RAMSEY PARTIAL ORDERS FROM ACYCLIC GRAPHS

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Abstract. We prove that finite partial orders with a linear extension form a Ramsey class. Our proof is based on the fact that the class of acyclic graphs has the Ramsey property and uses the partite construction.

§1. Introduction

Let $\mathcal{C}$ be a class of objects endowed with an isomorphism and a subobject relation. Given two objects P and Q from $\mathcal{C}$ we write $\binom{Q}{P}$ for the set of all subobjects of Q isomorphic to P. Also for $P' \in \binom{Q}{P}$ we will refer to an isomorphism $f\colon P \to P'$ as an *embedding* of P to Q.

For three objects P, Q, $R \in \mathcal{C}$ and a positive integer r the partition symbol

$$R \longrightarrow (Q)^P_r$$

means that no matter how $\binom{R}{P}$ gets colored by r colors there is some $\tilde{Q} \in \binom{R}{Q}$ for which $\binom{\tilde{Q}}{P}$ is monochromatic.

The class $\mathcal{C}$ is said to have the *P-Ramsey property* if for every $Q \in \mathcal{C}$ and every positive integer r there exists some $R \in \mathcal{C}$ with $R \longrightarrow (Q)^P_r$. Notice that this is equivalent to demanding that for every $Q \in \mathcal{C}$ there is some $R \in \mathcal{C}$ with $R \longrightarrow (Q)^P_2$. Therefore, we will from now on only discuss the case $r = 2$.

Finally, $\mathcal{C}$ is a *Ramsey class* if it has the P-Ramsey property for every $P \in \mathcal{C}$.

Ramsey classes form a fertile area of study. The original combinatorial motivation was complemented by the relationship to model theory, topological dynamics and ergodic theory.

Among the first combinatorial structures whose Ramsey properties were studied is the class $\mathcal{P}$ of partially ordered sets considered in [6] and in [9], where all partially ordered sets P for which $\mathcal{P}$ posseses the P-Ramsey property were characterised. These are precisely the partial orders P with the property that for any two linear extensions $P_1 = (P, \leqslant_1)$ and $P_2 = (P, \leqslant_2)$ of P there is an isomorphism between P_1 and P_2 which preserves both the partial and the linear order.

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Thus it is natural to consider partial orders with linear extensions. An *ordered* (finite) *poset* P is a poset (X, R) together with a linear extension $\leqslant$. We will write $P = (X, R, \leqslant)$ and also $X = X(P)$, $R = R(P)$, $\leqslant = \leqslant_P$.

An *embedding* of an ordered poset P into an ordered poset P' is an injective map $f\colon X(P) \to X(P')$ which satisfies

$$(x, y) \in R(P) \iff (f(x), f(y)) \in R(P')$$

and

$$x \leqslant_P y \iff f(x) \leqslant_{P'} f(y)\,.$$

As a consequence of the main result of this article, Theorem 1.8, we derive the following.

Theorem 1.1. *The class $\mathcal{P}$ of all ordered posets is a Ramsey class.*

This theorem was mentioned in the survey paper [3] without proof referring to [1] and [4] from which this result can be deduced (see also [6]). In this paper we carry out the details of such a proof. We mention that similar results were proved in [9] and [2] and the theorem was explicitly stated and proved in [10] (see also [11] and [12]). The method used in those four papers is different from the one we are using here.

In the proof we shall make use of the following notions:

- An *ordered acyclic graph* is an oriented graph (X, R) together with a linear order $\leqslant$ on X satisfying $(x, y) \in R \implies x < y$.
- By *ACYC* we denote the class of all ordered acyclic graphs with monotone embeddings.

As a special case of the result of [1] and [4] (see also [5]), *ACYC* is a Ramsey class. For the purposes of this article, it is actually more convenient to utilise a slight strengthening of this fact speaking about ordered structures with two graph relations rather than one. More precisely, these structures are defined as follows:

Definition 1.2. An *RN graph* $(X, R, N, \leqslant)$ consists of a linear order $\leqslant$ on X and two acyclic relations $R, N \subseteq X \times X$ which are

(i) disjoint (i.e., $R \cap N = \varnothing$) and
(ii) compatible with $\leqslant$ (i.e., both $R \subseteq \leqslant$ and $N \subseteq \leqslant$).

