H}$ may be extended to an automorphism of $\mathbf{H}$. The notion of ultrahomogeneous structure is one of the key notions of modern model theory and it is the source of the well known Classification Programme of Ultrahomogeneous Structures [46, 6, 4].

The ultrahomogeneous structures are characterised by the properties of finite substructures. For a structure $\mathbf{A}$ denote by Age($\mathbf{A}$) the class of all finite structures isomorphic to substructures of $\mathbf{A}$. For a class $\mathcal{K}$ of relational structures, we denote by Age($\mathcal{K}$) the class $\bigcup_{\mathbf{A}\in\mathcal{K}} \mathrm{Age}(\mathbf{A})$. The following is one of the cornerstones of model theory.

Theorem 3.1 (Fraïssé [24, 33])**.** *Let $\mathcal{K}$ be a class of finite structures with only countably many non-isomorphic structures.*

(a) Class $\mathcal{K}$ is the age of a countable ultrahomogeneous structure $\mathbf{H}$ if and only if $\mathcal{K}$ is an amalgamation class.

(b) If the conditions of (a) are satisfied then the structure $\mathbf{H}$ is unique up to isomorphism.

Recall that a structure $\mathbf{A}$ is *ω-categorical* if the automorphism group of $\mathbf{A}$ has only finitely many orbits on n-tuples, for every n. The ultrahomogeneous and ω-categorical classes are closely related to classes with Ramsey lifts as shown by the following easy proposition which exemplifies the Ramsey relevance of these model-theoretic notions.

Proposition 3.2 ([50])**.** *Let $\mathcal{K}$ be a hereditary Ramsey class with joint embedding property. Then $\mathcal{K}$ is an amalgamation class.*

This (by now) easy observation provided a link of combinatorics of Ramsey classes and their model-theoretic properties. It was discovered in order to characterise Ramsey classes of graphs. The link provided to be vital and a decade later it led to the characterisation programme for Ramsey classes [52] and to an important connection with topological dynamics [41].

3.2 Ramsey lifts and the Ramsey Classification Programme

Ages of most ultrahomogeneous structures are not Ramsey for trivial reasons (most frequently simply because they are not rigid enough). It is however often possible to extend the language by the order and produce a Ramsey lift. This we define now.

Let L^+ be a language containing language L. By this we mean $L \subseteq L^+$ and the arities of the relations both in L and L^+ are the same. Then every structure $\mathbf{X} = (X, (R_\mathbf{X}; R \in L^+)) \in \mathrm{Rel}(L^+)$ may be viewed as a structure $\mathbf{A} = (X, (R_\mathbf{X}; R \in L)) \in \mathrm{Rel}(L)$ together with some additional relations $R_\mathbf{X}$ for $R \in L^+ \setminus L$. We call $\mathbf{X}$ a *lift* of $\mathbf{A}$ and $\mathbf{A}$ is called the *shadow* of $\mathbf{X}$. In this sense the class $\mathrm{Rel}(L^+)$ is the class of all lifts of $\mathrm{Rel}(L)$. Conversely, $\mathrm{Rel}(L)$ is the class of all shadows of $\mathrm{Rel}(L^+)$. Note that a lift is also in the model-theoretic setting called expansion and a shadow is often called reduct. (Our terminology is motivated by a computer science context, see [44], and for our purposes we find it both intuitive and natural.) For the lift $\mathbf{X}$ we denote by $\mathrm{Sh}(\mathbf{X})$ its shadow. (Sh is also called a *forgetful functor*.) Similarly, for a class $\mathcal{K}^+$ of lifted objects we denote by $\mathrm{Sh}(\mathcal{K}^+)$ the class of all shadows of structures in $\mathcal{K}^+$ (assuming the language L^+ of lifts is specified). On the other hand for a class $\mathcal{K}$ of structures we often denote by $\mathcal{K}^+$ the class of lifted structures.

Given the large list of known ultrahomogeneous and ω-categorical structures (identified by the Classification Programme of ultrahomogeneous structures) it possible to ask if all those structures have Ramsey lifts.

The Ramsey Classification Programme [52, 34] has been completed for all ultrahomogeneous graphs [50] and digraphs [40]. Motivated by this line of research, Cherlin also recently extended the Classification Programme of Ultrahomogeneous Structures by the list of all ordered graphs [4] which, in turn, also all leads to Ramsey lifts. This paper can be seen as a contribution to the Ramsey Classification Programme.

It is easy to see that every class $\mathcal{K}$ has a Ramsey lift. (For example, we may extend the language by infinitely many unary relations and assign every vertex of every structure in $\mathcal{K}$ an unique unary relation. Such lift trivially prevents any embeddings and the Ramsey statement becomes vacuously true.) This is why we focus on Ramsey lifts using finitely many additional relations (where possible) or, more generally, on precompact lifts. This leads to the following definitions (see [69]).

Definition 3.1. Let a class $\mathcal{K}^+$ be a lift of $\mathcal{K}$. We say that $\mathcal{K}^+$ is a *precompact lift of* $\mathcal{K}$ if for every structure $\mathbf{A} \in \mathcal{K}$ there are only finitely many structures $\mathbf{A}^+ \in \mathcal{K}^+$ such that $\mathbf{A}^+$ is an lift of $\mathbf{A}$ (i.e. $\mathrm{Sh}(\mathbf{A}^+)$ is isomorphic to $\mathbf{A}$).

In a Ramsey setting the following is natural property (called in [69] expansion property).

Definition 3.2. Let a class $\mathcal{K}^+$ be a lift of $\mathcal{K}$. For $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we say that $\mathbf{B}$ has *lift property* for $\mathbf{A}$ if for every lift $\mathbf{B}^+ \in \mathcal{K}^+$ of $\mathbf{B}$ there is an embedding of every lift $\mathbf{A}^+ \in \mathcal{K}^+$ of $\mathbf{A}$ in $\mathbf{B}^+$.

$\mathcal{K}^+$ has the *lift property* with respect to $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ with the lift property for $\mathbf{A}$.

In the special case where the lift adds only the order the lift property is also called the *ordering property* (which is one of the classical Ramsey theory definitions [47, 54]).

Lifts with the lift property are used to compute Ramsey degrees and universal minimal flows [41]. Moreover it can be shown that every class has at most one Ramsey lift up to bi-definability. Ramsey lifts with the lift property can thus be considered to be the minimal lifts (see e.g. [69]).

In a Ramsey setting it is natural to work with classes that are not strong amalgamation classes of ordered structures themselves but can be turned into one by mean of a precompact lift. A good candidate for a class with a precompact Ramsey lift is the age of an ω-categorical structure: every ω-categorical structure can be turned to homogenous one by an appropriate precompact lift. This process is called the *standard homogenisation* [18] and the lift which turns a class to amalgamation class is the *homogenising lift*.

Given an age $\mathcal{K}$ of an ω-categorical structure $\mathbf{U}$ the homogenising lift $\mathcal{K}^+$ can always be constructed by, for every $n \geq 1$, considering the automorphism group of $\mathbf{U}$ and adding lifted relations of arity n denoting the individual orbits of n-tuples. The lift $\mathcal{K}^+$ is then the age of the ultrahomogeneous structure $\mathbf{U}^+$ created this way. Such a general description is rarely useful to obtain Ramsey property. We will focus on classes defined by forbidden homomorphism-embeddings because these, when homogenised, turns into strong amalgamation classes which are in heart of our Ramsey argument. First we give an explicit homogenisation of these classes. This is done in a fully constructive way which leads to an explicit description of Ramsey lifts and therefore also to a way to compute Ramsey degrees and universal minimal flows.

3.3 Lifts of $\mathrm{Forb_{he}}(\mathcal{F})$ with strong amalgamation

Let $\mathcal{F}$ be a family of finite structures. By $\mathrm{Forb_{he}}(\mathcal{F})$ we denote the class of all finite or countable structures $\mathbf{A}$ such that there is no homomorphism-embedding from any $\mathbf{F} \in \mathcal{F}$ to $\mathbf{A}$. Analogously, by $\mathrm{Forb_{h}}(\mathcal{F})$, $\mathrm{Forb_{e}}(\mathcal{F})$ and $\mathrm{Forb_{m}}(\mathcal{F})$ we shall denote the class of all finite or countable structures $\mathbf{A}$ such that there is no homomorphism, embedding and monomorphism from any $\mathbf{F} \in \mathcal{F}$ to $\mathbf{A}$ respectively.

Generalising [37, 38] we give a way to turn every class $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ into a lifted class $\mathcal{L}_{\mathcal{F}}$ which has strong amalgamation (and thus leads to a homogenisation of $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ and in turn to a Ramsey class).

3.3.1 Pieces of structures

For a structure $\mathbf{A}$ the *Gaifman graph* (in combinatorics often called *2-section*) is the graph $\mathbf{G_A}$ with vertices A and all those edges which are a subset of a tuple of a relation of $\mathbf{A}$: $\mathbf{G_A} = (V, E)$ where $\{x, y\} \in E$ if and only if $x \neq y$ and there exists tuple $\vec{t} \in R_{\mathbf{A}}, R \in L$ such that $x, y \in \vec{t}$. Structure $\mathbf{A}$ is *connected* if the Gaifman graph of $\mathbf{A}$ is a connected graph. A subset R of A is a *(vertex) cut of* $\mathbf{A}$ if $\mathbf{G_A}$ is disconnected by removing set R.

Given a structure $\mathbf{A}$ with cut R and two substructures $\mathbf{A}_1$ and $\mathbf{A}_2$, we say that R *separates* $\mathbf{A}_1$ and $\mathbf{A}_2$ if there are components $\mathbf{A}'_1 \neq \mathbf{A}'_2$ of $\mathbf{A}$ with cut R such that $A_1 \subseteq A'_1$ and $A_2 \subseteq A'_2$.

Given structure $\mathbf{A}$ and set of its vertices $S \subseteq A$, the *neighbourhood of* S *in* $\mathbf{A}$, denoted by $N_{\mathbf{A}}(S)$, is the set of all vertices in $N \setminus S$ connected to a vertex S by an edge in the Gaifman graph of $\mathbf{A}$.

Definition 3.3. Let R be a cut in a structure $\mathbf{A}$. Let $\mathbf{A}_1 \neq \mathbf{A}_2$ be two components of $\mathbf{A}$ with cut R. We call R *minimal separating cut* for $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$ if $R = N_{\mathbf{A}}(A_1) = N_{\mathbf{A}}(A_2)$.

For brevity, we can omit one or both components when speaking about a minimal separating cut: We also call a cut R minimal separating for $\mathbf{A}_1$ in $\mathbf{A}$ if there exists another structure $\mathbf{B}$ such that R is minimal separating for $\mathbf{A}_1$ and $\mathbf{B}$ in $\mathbf{A}$. A cut R is minimal separating in $\mathbf{A}$ if there exists structures $\mathbf{B}_1$ and $\mathbf{B}_2$ such that R is minimal separating for $\mathbf{B}_1$ and $\mathbf{B}_2$ in $\mathbf{A}$.

Example. Observe that every inclusion minimal cut is also minimal separating, but not vice versa. An example of minimal separating cut that is not inclusion minimal vertex cut is given in Figure 7.

Every minimal separating cut $R' \subset R$ that separates $\mathbf{A}_1$ and $\mathbf{A}_2$ is however also inclusion minimal cut that separates $\mathbf{A}_1$ and $\mathbf{A}_2$. One can say that this fine distinction is the core of our argument.

If R is a set of vertices then $\overrightarrow{R}$ will denote a tuple (of length $|R|$) formed by all the elements of R. Alternatively, $\overrightarrow{R}$ is an arbitrary linear ordering of R. A *rooted structure* $\mathfrak{P}$ is a pair $(\mathbf{P}, \overrightarrow{R})$ where $\mathbf{P}$ is a relational structure and $\overrightarrow{R}$ is a tuple consisting of distinct vertices of $\mathbf{P}$. $\overrightarrow{R}$ is called the *root* of $\mathfrak{P}$.

The following is our basic notion.

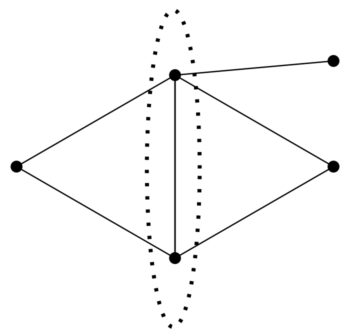


Figure 7: A minimal separating cut that is not inclusion minimal vertex cut.

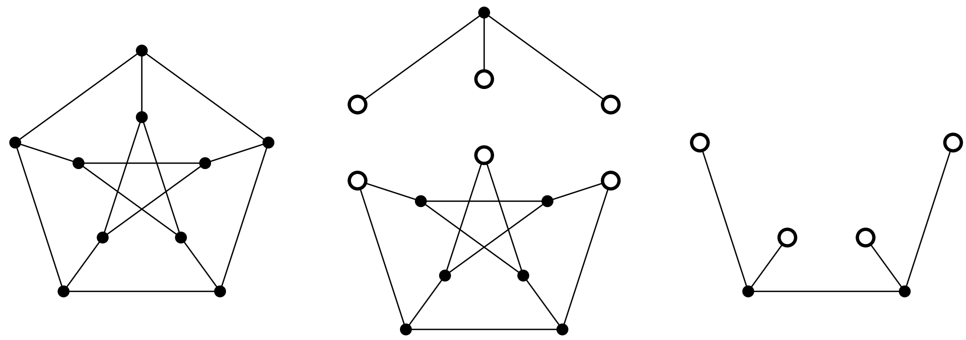


Figure 8: Pieces of the Petersen graph up to isomorphisms (and up to a permutations of roots) with white vertices denoting the roots.

Definition 3.4 ([37, 38])**.** Let $\mathbf{A}$ be a connected relational structure and R a minimal separating cut for component $\mathbf{A}_1$ in $\mathbf{A}$. A *piece* of a relational structure $\mathbf{A}$ is then a rooted structure $\mathfrak{P} = (\mathbf{P}, \overrightarrow{R})$, where the tuple $\overrightarrow{R}$ consists of the vertices of the cut R in a (fixed) linear order and $\mathbf{P}$ is a structure induced by $\mathbf{A}$ on $A_1 \cup R$. $|R|$ is called the *width* of $\mathfrak{P}$.

Note that every piece connected structure.

All pieces are considered as rooted structures: a piece $\mathfrak{P}$ is a structure $\mathbf{P}$ rooted at $\overrightarrow{R}$. Accordingly, we say that pieces $\mathfrak{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ and $\mathfrak{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ are *isomorphic* if there is a function $\varphi : P_1 \to P_2$ that is isomorphism of structures $\mathbf{P}_1$ and $\mathbf{P}_2$ and φ restricted to $\overrightarrow{R}_1$ is the monotone bijection between $\overrightarrow{R}_1$ and $\overrightarrow{R}_2$ (we denote this $\varphi(\overrightarrow{R}_1) = \overrightarrow{R}_2$).

Example. Observe that for relational trees, pieces are equivalent to rooted branches. Figure 8 shows all isomorphism types of pieces of the Petersen graph (up to a permutation of roots).

3.3.2 Regular families of structures

Let $\mathcal{F}$ be a finite set of connected finite relational structures of (finite) language L. For construction of an universal structure we use special lifts, called

$\mathcal{F}$-lifts.

Given rooted structures $(\mathbf{P}, \overrightarrow{R})$ and $(\mathbf{P}', \overrightarrow{R}')$ such that $|R| = |R'|$, denote by $(\mathbf{P}, \overrightarrow{R}) \oplus (\mathbf{P}', \overrightarrow{R}')$ the (possibly rooted) structure created as a free amalgamation of $\mathbf{P}$ and $\mathbf{P}'$ with corresponding roots being identified (in the order of $\overrightarrow{R}$ and $\overrightarrow{R}'$). Note that $(\mathbf{P}, \overrightarrow{R}) \oplus (\mathbf{P}', \overrightarrow{R}')$ is defined only if the rooted structure induced by $\mathbf{P}$ on $\overrightarrow{R}$ is isomorphic to the rooted structure induced by $\mathbf{P}'$ on $\overrightarrow{R}'$.

Definition 3.5. A piece $\mathfrak{P} = (\mathbf{P}, \overrightarrow{R})$ is *incompatible* with a rooted structure $\mathfrak{A}$ if $\mathfrak{P} \oplus \mathfrak{A}$ is defined and there exists $\mathbf{F} \in \mathcal{F}$ that is isomorphic to $\mathfrak{P} \oplus \mathfrak{A}$. (In other words, there exists $\mathbf{F}'$ isomorphic to some $\mathbf{F}'' \in \mathcal{F}$, such that $\mathfrak{P}$ is a piece of $\mathbf{F}'$ and $\mathfrak{A}$ is a structure induced on $F' \setminus (P \setminus R)$ by $\mathbf{F}'$ rooted by $\overrightarrow{R}$.)

Assign to each piece $\mathfrak{P}$ a set $\mathcal{I}_{\mathfrak{P}}$ of all rooted structures that are incompatible with $\mathfrak{P}$. For two pieces $\mathfrak{P}_1$ and $\mathfrak{P}_2$ put $\mathfrak{P}_1 \sim_{\mathcal{F}} \mathfrak{P}_2$ if and only if $\mathcal{I}_{\mathfrak{P}_1} = \mathcal{I}_{\mathfrak{P}_2}$. ($\sim_{\mathcal{F}}$ is called the *piece equivalence.*) Observe that every equivalence class of $\sim_{\mathcal{F}}$ contains pieces of the same width n. We also call n the *width* of an equivalence class of $\sim_{\mathcal{F}}$.

Definition 3.6. A family of finite structures $\mathcal{F}$ is *regular* if the equivalence $\sim_{\mathcal{F}}$ has only finitely many equivalence classes of width n, for every $n \geq 1$.

Remark. The notion of regular family of structures is a generalisation of that of a regular family of trees, introduced in [21] and it is motivated by the similarity to characterisation of regular languages by Myhill-Nerode Theorem. Definition 3.6 is a strengthening of the definition used in [37] for classes without a bound on the size of the cut.

3.3.3 Maximal $\mathcal{F}$-lifts

Now we are ready to explain the homogenising lift of the class $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$. We denote the language of the structures by L. In this section we define the lifted language L^+.

We fix the enumeration $\mathcal{P}^1_{\mathcal{F}}, \mathcal{P}^2_{\mathcal{F}}, \ldots$ of all equivalence classes of all pieces with respect to equivalence $\sim_{\mathcal{F}}$ corresponding to pieces of structures in $\mathcal{F}$. If there are only finitely many equivalence classes in $\sim_{\mathcal{F}}$, put $I = \{1, 2, \ldots, N\}$, where N denote the number of equivalence classes of $\sim_{\mathcal{F}}$. Otherwise put $I = \{1, 2, \ldots\}$.

The language L^+ extends language L by new relation L^i, $i \in I$. The arity of L^i corresponds to the width of $\mathcal{P}^i_{\mathcal{F}}$. (To make the distinction between languages more explicit, we use L^i to denote lifted relations instead of R^i.)

An $\mathcal{F}$*-lift* $\mathbf{X}$ of a structure $\mathbf{A} \in \mathrm{Rel}(L)$ is a structure $\mathbf{X} \in \mathrm{Rel}(L^+)$ and will be written as:

$$\mathbf{X} = (A, (R_{\mathbf{A}}; R \in L), (L^i_{\mathbf{X}}; i \in I)).$$

and, by an abuse of notation, briefly as:

$$\mathbf{X} = (\mathbf{A}, (L^i_{\mathbf{X}}; i \in I)).$$

For a relational structure $\mathbf{A}$, we define the *canonical* $\mathcal{F}$*-lift* of $\mathbf{A}$ as follows:

$$L_{\mathcal{F}}(\mathbf{A}) = (\mathbf{A}, (L^i_{L_{\mathcal{F}}(\mathbf{A})}; i \in I))$$

by putting $(v_1, v_2, \ldots, v_l) \in L^i_{L_{\mathcal{F}}(\mathbf{A})}$ if and only if there is a piece $\mathfrak{P} = (\mathbf{P}, \overrightarrow{R}) \in \mathcal{P}^i_{\mathcal{F}}$ and a homomorphism-embedding $f : \mathbf{P} \to \mathbf{A}$ such that:

1. $f(\overrightarrow{R}) = (v_1, v_2, \ldots, v_l)$ and
2. f is injective on vertices of $\overrightarrow{R}$.

We will use the following notion of maximal $\mathcal{F}$-lifts:

Definition 3.7. The canonical $\mathcal{F}$-lift $L_{\mathcal{F}}(\mathbf{A})$ of $\mathbf{A}$ is *maximal* on $B \subseteq A$ if for every $\mathbf{C} \in \mathrm{Forb_{he}}(\mathcal{F})$ such that $\mathbf{C}$ contains $\mathbf{A}$ as substructure, the $\mathcal{F}$-lift induced on B by $L_{\mathcal{F}}(\mathbf{A})$ is the same as the $\mathcal{F}$-lift induced on B by $L_{\mathcal{F}}(\mathbf{C})$. We say that an $\mathcal{F}$-lift $\mathbf{X}$ is *maximal* if there exists $\mathbf{A} \in \mathrm{Forb_{he}}(\mathcal{F})$ such that $\mathbf{X}$ is induced on X by $L_{\mathcal{F}}(\mathbf{A})$ and the canonical $\mathcal{F}$-lift $L_{\mathcal{F}}(\mathbf{A})$ of $\mathbf{A}$ is maximal on X.

Intuitively a maximal $\mathcal{F}$-lift contains all possible relations from all extensions. Because the extensions are not always compatible with each other, a maximal $\mathcal{F}$-lift is not unique. Maximal $\mathcal{F}$-lifts form the homogenization we seek for. Before stating the main result of this section we recall several notions.

