STRUCTURAL SPARSITY

JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ

To Anatoly Vershik, on the occasion of his 80*th birthday.*

Abstract. We discuss the notion of structural sparsity and how it relates to the nowhere dense/somewhere dense dichotomy introduced by the authors for classes of graphs. This will be the occasion of surveying the numerous facets of this dichotomy, as well as its connections to several concepts like stability, independence, VC-dimension, regularity partitions, entropy, class speed, low tree-depth decomposition, quasi-wideness, neighborhood covering, subgraph statistics, etc. as well as algorithmic complexity issues like fixed parameter tractability of first-order model checking.

1. Introduction

Every good theory starts with some key examples and motivating problems. To get insight into seemingly scattered facts is a prime goal of mathematics and, some feel, a principal feature of mathematical thinking. Here are then two problems, which motivated the research and theory surveyed in this paper:

Problem 1 (Embedding problem). For a fixed structure F and a given structure G, decide whether F is a substructure of G.

Problem 2 (Approximation problem). Given a structure G, can one reduce G to a relatively small structure H, which is locally similar to G?

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What is interesting here?

Both of these problems are complicated. For instance, although the complexity of the first problem is, for given F, bounded by a polynomial function of the size of G, the degree of the polynomial depends in general on the size of F (non-trivial bounds are known, which depend on fast matrix multiplication [65]). But if we specialize the problem to geometrically restricted classes (such as e.g. planar graphs) or to classes of graphs with (uniformly) bounded degrees, then the complexity of the problem becomes linear (with only a constant depending on F). For the second problem, we have an even more striking difference: In most cases there is no approximation which is known to exist, although for planar graphs or bounded degree graphs a good approximation can be computed in linear time.

What is behind these differences? This question motivated much of the research surveyed here, and so did particularly the following very concrete instance:

Problem 3 (Triangle problem). Does there exist a triangle-free graph H such that every triangle-free planar graph G is homomorphic to H?

This particular problem was solved by the authors in [49] and it provided a starting motivation for the theory surveyed here. In particular, it led to the key definitions of low tree-depth decomposition, bounded expansion class, and further, to the nowhere dense vs somewhere dense dichotomy.

Intuitively one expects that the answer to the Triangle problem is positive for graphs which are sparse. However, there is a warning example: for complete graph K_n, denote by K_n^{**} the graph obtained from K_n by subdividing each edge by two vertices. Clearly K_n^{**} has $2\binom{n}{2}+n$ vertices and $3\binom{n}{2}$ edges (as most of its vertices have degree two). However, K_n^{**} is not homomorphic to any triangle free graph H with less than n vertices. There are of course other examples of this phenomenon, some easy ones obtained by convenient gadget constructions. Nevertheless, the example of K_n^{**} is not as isolated and special as it might seem at first glance, as it leads to the following definition:

A class $\mathcal{C}$ of graphs is *nowhere dense* if there exists an integer p such that the pth subdivision of every complete graph arises as a subgraph of a member of $\mathcal{C}$; a class of graphs is *somewhere dense* if it fails to be nowhere dense.

We shall see that this notion fits to the intuitive meaning of sparsity. In fact, in its manifold characterizations it fits very well to several established and studied notions. In a broader perspective we shall review this aspect in the next section.

The study of the structural and algorithmic properties of classes of sparse graphs gained recently much interest. In this context, the authors introduced in 2006 [50, 51] the notion of classes with bounded expansion, which extends the notions of classes excluding a topological minor (as classes with bounded degree or classes excluding a minor like planar graphs) and, soon after [56, 57], the nowhere dense vs somewhere dense dichotomy for classes of graphs. Although this dichotomy looks arbitrary at first glance, it appeared to be very robust, expressible in a number of different (non obviously) equivalent way, and to reveal both a profound structural dichotomy (roughly speaking, between branching and homogeneous structures) and a profound algorithmic complexity dichotomy (for model checking, particularly), see for instance [61].

In this paper, we survey all of these aspects and their links with information theory, extremal graph theory, model theory, algorithmic complexity, etc. In an attempt to give a broader picture, we do not follow the chronological order of the discoveries but rather concentrate on recent development which brought the originally combinatorial setting in the realm of logic and information theory. This very general setting, which combines several areas, is introduced in Section 2 under the name of *Structural Sparsity*.

In Section 3, we tackle the notion of sparsity by the light of Shelah's model theoretic notions of independence and stability, and their connection to the notion of VC-dimension of learning theory.

In Section 4, we consider the point of view of structure approximation, by the light of Szemerédi's regularity lemma, the notion of left-convergence of graphs (and hypergraphs) introduced by Lovász and Szegedy, the notions of structural limits and modeling introduced by the authors, as well as Friedman's notion of totally Borel structures.

In Section 6, we relate the nowhere dense vs somewhere dichotomy to Bollobás's notion of class speed, and its connection to graph entropy and to the notion of logarithmic density.

In Section 7 we consider our original approach through the concept of shallow minors of Leiserson and Toledo, and its extensive use by the authors in the definition and study of classes with bounded expansion.

In Section 8 we consider structural properties related to decomposition, as the concept of low tree-depth decomposition initially introduced by the authors in their study of restricted homomorphism dualities in classes of sparse graphs, as well as coverings recently introduced by Grohe, Kreutzer, and Siebertz.

In Section 9, we consider Zhu's connection to the notion of generalized coloring numbers of Kierstead and Yang, as well as locally constrained orientations and augmentation at the heart of low tree-depth decomposition.

In Section 10 we eventually unveil the original motivation of our study of nowhere dense classes, which is the connection to Dawar's concept of quasi-wideness.

Then, in Section 11 we discuss algorithmic consequences of these works, in particular for first-order logic model checking problem.

It is amazing that all these concepts lead to equivalent definitions of nowhere dense classes. This interplay is schematically depicted on Fig. 1.

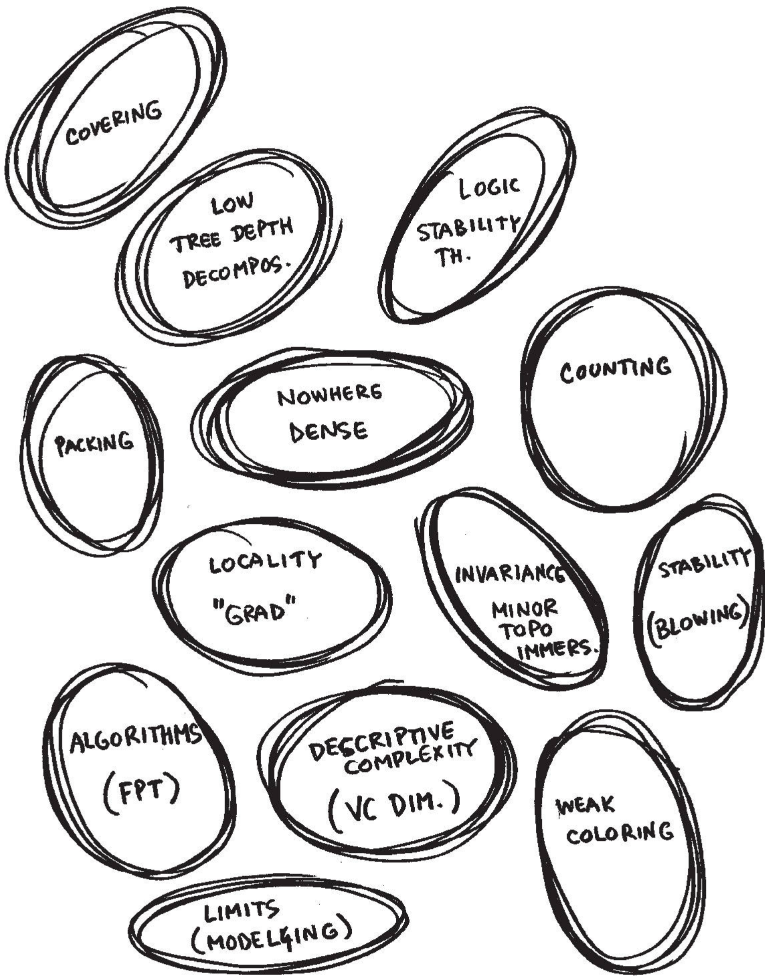

FIGURE 1. Manifold characterizations of nowhere dense classes (artist view)

Finally, we end this paper by some concluding remarks in Section 12.

This paper surveys the recent trends as well as new results, which appeared after publication of monograph [61]. The whole paradigm is shifting towards generality and involves new parts of mathematics, which we reflect in Section 2 and in the whole paper. But for technical details and full proofs the interested reader is referred to [61].

2. Structural Sparsity

The notion of *sparsity* is certainly not appearing sparsely across the diverse fields of mathematics. However, it appears that this notion might be more elusive than expected.

For instance, let us consider the notion of a sparse matrix. A *sparse matrix* is a matrix in which most of elements are zero. Compressed sensing (also known as sparse sampling) showed that the sparsity of a signal allows a reconstruction with much fewer samples than expected from Nyquist–Shannon sampling theorem. Nevertheless, compressed sensing may consider linear combinations of samples in a basis that is different from the basis in which the signal is known to be sparse. Indeed, the knowledge of the existence of a basis in which the signal is sparse being sufficient to reduce the number of samples needed for a reconstruction (see works of Candès [14], Romberg, Tao, and Donoho).