For an *RN* graph $A = (X, R, N, \leqslant)$ we will write $X = X(A)$, $R = R(A)$, $N = N(A)$, and $\leqslant = \leqslant_A$. Observe that the definition of *RN* graphs does not require

$$<_A = \{(x, y);\, x \leqslant_A y \text{ and } x \neq y\}$$

to be the union of $R(A)$ and $N(A)$. We call an ordered RN graph A *complete* if $<_A = R(A) \cup N(A)$ holds. Observe that any ordered poset $P = (X, R, \leqslant)$ can be expanded to a complete RN graph $(X, R, N, \leqslant)$ with $N = < - R$. This construction will allow us to regard ordered posets as complete RN graphs in Theorem 1.8 below.

Embeddings between RN graphs are defined in the expected way:

Definition 1.3. For two RN graphs A and B an embedding from A to B is an injective map $f \colon X(A) \to X(B)$ such that

- $(x, y) \in R(A) \iff \big(f(x), f(y)\big) \in R(B)$,
- $(x, y) \in N(A) \iff \big(f(x), f(y)\big) \in N(B)$,
- and $x \leqslant_A y \iff f(x) \leqslant_B f(y)$.

The following result is still a special case of the main theorems from [1] and [4], and its proof is not much harder than just showing that $ACYC$ is a Ramsey class.

Theorem 1.4. *The class $ACYC_{RN}$ of all RN graphs is a Ramsey class.*

The proof of Theorem 1.1 given below will utilise Theorem 1.4. It would be possible to base a very similar proof just on the fact that $ACYC$ is a Ramsey class, but at one place the details would be slightly more cumbersome and from today's perspective it does not seem to be worth the effort.

We refine the above Theorem 1.4 by means of the following concepts:

Definition 1.5. A *bad quasicycle* of length $j \geqslant 2$ in an RN graph $(X, R, N, \leqslant)$ consists of j vertices $x = x_1, x_2, \ldots, x_j = y$ with $(x_i, x_{i+1}) \in R$ for $i = 1, 2, \ldots, j-1$ and $(x, y) \in N$.

Definition 1.6. For an integer $\ell \geqslant 2$ the RN graph $(X, R, N, \leqslant)$ is called an *ℓ-RN graph* if it does not contain a bad quasicycle of length j for any $j \in [2, \ell]$.

Notice that due to condition *(i)* from Definition 1.2 every RN graph is also a 2-RN graph.

Definition 1.7. We will say that an RN graph is *good* if it contains for no $\ell \geqslant 3$ a bad quasicycle of length ℓ.

(Consequently, any RN graph $(X, R, N, \leqslant)$, where $(X, R, \leqslant)$ is a poset, is also good.) In the result that follows, ordered posets are regarded as complete RN graphs in the way that was explained after Definition 1.2.

Theorem 1.8. *Let A and B be two ordered posets viewed as complete RN graphs. There exists a sequence of RN graphs $C_2, C_3, \ldots$ such that for every $\ell \geqslant 2$*

(1) $C_\ell \to (B)^A_2$,

(2) *C_ℓ is an ℓ-RN graph,*

(3) *and there is a homomorphism $h_\ell\colon C_{\ell+1} \to C_\ell$.*

In particular, $h^*_\ell = h_{\ell-1} \circ \cdots \circ h_2$ is a homomorphism from C_ℓ to C_2.

We conclude this introduction by showing that Theorem 1.8 implies Theorem 1.1.

To this end, let A and B be two given ordered posets viewed as complete RN graphs. Consider a sequence $C_2, C_3, \dots$ as guaranteed by Theorem 1.8. Set $|X(C_2)| = \lambda$ and consider the λ-RN graph C_λ with homomorphism $h^*_\lambda\colon C_\lambda \to C_2$ just obtained.

Since C_λ contains no bad quasicycle of length $\ell \leqslant \lambda$, while due to the existence of the homomorphism $h^*_\lambda\colon C_\lambda \to C_2$ no direct path in C_λ has more than $\lambda = |X(C_2)|$ vertices, we infer that the transitive closure R^T of $R = R(C_\lambda)$ is disjoint with $N(C_\lambda)$. Consequently, if we take the transitive closure of $R(C_\lambda)$, all copies of A and B in C_λ (which are complete RN graphs) remain intact (i.e., contain no edges added by taking the transitive closure). In other words, the partial order $C = \bigl(X(C_\lambda), R^T\bigr)$ satisfies $\binom{C}{B} \supseteq \binom{C_\lambda}{B}$.

Consequently, $C \to (B)^A_2$ and Theorem 1.1 follows.