Recall that a structure $\mathbf{A} \in \mathcal{K}$ is *existentially complete in* $\mathcal{K}$ if for every structure $\mathbf{B} \in \mathcal{K}$ such that the identity mapping (of A) is an embedding $\mathbf{A} \to \mathbf{B}$, every existential statement ψ which is defined in $\mathbf{A}$ and true in $\mathbf{B}$ is also true in $\mathbf{A}$.

We say that a homomorphism-embedding f from L-structure $\mathbf{A}$ to L-structure $\mathbf{B}$ is *surjective* if $f(A) = \{f(x); x \in A\} = B$. Homomorphism-embedding f is *tuple-surjective* if for every $R \in L$ and every $\vec{u} \in R_{\mathbf{B}}$ there exists $\vec{v} = (v_1, v_2, \ldots, v_n) \in R_{\mathbf{A}}$ such that $f(\vec{v}) = (f(v_1), f(v_2), \ldots, f(v_n)) = \vec{u}$.

We say that a class $\mathcal{F}$ is *closed on homomorphism-embedding images* if for every $\mathbf{F} \in \mathcal{F}$, and every tuple-surjective homomorphism-embedding

$f : \mathbf{F} \to \mathbf{F}'$, there exists a substructure $\mathbf{F}''$ of $\mathbf{F}'$ such that $\mathbf{F}'' \in \mathcal{F}$. We shall prove the following result about the existence of homomogenization of classes $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$:

Theorem 3.3. *let $\mathcal{F}$ be a family of finite connected relational L-structures which is closed for homomorphism-embedding images. Denote by $\mathcal{L}_{\mathcal{F}}$ the class of all finite maximal $\mathcal{F}$-lifts. Then $\mathrm{Age}(\mathcal{L}_{\mathcal{F}})$ is an amalgamation class with strong amalgamations whose shadows are free amalgamations. If $\mathcal{F}$ is a regular family, then the $\mathcal{F}$-lift adds only finitely many new relations of every arity and therefore it is precompact.*

If $\mathcal{L}_{\mathcal{F}}$ is countable, denote by $\mathbf{U}'$ the Fraïssé limit of $\mathrm{Age}(\mathcal{L}_{\mathcal{F}})$. If $\mathcal{F}$ is regular and L is finite language, then the shadow $\mathbf{U} = \mathrm{Sh}(\mathbf{U}') \in \mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ is the ω-categorical existentially complete structure universal for $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$.

We can also show that the construction is tight. Family $\mathcal{F}$ is *upwards closed* if for every $\mathbf{F} \in \mathcal{F}$ we also have $\mathbf{F}' \in \mathcal{F}$ providing that $\mathbf{F}'$ is connected and there is a homomorphism-embedding $\mathbf{F} \to \mathbf{F}'$.

Theorem 3.4. *Let L be a finite language. Let $\mathcal{F}$ be a upwards closed family of finite connected relational L-structures. Then the following conditions are equivalent:*

(*a*) *$\mathcal{F}$ is a regular family of connected structures.*

(*b*) *There is an ultrahomogeneous lift $\mathbf{U}^+$ which extends L only by finitely many relations of any given arity. The shadow $\mathrm{Sh}(\mathbf{U}^+) \in \mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ is universal for $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$.*

(*c*) *$\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ contains an ω-categorical universal structure.*

Theorems 3.3 and 3.4 are proved in Sections 3.5 and 3.6 of this paper.

3.4 The existence of precompact Ramsey lifts

In this section we give an strenghtening of the following classical result:

Theorem 3.5 (Nešetřil-Rödl Theorem [67])**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be ordered hypergraphs, then there exists an ordered hypergraph $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.*

Moreover, if $\mathbf{A}$ and $\mathbf{B}$ do not contain an irreducible hypergraph $\mathbf{F}$ (as an non-induced sub-hypergraph) then $\mathbf{C}$ may be chosen with the same property.

In this original formulation (see [62]) the theorem speaks of hypergraphs (or set systems) with additional linear order on vertices. This linear order has no further constrains and is treated specially thorough the proof. In other words, the theorem states that for every family $\mathcal{E}$ of finite irreducible hypergraphs the lift of the class of all finite hypergraphs in $\mathrm{Forb_m}(\mathcal{E})$ adding a free linear order on vertices is a Ramsey class.

We first give a re-formulation of this theorem in the language of relational structures with a small strengthening. This is stated as Theorem 3.6 bellow. Then we proceed by the main result of this section (Theorem 3.7) which strenghten the Nešetřil-Rödl Theorem for classes with forbidden homomorphisms and closures.

In this section every Ramsey class $\mathcal{K}$ will always contain a binary relation $R^{\leq}$ which will (in every structure in $\mathcal{K}$) represent a linear order. We will also work with structures where $R^{\leq}$ is not a linear order and thus we say that structure $\mathbf{A}$ is *ordered* if the relation $R^{\leq}$ forms a linear order on A. If there is no restriction on $R^{\leq}$ then it is called *free ordering*.

We say that L-structure $\mathbf{F}$ is *is irreducible without order* if for every pair of distinct vertices $u, v \in F$ there exists a relation $R \in L$ other than $R^{\leq}$ and a tuple in $R_{\mathbf{F}}$ containing both u and v (in other words, the shadow of $\mathbf{F}$ removing the relation $R^{\leq}_{\mathbf{F}}$ is irreducible). Structure $\mathbf{F}$ is *ordered irreducible* structure if it is both ordered and irreducible without order.

Now we are ready to formulate Theorem 3.5 in our language:

Theorem 3.6 (Nešetřil-Rödl Theorem for relational structures)**.** *Let L be a language containing binary relation $R^{\leq}$ and $\mathcal{E}$ be a (possibly infinite) family of ordered irreducible L-structures. Then the class of all finite ordered structures in* $\mathrm{Forb_e}(\mathcal{E})$ *is a Ramsey class.*

There are two differences compared to the original formulation. First forbidden substructures in class $\mathcal{E}$ are ordered (we thus do not speak of a lift of the class adding a free order, but rather constrained relation $R^{\leq}$). This allows to use Theorem 3.6 to show, for example, the Ramsey property of acyclic graphs as shown in [60] (see Corollary 4.8). Second we speak of forbidden embeddings (and thus substructures). Both these strenghtenings follows by the same proof as presented in [62].

The linear order will continue to be special in our results, too. The following notion captures the properties of structures that can be forbidden as homomorphic images:

Definition 3.8. Let L be a language containing binary relation $R^{\leq}$. An L-structure $\mathbf{F}$ is *weakly ordered* if

1. $R^{\leq}_{\mathbf{F}}$ can be completed to linear order (in other words it forms a reflexive acyclic digraph), and,

2. for every pair of distinct vertices $a, b \in F$, either $(a, b) \in R^{\lneqq}_{\mathbf{F}}$ or $(b, a) \in R^{\lneqq}_{\mathbf{F}}$ if and only if there exists a relation $R \in L$ other than $R^{\leq}$ and a tuple in $R_{\mathbf{F}}$ containing both u and w.

(In other words $R^{\lneqq}_{\mathbf{F}}$ is an acyclic orientation of the Gaifman graph of shadow of $\mathbf{F}$ after removing the relation $R^{\lneqq}_{\mathbf{F}}$.)

Note that $R^{\lneqq}_{\mathbf{A}}$ is neither a partial order and nor a linear order. Weakly ordered structures typically arrise as a free amalgamation of ordered structures. In weakly a ordered structure $R^{\lneqq}_{\mathbf{A}}$ is a linear order if and only if $\mathbf{A}$ is irreducible without order.

In order to keep our clasees closed for free amalgamations in the shadow we restrict the closures. We say that closure description $\mathcal{U}$ is a *closure with ordered irreducible roots* if for every $(R^U, \mathbf{R}) \in \mathcal{U}$ the structure $\mathbf{R}$ is an ordered irreducible structure.

The sufficient conditions for the existence of a precompact Ramsey lift can now be formulated as follows.

Theorem 3.7. *Let*

1. *L be a language containing binary relation $R^{\leq}$,*

2. *$\mathcal{F}$ be a (possibly infinite) regular family of finite connected weakly ordered L-structures closed for homomorphism-embedding images,*

3. *$\mathcal{U}$ be a closure description in the language L with an ordered irreducible roots, and,*

4. *$\mathcal{E}$ be a (possibly infinite) family of $\mathcal{U}$-closed L-structures such that every $\mathbf{F} \in \mathcal{E}$ contains an ordered irreducible substructure $\mathbf{F}'$ such that $\mathrm{Cl}^{\mathcal{U}}_{\mathbf{F}}(F') = \mathbf{F}$.*

Further assume that the class of all finite ordered structures in $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ *is a locally finite subclass of the classs of all finite ordered L-structures. Then the class $\mathcal{K}$ of all finite ordered $\mathcal{U}$-closed structures in* $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F}) \cap \mathrm{Forb}_{\mathrm{e}}(\mathcal{E})$ *has a precompact Ramsey lift.*

More specifically: The class $\mathcal{K}_{\mathcal{F}}$ of all maximal $\mathcal{F}$-lifts of structures in $\mathcal{K}$ is a Ramsey class and for every pair of structures $\mathbf{A}, \mathbf{B}$ in $\mathcal{K}_{\mathcal{F}}$ there exists a structure $\mathbf{C} \in \mathcal{K}_{\mathcal{F}}$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

If $\mathcal{U}$ is empty, then the lift $\mathcal{K}_{\mathcal{F}}$ has the lift property with respect to $\mathcal{K}$.

Remark. The condition on the class of ordered structures in $\mathrm{Forb_{he}}(\mathcal{F})$ being locally finite subclass of the class of all ordered structures can be reformulated as follows: For every structure $\mathbf{C}_0$ there exists $n(\mathbf{C}_0)$ such that for every structure $\mathbf{C} \notin \mathrm{Forb_{he}}(\mathcal{F})$ with a homomorphism-embedding to $\mathbf{C}_0$ there is $\mathbf{F} \in \mathcal{F}$ with at most $n(\mathbf{C}_0)$ vertices and a homomorphism-embedding $\mathbf{F} \to \mathbf{C}_0$.

Remark. The closure with ordered irreducible roots makes it possible to complete order in a structure without introducing new roots of closures.

Theorem 3.7 may seem significantly more special than Theorem 2.2 as it gives, together with Theorem 3.3 an explicite description of Ramsey lift. For special case of empty closure description we arrive to the following characterisation:

Corollary 3.8 (Characterisation of Ramsey Classes with Forbidden Homomorphism-embeddings)**.** *Let L be a finite language containing binary relation $R^{\leq}$ and $\mathcal{F}$ be a family of finite connected L-structures. The following conditions are equivalent:*

1. $\mathrm{Age}(\mathrm{Forb_{he}}(\mathcal{F}))$ *has the precompact Ramsey lift with the lift property,*
2. $\mathrm{Forb_{he}}(\mathcal{F})$ *contains an ω-categorical universal structure,*
3. *there exists a regular family $\mathcal{F}'$ such that* $\mathrm{Forb_{he}}(\mathcal{F}) = \mathrm{Forb_{he}}(\mathcal{F}')$.

Proof. By Proposition 3.2 every precompact Ramsey Lift of $\mathrm{Age}(\mathrm{Forb_{he}}(\mathcal{F}))$ forms an amalgamation class and thus there exists the generic structure for this lifted class. Because the lift is precompact, the shadow of this structure is an ω-categorical universal structure and thus $1 \implies 2$. $2 \implies 3$ by Theorem 3.4. $3 \implies 1$ by Theorem 3.7. $\square$

3.5 Proof of Theorem 3.3

In this section we prove (essentially model-theoretic) Theorem 3.3 which gives a description of the homogenisation of classes $\mathrm{Forb_{he}}(\mathcal{F})$. This extends the construction in [37] for the case of regular infinite families $\mathcal{F}$ and particularly for families without an upper bound on the size of minimal separating cuts (completing our techniques to all classes with a precompact homogenisation). Note also that we use homomorphism-embeddings, instead of homomorphisms. By the use of the maximal lifts (introduced in [38]) we not only simplify the argument of [37], but more importantly, obtain the existentially complete homogenising lift. This, in turn, gives a lift property of

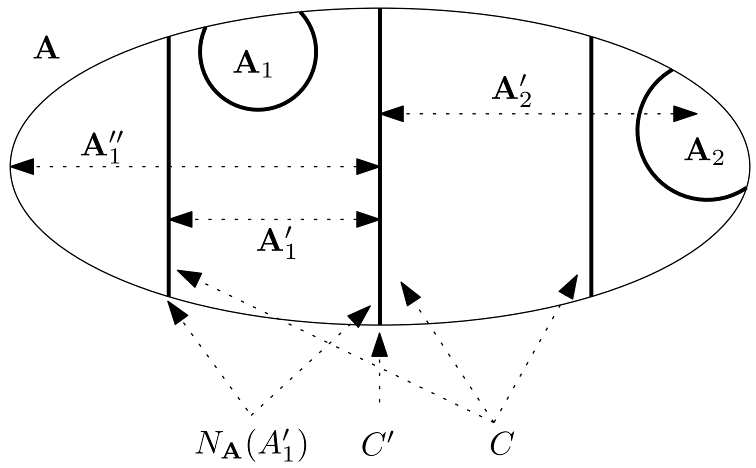


Figure 9: The construction of a minimal separating cut R' separating $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$.

the resulting Ramsey lift. All in all this part may be seen as an elaboration of [38, 37, 31].

We will make use and find it useful to single out the following simple (geometrical) observation about the neighbourhood and components in relational structures.

Observation 3.9. *Let $\mathbf{A}_1$ be a component of a connected structure $\mathbf{A}$ with cut R. Then the neighbourhood $N_{\mathbf{A}}(A_1)$ is a subset of R. Moreover $N_{\mathbf{A}}(A_1)$ is a cut and $\mathbf{A}_1$ is one of the components of $\mathbf{A}$ with cut $N_{\mathbf{A}}(A_1)$.*

Proof. Obvious. □

The name of minimal separating cut is justified by the following (probably folkloristic) proposition.

Proposition 3.10. *Let $\mathbf{A}$ be a connected relational structure, R a cut in $\mathbf{A}$ and let $\mathbf{A}_1$ and $\mathbf{A}_2$ be connected substructures of $\mathbf{A}$ separated by R. Then there exists a minimal separating cut $R' \subseteq R$ that separates $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$. Moreover if $N_{\mathbf{A}}(A_1) \subseteq R$ (or, equivalently, $\mathbf{A}_1$ is a component of $\mathbf{A}$ with cut R), then $R' \subseteq N_{\mathbf{A}}(A_1)$.*

Proof. We will construct a series of cuts and components as depicted in Figure 9.

Denote by $\mathbf{A}_1'$ the component of $\mathbf{A}$ with cut R containing $\mathbf{A}_1$ (and thus not containing $\mathbf{A}_2$). By Observation 3.9, $N_{\mathbf{A}}(A_1') \subseteq R$ is cut that separates $\mathbf{A}_1'$ and $\mathbf{A}_2$ (because $\mathbf{A}_1'$ is also a component of $\mathbf{A}$ with cut $N_{\mathbf{A}}(A_1')$ and $\mathbf{A}_1'$ do not contain $\mathbf{A}_2$).

Now consider the component $\mathbf{A}_2'$ of $\mathbf{A}$ with cut $N_{\mathbf{A}}(A_1')$ containing $\mathbf{A}_2$. Put $R' = N_{\mathbf{A}}(A_2')$. By Observation 3.9, $R' \subseteq N_{\mathbf{A}}(A_1') \subseteq R$ is cut and $\mathbf{A}_2'$ (not containing $\mathbf{A}_1$) is one of its components.

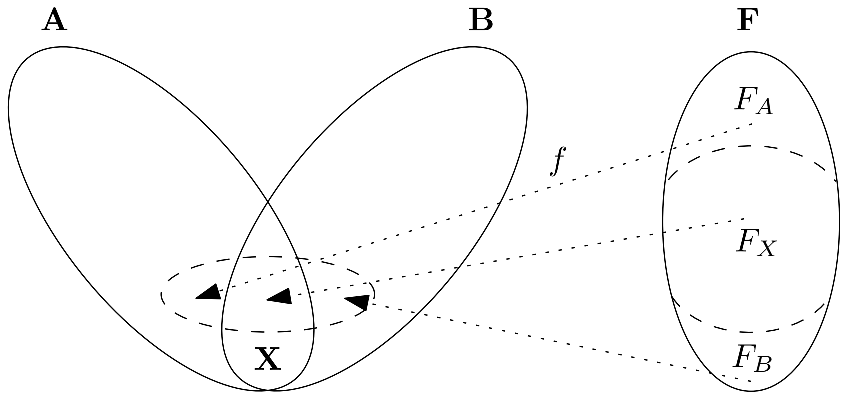


Figure 10: An amalgamation of maximal $\mathcal{F}$-lifts.

Denote by $\mathbf{A}_1''$ the component of $\mathbf{A}$ with cut R' containing $\mathbf{A}_1$. It follows that R' separates $\mathbf{A}_1''$ (that contains $\mathbf{A}_1$) and $\mathbf{A}_2'$ (that contains $\mathbf{A}_2$).

To see that R' is minimal a separating cut for $\mathbf{A}_1''$ and $\mathbf{A}_2'$ it remains to show that every vertex in $R' = N_{\mathbf{A}}(A_2')$ is also in $N_{\mathbf{A}}(A_1'')$. This is true because every vertex of R' is in $N_{\mathbf{A}}(A_1')$ and $\mathbf{A}_1'$ is substructure of $\mathbf{A}_1''$. $\square$

For a canonical lift $\mathbf{X} \in \mathcal{L}_{\mathcal{F}}$ we denote by $W(\mathbf{X})$ a structure $\mathbf{A} \in \mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ such that $\mathbf{X}$ is induced on X by $L_{\mathcal{F}}(\mathbf{A})$ and $L_{\mathcal{F}}(\mathbf{A})$ is maximal on X. $W(\mathbf{X})$ is called a *witness* of the fact that $\mathbf{X}$ belongs to $\mathcal{L}_{\mathcal{F}}$.

Given a piece $\mathfrak{P} = (\mathbf{P}, \overrightarrow{R})$ of structure $\mathbf{F}$, we call $\mathfrak{P}' = (\mathbf{P}', \overrightarrow{R}')$ a *sub-piece* of $\mathfrak{P}$ if $\mathfrak{P}'$ is piece of $\mathbf{F}$, $P' \subset P$.

The key technical part of our construction (and of proof of Theorem 3.3) is expressed by the following:

Lemma 3.11. *Let $\mathcal{F}$ be a family of connected structures closed for homomorphism-embedding images. Let $\mathbf{A}$ and $\mathbf{B}$ be both witnesses of $\mathcal{F}$-lift $\mathbf{X}$. Then the free amalgamation of $\mathbf{A}$ and $\mathbf{B}$ over the structure induced on X by both $\mathbf{A}$ and $\mathbf{B}$ is also a witness of $\mathbf{X}$.*

Proof. Denote by $\mathbf{D}$ the free amalgamation of $\mathbf{A}$ and $\mathbf{B}$ over X. From the maximality of $\mathbf{X}$ in both $\mathbf{A}$ and $\mathbf{B}$ we know that $\mathbf{D}$ is a witness of $\mathbf{X}$ if $\mathbf{D} \in \mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$. Assume, to the contrary, that $\mathbf{D} \notin \mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ and thus there is is an $\mathbf{F} \in \mathcal{F}$ and a homomorphism-embedding f from $\mathbf{F}$ to $\mathbf{D}$. Because $\mathcal{F}$ is closed for homomorphic images, we can also assume f to be injective. Mapping f partitions the vertex set of $\mathbf{F}$ into three sets defined as follows: F_X are vertices with image in X, F_A are vertices with image in $A \setminus X$ and F_B are vertices with image in $B \setminus X$. Without loss of generality we can assume that $\mathbf{F}$ and f was chosen in a way so $|F_A|$ is minimal, clearly $|F_A| \geq 1$. The situation is depicted in Figure 10.