In this spirit, we shall say that a structure is sparse if it may be reconstructed from a small amount of information, in particular if it can be obtained as an interpretation of a sparse structure (in a more usual sense of the word sparsity). Both notions are obviously linked: consider for instance a class of graphs $\mathcal{C}$ with the property that every graph of order n in $\mathcal{C}$ has at most $n^{1+o(1)}$ edges. Then, as noticed in [9], the number of graphs of order n in $\mathcal{C}$ is smaller (for every $\epsilon > 0$ and for sufficiently large n) than $2^{n^{1+\epsilon}}$. This means that graphs in such a class can be encoded with $n^{o(1)}$ bits per vertex, which is significantly smaller than general graphs, which require about n bits by vertex.

The structures that are inherently dense, meaning that they require much information to be fully reconstructed, are the structures that are close to be random. Indeed, a structure is random if none of its parts can be predicted from others. This aspect appears clearly in the duality of the notion of entropy in the theory of information: the entropy measures both the randomness and the information present in a system.

The aspect related to randomness is closely related to the general model theoretic notions of *independence property* introduced by Shelah [72], which expresses the possibility of encoding a random bipartite graph with a definable edge relation,and the notion of VC dimension, which arose in probability theory in the work of Vapnik and Chervonenkis [76]. These notions are deeply linked as, as observed by Laskowski [44], a complete first-order theory does not have the independence property if and only if, in each model, each definable family of sets has finite VC dimension.

The informational aspect not only relates to the efficiency of structure encoding but also to the ease to approximate the structure. Graph approximation is the basic aim of Szemerédi's regularity lemma, which gives a parsimonious description of a large graph as a random expansion of a bounded size weighted structure. Still the size of this structure is very large in general. However, in the case of graphs without the independence property, and in particular in the case of stable graphs, much smaller bounds can be achieved, as shown by Malliaris and Shelah [48]. Moreover, in such a case, the expansion is deterministic (this corresponds to edge densities between the parts that are either 0 or 1). Szemerédi's regularity lemma [75], the removal lemma (see extension in [68]), and the counting lemma (see [12]) are put into a new perspective by the concept of *left limit* introduced by Lovász and Szegedy [46]. In this context, limit of left-convergent sequences, the so-called *graphons*, are measurable functions from $[0,1]\times[0,1]$ to $[0,1]$, which encode (averaged) probabilistic properties of the graph. However, hereditary classes of graphs with bounded VC-dimension are random-free (as proved by Lovász and Szegedy [47]), in the sense that they can be represented by a $\{0,1\}$-valued graphon, which is essentially a *Borel graph* in the terminology introduced by Friedman in the late seventies. For a connection of the notions of random-free graphons, class speed of a hereditary class $\mathcal{C}$ (that is the function mapping each integer n to the number of graphs of order n in the class $\mathcal{C}$), and graphon entropy we refer the reader to [35]. (See also [4], [40] for connection of graphon entropy to the entropy of random graphs and [16], [15] for its use in the study of large deviations of random graphs and exponential models of random graphs)

With this general picture in mind we shall discuss the dichotomy introduced by the authors [56, 55, 59] between *nowhere dense* and *somewhere dense* classes of graphs. It seems that the results fit very well in this grand picture.

3. Sparsity, Cliques, VC-dimension, and Stability

The simplest setting for our purpose to define the notion of sparse structures is most probably the case of classes of graphs. In this particular setting several model-theoretical notions collapse (see for instance Theorem 6), allowing simpler definitions than the ones expected in the general framework of relational structures.

In the case of a monotone class, we have the following alternative characterization, which follows from an easy Ramsey argument: A monotone class of graphs $\mathcal{C}$ is nowhere dense if and only if there exists some integer $p\in\mathbb{N}$ such that $\mathcal{C}$ contains the p-th subdivision of every complete graph (hence of every graph). This easy reformulation leads the alternative description of monotone nowhere dense classes by means of a measure of descriptive complexity.

Let us now recall some definitions related to the notion of complexity of graphs. Introduced in [76] (and in [70] as a measure of density of a family of sets) the *Vapnik-Chervonenkis dimension* (or *VC-dimension* for short) of a hypergraph $\mathcal{H}$ is the maximum size of a shattered set of $\mathcal{H}$, where a set X of vertices is *shattered* if for every $X \subseteq X$ there exists a hyperedge e such that $e \cap X = X$.

The VC-dimension found many applications, as in learnability theory [36], in computational geometry [17], as well as in graph theory (see [13] for instance).

Definition 1. Let G be a graph and let d be an integer. The *d-distance VC-dimension* $\mathrm{VC}_d(G)$ of G is the VC-dimension of the hypergraph with vertex set $V(G)$, and hyper-edges $B_d(G, v)$ with $v \in V(G)$, where $B_d(G, v)$ stands for the set of vertices of G at distance at most d from vertex v (see Fig. 2).

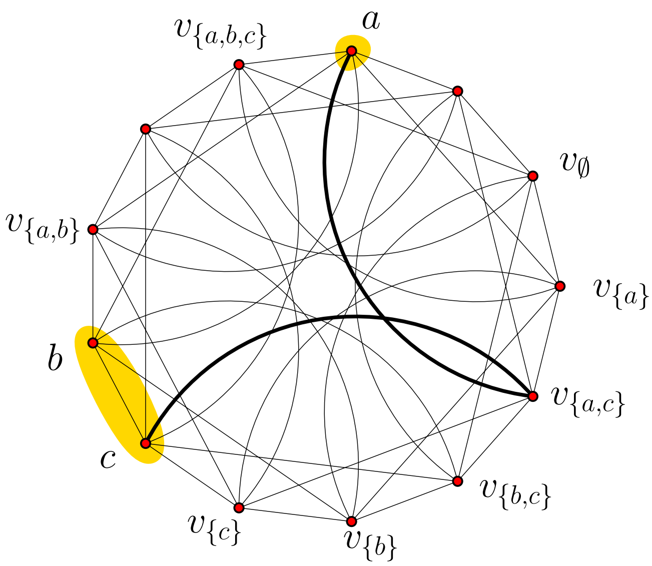

FIGURE 2. The set $\{a, b, c\}$ is shattered by the neighborhoods of the vertices $v_\emptyset, v_{\{a\}}, v_{\{b\}}, v_{\{c\}}, v_{\{a,b\}}, v_{\{a,c\}}, v_{\{b,c\}}, v_{\{a,b,c\}}$. For instance, among a, b, c, the vertex $v_{\{a,c\}}$ is exactly adjacent to a and c. It follows that the (1-distance) VC-dimension of this Paley graph is 3 (it does not have sufficiently vertices for 4).

Hereditary classes of graphs with uniformly bounded d-distance VC-dimension have nice properties (see [13]) and we have the following striking connection with the graph minor perspective. The *distance VC-dimension* of a graph is defined has the maximum over d of the d-distance VC-dimensions of the graph.

Theorem 2 ([13]). *A K_k-minor-free graph has distance VC-dimension at most $k-1$.*

(Note that this theorem extends to graphs with bounded rank-width: A graph with rank-width k has distance VC-dimension at most $3.2^{k+1}+2$ [13]).

The natural relaxation of the condition of bounded distance VC-dimension is to consider classes where the d-distance VC-dimension is bounded by some function of d. It appears that for monotone classes of graphs, this notion turns out to be equivalent to the notion of nowhere dense class.

Theorem 3 (Nowhere dense by d-distance VC-dimension). *Let $\mathcal{C}$ be a monotone class of graphs. The following are equivalent:*

(1) the class $\mathcal{C}$ is nowhere dense;
(2) for every integer d, it holds

$$\sup_{G\in\mathcal{C}} \mathrm{VC}_d(G) < \infty.$$

Proof. Assume $\mathcal{C}$ is somewhere dense. Then there exists an integer p such that every p-subdivision of graphs belong to $\mathcal{C}$. Hence the $(p-1)$-distance VC-dimension of $\mathcal{C}$ is not bounded.

Conversely, assume that $\mathcal{C}$ is nowhere dense. Then every model-theoretic interpretation of $\mathcal{C}$ has bounded VC-dimension [2]. In particular, the hypergraph of closed d-neighborhoods of G has bounded VC-dimension. □

In this context, the following is perhaps of interest:

Problem 4. Characterize monotone classes with bounded distance VC-dimension.

The connection of the existence of subdivided cliques to boundedness of d-distance VC-dimension also links to other model-theoretic notions, like independence property and stability.

Definition 4 (order property). A formula $\phi(x_1,\dots,x_l,y_1,\dots,y_r)$ has the *order property* with respect to some background theory T if there exists, in some sufficiently saturated model M of T, elements $\{a_1^i,\dots,a_l^i : i\in\mathbb{N}\}$ and $\{b_1^j,\dots,b_r^j : j\in\mathbb{N}\}$ such that $\models \phi(a_1^i,\dots,a_l^i,b_1^j,\dots,b_r^j)$ if and only if $i<j$.