§2. Proof of Theorem 1.8

Throughout this section we fix two ordered posets A and B, for which we want to prove Theorem 1.8.

The desired sequences of RN graphs (C_ℓ) and homomorphisms (h_ℓ) will be constructed recursively, beginning with the construction of C_2. For this purpose we invoke Theorem 1.4, which applied to A and B yields the desired RN graph C_2 with $C_2 \to (B)^A_2$.

Now suppose that for some integer $\ell \geqslant 3$ we have already managed to construct an $(\ell-1)$-RN graph $C_{\ell-1}$ with $C_{\ell-1} \to (B)^A_2$. To complete the recursive construction we are to exhibit an ℓ-RN graph C_ℓ satisfying $C_\ell \to (B)^A_2$ together with a homomorphism $h_{\ell-1}$ from C_ℓ to $C_{\ell-1}$.

To this end we employ the partite construction. In fact this proof is a variant of the proofs given in [7] and [8].

An essential component of the partite construction is a *partite lemma*, which will be described first.

2.1. **Partite Lemma.** Recalling that A is a good complete RN graph, we have a linear order $\leqslant_A$ on $X(A)$ extending $R(A)$. Let us write $X(A) = \{v_1, v_2, \dots, v_p\}$ in such a way that $v_1 <_A v_2 <_A \cdots <_A v_p$.

Definition 2.1. An *ordered A-partite RN graph E* is an RN graph with a distinguished partition $X(E) = X_1(E) \cup \ldots \cup X_p(E)$ of its vertex set satisfying

(*i*) $(x, y) \in R(E) \cap \big(X_i(E) \times X_j(E)\big) \implies (v_i, v_j) \in R(A)$,
(*ii*) $(x, y) \in N(E) \cap \big(X_i(E) \times X_j(E)\big) \implies (v_i, v_j) \in N(A)$,
(*iii*) and $X_1(E) <_E X_2(E) <_E \cdots <_E X_p(E)$.

Note that an ordered A-partite RN graph can also be viewed as an RN graph with a distinguished homomorphism into A. We observe the following:

Fact 2.2. For every A-partite RN graph E the following holds:

(*a*) If $(x, y) \in R(E) \cup N(E)$ and $(x, y) \in X_i(E) \times X_j(E)$, then $i < j$. In particular,

$$\big(R(E) \cup N(E)\big) \;\cap\; \big(X_i(E) \times X_i(E)\big) = \varnothing \quad \text{for all } i = 1, 2, \ldots, p\,.$$

(*b*) Any copy of A in E (i.e., any $\tilde{A} \in \binom{E}{A}$) is *crossing* in the sense that

$$\big|X\big(\tilde{A}\big) \cap X_i(E)\big| = 1 \quad \text{holds for all } i = 1, 2, \ldots, p\,.$$

(*c*) E is good.

Proof. Part (*a*) follows directly from Definition 2.1 (*i*) and (*ii*) as well as from our choice of the enumeration $\{v_1, v_2, \ldots, v_p\}$.

In order to deduce part (*b*) we note that the "in particular"-part of (*a*) entails $\big|V\big(\tilde{A}\big) \cap X_i(E)\big| \leqslant 1$ for all $i \in [p]$. Owing to $|X(\tilde{A})| = p$, we must have equality in all these estimates, so $\tilde{A}$ is indeed crossing.

To verify (*c*) we assume for the sake of contradiction that $\{x_1, x_2, \cdots, x_\ell\}$ is the vertex set of a bad quasicycle with $(x_i, x_{i+1}) \in R(E)$ for $i = 1, 2, \cdots, \ell - 1$, while $(x_1, x_\ell) \in N(E)$. Let $\psi\colon X(E) \longrightarrow X(A)$ be the projection sending for each $i \in [p]$ the set $X_i(E)$ to v_i. Due to the conditions (*i*) and (*ii*) from Definition 2.1 we get $\big(\psi(x_i), \psi(x_{i+1})\big) \in R(A)$ for $i \in [\ell - 1]$ while $\big(\psi(x_1), \psi(x_\ell)\big) \in N(A)$. In other words, $\{\psi(x_1), \ldots, \psi(x_\ell)\}$ is a bad quasicycle in A. This, however, contradicts the fact that A is a good RN graph. □

Definition 2.3. For two ordered A-partite RN graphs E and F an *embedding of E into F* is an injection $f\colon X(E) \to X(F)$ which is

(*i*) order preserving with respect to $<_E$ and $<_F$, and satisfies
(*ii*) $f\big(X_i(E)\big) \subseteq X_i(F)$ for all $i = 1, 2, \ldots, p$ as well as
(*iii*) $(x, y) \in R(E) \iff \big(f(x), f(y)\big) \in R(F)$ and
$(x, y) \in N(E) \iff \big(f(x), f(y)\big) \in N(F)$.