Observe that F_X is a cut of $\mathbf{F}$ separating F_A and F_B. Denote by $\mathfrak{P} = (\mathbf{P}, \overrightarrow{R})$ a piece with root contained in F_X containing a vertex of F_A (such

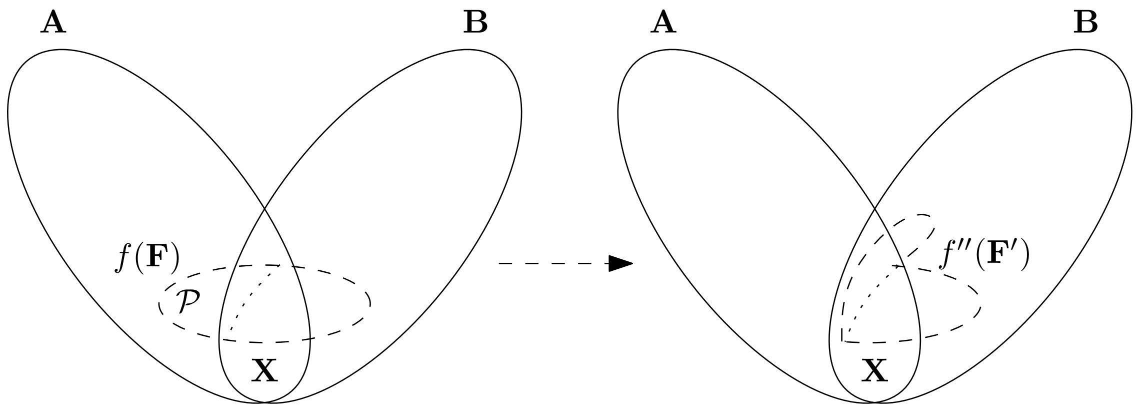


Figure 11: The flip operation.

piece can be obtained by Proposition 3.10). Denote by i the index such that $\mathfrak{P}$ belongs to the equivalence class $\mathcal{P}^i_{\mathcal{F}}$ of $\sim_{\mathcal{F}}$ (see Section 3.3.3). If $f(\overrightarrow{R}) \in L^i_{\mathbf{X}}$ (and thus also $f(\overrightarrow{R}) \in L^i_{\mathbf{L}_{\mathcal{F}}(\mathbf{A})}$ and $f(\overrightarrow{R}) \in L^i_{\mathbf{L}_{\mathcal{F}}(\mathbf{B})}$) then there exists a piece $\mathfrak{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2) \sim_{\mathcal{F}} \mathfrak{P}$ and a homomorphism-embedding $f' : \mathbf{P}_2 \to \mathbf{B}$ such that $f'(\overrightarrow{R}) = f(\overrightarrow{R})$. Consider $\mathbf{F}' \in \mathcal{F}$ created from $\mathbf{F}$ by replacing $\mathfrak{P}$ by $\mathfrak{P}_2$ and a function $f'' : F' \to D$ defined as follows:

$$f''(x) = \begin{cases} f'(x) \text{ for } x \in P_2, \\ f(x) \text{ otherwise.} \end{cases}$$

f'' is a homomorphism-embedding $\mathbf{F}' \to \mathbf{D}$ that uses fewer vertices of F_A and possibly more vertices of $F_X \cup F_B$. We call this *flip operation* (and we shall use it in the proof later again). When a piece has its root in F_X, flip operation moves the image of the piece from one part of the amalgamation to the other. This is schematically depicted by Figure 11.

By the minimality of F_A it thus follows that $f(\overrightarrow{R}) \notin L^i_{\mathbf{X}}$. If $P \subseteq A$, then by the definition of the canonical $\mathcal{F}$-lift we have $f(\overrightarrow{R}) \in L^i_{\mathbf{L}_{\mathcal{F}}(\mathbf{A})}$ a contradiction. We thus conclude that every piece with root in F_X containing a vertex of F_A must also contain a vertex of F_B.

Choose $\mathfrak{P}' = (\mathbf{P}', \overrightarrow{R}') \in \mathcal{P}^j_{\mathcal{F}}$ to be a piece containing both vertices of F_A and F_B with the minimal number of non-root vertices among pieces with this property. If $\mathfrak{P}'$ contains a sub-pieces with root in F_X contained in $F_X \cup F_B$, we can perform the flip operation, this time replacing vertices with images in F_B by vertices with image in $F_X \cup F_A$. If this procedure eliminates all vertices of $P' \cap F_B$ we get a homomorphism-embedding $f' : \mathbf{P}' \to \mathbf{A}$, $f'(\overrightarrow{R}') = f(\overrightarrow{R}')$, and therefore $f(\overrightarrow{R}') \in L^j_{\mathbf{L}_{\mathcal{F}}(\mathbf{A})}$ that contradicts the minimality of $|F_A|$.

Denote by $\mathbf{A}'$ a component of $\mathbf{F}$ with cut F_X contained in $\mathfrak{P}'$ consisting of vertices of F_A and by $\mathbf{B}'$ a component of $\mathbf{F}$ with cut F_X contained in

$\mathfrak{P}'$ consisting of vertices of F_B that can not be eliminated from $\mathfrak{P}'$ by the flip operations. Denote by F' the set vertices of any connected component of $\mathbf{F} \setminus P'$ such that $R' \subseteq N_{\mathbf{F}}(F')$ (such component exists because R' is a minimal separating cut). By the application of Proposition 3.10 on cut $F_X \cap P'$ with F' and B', one gets that $R' \subseteq N_{\mathbf{F}}(B')$. Otherwise one would obtain a sub-piece that would contradict the minimality of $\mathfrak{P}'$ or an assumption that $\mathbf{B}'$ can not be eliminated (and thus it is not a contained in a piece consisting only of vertices $F_X \cup F_B$). The symmetric argument gives $R' \subseteq N_{\mathbf{F}}(A')$. Now again by the application of Proposition 3.10 with cut F_X and components $\mathbf{A}'$ and $\mathbf{B}'$ we obtain a minimal separating cut C. Clearly $R' \subseteq C$ because $R' \subseteq N_{\mathbf{F}}(A') \cap N_{\mathbf{F}}(B')$. C must contain some additional vertices of $F_X \cup (P' \setminus R')$ because $\mathbf{P}' \setminus R'$ is connected and F_X separates A' and B'. The pieces obtained are thus a proper sub-pieces of $\mathfrak{P}'$ that either contain both vertices of F_A and F_B or they can be used for the flip operations. In all these cases this yields a contradiction. $\square$

Proof of Theorem 3.3. The class $\mathcal{L}_{\mathcal{F}}$ of all maximal $\mathcal{F}$-lifts is clearly hereditary, isomorphism closed and has the joint embedding property. Thus to show that $\mathcal{L}_{\mathcal{F}}$ is an amalgamation class it remains to verify that $\mathcal{L}_{\mathcal{F}}$ has the amalgamation property.

Consider $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{L}_{\mathcal{F}}$. Assume that structure $\mathbf{Z}$ is substructure induced by both $\mathbf{X}$ and $\mathbf{Y}$ on Z and without loss of generality assume that $X \cap Y = Z$.

Put

$$\begin{aligned} \mathbf{A} &= W(\mathbf{X}), \\ \mathbf{B} &= W(\mathbf{Y}), \\ \mathbf{C} &= \mathrm{Sh}(\mathbf{Z}). \end{aligned}$$

Now consider $\mathbf{D}$, the free amalgamation of $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{C}$. As shown by Lemma 3.11, $\mathbf{D}$ is a witness of $\mathbf{Z}$ and also a witness of $\mathbf{A}$ and $\mathbf{B}$. Now find $\mathbf{E} \in \mathrm{Forb_{he}}(\mathcal{F})$ containing $\mathbf{D}$ as a substructure such that $L_{\mathcal{F}}(\mathbf{E})$ is maximal on D. It follows that the structure induced on D on $L_{\mathcal{F}}(\mathbf{E})$ is the amalgamation of $\mathbf{X}$ and $\mathbf{Y}$ over $\mathbf{Z}$.

By the maximality condition it also follows that the Fraïssé limit constructed is existentially complete in the class of all structures in $\mathrm{Forb_{he}}(\mathcal{F})$. $\square$

3.6 Proof of Theorem 3.4

Theorem 3.4 gives a characterisation of those families $\mathcal{F}$ such that $\mathrm{Forb_{he}}(\mathcal{F})$ contains an ω-categorical universal structure. This is related to (and generalises) forbidden homomorphism theorem of [11]. This is also in contrast with forbidden monomorphisms where the corresponding characterisation is a well known problem conjectured to be undecidable [7].

For a family of relational structures $\mathcal{F}$ denote by $\overline{\mathcal{F}}$ the (complementary) class of all connected relational structures (over the same language) not isomorphic to some structure in $\mathcal{F}$. First we show that regular families are closed for complements:

Lemma 3.12. *$\mathcal{F}$ is regular if and only if $\overline{\mathcal{F}}$ is regular.*

Proof. Clearly it suffices to show only one implication. Assume that $\mathcal{F}$ is regular. Now consider $\overline{\mathfrak{P}}$, a piece of some $\overline{\mathbf{F}} \in \overline{\mathcal{F}}$. Denote by $\overline{\mathcal{I}}_{\overline{\mathfrak{P}}}$ to be the set of all rooted structures incompatible with $\overline{\mathfrak{P}}$ with respect to $\overline{\mathcal{F}}$ (see Definition 3.5). There are two cases:

1. $\overline{\mathfrak{P}}$ is not isomorphic to any piece $\mathfrak{P}$ of any structure $\mathbf{F} \in \mathcal{F}$. In this case for every rooted structure $\mathfrak{A}$ such that $\mathfrak{A} \oplus \overline{\mathfrak{P}}$ is defined we have that $\mathfrak{A} \oplus \overline{\mathfrak{P}}$ is not isomorphic to any structure in $\mathcal{F}$, consequently $\mathfrak{A} \oplus \overline{\mathfrak{P}} \in \overline{\mathcal{F}}$ and thus $\mathfrak{A} \in \overline{I}_{\overline{\mathfrak{P}}}$.

2. $\overline{\mathfrak{P}}$ is isomorphic to some piece $\mathfrak{P}$ of some $\mathbf{F} \in \mathcal{F}$. In this case for every rooted structure $\mathfrak{A}$ such that $\mathfrak{A} \oplus \overline{\mathfrak{P}}$ is defined we have $\mathfrak{A} \oplus \mathfrak{P}$ isomorphic to some structure in $\mathcal{F}$ if and only if $\mathfrak{A} \oplus \overline{\mathfrak{P}}$ is not isomorphic to any structure in $\overline{\mathcal{F}}$. It follows that $\mathfrak{A} \in \mathcal{I}_{\mathfrak{P}}$ if and only if $\mathfrak{A} \notin \overline{\mathcal{I}}_{\overline{\mathfrak{P}}}$.

We have shown sets $\overline{\mathcal{I}}_{\overline{\mathfrak{P}}}$ are, in a certain sense, complements of sets $\mathcal{I}_{\mathfrak{P}}$ and thus by the regularity of $\mathcal{F}$ there are finitely many different sets of $\overline{\mathcal{I}}_{\overline{\mathfrak{P}}}$ on pieces of $\overline{F}$ with any given width $n \geq 1$. It follows that $\overline{\mathcal{F}}$ is regular. $\square$

Proof of Theorem 3.4. $(a) \implies (b)$ follows from Theorem 3.3 for the class $\mathcal{F}$.

$(b) \implies (c)$ is immediate. The shadow of every ultrahomogeneous structure with finitely many relations of a given arity is ω-categorical.

To see that $(c) \implies (a)$ we first observe that for every ω-categorical structure $\mathbf{U}$ the family of relational structures $\mathcal{C}$ consisting of all connected structures in Age($\mathbf{U}$) is a regular family. Fix $n \geq 1$ and consider two pieces $\mathfrak{P} = (\mathbf{P}, \overrightarrow{R})$ and $\mathfrak{P}' = (\mathbf{P}, \overrightarrow{R})$. Denote by $O_{\mathfrak{P}}$ the set of all orbits of n-tuples of the automorphism group of $\mathbf{U}$ such that there exits a homomorphism-embedding $f : \mathbf{P} \to \mathbf{U}$ with tuple $f(\overrightarrow{R})$ being in the orbit. It is easy to see that $O_{\mathfrak{P}} = O_{\mathfrak{P}'}$ implies $\mathfrak{P} \sim_{\mathcal{F}} \mathfrak{P}'$. This gives the regularity of $\mathcal{F}$.

Consider upwards closed family $\mathcal{F}$ and such that $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ contains an ω-categorical universal structure $\mathbf{U}$. It is easy to see that $\overline{\mathcal{F}}$ is precisely the family of all connected structures in Age($\mathbf{U}$). Because the family $\overline{\mathcal{F}}$ is regular, the family $\mathcal{F}$ is regular by Lemma 3.12. $\square$

Theorem 3.4 clarifies the rôle of the necessity of the assumption of the regularity in Definition 2.8 and Theorem 3.7.

3.7 Proof of Theorem 3.7

By now this is an easy application of our lift construction together with the proof of Theorem 2.2.

Proof of Theorem 3.7. By Theorem 3.3 we obtain class $\mathcal{L}_\mathcal{F}$ which is a lift of $\mathrm{Forb_{he}}(\mathcal{F})$ with strong amalgamation. The class $\mathcal{K}_\mathcal{F}$ is then then a subclass of $\mathcal{L}_\mathcal{F}$ containing all maximal lifts of structures in $\mathcal{K}$.

Given $\mathbf{A}, \mathbf{B} \in \mathcal{K}_\mathcal{F}$ denote by $\overline{\mathbf{B}}$ a maximal lift of a witness of $\mathbf{B}$ (which is finite, because $\mathcal{F}$ is regular) and by Theorem 3.6 we obtain $\mathbf{C}'_0$ such that

$$\mathbf{C}'_0 \longrightarrow (\overline{\mathbf{B}})^{\mathbf{A}}_2.$$

By the application of Lemma 2.5 obtain an $\mathcal{U}$-closed ordered $\mathbf{C}_0$ such that

$$\mathbf{C}_0 \longrightarrow (\overline{\mathbf{B}})^{\mathbf{A}}_2$$

and moreover we have a homomorphism-embedding $\mathbf{C}_0 \to \mathbf{C}'_0$.

Now by the regularity of $\mathcal{F}$ there exists a finite $\mathcal{F}_0$ such that every structure $\mathbf{A} \in \mathrm{Forb_{he}}(\mathcal{F}_0)$ with a homomorphism-embedding to $\mathbf{C}'_0$ is also in $\mathrm{Forb_{he}}(\mathcal{F})$. Denote by n the size of the largest structure in $\mathcal{F}_0$ and construct $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ by the repeated application of Lemma 2.7 such that for every $1 \leq j \leq n$ the following holds:

1. $\mathbf{C}_j$ is an $\mathcal{U}$-closed L-structure,
2. $\mathbf{C}_j \in \mathrm{Forb_e}(\mathcal{E})$,
3. $\mathbf{C}_j \longrightarrow (\overline{\mathbf{B}})^{\mathbf{A}}_2$,
4. $\mathbf{C}_j$ has a homomorphism-embedding to $\mathbf{C}'_0$,
5. every substructure of $\mathbf{C}_j$ with at most j vertices has a completion in $\mathcal{L}_\mathcal{F}$.

We obtain $\mathcal{U}$-closed $\mathbf{C}_n$ where shadow of every substructure with at most n vertices has a completion in $\mathrm{Forb_{he}}(\mathcal{F})$. We conclude that the shadow of $\mathbf{C}_n$ is in $\mathrm{Forb_{he}}(\mathcal{F}_0)$ and because there is also a homomorphism-embedding from $\mathbf{C}_n$ to $\mathbf{C}'_0$ we know that the shadow of $\mathbf{C}_n$ is in $\mathrm{Forb_{he}}(\mathcal{F})$.

Let $\mathbf{C}$ be a maximal lift of the shadow of $\mathbf{C}_n$ with $R^{\leq}_{\mathbf{C}}$ completed to linear order. Because $\mathcal{F}$ is a family of weakly ordered structures we know that the shadow of $\mathbf{C}$ is in $\mathrm{Forb_{he}}(\mathcal{F})$. By the maximality of $\mathbf{B}$ in $\overline{\mathbf{B}}$ it follows that every copy of $\mathbf{B}$ which is maximal in a copy of $\overline{\mathbf{B}}$ in $\mathbf{C}_n$ is preserved in $\mathbf{C}$. It follows that

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

The lift property of $\mathcal{K}_\mathcal{F}$ follows from the maximality of lifts: given $\mathbf{A} \in \mathcal{K}_\mathcal{F}$ we construct $\mathbf{B}$ as the disjoint union of witnesses of all maximal lifts of $\mathbf{A}$ and apply the above proof. □

Remark. The second part of the proof (after the lift is constructed) is essentially the same as the proof of Theorem 2.2. It is however more convenient to give the proof by means of Lemma 2.5 and 2.7 because we do not need to go into a further analysis of the homogenising lift.

4 Examples of Ramsey classes

We believe Theorem 2.2 generalise most proofs of the Ramsey property of classes which are based on the Partite Construction. It is however often not obvious that a given class is a multiamalgamation class. We start by recalling some classical corollaries of the Nešetřil-Rödl Theorem (in Section 4.1) and then we show the Ramsey property of several classes (old and new) and thus illustrate multivariate use of Theorem 2.1 (in Section 4.2), Theorem 2.2 (in Section 4.3) and Theorem 3.7 (in Section 4.4).

Unless explicitely stated, all our examples of lifts are precompact and have the lift property.

4.1 Ramsey lifts of free amalgamation classes

By the Nešetřil-Rödl Theorem (Theorem 3.6) every hereditary class of ordered structures with amalgamation which is free in all relations except for $R^{\leq}$ defines a Ramsey class:

Definition 4.1. Let L be a language containing the order $R^{\leq}$. We say that a class $\mathcal{K}$ of ordered structures has *ordered free amalgamation* if for every $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ every ordered structure $\mathbf{C}$ created as a free amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with $R^{\leq}_{\mathbf{C}}$ completed arbitrarily to a linear order is in $\mathcal{K}$.

Corollary 4.1 (of Theorem 3.6)**.** *Let L be a language containing binary relation $R^{\leq}$ and $\mathcal{K}$ be an amalgamation class of ordered L-structures with ordered free amalgamation. Then $\mathcal{K}$ is a Ramsey class.*

Proof. Let $\mathcal{K}$ be an amalgamation class of ordered L-structures with ordered free amalgamation. Denote by $\mathcal{E}$ the class of ordered all L-structures $\mathbf{F}$ such that $\mathbf{F} \notin \mathcal{K}$ and every proper substructure of $\mathbf{F}$ is in $\mathcal{K}$ (i.e. the family of minimal obstacles). We claim that $\mathcal{E}$ is a family of ordered irreducible structures. This follows from the fact that every other structure can be constructed by means of an ordered free amalgamation. Consequently the Ramsey property follows by Theorem 3.6 □

As a warmup for many examples bellow let us give several special cases of Theorem 3.6 which also show two techniques to overcome its limitations by a suitable lift.

The easiest class we discuss is the class of directed graphs (digraphs) $\mathcal{D}$ and its lift $\overrightarrow{\mathcal{D}}$ adding a linear order on vertices. More precisely $\mathcal{D}$ consists of structures in the language containing single binary relation R^E with no restrictions. $\overrightarrow{\mathcal{D}}$ extends the language by binary relation $R^{\leq}$ and consists of all structures $\mathbf{A}$ in a way that the relation $R^{\leq}_{\mathbf{A}}$ represents a linear order on A. The class $\mathcal{G}$ of all (undirected) graphs may be viewed as a subclass of $\mathcal{D}$ of those structures $\mathbf{A}$ where $R^E_{\mathbf{A}}$ is symmetric and irreflexive. $\overrightarrow{\mathcal{G}}$ is a lift of $\mathcal{G}$ adding a free order on vertices. We immediately obtain:

Corollary 4.2. *The class $\overrightarrow{\mathcal{D}}$ of all finite directed graphs with free ordering of vertices is a Ramsey class. The class $\overrightarrow{\mathcal{G}}$ of all finite (simple) graphs with free ordering of vertices is a Ramsey class.*

Proof. Follows from Corollary 4.1 as both $\overrightarrow{\mathcal{D}}$ and $\overrightarrow{\mathcal{G}}$ are amalgamation classes with ordered free amalgamation. □

Remark. The same technique can be also used for the class of digraphs omitting a given set of tournaments (Henson graphs) or the class of graphs omitting K_n for a fixed n. Up to complementation this exhausts all ultrahomogeneous undirected graphs where the lift adding a free order on vertices forms a Ramsey class [50].

Equivalently we can say that $\overrightarrow{\mathcal{D}}$ is a Ramsey lift of class $\mathcal{D}$ and $\overrightarrow{\mathcal{G}}$ is a Ramsey lift of class $\mathcal{G}$.

It is a classical result that the lift $\overrightarrow{\mathcal{G}}$ has the lift property (Definition 3.2). Lift $\overrightarrow{\mathcal{D}}$ however does not: we can order directed graphs such that vertices with loops come before vertices without loops.

Consider class $\overrightarrow{\mathcal{D}}_0$ of all directed graphs ordered in a way that vertices with loops are before vertices without loops. It is easy to see that $\overrightarrow{\mathcal{D}}_0$ is also a Ramsey class. Given pairs of ordered directed graphs $\mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{D}}_0$

and an ordered directed graph $\mathbf{C}' \in \overrightarrow{\mathcal{D}}$ such that $\mathbf{C}' \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ (given by Corollary 4.2) we can construct $\mathbf{C} \in \overrightarrow{\mathcal{D}}_0$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ by reordering vertices of $\mathbf{C}'$ so all vertices with loops come first without breaking any of the embeddings of $\mathbf{B}$.

Clearly both $\overrightarrow{\mathcal{D}}$ and $\overrightarrow{\mathcal{D}}_0$ are Ramsey lifts of $\mathcal{D}$. One could claim that $\overrightarrow{\mathcal{D}}_0$ is better because there are fewer ways to lift a given directed graph and moreover it can be shown that $\mathcal{D}_0$ has the lift property with respect to $\mathcal{D}$.

We generalise this observation by the following concept of admissible ordering:

Definition 4.2. Let L be a language containing binary relation $R^{\leq}$. Denote by $\mathcal{O}_L$ the class of all isomorphism types of L-structures with one vertex and let $\leq_L$ denote a fixed linear order on $\mathcal{O}_L$. Given an ordered L-structure $\mathbf{A}$ we say that its order is $\leq_L$*-admissible* if for every pair of distinct vertices $u, v \in \mathbf{A}$ it holds that whenever $\mathbf{O}_u <_L \mathbf{O}_v$ then $(u, v) \in R^{\leq}_{\mathbf{A}}$. Here $\mathbf{O}_u$ and $\mathbf{O}_v$ are the structures in $\mathcal{O}_l$ isomorphic to structure inducted by $\mathbf{A}$ on $\{u\}$ and $\{v\}$ respectively.