Theories in which no formula has the order property are called *stable*. Such theories have been fundamental to model theory since Shelah's work in [73]. Stability is related to several global structural properties, such as number of models, existence of indiscernible sets, or number of types. By compactness, a formula has the order property (with respect to a background theory T) if and only if it has the k-order property for every natural number k,

Definition 5 (non-k-order property)**.** A graph G has the *non-k-order property* if G does not contain two disjoint sets A, B of size k such that the bipartite subgraph of G induced by the edges between A and B is a half-graph.

Note that stability property is a weaker form of the not independent property (NIP). However, for monotone classes of graphs these notions coincide.

Theorem 6 (Stability and NIP [2])**.** *For any monotone class of graphs $\mathcal{C}$ the following are equivalent:*

- *$\mathcal{C}$ is nowhere dense,*
- *$\mathcal{C}$ is stable,*
- *$\mathcal{C}$ is not independent (NIP).*

In yet another direction, one can relate nowhere dense classes to VC-dimensions of model-theoretical interpretations. Recall the definition:

Definition 7. Given two relational structures M, N with respective signatures σ and σ', we say that N is *interpretable* in M if for some $k \geq 1$ there exist

- a σ'-formula $\Delta(x)$ in k free variables;
- a σ'-formula $E(x, y)$ in $2k$ free variables;
- for each n-ary relation symbol $R \in \sigma$, a σ'-formula $\phi_R(x_1, \ldots, x_n)$ is nk free variables

such that:

- E^M is an equivalence relation on Δ^M;
- for each n-ary relation $R \in \sigma$, ϕ_R^M is an n-ary E^M-invariant relation on Δ^M;
- $(\Delta^M/E^M, ({}^M_R/E^M)_{R\in\sigma})$ is isomorphic to N.

An interpretation defines a mapping $I : M^k \to N$ with the property that the inverse image of a definable set of N is a definable set of M. It is easily checked that if the theory of M is stable and N is interpretable in M, then the theory of N is also stable.

The notion of interpretation allows to obtain the following corollary of Theorem 6, which again relates the concept of nowhere dense class of graphs to the VC-dimension of the neighborhoods.

Theorem 8 (Nowhere dense by interpretations [2]). *Let $\mathcal{C}$ be a monotone class of graphs. Then the following are equivalent:*

(1) $\mathcal{C}$ is nowhere dense;

(2) for every interpretation I of graphs in graphs it holds

$$\sup_{G\in\mathcal{C}} \mathrm{VC}(\mathsf{I}(G)) < \infty.$$

4. Regular partitions

Theorem 9 (Szemerédis regularity lemma). *For every ϵ, m there exist $N = N(\epsilon, m)$ and $m' = m'(\epsilon, m)$ such that given any finite graph G, of order at least N, there is k with $m \le k \le m'$ and a partition $V_1 \cup \dots \cup V_k$ of the vertex set of G satisfying:*

(1) $\big||V_i| - |V_j|\big| \le 1$ for all $i, j \le k$;

(2) all but at most ϵk^2 of the pairs (V_i, V_j) are ϵ-regular.

Note that in general the size of the partition can increase as a tower of 2s of height proportional to $\log(1/\epsilon)$ [33]. Moreover, several researchers (Lovász, Seymour, Trotter, as well as Alon, Duke, Leffman, Rödl, and Yuster in [7]) independently observed that the half-graph, i.e. the bipartite graph with vertex sets $\{a_i : i < n\} \cup \{b_i : i < n\}$ with a_i adjacent to b_j if $i < j$ shows that exceptional pairs are necessary, what led Malliaris and Shelah to ask whether a stronger regularity lemma could hold for classes of graphs which admit a uniform finite bound on the size of an induced sub-half-graph.

Malliaris and Shelah proved [48] the following strengthening of Szemerédi's regularity lemma for graphs with the non-k-order property:

Theorem 10. *For every $k \in \mathbb{N}$ there exists $k^* \le 2^{k+2}$ with the following property:*

For every $\epsilon > 0$ there exists $m = m(\epsilon)$ such that for every sufficiently large finite graph G with the non-k-order property there is a partition $V_1 \cup \dots \cup V_k$ of the vertex set of G into $k \le m$ parts so that

(1) the size of the parts differ by at most 1;

(2) the bipartite subgraph induced by the edges between V_i and V_j form (up to removal of at most $\epsilon|V_i|$ vertices in V_i and $\epsilon|V_j|$ vertices in V_j) define either an empty graph or a complete bipartite graph;

(3) if $\epsilon < \frac{1}{2^{k^*}}$ then $m \le (3 + \epsilon)\left(\frac{8}{\epsilon}\right)^{k^*}$.

Note that interpretation preserves the non k-order-property hence Theorem 10 applies to interpretations of nowhere dense classes. Szemerédi regularity lemma can be characterized by means of the compactness of a metric space related to graph limits.

Sparse structures, for which Szemerédi decomposition is trivial, need finer decomposition, and this leads us to a natural notion of convergence and to a nice conjecture to be explained in the next section.

5. Structural Limits

The study of limits of graphs recently gained much interest (see [45]). A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is said *left-convergent* if, for every graph F, the probability that a random map $f: V(F) \to V(G_n)$ is a *homomorphism* (that is an adjacency preserving map) converges as n grows to infinity. This convergence is deeply related to Szemerédi regularity, and the limit object can be represented by means of a *graphon*, that is a symmetric measurable function $W : [0,1]^2 \to [0,1]$. A graphon W is *random-free* if it is almost everywhere $\{0,1\}$-valued. A random-free graphon is essentially the same (up to isomorphism mod 0) as a *Borel graph* — that is a graph having a standard Borel space V as its vertex set and a Borel subset of $V \times V$ as its edge set — equipped with a non-atomic probability measure on V. A class of graph $\mathcal{C}$ is said to be *random-free* if every left-convergent sequence of graphs in $\mathcal{C}$ has a random-free limit. For instance, Janson derived from a structural characterization of random-free hereditary classes of graphs given by Lovász and Szegedy [47] that the class of cographs is random-free [39].

Lemma 11 ([47]). *For a hereditary class of graphs $\mathcal{C}$, the following are equivalent:*

- *$\mathcal{C}$ is random-free;*
- *there is a bipartite graph F with bipartition (U_1, U_2) such that no graph in $\mathcal{C}$ can be obtained from F by adding edges within U_1 and U_2.*
- $\mathrm{VC}(\mathcal{C}) < \infty$*, where*

$$\mathrm{VC}(\mathcal{C}) = \sup_{G\in\mathcal{C}} \mathrm{VC}(G).$$

For k-regular hypergraphs, left-limits have been constructed by Elek and Szegedy in landmark paper [27] using ultraproducts as measurable functions $W : [0,1]^{2^k-2} \to [0,1]$ (called *hypergraphons*), and have also been studied by Hoover [38], Aldous [5], and Kallenberg [41] in the setting of exchangeable random arrays.

Relational structures are a natural generalization of k-uniform hypergraphs. The authors introduced the notion of *structural limits* in [60]. This approach to limits of structures relies on a balance of model theoretic and functional analysis aspects. For a finite structure $\mathbf{A}$ and a first-order formula ϕ with free variables $x_1, \ldots, x_p$ we define the *Stone pairing* of ϕ and

$\mathbf{A}$ as

$$\langle \phi, \mathbf{A} \rangle = \frac{|\phi(\mathbf{A})|}{|A|^p},$$

where $\phi(\mathbf{A}) = \{(v_1, \dots, v_p) \in A^p : \ \mathbf{A} \models \phi(v_1, \dots, v_p)\}$. In other words, $\langle \phi, \mathbf{A} \rangle$ is the probability that ϕ is satisfied in $\mathbf{A}$ for a random (uniform independent) assignment of the free variables $x_1, \dots, x_p$ to elements of A. Note that this definition naturally extends to any modeling $\mathbf{L}$ (with probability measure $\nu_{\mathbf{L}}$) by putting

$$\langle \phi, \mathbf{L} \rangle = \nu_{\mathbf{L}}^{\otimes p}(\phi(\mathbf{A})),$$

where $\nu_{\mathbf{L}}^{\otimes p}$ is the product measure on L^p.

A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of σ-structure is FO-*convergent* if $(\langle \phi, \mathbf{A}_n \rangle)_{n\in\mathbb{N}}$ convergences for each first-order formula ϕ. The following representation theorem has been proved in [60].