Similarly as before the image $f(E) = \tilde{E}$ of such an embedding is called a *copy* of E and by $\binom{F}{E}$ we will denote the set of all copies of E in F.

The next lemma is an important component of partite amalgamation:

Lemma 2.4 (Partite Lemma)**.** *For every ordered A-partite RN graph E there exists an ordered A-partite RN graph F with $F \longrightarrow (E)^A_2$. In other words, F has the property that any 2-colouring of $\binom{F}{A}$ yields a copy $\tilde{E} \in \binom{F}{E}$ such that $\binom{\tilde{E}}{A}$ is monochromatic.*

We derive the partite Lemma 2.4 as a direct consequence of Theorem 1.4.

Proof of Lemma 2.4. Let E be ordered A-partite RN graph with the notation as in Definition 2.1

By Theorem 1.4 there exists an RN graph $\bar{F}$ with $\bar{F} \longrightarrow (E)^A_2$.

Let F be the ordered A-partite RN graph constructed as follows:

- Its partition classes are $X_i(F) = \{v_i\} \times X(\bar{F})$ for $i = 1, \dots, p$.
- The vertex set of F is ordered by the lexicographic ordering induced by $\leqslant_A$ and $\leqslant_{\bar{F}}$.
- Both $R(F)$ and $N(F)$ are obtained by taking the usual direct (or categorical) product of A and $\bar{F}$, i.e.,

$$\left.\begin{aligned} &\big((a,u),(a',u')\big) \in R(F) \iff (a,a') \in R(A) \text{ and } (u,u') \in R(\bar{F}) \\ \text{and}& \\ &\big((a,u),(a',u')\big) \in N(F) \iff (a,a') \in N(A) \text{ and } (u,u') \in R(\bar{F})\,. \end{aligned}\right\} \quad (\star)$$

We claim that $F \longrightarrow (E)^A_2$.

Indeed, consider an arbitrary 2-coloring of $\binom{F}{A}$ by red and blue. For each $A' \in \binom{\bar{F}}{A}$, where

$$X(A') = \{x_1 <_{\bar{F}} x_2 <_{\bar{F}} \cdots <_{\bar{F}} x_p\}\,,$$

the set $\big\{(v_i, x_i); i = 1, \dots, p\big\}$ induces a unique copy of A in F. Consequently, the coloring of $\binom{F}{A}$ yields an auxiliary coloring of $\binom{\bar{F}}{A}$ by red and blue. Since $\bar{F} \longrightarrow (E)^A_2$, there is a monochromatic $E' \in \binom{\bar{F}}{E}$. Due to property *(iii)* of Definition 2.1 we have

$$X_1(E') < X_2(E') < \cdots < X_p(E')\,,$$

and thus the set

$$\bigcup_{i=1}^{p} \{(v_i, x); x \in X_i(E'), i = 1, \dots, p\}$$

induces a monochromatic A-partite copy of E in F.

Finally we note that due to $(\star)$, F is an A-partite RN graph and consequently, due to Fact 2.2 (c), F is a good RN graph. $\square$

2.2. **Partite Construction.** Recall that within the proof of Theorem 1.8 we are currently in the situation that for some $\ell \geqslant 3$ an $(\ell-1)$-*RN* graph $C_{\ell-1}$ with $C_{\ell-1} \to (B)^A_2$ is given. We are to prove the existence of an ℓ-*RN* graph C_ℓ with $C_\ell \to (B)^A_2$ and the additional property that there exists a homomorphism $h_{\ell-1}$ from C_ℓ to $C_{\ell-1}$.

To accomplish this task we will utilise the partite construction (see e.g. [7], [8]). Set $D = C_{\ell-1}$ and let $\binom{D}{A} = \{A_1, \dots, A_\alpha\}$, $\binom{D}{B} = \{B_1, \dots, B_\beta\}$. Set $|X(D)| = d$ and without loss of generality assume that $X(D) = \{1, 2, \dots, d\}$.