The order $\leq_L$ will be usually understood from the context and thus we will just speak of an admissible order of the structure.

Proposition 4.3. *Let L be a language and $\mathcal{K}$, be a class of L-structures, and $\leq_L$ be a linear order of $\mathcal{O}_L$. If the lift $\overrightarrow{\mathcal{K}}$ of $\mathcal{K}$ adding a free order on vertices is a Ramsey class then the lift $\overrightarrow{\mathcal{K}}_0$ of $\mathcal{K}$ adding $\leq_L$-admissible order on vertices is also a Ramsey class.*

Proof. Let $\mathbf{A}, \mathbf{B}$ be structures in $\overrightarrow{\mathcal{K}}_0$ and $\mathbf{C} \in \overrightarrow{\mathcal{K}}$ such that $\mathbf{C}' \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ The $\leq_L$-admissibly ordered structure $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ is constructed by re-ordering the vertices of $\mathbf{C}'$ without breaking any of the desired embeddings of $\mathbf{B}$ (which is always possible). □

The phenomenon of admissible orderings is observed already in [50] and [41] in the context of bipartite graphs adding an unary relation R^L which denote one of the two bipartitions (here L comes from "left"). This representation of bipartite graphs forms a free amalgamation class and thus the lift adding a free order on vertices is Ramsey (by Corollary 4.1). Again this lift does not have the lift property which can be obtained by means of Proposition 4.3. Here the admissible ordering can be chosen in a way that all vertices in the unary relation R^L are before the remaining vertices. Such order, which respects the bipartition, is also sometimes called a *convex ordering* [41]. These observations can be further generalised.

Corollary 4.4. *Let* $\mathbf{H}$ *be a finite ordered structure. Denote by* $\mathrm{CSP}(\mathbf{H})$ *the class of all structures with a homomorphism to* $\mathbf{H}$*. Then the class all finite ordered structures in* $\mathrm{CSP}(\mathbf{H})$ *has a Ramsey lift adding only* $|H|$ *unary relations.*

Proof. Given $\mathbf{H}$ we lift the language by unary relations R^v, $v \in H$. For a finite ordered structure $\mathbf{A} \in \mathrm{CSP}(\mathbf{H})$ we construct lift $\mathbf{A}^+$ by choosing a homomorphism $c : \mathbf{A} \to \mathbf{H}$ arbitrarily and putting $(v) \in R^i_{\mathbf{A}}$ if and only if $c(v) = i$. (Our lifts explicitely fix the homomorphism to $\mathbf{H}$.) It is easy to see that the lifted class is a free amalgamation class and the Ramsey property follows by Corollary 4.1. □

It is easy to check that the described lift has the lift property with respect admissible orderings whenever there is no proper homomorphism $\mathbf{H} \to \mathbf{H}$ (i.e. $\mathbf{H}$ is a core [32]).

In special cases it is possible, for a given $\mathcal{F}$, construct a finite structure $\mathbf{H}$ such that $\mathrm{Forb_h}(\mathcal{F}) = \mathrm{CSP}(\mathbf{H})$. In such situation $\mathbf{H}$ is called the *homomorphism dual* of $\mathcal{F}$. All homomorphism dualities have been characterised in [64] and [21], see also [44]. In the context of universal structures, this can be further generalised to the notion of *monadic lifts* (i.e. homogenising lifts which add only finitely many unary relations). Classes $\mathrm{Forb_h}(\mathcal{F})$ with monadic lift are discussed in [38]: even if there is no homomorphism dual every monadic homogenising lift (see Section 3.5) is an amalgamation class of ordered structures with amalgamation which is free in all relations except for $R^{\leq}$ and thus Theorem 3.6 can still be applied.

Corollary 4.5. *Let* $\mathcal{F}$ *be a regular family of finite connected weakly ordered structures such that all minimal separating cuts consists of one vertex. Then there exists a Ramsey lift of* $\mathrm{Forb_h}(\mathcal{F})$ *adding only finitely many unary relations.*

Proof. This follows as a combination of Corollary 4.4 with [38] (as indicated above). □

Analogous proofs also give the corollaries for homomorphism-embeddings:

Corollary 4.6. *Let* $\mathbf{H}$ *be a finite ordered structure. Denote by* $\mathrm{CSP_{he}}(\mathbf{H})$ *the class of all finite ordered structures with a homomorphism-embedding to* $\mathbf{H}$*. Then the class* $\mathrm{CSP_{he}}(\mathbf{H})$ *has a Ramsey lift adding only* $|H|$ *unary relations.*

Again it is easy to show that the lift fixing a homomorphism embedding to $\mathbf{H}$ leads to a free amalgamation class. We omit the details. The following corollary represents the special (and easy) case of Theorem 3.7:

Corollary 4.7. *Let $\mathcal{F}$ be a regular family of finite connected weakly ordered structures such that all minimal separating cuts consists of one vertex. Then there exists a Ramsey lift of* $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ *adding only finitely many unary relations.*

Proof. This follows as a combination of Corollary 4.1 with Theorem 3.3. $\square$

Structures with minimal separating cuts (see Definition 3.3) of size one generalise graph trees [64]: every such structure can be constructed from a graph tree by replacing edges by arbitrary ordered irreducible structures (or, in other words, every two-connected component of its Gaifman graph is a complete graph). We know by Theorem 3.4 the regularity (Definition 3.6) is a necessary condition for the the existence of ω-categorical universal structure in $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ and is trivially satisfied for every finite family.

It may seem that by considering monadic lifts we exhausted all possible applications of Theorem 3.6. There is another case: [60] gives an example of an application of Theorem 3.6 which use order to give a Ramsey lift of the class of acyclic graphs. Because cycles are not irreducible structures it is necessary to use other means to describe the acyclicity. Instead of forbidding directed cycles we (dually) use the fact that every acyclic graph has linear extension. Finite acyclic graphs with linear extensions form a class with ordered free amalgamation and we immediately obtain:

Corollary 4.8 ([60])**.** *The class $\overrightarrow{\mathcal{A}}$ of all finite acyclic graphs with linear extension is a Ramsey class.*

One can verify the lift property and show that every Ramsey lift of the class of acyclic graphs always fix a linear extension. This shows that this technically looking trick (of adding a linear extension) is, in fact, necessary. The infinite linear order may be seen as an infinite dual of the class of all acyclic graphs.

4.2 Ramsey classes with strong amalgamation

In this section we focus on (more general) strong amalgamation classes that can be shown Ramsey by the application of Theorem 2.1. Recall that Theorem 2.1 states that the local finiteness is essentially the only condition which prevents us from showing the Ramsey property of every strong amalgamation class of ordered structures.

4.2.1 Partial orders with linear extension

We start with a classical example of a Ramsey class with non-trivial local finiteness. We interpret partial orders with linear extensions as structures in a language L_P containing two binary relations $R^{\leq}$ and $R^{\preceq}$ where $R^{\preceq}$ is a partial order and $R^{\leq}$ its linear extension.

In general to show that given class is locally finite subclass of a known Ramsey class, it is necessary to understand the minimal obstacles in the structures with no strong completion to the given class. It is easy to see that every L_P-structure $\mathbf{C}$ with completion to an ordered structure can be completed to a partial order with linear extension if and only if:

1. $\mathbf{C}$ contains no substructure with at most two vertices with no strong completion to a partial order with linear extension (for example, a substructure where $R^{\preceq}$ is not asymmetric, or not reflexive)

2. $\mathbf{C}$ contains no quasi-cycle as not necessarily induced substructure.

Here a *quasi-cycle* is a structure on vertices $u_1, u_2, \ldots, u_n$ such that

1. $(u_1, u_n) \in R^{\leq}_{\mathbf{A}}$ and $(u_1, u_n) \notin R^{\preceq}_{\mathbf{A}}$.

2. $(u_i, u_{i+1}) \in R^{\leq}_{\mathbf{A}}$ and $(u_i, u_{i+1}) \in R^{\preceq}_{\mathbf{A}}$, for every $1 \leq i < n$.

We use this fact to show the Ramsey property by the application of Theorem 2.1:

Theorem 4.9 ([60, 71]). *The class $\overrightarrow{\mathcal{P}}$ of all finite partial orders with linear extension is Ramsey.*

Proof. By Corollary 4.8 we know that the class of all $\overrightarrow{\mathcal{A}}_{\mathcal{P}}$ of acyclic graphs (in our language with binary relations $R^{\leq}$ and $R^{\preceq}$) is Ramsey. The class of all partial orders is a hereditary subclass of $\overrightarrow{\mathcal{A}}_{\mathcal{P}}$ with strong amalgamation (defined as the transitive closure of the free amalgamation).

We verify that $\overrightarrow{\mathcal{P}}$ is a locally finite subclass of $\overrightarrow{\mathcal{A}}_{\mathcal{P}}$: For acyclic graph $\mathbf{C}_0 \in \overrightarrow{\mathcal{A}}_{\mathcal{P}}$ put $n(\mathbf{C}_0) = |C_0|$. Let $\mathbf{C}$ be arbitrary structure with a homomorphism-embedding to $\mathbf{C}_0$. If $\mathbf{C}$ has no completion to partial order with linear extension we know that it contains a quasi-cycle $\mathbf{F}$ as a not necessarily induced substructure. Now because $\mathbf{C}_0$ is acyclic and because there is a homomorphism-embedding $\mathbf{F} \to \mathbf{C}_0$ we know that $|F|$ is no greater than the length of longest monotonous path in $\mathbf{C}_0$ and thus $|F| \leq n(\mathbf{C}_0)$.

By the application of Theorem 2.1 we get that $\overrightarrow{\mathcal{P}}$ is the Ramsey class. $\square$

Remark. By the same argument one can also observe that $\overrightarrow{\mathcal{P}}$ is a locally finite subclass of $\overrightarrow{\mathcal{G}}$ and thus it is not necessary to use Corollary 4.8. We can proceed directly from Theorem 3.6.

Remark. Note that for the local finiteness it is critical to use the linear extension. The class of all finite partial orders with free linear order is not Ramsey and moreover it is possible to verify the lift property of $\overrightarrow{\mathcal{P}}$ and show that the linear extension is actually necessary.

4.2.2 S-metric spaces with no jumps

In this section we strengthen results of [53] giving the Ramsey property the class of ordered finite rational metric spaces with free ordering of vertices, and, [20] giving the Ramsey property of the class of finite ordered graphs with free ordering of vertices and with respect to metric embeddings. Using the results of Sauer [72] we characterise, in a surprisingly simple way, (in Theorems 4.29 and Corollary 4.30) Ramsey classes of ordered metric spaces which are defined by a set S of possible distances.

We start by recalling the basic properties of S-metric spaces. It appears that it is useful to consider two main types of a distance set S: without jumps (Defined in Definition 4.6 and treated in this section) and with jumps (treated in Section 4.3.3 by means of closures).

Definition 4.3. Given $S \subseteq \mathbb{R}_{>0}$ (that is, the set of positive reals) an S-*metric space* $\mathbf{A}$ is a pair $(A, d_{\mathbf{A}})$ where A is the vertex set and d is a binary function $d_{\mathbf{A}} : A^2 \to S \cup \{0\}$ (the *distance function*) such that:

1. $d_{\mathbf{A}}(u, v) = 0$ if and only if $u = v$,
2. $d_{\mathbf{A}}(u, v) = d_{\mathbf{A}}(v, u)$, and,
3. $d_{\mathbf{A}}(u, w) \leq d_{\mathbf{A}}(u, v) + d_{\mathbf{A}}(v, w)$ (*the triangle inequality*).

We interpret an S-metric space as a relational structure $\mathbf{A}$ in the language L_S with (possibly infinitely many) binary relations R^s, $s \in S$, where we put, for every $u \neq v \in A$, $(u, v) \in R^l_{\mathbf{A}}$ if and only if $d(u, v) = l$. We do not explicitly represent that $d(u, u) = 0$ (i.e. no loops are added).

Definition 4.4. Every L_S-structure where all relations are symmetric and irreflexive and every pair of vertices is in at most one relation is S-*graph* which we define as a graph with edges coloured by S. Every non-induced substructure of an S-metric space is S-*metric graph* (S-metric graphs are structures with have a strong completion to S-metric space in the sense of Definition 2.3). Every S-graph that is not S-metric graph is *non-S-metric graph.*

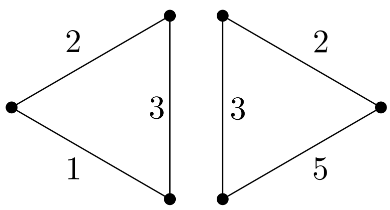


Figure 12: $\{1, 2, 3, 5\}$-metric spaces do not have the amalgamation property.

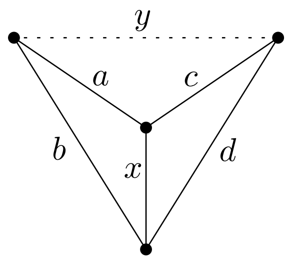


Figure 13: The 4-values condition.

Denote by $\mathcal{M}_S$ the class of all finite S-metric spaces. The universal ultrahomogeneous metric space was constructed by Urysohn [77] (by a Fraïssé type construction thus anticipating Fraïssé by more than 20 years). Generalising this result, when $\mathcal{M}_S$ forms an amalgamation class, we call this Fraïssé limit an *Urysohn S-metric space*. Not every choice of S leads an amalgamation class $\mathcal{M}_S$ (see Figure 12). The related concept is the following restriction on set S:

Definition 4.5 ([19]). A subset $S \subseteq \mathbb{R}_{>0}$ satisfies *4-values condition*, if for every $a, b, c, d \in S$ if there is some $x \in S$ such that triangles with distances a–b–x and c–d–x satisfies the triangle inequality there is also $y \in S$ such that that triangles with distances a–c–y and b–d–y satisfies the triangle inequality.

The 4-values condition describes the strong amalgamation of two 3-point metric spaces over a common 2-point subspace, see Figure 13. It follows that this is a sufficient and necessary condition for the amalgamation property of $\mathcal{M}_S$. Because sets S may be uncountable (and Theorem 3.1 can not be directly applied), the existence of an Urysohn S-metric space require S to be closed and have 0 as a limit:

Theorem 4.10 ([72]). *Let $S \subseteq \mathbb{R}_{>0}$ be a set with 0 as a limit of $S \cup \{0\}$. Then there exists an Urysohn S-metric space if and only if $S \cup \{0\}$ is a closed subset of $\mathbb{R}$ satisfying the 4-values condition.*

Let $S \subseteq \mathbb{R}_{>0}$ which does not have 0 as a limit of $S \cup \{0\}$. Then there exists an Urysohn S-metric space if and only if S is a countable subset of $\mathbb{R}$ satisfying the 4-value condition.

Any two Urysohn metric spaces having the same set of distance S are isometric.

We state the observations about the strong amalgamation as follows (and we include an easy proof):

Corollary 4.11 ([72]). *Let $S \subseteq \mathbb{R}_{>0}$ be a subset of positive reals. S satisfies the 4-values condition if and only if $\mathcal{M}_S$ has the strong amalgamation.*

Proof. We show that for every S satisfying the 4-values condition the class $\mathcal{M}_S$ has strong amalgamation. Let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{M}_S$ such that identity is an embedding of $\mathbf{A}$ to $\mathbf{B}_1$ and $\mathbf{B}_2$. We will construct the strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$. Because $\mathcal{M}_S$ is hereditary without loss of generality we can assume that $B_1 \setminus A = \{u\}$ and $B_2 \setminus A = \{v\}$. Let $w \in A$ be a vertex such that $a = d_{\mathbf{B}_1}(u, w) + d_{\mathbf{B}_2}(w, v)$ is minimised and let $w' \in A$ be a vertex such that $b = |d_{\mathbf{B}_1}(u, w') - d_{\mathbf{B}_2}(w', v)|$ is maximised. By the triangle-inequality a is the upper bound on the distance of u and v while b is the lower bound. If $w \neq w'$ we have the distance of u and v by 4-values condition. In case $w = w'$ we put the distance as $\max\{d_{\mathbf{B}_1}(u, w), d_{\mathbf{B}_2}(w, v)\} \in S$.

The opposite implication follows analogously. □

The 4-values condition can be expressed in the following neat algebraic way due to Sauer [73]. For $a, b \in S$ denote by $a \oplus_S b = \sup\{x \in S; x \leq a + b\}$. The algebraic characterisation of sets with the 4-values condition allows us to easily complete S-metric graphs to S-metric spaces.

Theorem 4.12. *A subset S of the positive reals satisfies the 4-values condition if and only if the operation $\oplus_S$ is associative.*

Let $\mathbf{G}$ be a S-metric graph and $\vec{w} = (w_1, w_2, \ldots, w_n)$ a set of vertices forming a walk in $\mathbf{G}$ (that is, for every $1 \leq i < n$ we have $w_i \neq w_{i+1}$ and the distance is defined in $\mathbf{G}$; i.e. $(w_i, w_{i+1}) \in R^l_{\mathbf{G}}$ for some $l \in S$). The *S-length* of walk $\vec{w}$ is $d_{\mathbf{G}}(w_1, w_2) \oplus_S d_{\mathbf{G}}(w_2, w_3) \oplus_S \ldots \oplus_S d_{\mathbf{G}}(w_{n-1}, w_n)$.

The following corollary summarise the results of [72] and [73] which are important for our construction. For the completeness we include a proof.

Corollary 4.13. *Let $S \subseteq \mathbb{R}_{>0}$ be a subset of positive reals that satisfies the 4-values condition.*

1. *Let $\mathbf{G}$ be a finite S-metric graph. Denote by $d'(u, v)$ the minimal S-length of a walk from u to v and by $\overrightarrow{W}(u, v)$ the corresponding walk. Then $\mathbf{G}$ can be completed to an S-metric space $\mathbf{A}$ by putting for every pair $u \neq v \in G$, $d_{\mathbf{A}}(u, v) = d'(u, v)$.*

2. *An S-graph $\mathbf{G}$ is a S-metric graph if and only if all of its cycles are S-metric.*

Proof. Both statements can be seen as an easy consequences of the associativity of $\oplus_S$:

1. First assume that $\mathbf{G}$ is S-metric. We show that the completion described will give S-metric space. We verify that d' satisfies the triangle inequality. Assume, to the contrary, the existence of vertices u, v, w such that $d'(u, v) > d'(u, w) + d'(w, v)$. Combine the walks $\overrightarrow{W}(u, w)$ and $\overrightarrow{W}(w, v)$ to the walk $\vec{p}$. By the associativity of $\oplus_S$ the S-length of $\vec{p}$ is $d'(u, w) \oplus_S d'(w, v) \geq d'(u, v)$. It follows that $d'(u, v) \leq d'(u, w) \oplus_S d'(w, v) \leq d'(u, w) + d'(w, v)$ which is a contradiction.

We have shown that d' forms an S-metric space on vertices of $\mathbf{G}$. We still need to check that $d_{\mathbf{G}}(u, v) = d'(u, v)$ whenever $d_{\mathbf{G}}(u, v)$ is defined. We show a stronger claim: let $\mathbf{B}$ be a completion of $\mathbf{G}$ to an S-metric space then $d_{\mathbf{B}}(u, v) \leq d'(u, v)$ for every $u \neq v \in G$.

We proceed by the induction on n which is the length of definition S-walk. For $n = 3$ this follows from the triangle inequality. For $n > 3$ denote by $(p_1, p_2, \ldots, p_{n-1}, p_n)$ the walk $\overrightarrow{W}(u, v)$. By the induction hypothesis we know that $d_{\mathbf{B}}(u, p_{n-1}) \leq d'(u, p_{n-1})$. The inequality then follows from the associativity of $\oplus_S$ and the triangle inequality. This finishes the proof of statement 1.

2. Assume that $\mathbf{G}$ is non-S-metric. In this case we have a pair of vertices u and v with the distance defined such that $d'(u, v) < d_{\mathbf{A}}(u, v)$. Because $\oplus_S$ is monotonous, it is easy to see that the walk $\overrightarrow{W}(u, v)$ can be turned into a path. A non-metric cycle is induced on vertices of this path. This is a contradiction. $\square$

In Section 2.1 we shown that $\{1, 3\}$-metric space is not locally finite in the class of all $\{1, 3\}$-graphs. This is an important example. It indicates that "large gaps" in the distance set S have to be treated with care. In the rest of this section we consider only those sets S where such scenario does not happen and $a \oplus_S b > \max(a, b)$. (Sets with graphs will be treated in Section 4.3.3.) Such sets are characterised by the absence of jump numbers:

Definition 4.6. Given $S \subseteq \mathbb{R}_{>0}$ and $a \in S$ we say that a is *jump number* if a is not maximal element of S and $2a \leq \min_{b \in S} (b > a)$.

We obtain a locally finite characterisation of $\mathcal{M}_S$ by means of the following structural observation about jumps in S.