Theorem 12. *There exists a compact standard Borel space S, a mapping k : FO $\to C(S)$ from the class of first-order formulas to the space of continuous functions on S, and a mapping $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ mapping finite σ-structure $\mathbf{A}$ to a probability measure $\mu_{\mathbf{A}}$ on S such that:*

- *The mapping $\mathbf{A} \mapsto \mu_{\mathbf{A}}$ is injective and for every first-order formula ϕ it holds*
$$\langle \phi, \mathbf{A} \rangle = \int k(\phi)\, \mathrm{d}\mu_{\mathbf{A}};$$
- *A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite structures is* FO-*convergent if and only if the sequence $(\mu_{\mathbf{A}_n})_{n\in\mathbb{N}}$ of probability measures converges weakly. Then, if $\mu_{\mathbf{A}_n} \Rightarrow \mu$, for every first-order formula ϕ it holds*
$$\int k(\phi)\, \mathrm{d}\mu = \lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle.$$

This representation theorem actually deals with S_∞-invariant measures, in the same spirit as infinite exchangeable graphs of Aldous [5] and Hoover [38], or S_∞-invariant measures in the space of symmetric matrices studied by Vershik [77].

The analog of random-free limit notion for left-convergence appears to be the notion of modeling limit for FO-convergence. This leads to the following notions:

Definition 13. A structure A is a *Borel structure* if its domain is a standard Borel space and if all its relations are Borel.

Note that this definition corresponds to the notion of *injective Borel structure* of [37], and of *Borel structure* of [74]. We introduced in [63] the following notions:

Definition 14. A *relational sample space* is a Borel structure A with the property that every first-order definable set is Borel (in the product space).

A *modeling* is a relational sample space equipped with a probability measure.

Note that the notion of *totally Borel structure* of [74] is intermediate between the notions of relational sample space and modeling: it is a special modeling with specified domain ($[0,1]$) and probability measure (Lebesgue measure).

Definition 15. An FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs has graph *modeling limit* $\mathbf{L}$ if $\mathbf{L}$ is a graph modeling and for every first order formula ϕ it holds (see Fig. 3:

$$\langle\phi,\mathbf{L}\rangle = \lim_{n\to\infty}\langle\phi,G_n\rangle.$$

A class of graphs $\mathcal{C}$ has *modeling limits* if every FO-convergent sequence of graphs in $\mathcal{C}$ has a modeling limit.

A first step toward the construction of a modeling limit could lie in the construction of a *weak modeling limit* L, that is a modeling L satisfying the property that for every first order formula ϕ it holds

$$\langle\phi,\mathbf{L}\rangle > 0 \quad\iff\quad \lim_{n\to\infty}\langle\phi,G_n\rangle > 0.$$

This means that properties satisfied with positive probability in L are exactly those which occur with a probability bounded away from zero in the sequence $(G_n)_{n\in\mathbb{N}}$. Existence of weak modeling limit can be carried out by considering the finite extension $\mathcal{L}(Q_m)$ of first-order logic obtained by adjoining a new quantifier Q_m, whose intended interpretation is "there exist non-measure 0 many". The axioms for $\mathcal{L}(Q_m)$ are all the usual axiom schema for first-order logic together with the following ones [74]:

- M0 $\neg(Q_m x)(x=y)$;
- M1 $(Q_m x)\psi(x,\dots)\leftrightarrow(Q_m y)\psi(y,\dots)$, where $\psi(x,\dots)$ is an $\mathcal{L}(Q_m)$-formula in which y does not occur and $\psi(y,\dots)$ is the result of replacing each free occurrence of x by y;
- M2 $(Q_m x)(\phi\vee\psi)\to(Q_m x)\phi\vee(Q_m x)\psi$;
- M3 $[(Q_m x)\phi\wedge(\forall x)(\phi\to\psi)]\to(Q_m x)\psi$;
- M4 $(Q_m x)(Q_m y)\phi\to(Q_m y)(Q_m x)\phi$.

The rules of inference for $\mathcal{L}(Q_m)$ are the same as for first-order logic: *modus ponens* and generalization. Let the system just described be denoted by $\mathcal{K}_m$.

In this context, the following completeness theorem has been proved by Friedman [32] (see also [74]):

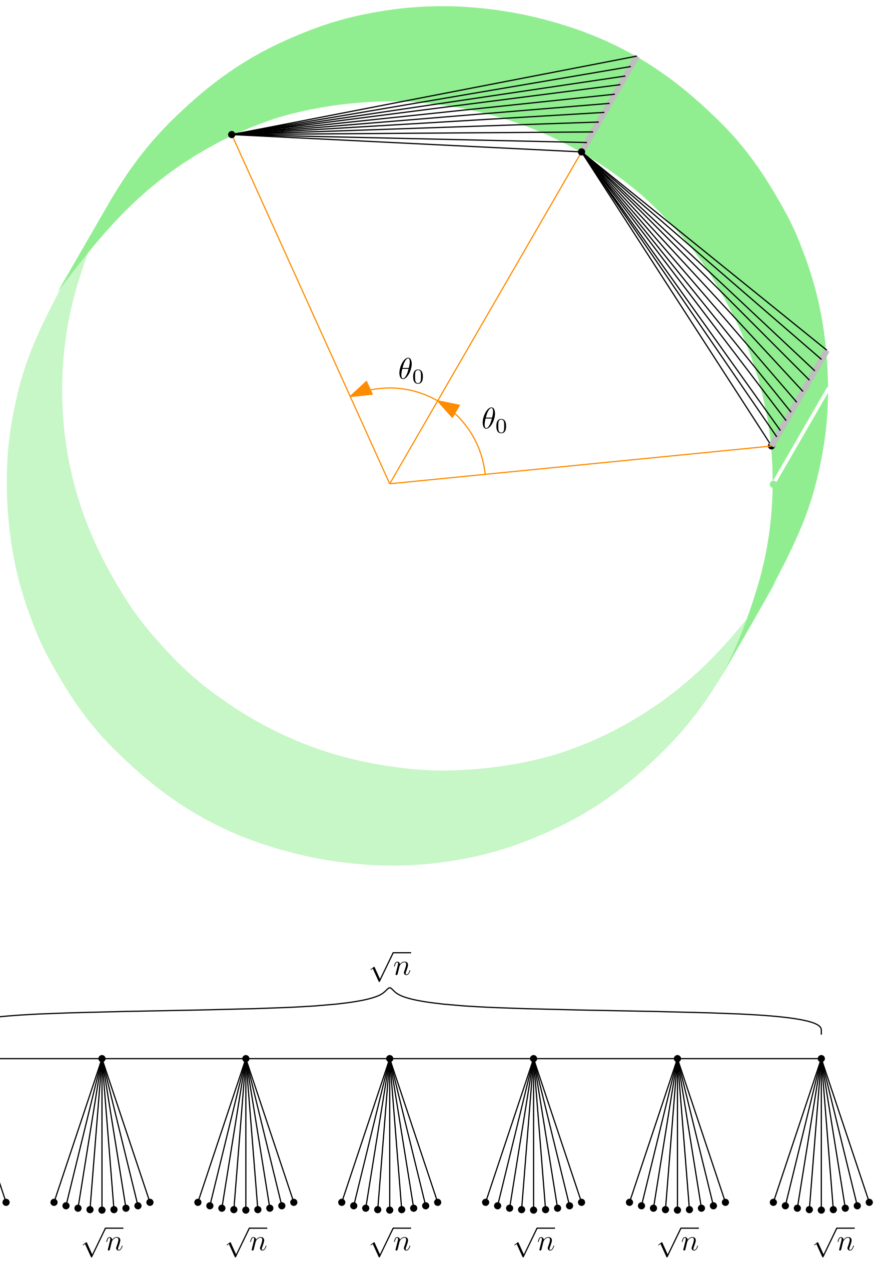

FIGURE 3. A modeling limit of a sequence of caterpillars. The domain of the modeling is $(S^2 \setminus \{0\}) \times [0, 1]$. In the picture, θ_0 stands for an irrational multiple of π. Note that the modeling has uncountably many connected components.

Theorem 16. *A set of sentences T in $\mathcal{L}(Q_m)$ has a totally Borel model if and only if T is consistent in $\mathcal{K}_m$.*

In particular, every FO-convergent sequence has a weak modeling limit.

However, we have no precise control on the measures assigned to first-order properties, and although weak modeling limits exist in general, modeling limits do not, as we shall see now.

By considering the fragment of first-order quantifier free formulas not using equality, we immediately deduce that every FO-convergent sequence

$(G_n)_{n\in\mathbb{N}}$ of graphs is also left-convergent. Also, every graph modeling defines a Borel graph. It follows that if FO-convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs has a graph modeling limit then it also has a random-free graphon as its left limit.

For instance, the existence of a weak modeling limit for a sequence of random graphs relates to the existence of a random-free graphon whose sampling gives is with high probability isomorphic to Rado graph (see [1] for more on this topic), although no random-free graphon is the limit of a (typical) sequence of random graphs.

From [2] and Lemma 11 follows the next theorem, which expresses a necessary condition for the existence of modeling FO-limits.

Theorem 17 ([62])**.** *Let $\mathcal{C}$ be a monotone class.*
If $\mathcal{C}$ has modeling limits then $\mathcal{C}$ is nowhere dense.

We actually conjectured that this condition is also sufficient [63]:

Conjecture 1*.* A monotone class of graphs $\mathcal{C}$ has modeling limits if and only if it is nowhere dense.

This may be regarded as the single most interesting problem raised in this paper.

6. Class speed, entropy and logarithmic densities

For a class of graphs $\mathcal{C}$ and an integer n we denote by $\mathcal{C}^n$ the set of all graphs in $\mathcal{C}$ with n vertices.