We are going to introduce D-partite ordered *RN* graphs $P_0, P_1, \dots, P_\alpha$, i.e., ordered *RN*-graphs with the property that for $j = 0, 1, \dots, \alpha$ the mapping $f_j \colon X(P_j) \to \{1, 2, \dots, d\}$, which maps each $x \in X_i(P_j)$ to i is a homomorphism from P_j to D.

The *RN* graph P_0 is formed by β vertex disjoint copies $\tilde{B}_1, \tilde{B}_2 \dots, \tilde{B}_\beta$ of B placed on the partite sets $X_i(P_0)$, $i = 1, 2, \dots, d$ of cardinalities $|X_i(P_0)| = |\{h \in [\beta]; i \in V(B_h\}|$ in such a way that for each $h = 1, 2, \dots, \beta$ we have

$$|X(\tilde{B}_h) \cap X_i(P_0)| = \begin{cases} 1 & \text{if } i \in X(B_h), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the mapping f_0 which for all $i \in \{1, 2, \dots d\}$ sends all elements $x \in X_i(P_0)$ to $\{i\}$ is a homomorphism.

Moreover, P_0 is a good *RN* graph, and thus, in particular, it is an ℓ-*RN* graph.

Next we assume that for some $j < \alpha$ a D-partite *RN* graph P_j together with a homomorphism $f_j \colon P_j \to D = C_{\ell-1}$ satisfying $X_i(P_j) = f_j^{-1}(i)$ for each $i \in X(D)$ has been constructed. We are going to describe the construction of P_{j+1}. To this end we consider the copy $A_{j+1} \in \binom{D}{A}$, let

$$X(A_{j+1}) = \{v_1 < v_2 < \dots < v_p\}$$

and let E_{j+1} be the ordered A-partite *RN* subgraph of P_j induced on the set $\bigcup_{t=1}^p X_{v_t}(P_j)$.

Applying the Partite Lemma to E_{j+1} yields an ordered A-partite *RN* graph F_{j+1} such that $F_{j+1} \to (E_{j+1})^{A_{j+1}}_2$.

Set $\mathcal{E}_{j+1} = \binom{F_{j+1}}{E_{j+1}}$ and extend each copy $E' \in \mathcal{E}_{j+1}$ to a copy $P'_j = P_j(E')$ of P_j in such a way that, for any $E', E'' \in \mathcal{E}_{j+1}$, the vertex intersection of $P'_j = P_j(E')$ and $P''_j = P_j(E'')$ is the same as the vertex intersection of E' and E''. In other words

$$X_i(P'_j) \cap X_i(P''_j) = \begin{cases} X_i(E') \cap X_i(E'') & \text{if } i \in X(A_{j+1}) \\ \varnothing & \text{otherwise.} \end{cases}$$

Finally, let P_{j+1} be the D-partite graph which is the union of all such copies of P_j, i.e., more formally

$$X_i(P_{j+1}) = \bigcup \{X_i\big(P_j(E)\big)\,;\, E \in \mathcal{E}_{j+1}\}$$

for all $i = 1, 2, \dots, d$ and

$$R(P_{j+1}) = \bigcup \{R\big(P_j(E)\big)\,;\, E \in \mathcal{E}_{j+1}\big\}\,,$$
$$N(P_{j+1}) = \bigcup \{N\big(P_j(E)\big)\,;\, E \in \mathcal{E}_{j+1}\}$$

and let $<_{P_{j+1}}$ be any linear order on $\bigcup_{i=1}^{d} X_i(P_{j+1})$ satisfying

$$X_1(P_{j+1}) <_{P_{j+1}} \dots <_{P_{j+1}} X_d(P_{j+1})\,.$$

Finally, let $f_{j+1}\colon X(P_{j+1}) \to X(D) = \{1, 2, \dots, d\}$ satisfy $f_{j+1}(x) = i$ for all $x \in X_i(P_{j+1})$ and $i = 1, 2, \dots, d$. Due to the construction above and the fact that $f_j\colon X(P_j) \to X(D)$ is a homomorphism, the mapping f_{j+1} is a homomorphism as well.