Lemma 4.14 ([72]). *Let $S \subseteq \mathbb{R}_{>0}$ be a finite set satisfying the 4-values condition and has no jump numbers. Then for every $a \in S$, $a \neq \max(S)$, there is $b \in S$ such that $a < b \leq a + \min(S)$.*

Lemma 4.14 gives the desired bound on size of obstacles in S-graphs (i.e. the local finiteness):

Lemma 4.15. *Let $S \subseteq \mathbb{R}_{>0}$ be a finite set satisfying the 4-values condition and has no jump numbers. Then there every non-S-metric cycle has at most $|S| + 1$ vertices.*

Proof. Assume, to the contrary, that there is a non-S-metric cycle $\mathbf{C}$ with $n > |S| + 1$ vertices. By Corollary 4.13 we know that $\mathbf{C}$ contains a pair of vertices whose distance is longer than the S-length of a path connecting them. We can thus order vertices of S as $v_1, v_2, \ldots, v_n$, such that $d_{\mathbf{C}}(v_1, v_n) > v_1 \oplus_S v_2 \oplus_S \ldots \oplus_S v_n$. Denote by

$$l_j = v_1 \oplus_S v_2 \oplus_S \ldots \oplus_S v_j$$

the S-length of the walk formed by the initial segment on j vertices. By Lemma 4.14 we know that $l_j < l_{j+1}$ for every $1 < j \leq n$. We thus obtain a sequence of $n-1$ different values in $|S|$. A contradiction with $n > |S|+1$. □

Corollary 4.16. *Let $S \subseteq \mathbb{R}_{>0}$ be a finite set satisfying the 4-values condition and has no jump numbers. Then the class of all finite S-metric spaces with free, i.e. arbitrary, ordering of vertices, $\overrightarrow{\mathcal{M}}_S$, is a Ramsey class.*

Proof. By Corollary 4.13 the weakly ordered structure has a strong completion in $\overrightarrow{\mathcal{M}}_S$ if and only if all its cycles are S-metric. By Lemma 4.15 the set of non-S-metric cycles is finite and and the statement follow by Theorem 2.1. □

Corollary 4.17. *Let S be a subset of the positive reals satisfying the 4-values condition and has no jump number. Then the class of all finite S-metric spaces with free ordering of vertices, $\overrightarrow{\mathcal{M}}_S$, is a Ramsey class.*

Proof. Fix $\mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{M}}_S$. Denote by S_0 the set of distances used in $\mathbf{B}$ (which is finite, because $\mathbf{B}$ is finite). Extend S_0 to a finite set $S_0' \subset S$, that has no jump number. Define S_1 to consist of all values l, $l \leq \max(S_0)$, that can be obtained sequences of values of S_0' summed by $\oplus_S$ operation. By Theorem 4.12 S satisfies the 4-value condition.

Having finite set S_1 satisfying the 4-values condition without jump numbers, and, both $\mathbf{A}$ and $\mathbf{B}$ are S_1-metric spaces we apply Corollary 4.16. □

Note that each of these corollaries implies [53] and [20]. Before extending our construction to sets S with jump numbers (which we will give in Section 4.3.3) we first discuss in a greater detail why Theorem 2.1 can not be applied directly in this case.

4.2.3 Ramsey classes with a locally finite interpretation

One of key elements of proof of Theorems 2.1 and 2.2 is the Iterated Partite Construction (Lemma 2.8) where the local finiteness condition yields a finite bound on the number of iterations. How to achieve local finiteness? In many cases it follows form the class of structures considered. There is also a more systematic way by means of a suitable interpretation. Examples of it we have seen already above (in Section 4.1) where we interpreted a given class as either CSP or $\mathrm{Forb}_{\mathrm{he}}$ class. This was related to addition of unary relations (i.e. monadic lifts). Here we generalise it to higher arities by means of the following standard (model-theoretic) way to treat such examples using the notion of the elimination of imaginaries [33, 74].

Let $\mathbf{A}$ be a relational structure. An *equivalence formula* is a first order formula $\phi(\vec{x}, \vec{y})$ that is symmetric and transitive on the set of all n-tuples $\vec{a}$ of vertices of $\mathbf{A}$ where $\phi(\vec{a}, \vec{a})$ holds (set of such n-tuples is called the *domain* of equivalence formula ϕ). An *imaginary* element $\vec{a}/\phi$ of $\mathbf{A}$ is an equivalence formula ϕ together with a representative $\vec{a}$ of some equivalence class of ϕ. So these are tuples ϕ-equivalent to $\vec{a}$.

Structure $\mathbf{A}$ *eliminate imaginary* $\vec{a}/\phi$ if there is first order formula $\Phi(\vec{x}, \vec{y})$ such that there is unique tuple $\vec{b}$ such that the equivalence class of $\vec{a}$ consists of all tuples $\vec{x}$, $\Phi(\vec{x}, \vec{b})$. $\vec{b}$ is thus a representative of the equivalence class $\vec{a}/\phi$.

Example. In the Urysohn {1,3}-metric space $\mathbb{U}_{\{1,3\}}$ there is an equivalence formula $\phi(x, y)$ which is satisfied for pair of vertices if and only if their distance is one. The imaginary element a/ϕ is then the set of all vertices of distance one from a. There is no way to eliminate these imaginaries. Observe that the same formula ϕ is not an equivalence formula in the Urysohn {1,2,3}-metric space.

For a given ordered structure $\mathbf{U}$ we say that ϕ is an *equivalence formula on copies of* $\mathbf{A}$ if and only if ϕ is an equivalence formula and moreover $\phi(\vec{a}, \vec{a})$ holds if and only if the structure induced by $\mathbf{U}$ on $\vec{a}$ is isomorphic to $\mathbf{A}$ and moreover order of vertices in $\vec{a}$ agrees with order $R^{\leq}_{\mathbf{U}}$.

Proposition 4.18. *Let $\mathcal{K}$ be a hereditary Ramsey class of ordered structures, $\mathbf{U}$ its Fraïssé limit, $\mathbf{A}$ be a finite substructure of $\mathbf{U}$ and ϕ an equivalence formula on copies of $\mathbf{A}$. Then ϕ has either one or infinitely many equivalence classes.*

Proof. Assume to the contrary that ϕ is an equivalence formula on copies of $\mathbf{A}$ which define k equivalence classes, $k > 1$. It is well known that from ultrahomogeneity we can assume that ϕ is quantifier free. Consequently there is a finite substructure $\mathbf{B} \in \mathcal{K}$ that contain a structure induced by two such copies of $\mathbf{A}$ that belongs to different equivalence classes of ϕ. We then claim that there is no $C \in \mathcal{K}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$ because copies of $\mathbf{A}$ in $\mathbf{C}$ can be coloured according to the equivalence classes of $\mathcal{K}$. $\square$

Remark. Recall that tuples $\vec{a}$ and $\vec{b}$ have the same *strong type* if $\phi(\vec{a}, \vec{b})$ holds for every equivalence formula ϕ with finitely many equivalence classes. By the above observation it follows that the automorphism group of the Fraïssé limit of a Ramsey class must also fix strong types (such automorphisms are considered, for example, in [39]).

For a given equivalence formula ϕ with finitely many equivalence classes it is possible to lift the language by explicit relations representing the individual equivalence classes. This will be demonstrated on two examples in this section. More generally, when given subclass $\mathcal{K}$ defines more equivalences than the class $\mathcal{R}$ it is contained in, Theorem 2.1 can not be applied directly. This holds also for the cases of equivalence formulas with infinitely many equivalence classes. In such cases we will add an artificial elements, closure relations and apply Theorem 2.2 as shown in Section 4.3.

Our first example is a simple class with perhaps surprising Ramsey lift. Consider structures with single quaternary relation $R^{\mathbb{E}}$. We say that a structure $\mathbf{A}$ is a *fat bipartite graph* if there exists a bipartite graph $G = (V, E)$ with vertex set formed by all 2-element subsets of A and:

$$(a, b, c, d) \in R^{\mathbb{E}}_{\mathbf{A}} \text{ if and only if } a \neq b, c \neq d, \text{ and } \{\{a, b\}, \{c, d\}\} \in E.$$

(Thus each $(a, b, c, d) \in R^{\mathbb{E}}$ has symmetries defined by partitions $\{a, b\}$ and $\{c, d\}$.) One can see that class of all fat bipartite graphs is not a locally finite subclass of the class of all finite structures with single quaternary relation $R^{\mathbb{E}}$. It is easy to see that an ω-categorical universal fat bipartite graph $\mathbf{U}_{\mathcal{FB}}$ can be constructed by assigning pairs to bipartitions at random and producing random bipartite graph spanning these partitions. There is also an equivalence defined on unordered pairs of vertices of $\mathbf{U}_{\mathcal{FB}}$ defined as follows: $\{u, v\} \sim \{u', v'\}$ if they are connected by a fat path of length two. By Proposition 4.18 we know that every Ramsey lift will thus have a binary relation denoting the bipartition. Consequently we can introduce binary relation $R^{\mathbb{L}}$ (denoting the class of bipartition) explicitely into our lifted language which yields the following:

Theorem 4.19. *The class $\mathcal{FB}$ of all finite fat bipartite graph has the following precompact Ramsey lift $\mathcal{FB}^+$ with the lift property:*

The language $L^+_{\mathcal{FB}}$ is extended by two binary relations $R^{\leq}$ and $R^{\mathbb{L}}$. The order $R^{\leq}$ is arbitrary. Relation $R^{\mathbb{L}}$ denote one of the two bipartitions.

Proof. The class $\mathcal{FB}^+$ is locally finite subclass of all finite ordered $L^+_{\mathcal{FB}}$-structures: if structure $\mathbf{A}$ has no $\mathcal{FB}^+$-completion then it either contains a tuple with duplicated vertices in $R^{\mathbb{L}}$ or $R^{\mathbb{E}}_{\mathbf{A}}$ or a tuple $(a,b,c,d) \in R^{\mathbb{E}}_{\mathbf{A}}$ such that either $(a,b) \notin R^{\mathbb{L}}_{\mathbf{A}}$ or $(c,d) \in R^{\mathbb{L}}_{\mathbf{A}}$. □

In fact Theorem 2.1 is not necessary here the Ramsey property also follows by application of Theorem 3.6. What is interesting about this lift? If one considers the shadow of $\mathbf{U}_{\mathcal{FB}}$ in the language containing only the relation $R^{\mathbb{L}}$ it will form the Rado graph. This shows that the precompact lifts with the lift property may give rise to rich structures and not only to orders and unary relations (as in most cases mentioned so far). It is easy to generalise this example further (giving fat analogies to Corollary 4.4, by forbidding a homomorphism from graph in the language R^L, or by introducing fat linear order as in Theorem 4.32).

As our second example, consider structures with a single ternary relation $R^{\mathbf{E}}$. We say that structure $\mathbf{A}$ is a *neighbourhood bipartite graph* if for every vertex $v \in R^{\mathbf{E}}$ the digraph $\mathbf{G}_v$ is bipartite graph. Here the graph $\mathbf{G}_v$ is defined on the vertex set $\mathbf{A} \setminus \{v\}$ where $(a,b) \in R^{E}_{\mathbf{G}_v}$ if and only if $(v,a,b) \in R^{\mathbf{E}}_{\mathbf{A}}$.

The class of all neighbourhood bipartite graphs is not a locally finite subclass of the class of all relational structures with single ternary relation $R^{\mathbf{E}}$. There is a definable equivalence on 2-tuples of vertices of the generic neighbourhood graph: $(u,v) \sim (u,v')$ if v and v' are connected by a path of length two in $\mathbf{G}_u$. This time however the number of equivalence classes is not finite and we can not apply Proposition 4.18 directly.

Let $\mathbf{U}_{\mathcal{NB}}$ be the ω-categorical universal neighbourhood graph. It is easy to observe that every Ramsey lift of $\mathbf{U}_{\mathcal{NB}}$ remains Ramsey even if one of vertices is denoted by a special unary relation (i.e. the automorphism group of $\mathbf{U}_{\mathcal{NB}}$ is forced to fix the vertex) [2]. Then the equivalences become definable in a sense of Proposition 4.18. Consequently every Ramsey lift of $\mathbf{U}_{\mathcal{NB}}$ must already represent this equivalence class (in the model-theoretic setting this correspond to the elimination of equivalences a parameter). We can eliminate these equivalences by means of a binary relation:

Theorem 4.20. *The class $\mathcal{NB}$ of all finite neighbourhood bipartite graph has the following precompact Ramsey lift $\overrightarrow{\mathcal{NB}}$ with the lift property:*

The language is extended by two binary relations $R^{\leq}$ and $R^{\mathbb{L}}$. The order $R^{\leq}$ is free. Relation $R^{\mathbb{L}}$ has the property that for every vertex v the set of *vertices connected to v by $R^{\mathbb{L}}$ denote one of bipartitions of the graph $\mathbf{G}_v$.*

This time the shadow of the Ramsey lift of the generic neighbourhood bipartite graph produce the generic digraph. We further develop the neighbourhood structures in Section 4.3.4. For this we however need to deal with the notion of closures.

4.3 Ramsey classes with closures

Essential part of more complex Ramsey classes is handling structures with equivalences defined on vertices (and even tuples of vertices). Such an equivalence may be present latently (as for example in S-metric spaces with jump numbers [72] or bowtie-free graphs [35] which we shall handle in Section 4.3.3 and 4.4.2). It is an important fact that such equivalence may have unboundedly many equivalence classes and thus one cannot assign labels to them and use Theorem 2.1.

In fact the equivalences with unboundedly many classes have to be interpreted so they can be viewed as relational structures over a finite language. This is a place where closure operations may be used effectively. We can proceed as follows:

Let $\sim$ be an equivalence on set X. To every equivalence class E of $\sim$ we assign a vertex v_E of E and a mapping $L_E : E \to E$ which maps every vertex of E to v_E: $L_E(v) = v_E$ for every $v \in E$. What we obtain is the relational structure $\mathbf{A}(\sim)$ in the language $L_{\mathcal{P}E}$ consisting from a binary relational symbol R^U and unary symbol R^S. The class of all structures $\mathbf{A}(\sim)$ is denoted by $\mathcal{PE}$. Explicitly, class $\mathcal{PE}$ contains all finite $L_{\mathcal{PE}}$-structures $\mathbf{A}$ where for every pair $(u, v) \in R^U_{\mathbf{A}}$ it holds that $(u) \notin R^S_{\mathbf{A}}$ and $(v) \in R^S_{\mathbf{A}}$ and moreover every non-special vertex u is in precisely one pair $(u, v) \in R^U_{\mathbf{A}}$.

$\mathcal{PE}$ stands for pointed equivalences: in every equivalence class we selected a special vertex (thus obtaining a "pointed set"). Clearly embeddings of pointed equivalences $\mathbf{A}(\sim_1)$ to $\mathbf{A}(\sim_2)$ corresponds to embeddings of $\sim_1$ into $\sim_2$ (as relation) with the additional property that special vertices are mapped to special vertices. Thus we have an interpretation of the class of equivalences and their embeddings. Combining this with Theorem 2.2 we obtain:

Theorem 4.21. *The lift $\overrightarrow{\mathcal{PE}}$ of $\mathcal{PE}$ which adds a free order to vertices is a Ramsey class.*

Proof. In the setting of Theorem 2.2 $\overrightarrow{\mathcal{PE}}$ forms a multiamalgamation class (Definition 2.8 where $\mathcal{R}$ is the class of all ordered structures in the language of $\overrightarrow{\mathcal{PE}}$ and the closure description $\mathcal{U}$ contains a single pair $(\mathbf{R}, R^U)$ where $\mathbf{R}$ is a structure containing single non-special vertex.

The Ramsey property of $\overrightarrow{\mathcal{PE}}$ then follows by Theorem 2.2. $\square$

To obtain the lift with the lift property it is necessary to order vertices in a convex way where every equivalence class forms an interval and in each of the interval the special vertex must be the first one.

Theorem 4.21 is top of an iceberg and it is important that we can generalise it in combination with other structures. For example we have the following result whose proof is a similar consequence of Theorem 2.2 as in the above proof of Theorem 4.21.

Theorem 4.22. *Let L be a finite relational language. Let L^+ denote the language $L \cup L_{\mathcal{PE}}$. Let $\mathcal{PE}\,\mathrm{Rel}(L)$ be the class of all L-structures together with a pointed equivalence. The embeddings of these structures preserve embeddings of both $\mathrm{Rel}(L)$ and $\mathcal{PE}$. Then the class $\mathcal{PE}\,\mathrm{Rel}(L)$ has a Ramsey lift adding a free order to vertices.*

We do not know an easy proof of this result even for the simplest case of one binary relation. It seems that the closure description (and thus Theorem 2.2) is capturing the complexity of Theorem 4.22. However equivalences (interpreted as unary functions) presents an interesting interplay with other structures and this is the contents of the next section. This turns our attention to unary and m-ary functions.

4.3.1 Unary functions (only) are easy

We first consider unary functions (of which Theorem 4.21 is a particular example). Despite the seeming complexity (as exemplified by [75]) the basic result is deceptively easy and can be formulated as follows.

Consider a structure $\mathbf{A}$ with (unary) function symbols, $\mathbf{A} = (A, f^1_{\mathbf{A}}, f^2_{\mathbf{A}}, \ldots, f^m_{\mathbf{A}})$, where each f_i is a function $A \to A$. Such structures represents the most natural example of a class with a closure. For example, given a structure $\mathbf{B} = (\{u, v\}, f^1_{\mathbf{B}})$ where $f^1_{\mathbf{B}}(u) = v$ and $f^1_{\mathbf{B}}(v) = v$, the closure of vertex v in $\mathbf{B}$ is $\mathbf{B}$: there is no structure induced by $\mathbf{B}$ on $\{u\}$ because the function $f^1_{\mathbf{B}}$ wold become partial.

Denote by $\mathcal{F}^m_1$ the the class of all finite structures with m unary functions. The ordered structure with unary functions adds binary relation $R^{\leq}_{\mathbf{A}}$ representing the linear order of vertices as usual. The class of all finite ordered structures with m unary functions will be denoted by $\overrightarrow{\mathcal{F}}^m_1$.

Given vertex v of structure $\mathbf{A}$, its *vertex closure* is the smallest substructure of $\mathbf{A}$ containing v.

The Ramsey property of class $\overrightarrow{\mathcal{F}}^m_1$ follows by a simple direct argument:

Theorem 4.23. *Let* $\mathbf{A}$ *be a finite ordered (closed) structure with* m *unary functions and* $\mathbf{B}$ *a finite or countably infinite ordered structure with* m *unary functions, then there exists an ordered structure with* m *unary functions* $\mathbf{C}$ *such that* $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Moreover if $\mathbf{B}$ *is finite, then* $\mathbf{C}$ *is finite, too. If all vertex closures of vertices of* $\mathbf{B}$ *are finite, then* $\mathbf{C}$ *is countable.*

Proof. Fix closed ordered structures with m unary functions $\mathbf{A}$ and $\mathbf{B}$. Without loss of generality assume that $B = \{1, 2, \ldots, b\}$ or $B = \mathbb{N}$ and is ordered naturally by $R^{\leq}_{\mathbf{B}}$. Obtain $N \longrightarrow (b)^{|A|}_2$ by the Ramsey Theorem. Consider lifted a language adding unary relations R^i for every $1 \leq i \leq N$.

Now construct a structure $\mathbf{P}$ as follows: For each b-tuple $\vec{v} = (v_1, v_2, \ldots v_b)$ of elements of $\{1, 2, \ldots, N\}$ such that $v_1 < v_2 < \ldots < v_b$ add a disjoint copy $\mathbf{B}_{\vec{v}}$ of $\mathbf{B}$ to $\mathbf{P}$ and for every n, $1 \leq n \leq b$, put n-th vertex v_n of $\mathbf{B}_{\vec{v}}$ into $R^{v_n}_{\mathbf{P}}$. Order vertices of $\mathbf{P}$ in a way that for $1 \leq i < j \leq N$ every vertex $v \in R^i_{\mathbf{P}}$ is before every vertex $v' \in R^j_{\mathbf{P}}$. (Note that this is essentially the Picture zero of the Partite Construction cf. Section 2.4.)

Construct structure by identifying every pair of vertices of $\mathbf{P}$ with isomorphic vertex closures (both by the unary functions the unary relations). Finally remove the unary relations and call the resulting structure $\mathbf{C}$. ($\mathbf{C}$ is an ordered structure with m unary functions.) There is a homomorphism from $\mathrm{Sh}(\mathbf{P})$ to $\mathbf{C}$ that is an embedding on every $\mathrm{Sh}(\mathbf{B}_{\vec{v}})$.

It is easy to verify that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_2$. Colouring of copies of $\mathbf{A}$ in $\mathbf{C}$ gives a colouring of $|A|$-tuples of $\{1, 2, \ldots N\}$ (note that there is only one copy of $\mathbf{A}$ for every $|A|$-tuple of elements of $\{1, 2, \ldots, N\}$) and the Ramsey Theorem gives a monochromatic b-tuple witch corresponds to a copy of $\mathbf{B}$ in $\mathbf{P}$ and thus also to a copy of $\mathbf{B}$ in $\mathbf{C}$. $\square$

As a consequence we obtain the Ramsey property of $\mathcal{F}^m_1$.

Corollary 4.24 ([75]). $\overrightarrow{\mathcal{F}}^m_1$ *is a Ramsey lift of* $\mathcal{F}^m_1$

Remark. Note that $\overrightarrow{\mathcal{F}}^m_1$ does not have lift property with respect to $\mathcal{F}^m_1$. If one interprets structures in $\mathcal{F}^1_1$ as oriented graphs (with edges are pointing from v to $f^1(v)$), then these graphs form a forest of "graph trees" oriented towards a root where the rood may be an oriented cycle. To obtain the lift property the order needs to be convex with respect to the individual connected components, it needs to order the cycles of a given size in a unique way and the vertices of trees needs to be ordered convexly level wise with children of a vertex forming a linear interval. See [75] for details. The lift property becomes even more involved for classes $\mathcal{F}^m_1$, $m > 1$. Precise description of this order will appear in [23].

Remark. The unary functions can be seen as a generalisation of structures with unary relations: every unary relation R can be represented by an unary function f and two artificial vertices 0, 1 by putting $f(v) = 1$ if $(v) \in R$ and $f(v) = 0$ otherwise. This gives an intuition why the Ramsey property of classes with unary functions follows by a simple argument and why this argument can not be easily generalised to non-unary functions. Still the proof of Theorem 4.23 can be seen as a basic case of the Partite Construction where the Partite Lemma is replaced by identification of all copies of $\mathbf{A}$ with a given projection to one.