The following observation provides us much information.

Lemma 18 ([9])**.** *Let $\epsilon > 0$ and $0 < c \leq 2$. There is an N such that for all $n > N$, if S is a set of graphs on n vertices and $|S| > 2^{n^{2-c+\epsilon}}$, then there is a graph $G \in S$ with $\|G\| > 2^{2-c}$.*

The following trichotomy theorem was proved by the authors in [59]:

Theorem 19 (Trichotomy theorem)**.** *Let $\mathcal{C}$ be an infinite class of graphs (containing arbitrarily large non-discrete graphs). Then*

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\,r} \frac{\log\|G\|}{\log|G|} \in \{0,1,2\}.$$

Moreover, this limit is 2 if $\mathcal{C}$ is somewhere dense and at most 1 if $\mathcal{C}$ is nowhere dense.

The *entropy* $h(\mathcal{C})$ of a class of graphs $\mathcal{C}$ is defined as

$$h(\mathcal{C}) = \lim_{n\to\infty} \frac{\log|\mathcal{C}^n|}{\binom{n}{2}}.$$

It is known that for every hereditary property this limit exists and that it can only reach particular range of values (see [6]). It follows from Theorem 19 and Lemma 18 that the speed of a nowhere dense class $\mathcal{C}$ is bounded by

$$|\mathcal{C}^n| < 2^{n^{1+\epsilon}}$$

for every $\epsilon > 0$ and every sufficiently large n (i.e. $n > N(\mathcal{C}, \epsilon)$). It follows that the entropy of a nowhere dense class of graphs is 0. Precisely:

Theorem 20 (Nowhere dense by entropy). *Let $\mathcal{C}$ be an infinite monotone class of graphs. Then*

$$\lim_{r\to\infty} h(\mathcal{C} \,\widetilde{\nabla}\, r) \in \{0, 1\}.$$

Moreover, this value is 0 if $\mathcal{C}$ is nowhere dense and 1 if $\mathcal{C}$ is somewhere dense.

Note that the bound on $|\mathcal{C}^n|$ for a nowhere dense class $\mathcal{C}$ can be improved under stronger assumptions: If $\mathcal{C}$ is a class with expansion bounded by the function $f(r) = c^{r^{1/3-\epsilon}}$ (for arbitrary constants $c, \epsilon > 0$) then $|\mathcal{C}^n| < n!\,\alpha^n$ for some constant $\alpha > 0$, as proved by Dvořák and Norine [24].

Theorem 19 was generalized in [58] to logarithmic densities of arbitrary graph F:

Theorem 21 (Nowhere dense by counting). *Let $\mathcal{C}$ be an infinite class of graphs and let F be a graph with at least one edge. Then*

$$\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\,\widetilde{\nabla}\, r} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \ldots, \alpha(F), |F|\},$$

where $\alpha(F)$ is the stability number of F.

Moreover, this limit is $|F|$ if $\mathcal{C}$ is somewhere dense and at most $\alpha(F)$ if $\mathcal{C}$ is nowhere dense.

For a nowhere dense class of graphs, the integrality of the supremum of the logarithmic density holds without need for considering shallow topological minors:

Theorem 22 (Degree of freedom, [58]). *Let $\mathcal{C}$ be an infinite nowhere dense class of graphs and let F be a graph. Then*

$$\limsup_{G\in\mathcal{C}} \frac{\log(\#F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \ldots, \alpha(F)\};$$

this integer is called degree of freedom *of F in $\mathcal{C}$.*

7. Shallow Minors, Cliques and Density

Classes with bounded expansion, which have been introduced in [53], may be viewed as a relaxation of the notion of proper minor closed class. The original definition of classes with bounded expansion relates to the notion of shallow minor, as introduced by Plotkin, Rao, and Smith [67].

Definition 23. Let G, H be graphs with $V(H) = \{v_1, \ldots, v_h\}$ and let r be an integer. A graph H is a *shallow minor* of a graph G *at depth* r, if there exists disjoint subsets $A_1, \ldots, A_h$ of $V(G)$ such that (see Fig. 4)

- the subgraph of G induced by A_i is connected and as radius at most r,
- if v_i is adjacent to v_j in H, then some vertex in A_i is adjacent in G to some vertex in A_j.

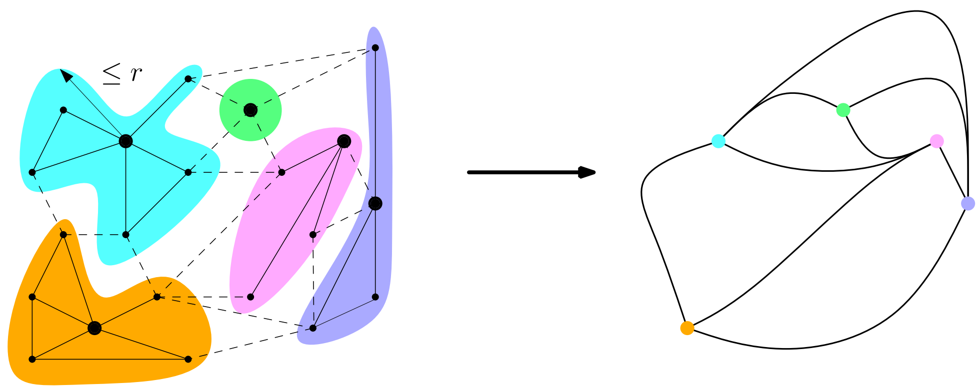

Figure 4. A shallow minor

We denote [53, 61] by $G \triangledown r$ the class of the (simple) graphs which are shallow minors of G at depth r, and we denote by $\nabla_r(G)$ the maximum density of a graph in $G \triangledown r$, that is:

$$\nabla_r(G) = \max_{H \in G \triangledown r} \frac{\|H\|}{|H|}$$

A class $\mathcal{C}$ has *bounded expansion* if $\sup_{G \in \mathcal{C}} \nabla_r(G) < \infty$ for each value of r.

Considering shallow minors may, at first glance, look arbitrary. Indeed one can define as well the notions of shallow topological minors and shallow immersions:

Definition 24. A graph H is a *shallow topological minor* at depth r of a graph G if some subgraph of G is isomorphic to a subdivision of H in which every edge has been subdivided at most $2r$ times (see Fig. 5).

We denote [53, 61] by $G \,\widetilde{\triangledown}\, r$ the class of the (simple) graphs which are shallow topological minors of G at depth r, and we denote by $\widetilde{\nabla}_r(G)$ the

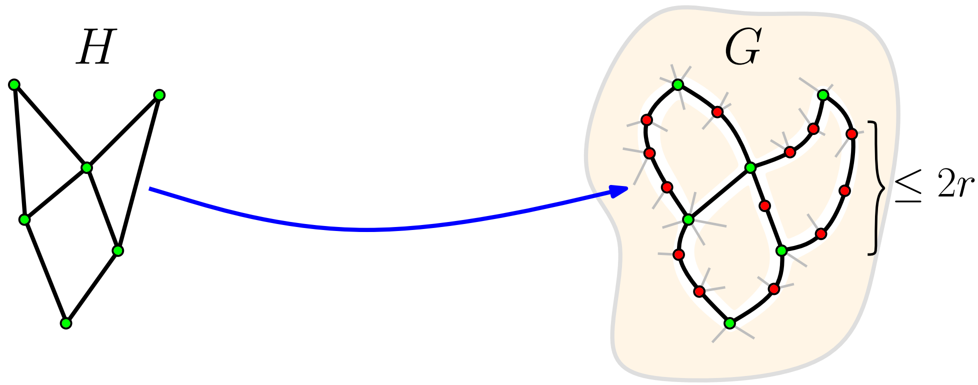

FIGURE 5. H is a shallow topological minor of G at depth r

maximum density of a graph in $G \,\widetilde{\triangledown}\, r$, that is:

$$\widetilde{\nabla}_r(G) = \max_{H \in G \,\widetilde{\triangledown}\, r} \frac{\|H\|}{|H|}$$

Note that shallow topological minors can be alternatively defined by considering how a graph H can be topologically embedded in a graph G: a graph H with vertex set $V(H) = \{a_1, \dots, a_k\}$ is a shallow topological minor of a graph G at depth r is there exists vertices $v_1, \dots, v_k$ in G and a family $\mathcal{P}$ of paths of G such that

- two vertices a_i and a_j are adjacent in H if and only if there is a path in $\mathcal{P}$ linking v_i and v_j;
- no vertex v_i is interior to a path in $\mathcal{P}$;
- the paths in $\mathcal{P}$ are internally vertex disjoint;
- every path in $\mathcal{P}$ has length at most $2r + 1$.

We can similarly define the notion of shallow immersion:

Definition 25. A graph H with vertex set $V(H) = \{a_1, \dots, a_k\}$ is a *shallow immersion* of a graph G at depth r is there exists vertices $v_1, \dots, v_k$ in G and a family $\mathcal{P}$ of paths of G such that

- two vertices a_i and a_j are adjacent in H if and only if there is a path in $\mathcal{P}$ linking v_i and v_j;
- the paths in $\mathcal{P}$ are edge disjoint;
- every path in $\mathcal{P}$ has length at most $2r + 1$;
- no vertex of G is internal to more than r paths in $\mathcal{P}$.