The crucial part of our argument will be the verification of the following

Claim 2.5. *If P_j is an ℓ-RN graph, then so is P_{j+1}.*

Once this is shown we will know that, in particular, P_α is an ℓ-RN graph. Moreover, a standard argument (see e.g. [8]) shows that $P_\alpha \to (B)^A_2$. Indeed, any red/blue colouring of $\binom{P_\alpha}{A}$ yields a copy of $P_{\alpha-1}$ in which all copies $\tilde{A}$ of A with $f_\alpha(\tilde{A}) = A_\alpha$ are the same colour. By iterating this argument we eventually obtain a copy $\tilde{P}_0$ of P_0 such that the colour of any crossing copy $\tilde{A} \in \binom{\tilde{P}_0}{A}$ depends only on $f_\alpha(\tilde{A})$. Owing to $C_{\ell-1} \to (B)^A_2$ this leads to a monochromatic copy of B in P_α.

For these reasons, the recursion step in the proof of Theorem 1.8 can be completed with the stipulations $C_\ell = P_\alpha$ and $h_{\ell-1} = f_\alpha$.

Proof of Claim 2.5. Assume that $(x, y) \in N(P_{j+1})$ and that there is an oriented path $x = x_1, \dots, x_{\ell'} = y$ in $R(P_{j+1})$, where $\ell' \leqslant \ell$. Note that since $f_{j+1} : P_{j+1} \to D = C_{\ell-1}$ is a homomorphism into the $(\ell-1)$-RN graph $C_{\ell-1}$ (containing no bad quasicycle of length $\leqslant \ell - 1$) we can assume that $\ell' = \ell$.

By the definition of $N(P_{j+1})$ there exists a copy $E' \in \mathcal{E}_{j+1}$ such that $x, y \in X\big(P_j(E')\big)$. On the other hand, since P_j is an ℓ-RN graph by assumption, not all edges of the path $x_1, \dots, x_\ell$ belong to $P_j(E')$. This together with the fact that x and y are in the same copy of P_j implies that the set

$$S = \big\{f_{j+1}(x_i)\,;\, i = 1, \dots, \ell\big\} \cap X(A_{j+1})$$

satisfies $|S| \geqslant 2$.

We further claim that for some r and s with $s - r \geqslant 2$ both $f_{j+1}(x_r)$ and $f_{j+1}(x_s)$ belong to $X(A_{j+1})$. Otherwise for some r we would have $S = \{f_{j+1}(x_r), f_{j+1}(x_{r+1})\}$. This, however, would mean that all vertices of the quasicycle would have to belong to $P_j(E')$, contrary to the assumption that P_j is an ℓ-RN graph.

Now consider $\{f_{j+1}(x_r), f_{j+1}(x_s)\} \subseteq X(A_{j+1})$ with $s-r \geqslant 2$. Due to the fact that A_{j+1} is a complete RN graph either $\big(f_{j+1}(x_r), f_{j+1}(x_s)\big) \in R(A_{j+1})$ or $\big(f_{j+1}(x_r), f_{j+1}(x_s)\big) \in N(A_{j+1})$.

If the former holds, then we get a contradiction, since

$$f_{j+1}(x_1), f_{j+1}(x_2), \ldots, f_{j+1}(x_r), f_{j+1}(x_s), \ldots, f_{j+1}(x_\ell)$$

would be a quasicycle of length $\leqslant \ell - 1$ in $C_{\ell-1}$.

This argument proves that for any $r, s \in \{1, 2, \ldots, \ell\}$ with

$$s - r \geqslant 2 \quad \text{and} \quad \{f_{j+1}(x_r), f_{j+1}(x_s)\} \subseteq X(A_{j+1})$$

we have $\big(f_{j+1}(x_r), f_{j+1}(x_s)\big) \in N(A_{j+1})$.

Now suppose that there is a pair (r, s) with the above properties satisfying in addition $(r, s) \neq (1, \ell)$. Then $f_{j+1}(x_r), \ldots, f_{j+1}(x_s)$ would be a bad quasicycle in $C_{\ell-1}$ whose length is at most $\ell - 1$, which is again a contradiction.

Thus either $\ell = 3$ and $S = \{f_{j+1}(x_1), f_{j+1}(x_2), f_{j+1}(x_3)\}$ or $S = \{f_{j+1}(x_1), f_{j+1}(x_\ell)\}$. The first alternative cannot happen, since A is good. If the second possibility happens, there is a copy $E'' \in \mathcal{E}_{j+1}$ such that all the vertices $x_1, \ldots, x_\ell$ belong to $P_j(E'')$. But, since $P_j(E'')$ is an induced copy of P_j in P_{j+1}, this means that there is a bad quasicycle of length ℓ in $P_j(E'')$, which contradicts our assumption about P_j. □

As we observed after stating Claim 2.5, the proof of Theorem 1.8 is thereby complete.

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