In a way structures with unary functions are a misleading example. The easy proof of Theorem 4.23 should be contrasted with situation of function symbols with higher arities where we need our main theorem. That is subject of the next section.

4.3.2 The general case of n-ary functions and finite models

We now turn our attention structures with general functions and relations and to finite models. For brevity we now restrict our attention to structures with one function symbol assigning every n-tuple of vertices without repeated entries a single vertex. The general case of structures with multiple function symbols from n-tuples to k-tuples follows analogously.

Consider structures with one function symbol f of arity n. Denote by $\mathcal{F}^1_n$ the class of all finite structures $\mathbf{A} = (A, f_\mathbf{A})$ where $f_\mathbf{A}$ is an n-ary function from n-tuples of vertices $\mathbf{A}$ without repeated vertices to A.

Proposition 4.25. *The class $\overrightarrow{\mathcal{F}}^1_n$ of freely ordered structures in $\mathcal{F}^1_n$ is Ramsey for every $n \geq 1$.*

Proof. We interpret structures $\mathbf{A} \in \mathcal{F}^1_n$ as relational structures in the language $L_{\mathcal{F}^1_n}$ with $(n+1)$-ary relation R^U, where $(a_1, a_2, \dots, a_n, b) \in R^U_\mathbf{A}$ if and only if $f_\mathbf{A}(a_1, a_2, \dots a_n) = b$. Denote by $\mathcal{F}'_n$ the class of all such interpretations of finite structures in $\mathcal{F}^1_n$. To show that the class $\overrightarrow{\mathcal{F}}'_n$ of all ordered structures in $\mathcal{F}^1_n$ is a Ramsey class we appl Theorem 3.7. The closure description $\mathcal{U}_{\mathcal{F}^1_n}$ contains single pair $(R^U, \mathbf{R})$ where $\mathbf{R}$ is a structure on n vertices and no tuples in any relation. By the application of Theorem 3.6 we obtain Ramsey class $\mathcal{R}_{\mathcal{F}^1_n}$ of structures in language $L_{\mathcal{F}^1_n}$ with no further constrains. It is easy to see that $\overrightarrow{\mathcal{F}}'_n$ is $(\mathcal{R}_B, \mathcal{U}_B)$-multiamalgamation class. For $n = 1$ the completion property follows trivially, because every vertex is $\mathbf{C}$ is contained in a copy of $\mathbf{B}$ and thus properly closed. For $n \geq 1$ it is necessary to complete the structure in an arbitrary way. $\square$

Remark. Every structure with an n-ary relation R, $n > 1$, which contains only n-tuples without repeated vertices can be interpreted as a structure with an n-ary function f in $\mathcal{F}^1_n$: put $f(v_1, v_2, \dots, v_n) = v_1$ if $(v_1, v_2, \dots, v_n) \in R$ and $f(v_1, v_2, \dots, v_n) = v_n$ if $(v_1, v_2, \dots, v_n) \notin R$. By means of this re-interpretation one can also show that class $\overrightarrow{\mathcal{F}}^1_n$ has the lift property for $n > 1$. If one however consider functions from n-tuples with repeated vertices, the order of such class needs to extend the order described for unary functions: n-ary function from tuples with repeated vertices can be seen as a family of functions from tuples without repeated vertices of arities $1, 2, \dots, n$.

The above results generalise to finite models (that is structures with both relational and function symbols) leading Theorem 4.26. Let us invoke it in the following general result. Recall a formal definition of a structure involving both relations and functions (compare with the Section 2.1). Let $L = L_R \cup L_F$ be a language involving relations $R \in L_R$ and function symbols $f \in L_F$ (each coming with corresponding arities denoted by $a(R)$ and $d(f), r(f)$). A finite *model* $\mathbf{A}$ is a structure with functions $f_{\mathbf{A}} : A^{d(f)} \to A^{r(f)}$, $f \in L_F$ and relations $R \subseteq A^{a(R)}$, $R \in L_R$. Embeddings, homomorphisms (and homomorphism-embeddings) can be defined in the expected manner. The class of all finite models in the language L is denoted by $\mathrm{Mod}(L)$.

For L containing a binary relation $R^{\leq}$ we denote by $\overrightarrow{\mathrm{Mod}}(L)$ the class of all finite models $\mathbf{A} \in \mathrm{Mod}(L)$ where the set A is linearly ordered by the relation $R^{\leq}$. We have the following result:

Theorem 4.26 (Ramsey Theorem for Finite Models)**.** *For every language L involving both relations and functions and containing a binary relation $R^{\leq}$ the class $\overrightarrow{\mathrm{Mod}}(L)$ is a Ramsey class.*

Proof. Put explicitly $L = \{R; R \in L_R\} \cup \{f; f \in L_F\}$. We know how to handle $\{R; R \in L_R\}$ by the Nešetřil-Rödl Theorem 3.6 and we know how to handle L_f by the above proof. However we have to consider the degree condition of each closure relation interpreting function symbol f. Above (in Proposition 4.25), this was done under assumption that f acts on tuples of distinct vertices only. In general we can proceed as follows:

If f is an $(d(f), r(f))$-ary function symbol then we consider f as coded by functions $f^\pi : A^\pi \to A^{r(f)}$ where π is an equivalence on $\{1, 2, \dots, d(f)\}$ and $f^\pi(v_1, v_2, \dots .v_{d(f)})$ is defined only on those tuples $(v_1, v_2, \dots .v_{d(f)})$ satisfying $v_n = v_m$ if and only if $\pi(n, m)$. In other words we split the domain $A^{d(f)}$ into blocks with prescribed repetition of coordinates. In this extended language of f^π we can then proceed analogously as in the proof of Proposition 4.25. All the assumptions of Theorem 2.2 are obviously satisfied and thus we get the result. $\square$

Remark. The proof of the Ramsey Theorem for Finite Models involves most of the techniques introduced in this paper. Of course it can be generalised further (for example, to forbidden homomorphisms and to total orderings) but we decided to formulate it in this concise form. We believe it nicely complements results for relational structures (Abramson-Harrington [1], Nešetřil-Rödl [67]).

Remark. A Ramsey theorem for structures involving both relations and functions is given in [76]. The notion of functions used in [76] is however different from the standard model-theoretic sense and can be interpreted by a combination of relations and unary closures. The Ramsey property proved in [76] then follows by Theorem 2.2.

4.3.3 S-metric spaces

We are now ready to further develop results of Section 4.2.2 and complete the classification of Ramsey properties of general S-metric spaces (i.e. even for sets S containing jump numbers). This generalise results of [68] where Ramsey property of S-metric spaces was shown for all sets S containing at most 3 distances and confirms the hypothesis stated there about every S-metric space, S finite, having a precompact Ramsey lift. Our analysis is based on (and refines) [72] which gives a family of definable equivalences on the S-metric spaces where S contains jump numbers:

Definition 4.7. [72] Let $S \subseteq \mathbb{R}_{>0}$ be a subset satisfying the 4-values condition. A block B of S is any inclusion maximal subset of S satisfying the 4-values condition that has no jump number.

It is shown in [72] that for every S satisfying the 4-values condition can be decomposed to mutually disjoint blocks and that for every block $B \subseteq S$ value of $\max(B)$ is either a jump number of S or $\max(S)$. More precisely, this decomposition decomposition is based on the following notion:

Definition 4.8. [72] Let $\mathbf{A}$ be an S-metric space and B block of S. We define a *block equivalence* $\sim_B$ on vertices of $\mathbf{A}$ by putting $u \sim_B v$ whenever $d(u, v) \leq \max(B)$.

It is easy to see that paths where every pair of vertices has distance at most $\max(B)$ can only be completed by distances at most $\max(B)$ and thus $\sim_B$ is indeed an equivalence relation. By Proposition 4.18 it is thus necessary to consider lift of $\mathcal{M}_S$ which represent these classes explicitly.

The following definition and technical lemma is the key to obtaining a locally finite description of $\mathcal{M}_S$ (needed for Theorem 2.2):

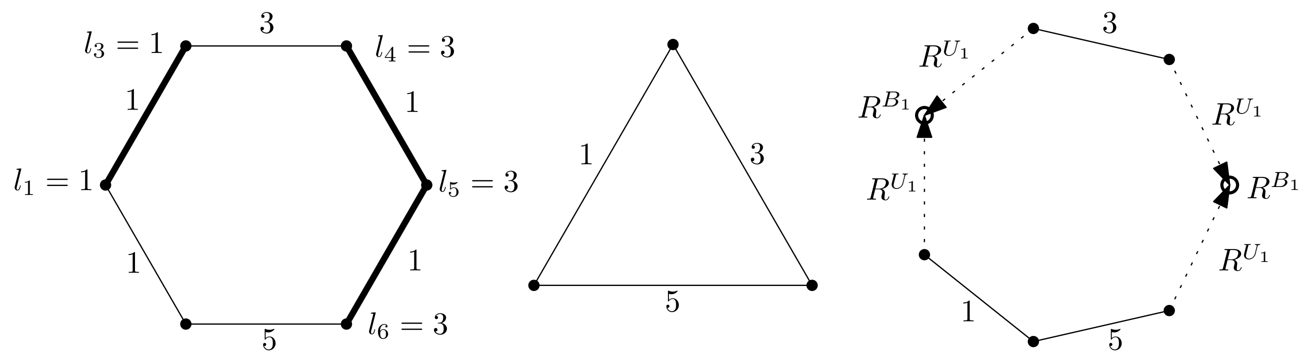


Figure 14: The family of unimportant paths in a non-$\{1, 3, 5\}$-metric cycle (left), the non-$\{1, 3, 5\}$-metric cycle created after the concatenation of unimportant paths (middle) and the corresponding forbidden substructure in $\mathcal{F}_{\{1,3,5\}}$ (right).

Definition 4.9. Let $\mathbf{P}$ be a S-metric path. Denote by $B(\mathbf{P})$ the block of S containing the maximal distance of an edge in $\mathbf{P}$. Let $\mathbf{P}'$ be any S-metric path. We say that $\mathbf{P}' \preceq_S \mathbf{P}$ if all the distances in $\mathbf{P}'$ are at most $\max(B(\mathbf{P}))$.

Lemma 4.27. *Let $S \subseteq \mathbb{R}_{>0}$ be a finite set satisfying the 4-values condition and let $\mathbf{C}$ be a non-S-metric cycle. Then there exists paths $\mathbf{P}^i$, $1 \leq i \leq p$, of vertices of $\mathbf{C}$ such that the cycle created by identifying all vertices of each path into a single vertex is non-S-metric cycle with at most $2|S|$ vertices, and moreover every cycle created from $\mathbf{C}$ by replacing each of the paths $\mathbf{P}^i$ by arbitrary path $\mathbf{P}'^i$, $\mathbf{P}'^i \preceq_S \mathbf{P}^i$ is non-S-metric.*

We will call paths in the family $\mathbf{P}^i$ *unimportant*. An example is given in Figure 14.

Proof. Let $\mathbf{C}$ be a non-S-metric cycle with $n > |S| + 1$ vertices. By Corollary 4.13 we know that $\mathbf{C}$ contains a pair of vertices whose distance is longer than the S-length of path connecting them. We can thus order vertices of S as $v_1, v_2, \ldots, v_n$, such that $d_{\mathbf{C}}(v_1, v_n) > v_1 \oplus_S v_2 \oplus_S \ldots \oplus_S v_n$. Denote by

$$l_j = v_1 \oplus_S v_2 \oplus_S \ldots \oplus_S v_j$$

the S-length of the walk formed by the initial segment on j vertices. We know that $l_j \leq l_{j+1}$ for every $1 < j \leq n$.

A path induced by $\mathbf{C}$ on vertices $(v_j, v_{j+1}, \ldots v_k)$ is unimportant if $l_j = l_k$. The paths $\mathbf{P}^i$, $1 \leq i \leq p$, will consist of all inclusion maximal unimportant paths. As a special case, if is only one inclusion maximal unimportant path on vertices $v_3, v_4, \ldots v_n$, put $\mathbf{P}^1$ to be the path on vertices $v_3, v_4, \ldots, v_{n-1}$ so the result of identification is a triangle.

Because there are only finitely many values in S, we know that there are are at most $|S|$ choices of j such that $l_j < l_{j+1}$. We thus know that there are at most $|S|$ pairs v_j, v_{j+1} which are not part of an unimportant path. It follows that the graph created by concatenating all unimportant paths has at most $2|S|$ vertices (and, because of the special case, at least three vertices). It is also easy to verify that the resulting graph is non-S-metric cycle because the S-length of walk connecting v_1 and v_n was not affected by such replacements: if path $\mathbf{P}^i$ is unimportant then every path $\mathbf{P}'^i$, $\mathbf{P}'^i \preceq_S \mathbf{P}^i$ is also unimportant. □

The following equivalences can be defined on S-metric spaces for S containing jump numbers.

Definition 4.10 ([68]). Given S satisfying the 4-value condition, denote by J_S the set of all jump numbers and for every $j \in J_S$ denote by B_j the block of S containing j. We say that an order $R^{\leq}_{\mathbf{A}}$ of ordered S-metric space $\mathbf{A}$ is *convex with respect to block equivalences* if every equivalence class of every $\sim_{B_j}$, $j \in J_S$, is an interval of $R^{\leq}_{\mathbf{A}}$. An ordered S-metric space which order is convex with respect to block equivalences is also called *convexly ordered S-metric space.*

The following result is a direct analogy of Corollary 4.16 permitting jump numbers. We represent equivalence classes by closure vertices.

Lemma 4.28. *Let $S \subseteq \mathbb{R}_{>0}$ be a finite set satisfying the 4-values condition and has finitely many blocks. Then the class $\overrightarrow{\mathcal{M}}_S$ of all convexly ordered S-metric spaces is a Ramsey class.*

Proof. We lift the language L_S to L^+_S by adding the order $R^{\leq}$, unary relations R^{E_j} and binary relations R^{U_j}, $j \in J_S$. For a given ordered metric space $\mathbf{A} \in \overrightarrow{\mathcal{M}}_S$ where the order is convex with respect to block equivalences denote by $L(\mathbf{A})$ the lift of $\mathbf{A}$ created by the following procedure:

1. For every $j \in J_S$ enumerate equivalence classes of $\sim_{B_j}$ in $\mathbf{A}$ as $E^1_j, E^2_j, \ldots, E^{n_j}_j$. Moreover order them in a way so for $v \in E^i_j$, $v' \in E^{i'}_j$ we have $v <_{\mathbf{A}} v'$ whenever $1 \leq i < i' \leq n_j$.

2. For every $j \in J_S$ and $1 \leq j \leq n_j$ add a new vertex v^i_j and tuple (v^i_j) to $R^{E_j}_{L(\mathbf{A})}$.

3. For every $j \in J_S$, $1 \leq i \leq n_j$ and $v \in E^i_j$ add tuple (v, v^i_j) to $R^{U_j}_{L(\mathbf{A})}$.

The linear order $R^{\leq}_{L(\mathbf{A})}$ extends the linear order of $\mathbf{A}$ by putting vertices v^i_j last in a way that $v^i_j <_{L(\mathbf{A})} v^{i'}_{j'}$ whenever $j < j'$ or $j = j'$ and $i < i'$.

The vertices in relations $R^{E_i}_{L(\mathbf{A})}$ represent individual equivalence classes and every vertex of $\mathbf{A}$ has an unique vertex in each of relations R^{E_j}, $j \in J_S$ connected to it by the closure tuple. We will call vertices of $\mathbf{A}$ the *original* vertices and vertices v^i_j the *closure* vertices.

Denote by $\overrightarrow{\mathcal{M}}^+_S$ the class of all structures $L(\mathbf{A})$, $A \in \overrightarrow{\mathcal{M}}_S$. Because for every ordered S-metric subspace $\mathbf{A}$ of convexly ordered S-metric space $\mathbf{B}$ we also know that $L(\mathbf{A})$ is a substructure of $L(\mathbf{B})$ the Ramsey property of $\overrightarrow{\mathcal{M}}^+_S$ implies the Ramsey property of $\overrightarrow{\mathcal{M}}_S$. We show that $\overrightarrow{\mathcal{M}}^+_S$ is an $(\mathcal{R}_S,\mathcal{U}_S)$-multiamalgamation class where:

1. The class $\mathcal{R}_S$ consists of all finite ordered L^+_S structures. This class is Ramsey by the application of Theorem 3.6.

2. The closure description $\mathcal{U}_S$ consists of pairs $(R^{U_j}, \mathbf{R})$, $j \in J_S$, where $\mathbf{R}$ is structure containing one vertex and no tuples.

It remains to verify the locally finite completion property (see Definition 2.8). We put $n = 4|S|$: Let $\mathbf{B} \in L(S)$ and $\mathbf{C}'$ be an $\mathcal{U}_S$-closed structure with a homomorphism-embedding to $\mathbf{C}_0 \in \mathcal{R}_S$ such that every n-element substructure of $\mathbf{C}'$ has a completion in $\overrightarrow{\mathcal{M}}^+_S$. Without loss of generality we may further assume that every vertex as well as every tuple in every relation of $\mathbf{C}'$ is contained in $\mathbf{B}$. We verify that $\mathbf{C}'$ has a strong completion in $\overrightarrow{\mathcal{M}}^+_S$. We proceed in three steps:

1. For every pair of original vertices $u, v \in \mathbf{C}'$ and jump number $j \in J_S$ such that there exists a walk from u to v where every distance is at most j there is c such that $(u, c), (v, c) \in R^{U_j}_{\mathbf{C}'}$ (that is, the u and v have the same $R^{U_j}_{\mathbf{C}'}$-closure):

   Let u' and w' be two neighbouring vertices in the walk. Because $(u', w') \in R^{d(u',w')}_{\mathbf{C}'}$ and every tuple of $\mathbf{C}'$ is part of a copy of $\mathbf{B}$ we know that there is a copy $\widetilde{B}$ in $\mathbf{C}'$ containing both u', w'. It follows that the unique vertex connected by $R^{U_j}_{\mathbf{C}'}$ to u' must be the same as the unique vertex connected by $R^{U_j}_{\mathbf{C}'}$ to v'. Consequently all vertices of the walk have the same $R^{U_j}_{\mathbf{C}'}$-closure.

2. Denote by $\mathbf{G}$ the S-graph induced by $\mathbf{C}'$ on the set of original vertices. We show that there exists an S-metric space $\mathbf{G}'$ which is a strong completion of $\mathbf{G}$ (i.e. $\mathbf{G}$ is S-metric) and moreover every pair u and v

of vertices of $\mathbf{C}'$ which have the same $R^{U_j}_{\mathbf{C}'}$-closure has distance in $\mathbf{G}'$ at most j.

To simplify the discussion we assume that every pair of vertices u and v with same $R^{U_j}_{\mathbf{C}'}$-closure is connected by a walk where every distance is at most j. If that is not the case the distance of u and v in $\mathbf{G}$ can be set to j.

To the contrary now assume that $\mathbf{G}$ is non-S-metric. By Corollary 4.13 there exists non S-metric cycle $\mathbf{K}$ in $\mathbf{G}$. Consider the family of unimportant paths in $\mathbf{K}$ given by Lemma 4.27. Let $\mathbf{P}$ be an unimportant path and j the smallest jump number of S such that all distances in $\mathbf{P}$ are at most j. Then we know that there exists closure vertex $c \in C'$ such that $(u, c) \in R^{U_j}_{\mathbf{C}'}$ for every $u \in P$. We call c the *common closure of the path* $\mathbf{P}$.

Create $\mathbf{K}'$ as the structure induced by $\mathbf{C}'$ on the set of all vertices of $\mathbf{K}$ which are not in unimportant paths, all initial and terminal vertices of unimportant path and the common closures of unimportant paths. By Lemma 4.27 there is no completion of $\mathbf{K}'$.

3. Finally we complete the order of $\mathbf{C}'$ in a way that $\mathbf{G}'$ is ordered convexly. This can be done by first fixing the order of vertices in $R^{U_j}_{\mathbf{C}'}$ for j being the largest jump number and then proceed by second largest keeping the convex order and finally completing the order of original vertices.

   It follows that $\mathbf{C} = L(\mathbf{G}')$ is the (strong) completion of $\mathbf{C}'$ in $\overrightarrow{\mathcal{M}}'_S$.

$\square$

Now we are ready to characterise S-metric Ramsey classes (extending Corollary 4.17):

Theorem 4.29 (Ramsey S-metric Spaces Theorem)**.** *Let S be a subset of positive reals. Then the following conditions are equivalent:*

1. *S satisfies the 4-values condition,*
2. *$\mathcal{M}_S$ has the strong amalgamation property,*
3. *$\mathcal{M}_S$ has the amalgamation property, and,*
4. *the class $\overrightarrow{\mathcal{M}}_S$ of all convexly ordered S-metric spaces is Ramsey.*

Proof. 1 $\iff$ 2 by Corollary 4.11. Clearly 2 $\implies$ 3. To see that 3 $\implies$ 2 consider S which fails to satisfy the 4-values condition for a, b, c, d and x. Assume to the contrary that $\mathcal{M}_S$ has the amalgamation property. It follows that the amalgamation of triangles with distances a–b–x and c–d–x over an edge of distance x must identify vertices. To make this possible, it follows that $a = c$ and $b = d$. It follows that 4-values condition is then trivially satisfied by putting $y = \min(a, b)$. A contradiction.