We denote [53, 61] by $G \,\overset{\propto}{\triangledown}\, r$ the class of the (simple) graphs which are shallow immersions of G at depth r, and we denote by $\overset{\propto}{\nabla}_r(G)$ the maximum density of a graph in $G \,\overset{\propto}{\triangledown}\, r$, that is:

$$\overset{\propto}{\nabla}_r(G) = \max_{H \in G \,\overset{\propto}{\triangledown}\, r} \frac{\|H\|}{|H|}$$

It appears that although minors, topological minors, and immersions behave very differently, their shallow versions are deeply related, as witnessed by the following theorem:

Theorem 26 (Bounded expansion invariance [61]). *Let $\mathcal{C}$ be a class of graphs. Then the following are equivalent:*

(1) the class $\mathcal{C}$ has bounded expansion;
(2) for every integer r it holds $\sup_{G\in\mathcal{C}} \nabla_r(G) < \infty$;
(3) for every integer r it holds $\sup_{G\in\mathcal{C}} \widetilde{\nabla}_r(G) < \infty$;
(4) for every integer r it holds $\sup_{G\in\mathcal{C}} \overset{\propto}{\nabla}_r(G) < \infty$;
(5) for every integer r it holds $\sup_{H\in\mathcal{C}\,\triangledown\, r} \chi(H) < \infty$;
(6) for every integer r it holds $\sup_{H\in\mathcal{C}\,\widetilde{\triangledown}\, r} \chi(H) < \infty$;
(7) for every integer r it holds $\sup_{H\in\mathcal{C}\,\overset{\propto}{\triangledown}\, r} \chi(H) < \infty$.

In the above theorem, we see that not only shallow minors, shallow topological minors, and shallow immersions behave closely, but that the (sparse) graph density $\|G\|/|G|$ and the chromatic number $\chi(G)$ of a graph G are also related. This last relation is intimately related to the following result of Dvorák [22].

Lemma 27. *Let $c \geq 4$ be an integer and let G be a graph with average degree $d > 56(c-1)^2\frac{\log(c-1)}{\log c - \log(c-1)}$. Then the graph G contains a subgraph G' that is the 1-subdivision of a graph with chromatic number c.*

It follows from Theorem 26 that the notion of class with bounded expansion is quite robust. Not only classes with bounded expansion can be defined by edge densities and chromatic number, but, as we shall see shortly, also by virtually all common combinatorial parameters [61].

Similarly to Theorem 26, we have several characterizations of nowhere dense classes.

Theorem 28 (Nowhere dense invariance [61]). *Let $\mathcal{C}$ be a class of graphs. Then the following are equivalent:*

(1) the class $\mathcal{C}$ is nowhere dense;
(2) for every integer r it holds $\limsup_{G\in\mathcal{C}} \frac{\log \nabla_r(G)}{\log |G|} = 0$;
(3) for every integer r it holds $\limsup_{G\in\mathcal{C}} \frac{\log \widetilde{\nabla}_r(G)}{\log |G|} = 0$;
(4) for every integer r it holds $\limsup_{G\in\mathcal{C}} \frac{\log \overset{\propto}{\nabla}_r(G)}{\log |G|} = 0$;
(5) for every integer r it holds $\sup_{H\in\mathcal{C}\,\triangledown\, r} \omega(H) < \infty$;
(6) for every integer r it holds $\sup_{H\in\mathcal{C}\,\widetilde{\triangledown}\, r} \omega(H) < \infty$;
(7) for every integer r it holds $\sup_{H\in\mathcal{C}\,\overset{\propto}{\triangledown}\, r} \omega(H) < \infty$.

8. Low tree-depth decomposition and covering

After the general grows of classes it is perhaps surprising that nowhere dense and bounded expansion classes can be described by the existence of decomposition and covering of very special type. We are back to simple (and algorithmic) graph theory.

The *tree-depth* of a graph is a minor monotone graph invariant that has been defined in [52], and which is equivalent or similar to the *rank function* (used for the analysis of countable graphs, see e.g. [66]), the *vertex ranking number* [20, 71], and the minimum height of an elimination tree [10]. Tree-depth can also be seen as an analog for undirected graphs of the cycle rank defined by Eggan [25], which is a parameter relating digraph complexity to other areas such as regular language complexity and asymmetric matrix factorization. This parameter is also deeply related to the fill-in of a sparse matrix during a Cholesky decomposition. The notion of tree-depth found a wide range of applications, from the study of non-repetitive coloring [28] to the proof of the homomorphism preservation theorem for finite structures [69]. Recall the definition of tree-depth (see Fig. 6):

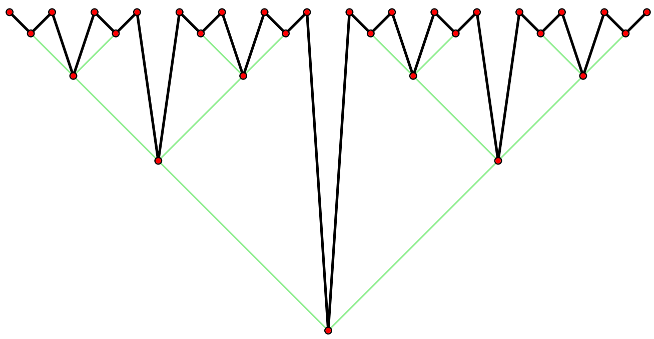

Figure 6. The tree-depth of the path P_{2^n-1} is n.

Definition 29. The *tree-depth* $\mathrm{td}(G)$ of a graph G is defined as the minimum height[1] of a rooted forest Y such that G is a subgraph of the closure of Y (that is of the graph obtained by adding edges between a vertex and all its ancestors). In particular, the tree-depth of a disconnected graph is the maximum of the tree-depths of its connected components.

Perhaps the most useful (certainly from the algorithmic point of view) is the following notion:

Definition 30. A *low tree-depth decomposition* with parameter p of a graph G is a coloring of the vertices of G, such that any subset I of at most p colors

[1]Here the height is defined as the maximum number of vertices in a chain from a root to a leaf

induce a subgraph with tree-depth at most $|I|$ (see Fig. 7. The minimum number of colors in a low tree-depth decomposition with parameter p of G is denoted by $\chi_p(G)$.

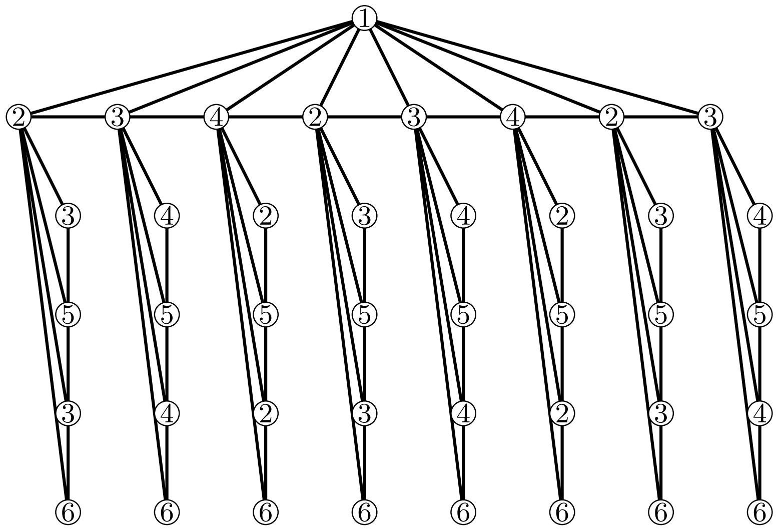

FIGURE 7. A low tree-depth decomposition with parameter $p = 2$: each color class induces a stable set (i.e. a graph with tree-depth 1) and every two color classes induce a star forest (i.e. a graph with tree depth 2).

For instance, $\chi_1(G)$ is the (standard) chromatic number of G, while $\chi_2(G)$ is the *star chromatic number* of G, that is the minimum number of colors in a proper vertex-coloring of G such that any two colors induce a star forest (see e.g. [8, 49]). For a more exhaustive survey on low tree-depth decomposition we refer the reader to [61, 64].

Theorem 31 (Nowhere dense by decomposition [61])**.** *Let $\mathcal{C}$ be a class of graphs, then the following are equivalent:*

(1) for every integer p it holds $\limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} = 0$;
(2) the class $\mathcal{C}$ is nowhere dense.

Moreover, if $\limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} \neq 0$ for some integer p, that is if $\mathcal{C}$ is somewhere dense, then for some integer p it holds $\limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log |G|} \geq 1/2$.

In a low tree-depth decomposition of a graph G by N colors and for parameter t, the subsets of t colors define a disjoint union of clusters that cover the graph, such that each cluster has tree-depth at most t, every vertex belongs to at most $\binom{N}{t}$ clusters, and every connected subgraph of order t is included in at least one cluster.

