ITI Series (IUUK)

Institut teoretické informatiky
Institute for Theoretical
Computer Science

Informatický ústav Univerzity Karlovy
Computer Science Institute
of Charles University

2024-687

John Gimbel

A conjecture on rainbow connectivity

Institute for Theoretical
Computer Science (ITI)
Charles University

Computer Science Institute
of Charles University
(IUUK)

Malostranské náměstí 25
118 00 Praha 1
Czech Republic

http://iti.mff.cuni.cz/series/

http://iuuk.mff.cuni.cz/series/

A conjecture on rainbow connectivity

By John Gimbel

University of Alaska

Abstract

We show that if *G* is a graph of order $n \geq 3$ with minimum degree at least $\frac{n}{2} + \log_2(n) - 1$ then the edges of *G* can be colored with two colors so that every pair of vertices is joined by a path having distinct colors on each edge.

Introduction. In the following all graphs will be finite and simple, that is with no loops, arcs nor multiple edges. For undefined terms and concepts the reader is referred to [2]. In this missive, a *coloring* will be an assignment of colors to the edges of a graph. We will say a path in a colored graph is *rainbow* if each edge in the path is assigned a distinct color. A coloring of a graph is *rainbow connected* if each pair of distinct vertices is joined by a rainbow path. This missive is motivated by the problem: is there a constant *c* with the property that any graph of order *n* with minimum degree at least $\frac{n}{2} + c$ can be rainbow connected. Professor Xueliang Li asked this in a private communication and it appears in [1]. We do not answer this question but would like to offer the affirmative answer as a conjecture. It is unknown if the conjecture is true even in the case where $c=0$. We can, however, note that if true, $c \geq -\frac{1}{2}$. So essentially *c* must be nonnegative. We establish this with two constructions.

First, we note that if *G* can be rainbow colored with *k* colors then *k* must be at least as large as the diameter of *G*. Hence, if *G* is the complete bipartite graph $K_{m,m}$ minus an edge, it cannot be rainbow colored with only two colors. In this example, with order $n=2m$, we see that the minimum degree of G is $\frac{n}{2} - 1$.

For odd values of *n* consider the following. Let *A* and *B* be disjoint sets of *m* vertices that induce cliques. Add adjacent vertices *u* and *v* where *u* is adjacent to all vertices in *A* and *v* is adjacent to all vertices in *B*. Now add one more vertex *w* that is adjacent to all vertices except *u* and *v*. Note that this graph cannot be rainbow connected with just two colors. Further, the order is $n=2m+3$. The minimum degree is $m+1=(n-1)/2$.

While it is unclear if the previous conjecture is correct, the following shows that the constant can be replaced with a slowly growing function.

Theorem. If *G* is a graph of order *n*≥3 with minimum degree at least $\frac{n}{2} + \log_2(n) - 1$ then *G* has a rainbow connected coloring with two colors.

Proof. Suppose *G* is such a graph. If *n*=3 then *G* has minimum degree two and is thus complete. In which case, we reach our desired conclusion. So suppose *n*≥4 and let *G* be a graph with minimum degree at least $\frac{n}{2} + \log_2(n) - 1$. Let us two color the edges of *G* with colors red and blue so that each edge is colored red with a probability of one half.

Let us say a pair of vertices is "bad" if there is no rainbow path that connects them. Otherwise, we will say the pair is "good." We will show that the expected number of bad pairs is less than one. Hence, there must be some coloring where no bad pair is present.

Suppose *u* and *v* are a pair of distinct vertices in *G*. If *u* and *v* are adjacent, then clearly there is a rainbow path joining them. So suppose *u* and *v* aren't adjacent. By the Inclusion-Exclusion Principle, these vertices have at least $2\log_2(n)$ neighbors in common.

Suppose *w* is a common neighbor. The probability that *uw* and *vw* share the same color is one half. Accordingly, the probability and u and v are bad is at most $\left(\frac{1}{2}\right)^{2\log_2(n)} = \frac{1}{n^2}$. Hence, the expected number of bad vertices is at most $\binom{n}{2}\frac{1}{n^2} < 1$. Thus, our desired conclusion. □

Bibliography

[1] G. Chartrand, 2016. Highly irregular. *Graph Theory: Favorite Conjectures and Open Problems-1*, pp.1-16.

[2] G. Chartrand, L. Lesniak and P. Zhang, 2010. *Graphs & Digraphs* (Vol. 39). CRC press.

[3] Y. Caro, A. Lev, Z. Roddity, Z. Tuza and R. Yuster, 2008, On Rainbow Connection, Electron J. Combiin. 15 No. 1, Research paper 57.

This *Conjecture on Rainbow Connectity* is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 810115).

Author's Note. Since this manuscript was submitted, the author has learned of similar work performed in [3].