We show that 1 $\implies$ 4: Fix $\mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{M}}_S$. Denote by S_0 the set of distances used in $\mathbf{B}$ (which is finite, because $\mathbf{B}$ is finite). Extend S to a finite set $S'_0 \subset S$, such that $J_{S'_0} \subseteq J_S \cup \{\max(S_0)\}$. Define S_1 as set of all values l, $l \leq \max(S_0)$ that can be obtained sequences of values of S'_0 summed by $\oplus_S$ operation. By Theorem 4.12 S satisfies the 4-value condition.

Having finite set S_1 satisfying the 4-values condition and both $\mathbf{A}$ and $\mathbf{B}$ are ordered S_1-metric spaces, we observe that because $J_{S_1} \subseteq J_S \cup \{\max(S_1)\}$ they are also convexly ordered S_1-metric spaces and apply Lemma 4.28.

Finally we show 4 $\implies$ 3. By Proposition 3.2 we know that $\overrightarrow{\mathcal{M}}_S$ forms an amalgamation class. Because the convex order is definable by quantifier-free first order formula the same amalgamation procedure can be applied to $\mathcal{M}_S$. $\square$

Applying Theorem 4.10 we can state the results compactly in terms of Fraïssé limits:

Corollary 4.30. *Let S be a set of positive reals such that there exists an Urysohn S-metric space. Then the class of all convexly ordered S-metric spaces is a Ramsey lift of the class of all S-metric spaces with the lift property.*

4.3.4 Ramsey classes with multiple linear orders

In this section we focus more on the special rôle of the order in our constructions.

Consider the class of structures with two linear orders $R^{\leq}$ and $R^{\preceq}$ (or, equivalently, the class of permutations: the order $R^{\leq}$ represent original order and order $R^{\preceq}$ the permutation). It is not obvious how to describe this class as a multiamalgamation class (technique of Section 4.2.1 would apply only for classes where $R^{\leq}$ agrees with $R^{\preceq}$). In the following proposition we show a way of effectively splitting the order $R^{\leq}$ into multiple linear orders free to each other. We proceed more generally.

Let $\mathcal{K}_1$, $\mathcal{K}_2$ be classes of finite structures in disjoint languages L_1 and L_2 respectively. Denote by L the language $L_1 \cup L_2$. The *free interposition of* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ is the class $\mathcal{K}$ containing all structures $\mathbf{A}$ such that the L_1-shadow of $\mathbf{A}$ is in $\mathcal{K}_1$ and the L_2-shadow of $\mathbf{A}$ is in $\mathcal{K}_2$.

Proposition 4.31. *Let L_1 and L_2 be disjoint languages containing order (the order of L_1 is standard order $R^{\leq}$ together with order of L_2 denoted by $R^{\leq_2}$). let $\mathcal{R}_1$ be the class of finite ordered L_1-structures, $\mathcal{R}_2$ be the class of finite ordered L_2-structures, and let $\mathcal{K}_1$ be an $(\mathcal{R}_1,\mathcal{U}_1)$-multiamalgamation class and $\mathcal{K}_2$ be an $(\mathcal{R}_2,\mathcal{U}_2)$-multiamalgamation class. Then the free interposition $\mathcal{K}$ of $\mathcal{K}_1$ and $\mathcal{K}_2$ is Ramsey.*

Proof. We further extend our language $L = L_1 \cup L_2$ to L^+ by two unary relations R^1 and R^2 and two binary closure relations R^{U_1} and R^{U_2}.

The basic idea of the proof is to split every structure of $\mathbf{A} \in \mathcal{K}$ into its L_1-shadow $\mathbf{A}_1$ and L_2-shadow $\mathbf{A}_2$. Then take a "disjoint union" of $\mathbf{A}_1$ and $\mathbf{A}_2$ in the language L^+ where vertices of $\mathbf{A}_1$ are coded by R^1 and vertices of $\mathbf{A}_2$ by R^2 and use the closure relations to define a bijection between vertices of $\mathbf{A}_1$ and vertices of $\mathbf{A}_2$. This construction preserves substructures and thus the Ramsey property of such split structures implies the Ramsey property of $\mathcal{K}$.

The class of such split structures is described as an $(\mathcal{R},\mathcal{U})$-multiamalgamation class as follows:

1. $\mathcal{R}$ consist all ordered L^+-structures. The Ramsey property of $\mathcal{R}$ follows by Theorem 3.6.

2. Closure description $\mathcal{U}$ consist of all pairs $(R^{U_i}, \mathbf{R}_i^+)$ such that $(R^{U_i}, \mathbf{R}_i) \in U_1$ where $\mathbf{R}_i^+$ is a lift of $\mathbf{R}_i$ adding every vertex to relation R^1 and pairs $(R^{U_j}, \mathbf{R}_j^+)$ such that $(R^{U_j}, \mathbf{R}_j) \in U_2$ where $\mathbf{R}_j^+$ is a lift of $\mathbf{R}_j$ adding every vertex to relation R^2.

   Moreover we extend the closure description to define a bijection between vertices in relation R^1 and vertices in relation R^2: every vertex in R^1 has a closure defined by R^{U_1} and every vertex in R^2 has a closure defined by R^{U_2}.

By combining the completion properties of $\mathcal{K}_1$ and $\mathcal{K}_2$ it easily follows that the class described is an $(\mathcal{R},\mathcal{U})$-multiamalgamation class and thus by Theorem 2.2 we get Proposition 4.31. □

Remark. A variant of Proposition 4.31 was proved in [2] for strong amalgamation classes. For the first time we however show that even free interposition of classes with a closure is Ramsey.

4.3.5 Totally ordered structures via incidence closure

Let $\mathbf{A}$ be a relational structure in finite language $L = \{R^i; i \in I\}$ with order of its vertices $\leq_{\mathbf{A}}$ (this order is not part of the language L). Here we show how to handle the structures with each relation viewed as an ordered set and embeddings of relational structures which moreover preserve these embeddings.

Assume that each of the sets $R^i_{\mathbf{A}}$, $i \in I$, is linearly ordered by $\leq^i_{\mathbf{A}}$. For the time being we call $\mathbf{A}$ together with orderings $\leq_{\mathbf{A}}$ and $\leq^i_{\mathbf{A}}$ (for each $i \in I$) *totally ordered structure* $\overrightarrow{\mathbf{A}}$. For two totally ordered structures $\overrightarrow{\mathbf{A}}$ and $\overrightarrow{\mathbf{B}}$ we define an embedding $f : \overrightarrow{\mathbf{A}} \to \overrightarrow{\mathbf{B}}$ as an embedding $f : \mathbf{A} \to \mathbf{B}$ which is also an embedding of all orders $\leq_{\mathbf{A}}$ and $\leq^i_{\mathbf{A}}$, $i \in I$. Explicitely $f : A \to B$ is an embedding provided:

1. For every $i \in I$ it holds that
$$(u_1, u_2, \ldots, u_{a(R^i)}) \in R^i_{\mathbf{A}}$$
if and only if
$$(f(u_1), f(u_2), \ldots, f(u_{a(R^i)})) \in R^i_{\mathbf{B}},$$
and
$$(u_1, u_2, \ldots, u_l) \leq^i_{\mathbf{A}} (v_1, v_2, \ldots, v_l)$$
if and only if
$$(f(u_1), f(u_2), \ldots, f(u_l)) \leq^i_{\mathbf{B}} (f(v_1), f(v_2), \ldots, f(v_l)),$$
2. $u \leq_{\mathbf{A}} v$ if and only if $f(u) \leq_{\mathbf{B}} f(v)$.

Totally ordered structures fail to be relational structures per se. However such structures may be interpreted easily as ordered relational structures and this interpretation paves the way to our approach:

For any relation $R^i_{\mathbf{A}}$ of arity $a(R^i)$ of a totally ordered structure $\overrightarrow{\mathbf{A}}$ we may consider a relation $R^{\leq_i}_{\mathbf{A}}$ of arity $2a(R^i)$ defined by:
$$(x_1, x_2, \ldots, x_l, y_1, y_2, \ldots, x_l) \in R^{\leq_i}_{\mathbf{A}}$$
if and only if
$$(x_1, x_2, \ldots, x_l) \leq^i_{\mathbf{A}} (y_1, y_2, \ldots, y_l).$$
The order $R^{\leq}_{\mathbf{A}}$ is the order $\leq_{\mathbf{A}}$. The language of such interpretations is L together with relations $R^{\leq_i}_{\mathbf{A}}$ for every $i \in I$. Call this relational structure $\mathrm{TO}(\overrightarrow{\mathbf{A}})$. Observe that $f : \overrightarrow{\mathbf{A}} \to \overrightarrow{\mathbf{B}}$ is an embedding $\overrightarrow{\mathbf{A}} \to \overrightarrow{\mathbf{B}}$ if and only if $f : \mathrm{TO}(\overrightarrow{\mathbf{A}}) \to \mathrm{TO}(\overrightarrow{\mathbf{B}})$ is an embedding.

We can denote the extended language by $L, 2L$. Denote by $\mathrm{TO}(L)$ the class of all structures $\mathrm{TO}(\mathbf{A})$ in the language $L, 2L$. We claim the following:

Theorem 4.32 (Ramsey Theorem for Totally Ordered Structures)**.** *Let L be a finite language, then* $\mathrm{TO}(L)$ *is a Ramsey class of relational structures.*

Before the proof of Theorem 4.32 let us add the following remark.

Remark. It may seem on the first glance that the natural way to prove this is to show that $\mathrm{TO}(L)$ is a locally finite subclass of the class of all finite ordered L,2L-structures. This is however not the case. We proceed differently using a particular closure operation.

Fix arbitrary order $\leq_L$ of L. Given a structure $\mathbf{A} \in \mathrm{TO}(L)$ we denote the following lift $\mathbf{A}^+$ which we call *incidence closure*:

1. The vertex set of $\mathbf{A}^+$ extends the vertex set of $\mathbf{A}$ by a new vertex for every tuple in every relation. More precisely:
$$\mathbf{A}^+ = \mathbf{A} \cup \bigcup_{i \in I} \{i\} \times R^i_{\mathbf{A}}.$$
2. The order $\leq^+_{\mathbf{A}}$ extends the order of $\leq_{\mathbf{A}}$ as follows:
   (a) For every $i \in I$ we put
   $$(i, u_1, u_2, \ldots, u_{a(R^i)}) \leq_{\mathbf{A}^+} (i, v_1, v_2, \ldots, v_{a(R^i)})$$
   if and only if
   $$(u_1, u_2, \ldots, u_{a(R^i)}) \leq^i_{\mathbf{A}} (v_1, v_2, \ldots, v_{a(R^i)}).$$
   (b) For every $u \in A$ and $v \notin A$ we put $u \leq_{\mathbf{A}^+} v$.
   (c) For every $i, j \in I$ such that $i <_L j$ and every
   $$(i, u_1, u_2, \ldots, u_{a(R^i)}), (j, v_1, v_2, \ldots, v_{a(R^j)}) \in A^+$$
   we put
   $$(i, u_1, u_2, \ldots, u_{a(R^i)}) \leq_{\mathbf{A}^+} (j, v_1, v_2, \ldots, v_{a(R^j)}).$$
3. For every $i \in I$ we add a relation $R^{U_i}_{\mathbf{A}^+}$ of arity $a(R^i)$ and we put
$$(u_1, u_2, \ldots, u_{a(R^i)}, (i, u_1, u_2, \ldots, u_{a(R^i)})) \in R^{U_i}_{\mathbf{A}^+}$$
if and only if
$$(u_1, u_2, \ldots, u_{a(R^i)}) \in R^i_{\mathbf{A}}.$$

4. In addition we have L unary relations $R^{M_i}_{\mathbf{A}^+}$, $i \in I$ and we put

$$((i, u_1, u_2, \ldots, u_{a(R^i)})) \in R^{M_i}_{\mathbf{A}^+}$$

for every $(i, u_1, u_2, \ldots, u_{a(R^i)}) \in A^+$.

Proof. Denote class of such lifts (with the incidence closure) by $\mathrm{TO}^+(L)$. One sees that the relations R^{U_i} forms a closure. Now after this reformulation we get that $\mathrm{TO}^+(L)$ is a Ramsey lift by a routine application of Theorem 2.2. $\square$

This example may be summarised by saying that relational structures with all relations ordered have a Ramsey lift. We do not know of a simple direct proof of this fact. Like Theorem 4.26 it involves essential part of the Theorem 2.2.

Remark. The incidence closure can be used to give an order on n-tuples in general. For example, the following class giving linear order to the neighbourhood of every vertex can be shown to be Ramsey by essentially the same argument:

Denote by $\mathcal{QQ}$ the class of finite structures $\mathbf{A}$ with one binary relation $R^{\leq}_{\mathbf{A}}$ and one ternary relation $R^{\preceq}_{\mathbf{A}}$ with the following properties:

1. the relation $R^{\leq}_{\mathbf{A}}$ forms a linear order on A, and,
2. for every vertex $a \in A$ the relation $\{(b, c) : (a, b, c) \in R^{\preceq}_{\mathbf{A}}\}$ forms a linear oder on $A \setminus \{a\}$ (that is independent to $R^{\leq}_{\mathbf{A}}$).

($\mathcal{QQ}$ may be viewed as the class of all structures endowed with local order on neighbourhoods.)

4.4 Ramsey lifts of ages of ω-categorical structures

We end this paper by considering particlar examples of homomorphism defined classes which in fact provided the original motivation for this paper. This section provides a rich spectrum of Ramsey classes defined by means of forbidden substructures. We start with a detailed description of the Ramsey lift of the class of finite graphs with a given odd girth and show how this particular (but key) example fits both Theorem 2.1 and Theorem 3.7. These results indicate that the case of forbidden homomorphism is well understood. We then (in Section 4.4.2) turn our attention to classes defined by forbidden monomorphisms (such as forbidden subgraphs). Here the situation is much more complicated even on the model-theoretic side (see e.g.[7]) and this is where (again) we have to use closures.

4.4.1 Graphs omitting odd cycles of length at most l

Perhaps the simplest example of graphs defined by means of forbidden homomorphism is the class of all (undirected) graphs $\mathbf{G}$ such that there is no homomorphism $\mathbf{C}_l \to \mathbf{G}$, where $\mathbf{C}_l$ is a graph cycle on l vertices. Of course we assume l odd. Equivalently, we will consider this to be the class of all graphs in the class $\mathrm{Forb_{he}}(\mathbf{C}_l)$.

By Proposition 3.2 we know that the Ramsey lift must have the amalgamation property. It is easy to see that the class of all graphs in $\mathrm{Forb_{he}}(\mathbf{C}_l)$ is not an amalgamation class for any odd $l \geq 5$ so a convenient lift is needed. We illustrate this by means of the smallest non-trivial example $l = 5$. There is no amalgamation of a path of length two and a path of length three over the endpoints. It follows that every non-trivial graph omitting 5-cycle has (at least) two types of pairs of independent vertices — vertices connected by a path of length two and vertices connected by a path of length three. The Ramsey lift thus needs to distinguish those pairs.

An explicite homogenising (and also Ramsey) lift can be described as follows: Fix an odd l. The language of graphs is extended to language L_l by a linear order $R^{\leq}$ and binary relations $R^2, R^3, \ldots, R^{(l-1)/2}$. Given finite graph $\mathbf{G} \in \mathrm{Forb_{he}}(\mathbf{C}_l)$ we define its lift $\mathbf{G}^+$ as follows:

1. $R^{\leq}_{\mathbf{G}^+}$ is (arbitrary) linear order of G.
2. $u, v \in R^i_{\mathbf{G}^+}$ if and only if the graph distance of u and v is i, for every $1 < i \leq \frac{l-1}{2}$. (The distance one is already represented by relation $R^E_{\mathbf{G}}$.)

We call this lift the *distance lift* of the graph $\mathbf{G}$.

The lifted class $\mathcal{K}_{\mathbf{C}_l}$ then consist of all possible substructures of all above lifts of finite graphs in $\mathrm{Forb_{he}}(\mathbf{C}_l)$.

Remark. The homogenisation of the class of all graphs in $\mathrm{Forb_{he}}(\mathbf{C}_l)$ was first given by Komjáth, Mekler and Pach [43] (corrected proof appears in [42]). This, in fact, presented an early example of universal graphs defined by forbidden homomorphisms (this result was generalised in [11]; see also [8, 14] for negative results). Alternative homogenisation (in the form of even-odd metric spaces) is given in [38]. Homogenisation presented here appears in the catalogue of metrically ultrahomogeneous graphs [5] and is the only one (up to bi-definability) leading to the homogenisation of existentially complete ω-categorical universal graph for the class of all graphs in $\mathrm{Forb_{he}}(\mathbf{C}_l)$.

Theorem 4.33. *The class $\mathcal{K}_{\mathbf{C}_l}$ is a Ramsey class. Every lift $\mathbf{A} \in \mathcal{K}_{\mathbf{C}_l}$ can be viewed as a metric space with distances truncated by $\frac{l+1}{2}$. More precisely* *the following function* $d_{\mathbf{A}}: A \times A \to \{0, 1, 2, \dots \frac{l+1}{2}\}$ *is metric:*

$$d_{\mathbf{A}}(u,v) = \begin{cases} 0 & \textit{if } u = v, \\ 1 & \textit{if } (u,v) \in R^E_{\mathbf{A}}, \\ d & \textit{if } (u,v) \in R^d_{\mathbf{A}}, 2 \leq d \leq \frac{d-1}{2}, \textit{and}, \\ \frac{(l+1)}{2} & \textit{otherwise.} \end{cases}$$

As a further illustration of our technique we give two proofs of Theorem 4.33.

Proof (by the application of Theorem 2.1). We give a strong amalgamation procedure for $\mathcal{K}_{\mathbf{C}_l}$. Let $\mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}_{\mathbf{C}_l}$. Without loss of generality we can assume that both $\mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}_{\mathbf{C}_l}$ are both distance lifts of graphs in $\mathrm{Forb}_{\mathrm{he}}(\mathbf{C}_l)$ and $\mathbf{A}$ is a structure induced by both $\mathbf{B}_1$ and $\mathbf{B}_2$ on $A = B_1 \cap B_2$. Construct a graph $\mathbf{G}$ as the free amalgamation of $\mathrm{Sh}(\mathbf{B}_1)$ and $\mathrm{Sh}(\mathbf{B}_2)$ over $\mathrm{Sh}(\mathbf{A})$. That is $G = B_1 \cup B_2$ and $(u,v) \in R^E_{\mathbf{G}}$ if and only if either $(u,v) \in R^E_{\mathbf{B}_1}$ or $(u,v) \in R^E_{\mathbf{B}_2}$. Denote by $\mathbf{C}$ the distance lift of $\mathbf{G}$. We claim that $\mathbf{C}$ is a strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$. Because for every pair of vertices $u, v \in A$ we have $l_{\mathbf{B}_1}(u,v) = l_{\mathbf{B}_2}(u,v) = d_{\mathbf{A}}(u,v)$ it is easy to see that identities are embeddings from $\mathbf{B}_1$ and $\mathbf{B}_2$ to $\mathbf{C}$. It remains to verify that $\mathbf{G}$ does not contain any odd cycles of length at most l.

Assume, to the contrary, that there exists a cycle $\widetilde{\mathbf{C}}_k$, $k \leq l$ odd, that is a subgraph of $\mathbf{G}$. Among all choices of $\widetilde{\mathbf{C}}_k$ pick one such that k is minimal. Because both $\mathbf{B}_1$ and $\mathbf{B}_2$ have no homomorphic images of $\mathbf{C}_l$ we know that $\widetilde{\mathbf{C}}_k$ contains some vertices of $B_1 \setminus A$ and some of $B_2 \setminus A$. Because $\mathbf{G}$ is a free amalgamation and $\widetilde{\mathbf{C}}_k$ connected, there are also some vertices in $A \cap \widetilde{C}_k$ which forms a vertex cut of $\widetilde{\mathbf{C}}_k$.

Now consider a path in $\widetilde{\mathbf{C}}_k$ on vertices $v_1, v_2, \dots, v_n$, such that $n \leq \frac{k-1}{2}$, $v_1, v_n \in A$ and $v_2, v_3, \dots, v_{n-1} \notin A$. Without loss of generality assume that the whole path is contained in $\mathbf{B}_1$. We show that $d_{\mathbf{C}}(v_1, v_n) = n$ in $\mathbf{A}$:

1. Clearly $l_{\mathbf{B}_1}(v_1, v_n) = l_{\mathbf{B}_2}(v_1, v_n) = d_{\mathbf{A}}(v_1, v_n) \leq n$.

2. Assume $d_{\mathbf{A}}(v_1, v_n) < n$. In this case create $\widetilde{\mathbf{C}}'$ from $\widetilde{\mathbf{C}}_k$ by replacing vertices $v_1, v_2, \dots, v_n$ of $\widetilde{C}_k$ with the path of length $d_{\mathbf{C}}(v_1, v_n)$ in $\mathbf{G}$. $\widetilde{\mathbf{C}}'$ is a homomorphic image of a cycle of length $k' = k - n + d(v_1, v_n)$ in $\mathbf{G}$. Because k is minimal we know that k' is even. It follows that $n + d(v_1, v_n)$ is odd and vertices v_1 and v_n are connected in $\mathbf{B}_1$ both by a path of length n and a path of length $d(v_1, v_n)$. Combining these paths together leads to a homomorphic image of odd cycle in $\mathbf{B}_1$ of length $d(v_1, v_n) + n \leq k$ a contradiction with $\mathbf{B}_1 \in \mathrm{Forb}_{\mathrm{he}}(\mathbf{C}_l)$.