It is natural to ask whether the condition that such a covering comes from a coloring could be dropped. This is indeed so:

Theorem 32 ([64]). *A hereditary class $\mathcal{C}$ is nowhere dense if there exists a function f such that for every integer t and every $\epsilon > 0$, every graph $G \in \mathcal{C}$ of order $n \geq f(t, \epsilon)$ has a covering $C_1, \ldots, C_k$ of its vertex set such that*

- *each C_i induces a connected subgraph with tree-depth at most t;*
- *every vertex belongs to at most n^ϵ clusters;*
- *every connected subgraph of order at most t is included in at least one cluster.*

A similar statement holds if we weaken the condition that each cluster has tree-depth at most t while we strengthen the condition that every connected subgraph of order at most t is included in some cluster. Namely, we consider the question whether a similar statement holds if we allow each cluster to have radius at most $2t$ while requiring that every t-neighborhood is included in some cluster. In the context of their solution of model checking problem for nowhere dense classes (cf Theorem 45), Grohe, Kreutzer and Siebertz introduced in [34] the notion of r-neighborhood cover and proved that nowhere dense classes admit such cover with small maximum degree.

Precisely, for $r \in \mathbb{N}$, an *r-neighborhood cover* $\mathcal{X}$ of a graph G is a set of connected subgraphs of G called *clusters*, such that for every vertex $v \in V(G)$ there is some $X \in \mathcal{X}$ with $N_r(v) \subseteq X$. The *radius* $\mathrm{rad}(\mathcal{X})$ of a cover $\mathcal{X}$ is the maximum radius of its clusters. The *degree* $d^{\mathcal{X}}(v)$ of v in $\mathcal{X}$ is the number of clusters that contain v. The *maximum degree* $\Delta(\mathcal{X}) = \max_{v\in V(G)} d^{\mathcal{X}}(v)$. For a graph G and $r \in \mathbb{N}$ we define $\tau_r(G)$ as the minimum maximum degree of an r-neighborhood cover of radius at most $2r$ of G.

Theorem 33 ([34]). *Let $\mathcal{C}$ be a nowhere dense class of graphs. Then there is a function f such that for all $r \in \mathbb{N}$ and $\epsilon > 0$ and all graphs $G \in \mathcal{C}$ with $n \geq f(r, \epsilon)$ vertices, it holds $\tau_r(G) \leq n^\epsilon$.*

In other words, every infinite nowhere dense class of graphs $\mathcal{C}$ is such that

$$\sup_{r\in\mathbb{N}} \limsup_{G\in\mathcal{C}} \frac{\log \tau_r(G)}{\log |G|} = 0.$$

The following characterization of nowhere dense classes of graphs follows [64]:

Theorem 34 (Nowhere dense by covering)**.** *Let $\mathcal{C}$ be an infinite monotone class of graphs. Then*

$$\sup_{r\in\mathbb{N}} \limsup_{G\in\mathcal{C}} \frac{\log \tau_r(G)}{\log |G|}$$

is either 0 *if* $\mathcal{C}$ *is nowhere dense, at at least* 1/3 *if* $\mathcal{C}$ *is somewhere dense.*

9. Ordering and Locally Constrained Orientations

Here is a game theory relevant characterization. Kierstead and Yang introduced the r-coloring number of a graph for the purpose of studying coloring games and marking games on graphs [43].

Denote by $\Pi(G)$ the set of linear orders on the vertex set of a graph G. A vertex u is *weakly r-reachable* from v with respect to an order $<\in \Pi(G)$, if there exists a path P of length $0 \leq \ell \leq r$ between u and v such that u is minimum in $V(P)$. Let $\mathrm{Wreach}_r[G,<,v]$ be the set of vertices that are weakly r-reachable from v with respect to $<$.

Vertex u is *strongly r-reachable* from v with respect to an order $<\in \Pi(G)$, if there is a path P of length $0 \leq \ell \leq r$ connecting u and v such that $u \leq v$ and such that all inner vertices w of P satisfy $w > v$. Let $\mathrm{Sreach}_r[G,<,v]$ be the set of vertices that are strongly r-reachable from v with respect to $\leq$.

The *weak r-coloring number* $\mathrm{wcol}_r(G)$ of G is defined as

$$\mathrm{wcol}_r(G) = \min_{<\in\Pi(G)} \max_{v\in V(G)} |\mathrm{Wreach}_r[G,\leq,v]|,$$

and the r-coloring number $\mathrm{col}_r(G)$ of G is defined as

$$\mathrm{col}_r(G) = \min_{<\in\Pi(G)} \max_{v\in V(G)} |\mathrm{Sreach}_r[G,\leq,v]|.$$

These invariants are easily shown to be polynomially equivalent via the monotone path segmentation [43]:

$$\mathrm{col}_r(G) \leq \mathrm{wcol}_r(G) \leq \mathrm{col}_r(G)^r.$$

Zhu proved that these invariants are related to shallow minor densities as follows:

Theorem 35 ([78])**.** *For every integer r there exists a polynomial F_r such that for every graph G it holds*

$$\nabla_{\frac{r-1}{2}}(G) + 1 \leq \mathrm{wcol}_r(G) \leq F_r(\nabla_{\frac{r-1}{2}}(G)).$$

Lemma 36 ([78])**.** *For every graph G and every integer r it holds*

$$\chi_r(G) \leq \mathrm{wcol}_{2^{r-1}}(G).$$

Theorem 35 and Lemma 36 gives a bound on $\chi_r(G)$ in terms of $\nabla_{2^{r-2}-1/2}(G)$ A first (weaker) bound for χ_r was originally obtained by the authors by means of transitive fraternal augmentations in [53]. A more precise analysis of fraternal augmentations led to the following improved bounds

Theorem 37 ([61]). *For every integer r there exists a polynomial P_r such that for every graph G it holds*

$$\chi_r(G) \le P_r(\widetilde{\nabla}_{2^{p-2}+1/2}(G))$$

On the other hand, the following inequality is easily checked

Theorem 38 ([52]). *For every integer r and every graph G it holds*

$$\nabla_r(G) \le (2r+1)\binom{\chi_{2r+2}(G)}{2r+2}.$$

Definition 39. Let $\vec{G}$ be a directed graph and let k be an integer. A *fraternal augmentation* of $\vec{G}$ is an edge-labeled super-digraph $\vec{G}^+$ of $\vec{G}$ such that, denoting label(x,y) the (integer) label of arc (x,y) and putting

$$\ell(x,y) = \min\{\text{label}(x,z), \text{label}(y,z) :\ (x,z),(y,z) \in E(\vec{G}^+)\},$$

it holds:

(1) the arcs of $\vec{G}^+$ with label 1 are exactly the arcs of $\vec{G}$;
(2) for every distinct vertices u,v of $\vec{G}$, either $\ell(x,y) > k$ or

$$\min(\text{label}(x,y), \text{label}(y,x)) = \ell(x,y);$$

Given a fraternal augmentation $\vec{G}^+$ of a directed graph $\vec{G}$, we denote by $\Delta_i^-(\vec{G}^+)$ the maximum indegree of a vertex when considering arcs with label at most i only.

Theorem 40 ([61]). *For every integer k there exists a polynomial $Q_k(x,y)$ such that every directed graph $\vec{G}$ has a fraternal augmentation $\vec{G}^+$ such that*

$$\Delta_k^-(\vec{G}^+) \le Q_k(\Delta^-(\vec{G}), \widetilde{\nabla}_{\frac{k-1}{2}}(G)),$$

where G is the underlying undirected graph of $\vec{G}$.

In some sense, Theorem 40 can be seen as a local version of Theorem 35. In this version no condition of acyclicity is required on the orientation of the augmentation, but any arbitrary indegree bounded orientation can be required for the graph to be augmented.

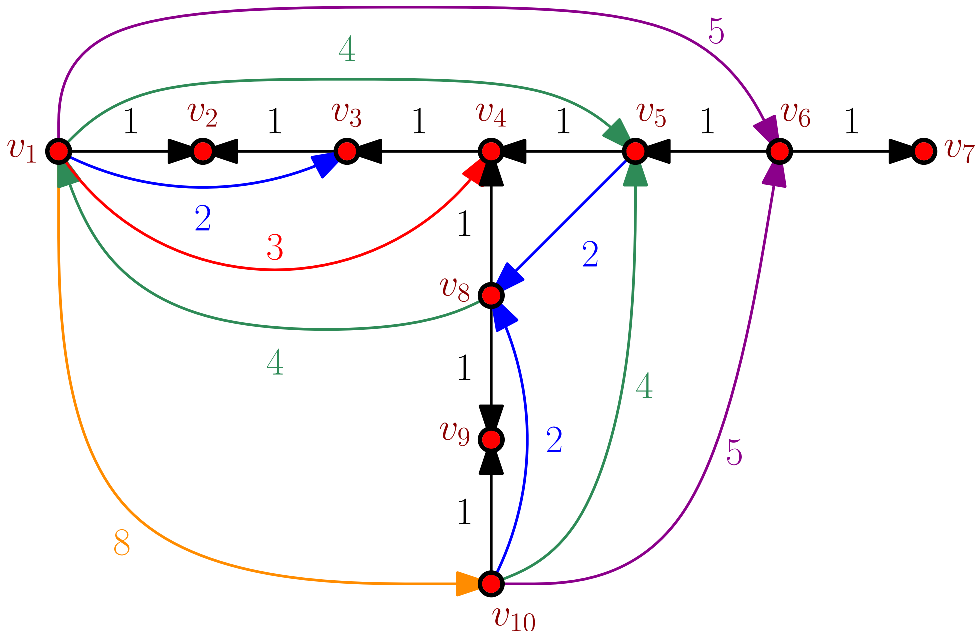

FIGURE 8. Example of a fraternal augmentation of a directed graph.

10. QUASI-WIDENESS AND RELATED

Atserias and Dawar defined the notions of wide, almost-wide and quasi-wide classes of graphs in the context of logic (cf. [18] for instance). Let $d \in \mathbb{N}$. A subset A of vertices of a graph G is *d-independent* if the distance between any two distinct vertices in A is strictly greater than d. A class $\mathcal{C}$ of graphs is *quasi-wide* if there exists functions $s : \mathbb{N} \to \mathbb{N}$ and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following (rather technical) property holds:

> For every integer d and every integer m, every graph $G \in \mathcal{C}$ with order at least $f(d, m)$ contains a subset S of size at most $s(d)$ so that $G - S$ has a d-independent set of size m.

Note that the key point of this property is that the bound $s(d)$ on the size of S does not depend on m.

The following is proved in [55]:

Theorem 41 (Nowhere dense by independence)**.** *Let $\mathcal{C}$ be a hereditary class of graphs. Then $\mathcal{C}$ is nowhere dense if and only if it is quasi-wide.*

Another variant of this characterization may be formulated as follows:

Theorem 42 (Nowhere dense by spreading)**.** *Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ is nowhere dense if and only if for every integer d and every $\epsilon > 0$ there is an integer N with the following property: for every graph $G \in \mathcal{C}$, and every subset A of vertices of G, there is $S \subseteq A$ with $|S| \le N$ such that no ball of radius d in $G[A \setminus S]$ has order greater than $\epsilon\,|A|$.*

Proof. Suppose that $\mathcal{C}$ is somewhere dense, that is that there exists $p \in \mathbb{N}$ such that for every integer n, the p-subdivision of K_n is a subgraph of some $G \in \mathcal{C}$. Assume for contradiction that there is an integer N with the property that for every graph $G \in \mathcal{C}$, and every subset A of vertices of G, there is $S \subseteq A$ with $|S| \leq N$ such that no ball of radius $2p$ in $G[A \setminus S]$ has order greater than $|A|/2$. Let $G \in \mathcal{C}$ be a graph that includes a p-subdivision of K_n as a subgraph, where $n > 2N$, and let A be the vertex set of this subgraph. Then, for every subset S of cardinality at most N, $G[A \setminus S]$ has order at least $|A| - N - \binom{N}{2}p > |A|/2$, what contradicts our hypothesis.

Conversely, considering the hereditary closure of $\mathcal{C}$, we can restrict ourselves to the case where $A = V(G)$. Let d be an integer and let $\epsilon > 0$ be a positive real. By Theorem 41, there is s and C such that if $|X| > C$ then one can find a subset Y of vertices of G with $|Y| \leq s$ such that in $G - Y$ at least $m = \lfloor 1/\epsilon \rfloor + s + 1$ vertices in $X \setminus Y$ are pairwise at distance $> 4d$. Let us prove by contradiction that the value $N = C$ has the property required by the Lemma: For $G \in \mathcal{C}$, let X be a subset of $V(G)$ with minimum cardinality, such that no ball of radius d in $G - X$ has order greater than $\epsilon\,|G|$. By minimality, every ball of radius $2d$ centered at a vertex in S contains at least $\epsilon\,|G|$ vertices. There exists a set Y of at most s vertices and a subset $A \subseteq X \setminus Y$ of at least m vertices pairwise at distance at least $4d$ in $G - Y$. However, in A there are at most $\lfloor 1/\epsilon \rfloor$ vertices that are centers of balls of radius $2d$ containing more than $\epsilon\,|G|$ vertices (as these balls are disjoint). As $m > \lfloor 1/\epsilon \rfloor + s$ there is a subset $A' \subseteq A$ such that $|A'| > s$ and for every $v \in A'$ it holds $|B_{2d}(G, v)| \leq \epsilon |G|$. Let $S = (A \setminus A') \cup Y$. Then no ball of radius d in $G - S$ has order greater than $\epsilon\,|G|$. However, one checks that $|S| < |X|$, what contradicts the minimality of X. □

In particular, this theorem implies that the removal of $N(d, \epsilon)$ vertices in a graph G in $\mathcal{C}$ results in a graph such that no ball of radius d has order greater than $\epsilon\,|G|$.

11. Algorithmic Consequences

From the point of view of theoretical computer science, of particular importance is the program of establishing fixed-parameter tractability (FPT) of model checking first order logic on sparse graphs. A long line of work resulted in FPT algorithms for model checking first order formulas on more and more general classes of sparse graphs. Finally, FPT algorithms for the problem have been given for graph classes of bounded expansion by Dvořák *et al.* [23], and very recently for nowhere dense graph classes by Grohe *et al.* [34]. This is the ultimate limit of this program: as proven in [23], for any monotone somewhere dense class $\mathcal{C}$, model checking first order formulas on $\mathcal{C}$ is not fixed-parameter tractable (unless FPT = W[1]).

Theorem 43 ([23]). *For every class $\mathcal{C}$ with bounded expansion, every property of graphs definable in first-order logic can be decided in time $O(n)$ on $\mathcal{C}$.*

The above theorem relies on low tree-depth decomposition. However, the next result, due to Kazana and Segoufin, is based on the notion of transitive fraternal augmentation, which was introduced in [53, 54] to construct low-tree depth decomposition.

Theorem 44 ([42]). *Let $\mathcal{C}$ be a class of graphs with bounded expansion and let ϕ be a first-order formula. Then, for all $G \in \mathcal{C}$, we can compute the number $|\phi(G)|$ of satisfying assignments for ϕ in G in in time $O(|G|)$.*

Moreover, the set $\phi(G)$ can be enumerated in lexicographic order in constant time between consecutive outputs and linear time preprocessing time.

The following result is also based on the transitive-fraternal augmentation algorithm.

Theorem 45 ([34]). *For every nowhere dense class $\mathcal{C}$ and every $\epsilon > 0$, every property of graphs definable in first-order logic can be decided in time $O(n^{1+\epsilon})$ on $\mathcal{C}$.*

This is a natural question whether Theorem 45 has a counting and enumeration version in the spirit of Theorem 44.

Fixed-parameter tractability of Dominating Set on nowhere dense graph classes follows immediately from the result of Grohe et al., since the problem is definable in first order logic. However, an explicit algorithm was given earlier by Dawar and Kreutzer [19]. The search for linear kernels for the Dominating Set problem on classes of graphs also followed a long line of results on more and more general classes of sparse graphs, from the work of Alber *et al.* [3] that established a linear kernel for the problem on planar graphs, linear kernels have been given for bounded-genus graphs [11], apex-minor-free graphs [29], H-minor-free graphs [30], and H–topological-minor-free graphs [31]. These efforts culminated in the following results (where $\mathrm{ds}(G)$ denotes the size of a minimum dominating set of graph G):

Theorem 46 ([21]). *Let $\mathcal{C}$ be a graph class of bounded expansion. There exists a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ and an integer k, either correctly concludes that $\mathrm{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size $O(k)$ with the property that $\mathrm{ds}(G) \leq k$ if and only if $\mathrm{ds}(G[Y])| \leq k$.*

Theorem 47 ([21]). *Let $\mathcal{C}$ be a nowhere dense class of graphs and let $\epsilon > 0$ be a real number. There exists a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ and an integer k, either correctly concludes that $\mathrm{ds}(G) > k$ or finds a subset of vertices $Y \subseteq V(G)$ of size $O(k^{1+\epsilon})$ with the property that $\mathrm{ds}(G) \leq k$ if and only if $\mathrm{ds}(G[Y]) \leq k$.*

12. Concluding Remarks

The research covered in this survey is by no means closed, and in fact it is reflecting fast development. Particularly Conjecture 1 seems to be of central importance and may shed some light on the Aldous-Lyons conjecture, as well as problems in group theory (see e.g. [26]). We tried to reflect this in extensive references.

We believe that this is an exciting topic, which spans several disciplines, and we hope that an interested reader will share this view.

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Jaroslav Nešetřil, Computer Science Institute of Charles University (IUUK and ITI), Malostranské nám. 25, 11800 Praha 1, Czech Republic
E-mail address: nesetril@iuuk.mff.cuni.cz

Patrice Ossona de Mendez, Centre d'Analyse et de Mathématiques Sociales (CNRS, UMR 8557), 190-198 avenue de France, 75013 Paris, France — and — Computer Science Institute of Charles University (IUUK), Malostranské nám.25, 11800 Praha 1, Czech Republic
E-mail address: pom@ehess.fr