It follows that for every two vertices $v_1, v_n \in \widetilde{C}_l \cap C$ such that the distance k within $\widetilde{\mathbf{C}}_k$ is most $\frac{k-1}{2}$ there is a path of length l in both $\mathbf{B}_1$ and $\mathbf{B}_2$.

Because there is no copy of $\widetilde{C}_k$ in $\mathbf{B}_1$ or $\mathbf{B}_2$ we conclude that there is a path $w_1, w_2, \dots, w_m$, such that $m > \frac{k-1}{2}$, $w_1, w_m \in A$, $w_2, w_3, \dots w_{m-1} \notin A$. Again without a loss of generality assume this path is in $\mathbf{B}_1$. Because there is only one such long path in $\widetilde{C}_k$ we obtain a homomorphic copy of $\widetilde{C}_k$ in $\mathbf{B}_1$, a contradiction with $\mathbf{B}_1 \in \mathrm{Forb}_{\mathrm{he}}(\mathbf{C}_l)$. This finishes the proof that $\mathbf{C}$ is the strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

To apply Theorem 2.1 we observe that $\mathcal{K}_{\mathbf{C}_l}$ is locally finite subclass of the class of all ordered structures in the language L_l. □

Now we show how the same lift can be shown Ramsey by the application of Theorem 3.7.

Proof (by the application of Theorem 3.7). Denote by $\mathcal{C}_l$ the family of all possible weak orderings of $\mathbf{C}_l$ along with a structure containing one vertex with loop and two structures containing two vertices and a directed edge (in both possible orderings). Those additional structures describe the class of all unoriented graphs.

It immediately follows that the class of all finite ordered structures in $\mathrm{Forb}_{\mathrm{he}}(\mathcal{C}_l)$ is the class of all ordered graphs with no homomorphic image of $\mathbf{C}_l$ and the existence of precompact Ramsey lift is given by Theorem 3.7. However this result claims more in that it derives a particular lift in the form of maximal $\mathcal{F}$-lifts as given by Definition 3.7. It remains to check that this homogenisation is equivalent to one described by statement of Theorem 3.7.

The pieces of $\mathbf{C}_l$ (see Definition 3.4) are all paths of lengths $2, 3, \dots, l-2$ rooted in the endpoints. The homomorphism-embeddings from a path of length k rooted in the endpoints then describe a walk of length k. The pieces of structures in $\mathcal{C}_l$ are weakly ordered paths, but because we consider all possible weak orders, we know that all weakly ordered paths of the same length are $\sim$-equivalent. In the following we can thus speak only of a pieces formed by paths of given length.

Because the construction of homogenising lift adds relations describing individual pieces and tuples in these relations describe roots of homomorphism-embeddings, at first glance it seems that the lift constructed is thus more expressive than one we ask for: we measure the distance of walks of length up to $l-2$ (instead of $\frac{l+1}{2}$) and in addition every pair of vertices (u, v) can be in many binary relations. Here we need to use the maximality (as defined in Definition 3.7).

To see this we proceed as follows. Given a pair of vertices (u, v) of a maximal lift $\mathbf{A}^+$ and its witness $\mathbf{W}$, we verify that the set of relations (i.e. the set of lengths of permitted walks between u and v in $\mathbf{A}$) is fully determined by the graph distance $l_{\mathbf{W}}(u,v)$ in $\mathbf{W}$ and that $l_{\mathbf{W}}(u,v) \leq \frac{(l+1)}{2}$:

1. If the distance $l_{\mathbf{W}}(u,v) = k$ is even, the existence of walks of all even distances greater than k follows trivially; there is always a homomorphism from the path of length $k+2$ to the path of length k mapping an endpoint to an endpoint. By the maximality there are also all odd walks of distances greater or equal to $l-k+2$. If such walk would be missing, it would be possible to extend $\mathbf{W}$ by a path of length $l-k+2$ connecting u and v without obtaining a homomorphism-embedding copy of $\mathbf{C}_l$. This would contradict the maximality of $\mathbf{A}^+$. We also know that there are no shorter odd walks because every combination of two walks between u and v of length l and $l-k$ produce a homomorphism-embedding copy of $\mathbf{C}_l$.

   It follows that (for given even distance k) there is only one possible set of relations in between vertices u and v in the maximal lift.

2. The case of odd distance follows in full analogy.

3. There are no pairs of vertices of $\mathbf{A}^+$ with distance greater than $\frac{(l+1)}{2}$ in $\mathbf{W}$: for any pair of vertices in a greater distance one can add a path of length $\frac{(l+1)}{2}$ without introducing a short cycle again contradicting the maximality of $\mathbf{A}$.

□

Remark. While in this simple case both proofs appears similarly complex, in less trivial scenarios it is often a lot easier to analyse the structure of pieces rather than give an explicit homogenisation and amalgamation procedure. Consider, for example, the class of graphs having no homomorphic image of the Petersen graph. The pieces of this graph are given in Figure 8.

4.4.2 Forbidden monomorphisms (Cherlin-Shelah-Shi classes)

The classes defined by forbidden homomorphism-embeddings (i.e. classes $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ used in Section 3) can be seen as a special case of classes defined by forbidden monomorphisms (or, equivalently, by forbidden non-induced substructures). In this section we treat those monomorphism defined classes which can be handled by application of Theorem 3.7. (Recall: a monomorphisms indicates weak (not necessarily induced) substructures).

Recall that we denote by $\mathrm{Forb}_{\mathrm{m}}(\mathcal{M})$ the class of all finite or countable structures $\mathbf{A}$ such that there is no monomorphism from some $\mathbf{M} \in \mathcal{M}$ to

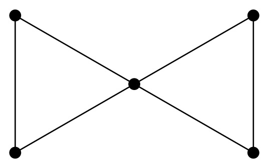


Figure 15: The bowtie graph.

A. The question of the existence of an ω-categorical universal structure in $\mathrm{Forb_m}(\mathcal{M})$ is considered by Cherlin, Shelah and Shi [11]. A sufficient and necessary condition is given in the form of local finiteness of the algebraic closure stated below as Theorem 4.34. While the existence of an universal structure in monomorphism defined classes was intensively studied in the series of papers [43, 8, 14, 16, 26, 17, 9, 13, 15, 7, 10] it even remains open if the problem of the existence of an universal structure in the class of all graphs in $\mathrm{Forb_m}(\mathcal{M})$ is decidable for family $\mathcal{M}$ consisting of single finite graph. On the positive side [11] proves that for every finite family $\mathcal{M}$ of finite connected structures which is closed for homomorphic images the class of all graphs in $\mathrm{Forb_m}(\mathcal{M})$ contains an universal structure. Theorem 3.4 generalises this result for infinite families.

It was the analysis of the bowtie-free graphs [35] (a bowtie is the graph depicted in Figure 15) which led to the notion of a closure description (Definition 2.6). Here we use it to obtain a Ramsey lifts of classes defined by forbidden monomorphisms in a greater generality. This extends the family of known Ramsey classes by non-trivial new examples, such as forbidden 2-bouquets [17], paths [43, 11], complete graphs adjacent to a path [43, 11], bowties adjacent to a path [11] and all known cases in the work in progress catalogue [12]. Some of these classes are really exotic ones. For example, the class of all graph omitting the graph depicted in Figure 16 contains ω-categorical universal graph and this is a singular example: it is not possible to change size of one clique in the picture and obtain class containing an universal graph again! While in the case of the bowtie-free graphs it is possible to manually analyse the structure of graphs in the class (and this analysis is a core of [35]), it is hard to imagine to perform such analysis in the case of graph in Figure 16.

It is only fitting that we end this paper by reviewing the relevance of Ramsey classes described here and of perhaps the most successful result for the existence of ω-categorical objects provided by [11]. In fact this fits together very well. First we briefly review the terminology of [11].

Definition 4.11. Let $\mathbf{A}$ be an L-relational structure and S a finite subset of A. The *algebraic closure of S in* $\mathbf{A}$, denoted by $\mathrm{Acl}_{\mathbf{A}}(S)$, is the set all vertices $v \in A$ for which there is a formula ϕ in the language L with $|S|+1$

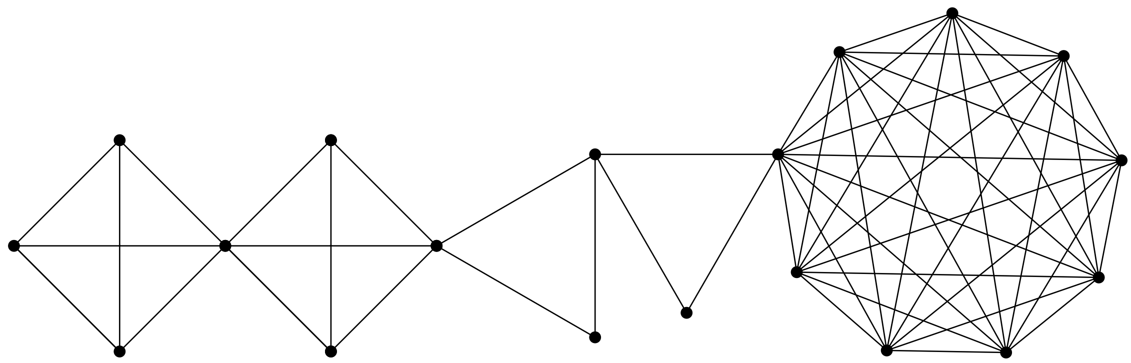


Figure 16: An example of forbidden subgraph (by monomorphism).

variables such that $\phi(\vec{S}, v)$ is true and there is only finitely many vertices $v' \in A$ such that $\phi(\vec{S}, v')$ is also true. (Here $\vec{S}$ is an arbitrary ordering of vertices of S.)

The algebraic closure when applied to structures involving orderings is closely related to our notion of a closure description (Definition 2.5) and thus in our setting both these approaches are equivalent.

We say that structure $\mathbf{A}$ is *algebraically closed in* $\mathbf{U}$ if for every embedding $e : \mathbf{A} \to \mathbf{U}$, $\mathrm{Acl}_{\mathbf{U}}(e(A)) = e(A)$. ($e(A) = \{e(x); X \in A\}$.) Algebraic closure in $\mathbf{A}$ is *locally finite* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $|\mathrm{Acl}_{\mathbf{A}}(S)| \leq f(|S|)$.

Theorem 4.34 (Cherlin, Shelah, Shi [11])**.** *Let $\mathcal{M}$ be a finite family of finite connected relational structures. There is an ω-categorical universal structure in* $\mathrm{Forb_m}(\mathcal{M})$ *if and only if the algebraic closure in existentially complete structures in* $\mathrm{Forb_m}(\mathcal{M})$ *is locally finite.*

We will make use of two observations about the amalgamation in classes $\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$.

Lemma 4.35. *Let $\mathcal{M}$ be a finite family of finite connected structures. If there is a strong amalgamation of* $\mathbf{B}$ *and* $\mathbf{B}'$ *over* $\mathbf{A}$ *in* $\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$ *then there is also a free amalgamation of* $\mathbf{B}$ *and* $\mathbf{B}'$ *over* $\mathbf{A}$ *in* $\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$.

Proof. This follows immediately from the fact that $\mathrm{Forb_m}(\mathcal{M})$ is defined by forbidden monomorphisms and thus is closed for removal of tuples from relation (or, putting otherwise, $\mathrm{Forb_m}(\mathcal{M})$ is a monotone class). □

The following model-theoretic lemma lies in the background of efficiency of Theorem 3.7 formulated bellow as Theorem 4.37.

Lemma 4.36. *Let $\mathcal{M}$ be a class of finite connected structures such that* $\mathrm{Forb_m}(\mathcal{M})$ *contains an ω-categorical universal structure* $\mathbf{U}$. *Let* $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in$

$\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$ *with inclusion embeddings* $\alpha_1 : \mathbf{A} \to \mathbf{B}_1$, $\alpha_2 : \mathbf{A} \to \mathbf{B}_2$. *Further assume that* $\mathbf{A}$ *is algebraically closed in* $\mathbf{U}$. *Moreover let us assume that there is an amalgamation of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *over* $\mathbf{A}$ *in* $\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$. *The there is also a free amalgamation of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *over* $\mathbf{A}$ *in* $\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$.

Proof. Fix $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2$, and, $\mathbf{U}$. Denote by $\mathbf{C} \in \mathrm{Forb_m}(\mathcal{M})$ an (not necessarily free) amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$. Let e_1 be an embedding of $\mathbf{C}$ to $\mathbf{U}$ (which is known to exist by universality of $\mathbf{U}$). To obtain a strong amalgamation, we find an embedding $e_2 : \mathbf{B}_2 \to \mathbf{U}$ such that $e_1(\mathbf{A}) = e_2(\mathbf{A})$ and $e_2(B_2) \cap e_1(B_1 \setminus A) = \emptyset$.

Assume (for simplicity) that $|B_2 \setminus A| = 1$. Because $\mathbf{A}$ is algebraically closed, there are infinitely many embeddings from $\mathbf{B}_2 \to \mathbf{U}$ that agrees on with e_1 on $\mathbf{A}$. Because $\mathbf{B}_1$ is finite, we can chose embedding where the image of the additional vertex of $\mathbf{B}_2$ is not contained in $e_1(B_1)$.

Now consider case where $|B_2 \setminus A| = 2$. Denote by u and v the vertices of $B_2 \setminus A$. By the same argument as before, there are infinitely many possible images of u. If v is not in the algebraic closure of $\mathbf{B}_2 \setminus \{v\}$ for each of this image we have infinitely many images of v. This makes it possible to chose e_2. We thus consider the case where v is in the algebraic closure of $\mathbf{B}_2 \setminus \{v\}$ and for each possible choice of image of u, the possible choices of images of v lie in $e_1(B_1)$. Because $e_1(B_1)$ is finite and because it is possible to write a formula describing all such u taking vertices of $\mathbf{A}$ as parameters, we know that all vertices in $e_1(B_1 \setminus A)$ are in $\mathrm{Acl}_{\mathbf{U}}(e_1(A))$. A contradiction with an assumption that $\mathbf{A}$ is closed.

The case of $|B_2 \setminus A| > 2$ follows in a complete analogy.

The structure induced on vertices of $e_1(B_1) \cup e_2(B_2))$ is a strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$. Using Lemma 4.35 it is also a free amalgamation. □

Theorem 4.37. *Let* $\mathcal{M}$ *be a set of finite connected structures such that* $\mathrm{Forb_m}(\mathcal{M})$ *contains an* ω*-categorical universal structure* $\mathbf{U}$. *Further assume that for every* $\mathbf{M} \in \mathcal{M}$ *at least one of the conditions holds:*

1. *there is no homomorphism-embedding of* $\mathbf{M}$ *to* $\mathbf{U}$, *or,*

2. $\mathbf{M}$ *can be constructed from irreducible structures by a series of free amalgamations over irreducible substructures.*

Then the class of finite algebraically closed substructures of $\mathbf{U}$ *has precompact Ramsey lift. (By the standard homogenisation argument it also follows that* $\mathrm{Age}(\mathrm{Forb_m}(\mathcal{M}))$ *has precompact Ramsey lift.)*

Proof. We first expand language L by an order. Let $\overrightarrow{\mathbf{U}}$ be the generically linearly ordered $\mathbf{U}$. We further extend language L by necessary relations to represent the algebraic closure of every finite ordered irreducible substructure of $\overrightarrow{\mathbf{U}}$. Denote by L^+ the resulting language and by $\mathcal{U}_\mathcal{M}$ the resulting closure description (which exists and is finite by Theorem 4.34). Denote by $\overrightarrow{\mathbf{U}}^+$ the lift of $\overrightarrow{\mathbf{U}}$ adding the newly introduced relations representing closures and relations describing an isomorphism type of a closure of every k-tuple. Thus L^+ is the language of $\overrightarrow{\mathbf{U}}^+$. By the local finiteness of the algebraic closure we know we introduced only finitely many relations of a given arity.

Denote by n the size of the largest structure in $\mathcal{M}$ and by N the bound on size of algebraic closure of a structure on n vertices. Let $\mathcal{F}_\mathcal{M}$ denote the class of all structures with at most N vertices whose all homomorphism-embeddings are forbidden in $\overrightarrow{\mathbf{U}}^+$ and $\mathcal{E}_\mathcal{M}$ denote the class of all structures $\mathbf{A} \notin \mathrm{Age}(\overrightarrow{\mathbf{U}}^+)$ containing irreducible substructure $\mathbf{A}' \in \mathrm{Age}(\overrightarrow{\mathbf{U}}^+)$ such that $\mathrm{Cl}^{\mathcal{U}}_{\mathbf{A}}(A') = \mathbf{A}$. Now apply Theorem 3.7 to obtain precompact Ramsey lift $\mathcal{K}^+_\mathcal{M}$ of the class $\mathcal{K}_\mathcal{M}$ of all finite ordered $\mathcal{U}_\mathcal{M}$-closed structures in $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F}_\mathcal{M}) \cap \mathrm{Forb}_{\mathrm{e}}(\mathcal{E}_\mathcal{M})$. We claim that $\mathcal{K}_\mathcal{M}$ is the class of all $\mathcal{U}_\mathcal{M}$-closed structures in $\mathrm{Age}(\overrightarrow{\mathbf{U}}^+)$ and consequently $\mathcal{K}^+_\mathcal{M}$ is the precompact Ramsey lift of $\mathrm{Age}(\mathrm{Forb}_{\mathrm{m}}(\mathcal{M}))$.

Let $\mathbf{C} \in \mathcal{K}_\mathcal{M}$ and assume, to the contrary, the existence of $\mathbf{M} \in \mathcal{M}$ such that there is a monomorphism m from $\mathbf{M}$ to shadow $\mathrm{Sh}(\mathbf{C})$. Because $\mathrm{Sh}(\mathbf{C}) \in \mathrm{Forb}_{\mathrm{he}}(\mathcal{F}_\mathcal{M})$ there is a monomorphic image $\mathbf{M}$ in $\mathrm{Forb}_{\mathrm{m}}(\mathcal{M})$ and thus denote by $\mathbf{M}_1, \mathbf{M}_2, \ldots, \mathbf{M}_n$ the irreducible structures used to build $\mathbf{M}$. Denote by $\mathbf{M}'_1, \mathbf{M}'_2, \ldots, \mathbf{M}'_n$ the closures of $m(\mathbf{M}_1), m(\mathbf{M}_2), \ldots, m(\mathbf{M}_n)$ in $\mathbf{C}$. Because the structures are irreducible we know that $\mathbf{M}'_1, \mathbf{M}'_2, \ldots, \mathbf{M}'_n \in \mathcal{K}_\mathcal{M}$: if $\mathbf{M}'_i \notin \mathcal{K}_\mathcal{M}$ we also get $\mathbf{M} \in \mathcal{F}_\mathcal{M}$ or $\mathbf{M}'_i \in \mathcal{E}_\mathcal{M}$ (because $\mathbf{M}_i$ is irreducible in $\mathcal{K}_\mathcal{M}$). It follows that, by Lemma 4.36, we can use the same series of free amalgamations over structures $\mathbf{M}'_1, \mathbf{M}'_2, \ldots, \mathbf{M}'_n$ to build a structure in $\mathcal{K}_\mathcal{M}$ containing a monomorphic copy of $\mathbf{M}$. A contradiction. $\square$

Remark. The order needs to be handled carefully in the proof. It may seem more natural to first homogenise $\mathbf{U}$ and then add the order. This however often leads to more complex structure. If the order is introduced first and assuming that the language L contains no relations of arity greater than k, the isomorphism type of closure of a substructure can be uniquely determined by the isomorphism type of its substructures of size at most k [35].

In addition, lifting by the free order will not give a lift with the lift property for classes with non-trivial closure. Such lifts needs more detailed analysis of the structure of this closure. The special case of the bowtie-free graphs is analysed in [35].

Remark. It is conjectured in [10, 7] that every graph $\mathbf{G}$ such that there exists an ω-categorical universal graph for the class of all graphs in $\mathrm{Forb_m}(\{\mathbf{G}\})$ has all 2-connected components irreducible. If this conjecture is true, then Theorem 4.37 shows the existence of a precompact Ramsey lift for every class of graphs $\mathrm{Forb_m}(\{\mathbf{G}\})$ with an ω-categorical universal graph. So it seems this is as far as we can go: the existence of Ramsey lift is here equivalent to ω-categoricity of universal graph.

Remark. If $\mathcal{M}$ consists only of structures constructed from irreducible structures by a series of free amalgamations over an irreducible substructures the existence of ω-categorical universal structure is actually necessary in Theorem 4.37 only to establish the precompactness of the lift. Even in the cases where algebraical closure is not locally finite, the same technique as above can be used for the class of homogenising lifts of the structures (which is not precompact and the resulting Fraïssé limit will not be universal, only universal for finite structures of the age). The resulting Ramsey lift will be precompact lift of this homogenising lift.

On the other hand consider class $\mathrm{Forb_m}(\mathbf{C}_4)$. This class has a binary closure: for every pair of vertices there is at most one vertex connected to both of them. It is easy to consider a lift by adding binary relation R^C which denote every pair of vertices for which there exists a vertex connected to both and ternary relation R^U representing the closure (note that the closure is not locally finite [25] and there is no ω-categorical universal graph for $\mathrm{Forb_m}(\mathbf{C}_4)$). This class has however a strong amalgamation over the closed structures. It seem to fail to have locally finite completion property and the existence of Ramsey lift of this class is an open (and well known) problem (see e.g. [51]).

Remark. The condition of Theorem 4.37 given on the family $\mathcal{M}$ can be strengthened. Cherlin [7] give an example of class $\mathrm{Forb_m}(\mathcal{M})$ with non-unary algebraic closure. It is easy to show that the techniques used in proof of Theorem 4.37 apply for this class, too.

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