UNIVERSAL TOPOLOGICAL UNARY VARIETIES

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Abstract. For a regular variety $\mathbb{V}$ of unary algebras, we consider the category $\mathbb{PAR}(\mathbb{V})$ whose objects are algebras from $\mathbb{V}$ endowed by connected paracompact topology in which all operations are continuous, and whose morphisms are all continuous homomorphisms between these algebras. Then $\mathbb{PAR}(\mathbb{V})$ is a universal category (a category $\mathcal{K}$ is universal if any concrete category has a full embedding into $\mathcal{K}$) if and only if $\mathbb{V}$ contains an algebra $\mathbf{A}$ with no singleton subalgebra and such that the identity is the only derived operation of $\mathbf{A}$ which is also an endomorphism of $\mathbf{A}$.

R. Frucht [8] proved that any finite group is isomorphic to the automorphism group of a finite graph. This is the first result of a "universal representation" of an algebraic structure by morphisms of some mathematical structure. This fact was generalized by G. Birkhoff [3] to a representation of groups by automorphisms of distributive lattices and by G. Sabidussi [29] who proved that any group is isomorphic to the automorphism group of a graph. The natural generalization is to replace groups by monoids and automorphisms by endomorphisms. The pioneering papers in this direction are due to Z. Hedrlín and A. Pultr [14] and [15] who showed that any monoid is isomorphic to the endomorphism monoid of a graph, and to M. Armbrust and J. Schmidt [2] who proved that any monoid is isomorphic to the endomorphism monoid of an algebra. An effort to represent monoids as endomorphism monoids of other mathematical structures led to the following generalization. A category $\mathcal{K}$ is <u>alg-universal</u> if any category of algebras and all homomorphisms between them is isomorphic to a full subcategory of $\mathcal{K}$. The first known examples of alg-universal categories are due to Z. Hedrlín and A. Pultr. They proved that categories of all directed graphs and of all undirected graphs are alg-universal [14], and that the category of all algebras of a given type Δ and all their homomorphisms is alg-universal if and only if the sum of all arities in Δ is at least 2, [16]. Many other conventional categories are alg-universal, for example the variety of semigroups [13], the variety of rings [6], the variety of $(0,1)$-lattices [10], the category of all integral domains of characteristic zero [7], the variety of de Morgan algebras [1]. A representative list of alg-universal categories is given in the monograph by A. Pultr and V. Trnková [27]. More recent results give a complete characterization of all alg-universal varieties in certain classes of varieties. The first result of this kind is due to J. Sichler [30]

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who characterized all alg-universal varieties of unary algebras. Another result says that a variety of semigroups is alg-universal if and only if it contains all commutative semigroups and fails the identity $(xy)^n = x^n y^n$ for all $n > 1$ [22], or that a variety of $(0,1)$-lattices is alg-universal if and only if it contains a simple $(0,1)$-lattice with at least three elements or equivalently, just when it contains a lattice with at least three elements and no prime ideal, [9]. Furthermore, the alg-universal varieties of finitely generated double p-algebras [23], the alg-universal varieties of bands with added constants [5], the alg-universal varieties of semigroups with an added unary operation of generalized inverse [21] were characterized.

The fact that many conventional categories are alg-universal leads to an investigation of properties of alg-universal categories. A combination of results of Z. Hedrlín and A. Pultr shows that any monoid can be represented as an endomorphism monoid of a $\mathcal{K}$-object for any alg-universal category $\mathcal{K}$. This fact was strengthened by Z. Hedrlín and J. Sichler to

Theorem 1. [17] *or* [27] *Let $\mathcal{K}$ be an alg-universal category and let M be a monoid. Then*

(1) *there is a proper class of non-isomorphic $\mathcal{K}$-objects whose endomorphism monoids are isomorphic to M;*
(2) *for any cardinal α there exists a set C of a cardinality α consisting of $\mathcal{K}$-objects such that endomorphism monoid of any object from C is isomorphic to M and there exists no $\mathcal{K}$-morphism between distinct objects from C.* $\square$

This fact is often used in proofs that a category is not alg-universal. Another important properties of alg-universal categories were stated by J. Rosický [28] and V. Koubek and J. Sichler [24] and [25].

The category $\mathcal{K}$ is called universal if any concrete category can be fully embedded into $\mathcal{K}$. Clearly, any universal category is also alg-universal. Z. Hedrlín and L. Kučera [27] proved that the reverse implication is true under the validity of the set axiom

(M) there exists only a set of measurable cardinals.

L. Kučera and A. Pultr [26] then showed that the set axiom (M) holds if and only if the category of all graphs is universal, or equivalently, exactly when a variety of algebras is universal. On the other hand, L. Kučera [27] proved that a category of all hypergraphs is universal without the assumption of any special set axioms. Another example of universal category is the category of all topological spaces and continuous open mappings, see [27]. A list of known universal categories is substantially shorter than a list of alg-universal categories and contains only a few conventional categories. Any universal category satisfies Theorem 1 because it is alg-universal. In fact, any universal category $\mathcal{K}$ fulfils the following stronger form of Theorem 1, [27]: if M is a monoid then there exists a proper class of $\mathcal{K}$-objects whose endomorphism monoids are isomorphic to M and there exists no $\mathcal{K}$-morphism between any two distinct objects from this class.

The question asked by J. R. Isbell [19] whether or not there exists a full embedding of the category of Hausdorff spaces and continuous mappings into some category of algebras and all their homomorphisms is closely related to the relation between alg-universality and universality. From the foregoing results we obtain that the category of all Hausdorff spaces and continuous mappings has a full embedding into a category of algebras and all their homomorphisms if and only if the set axiom (M) holds.

The category $\mathbb{TOP}$ of all topological spaces and continuous mappings is neither universal nor alg-universal. Indeed, any constant mapping is continuous, and hence any rigid topological space (an object is called <u>rigid</u> if its endomorphism monoid consists of the identity mapping alone) has at most one element. Thus $\mathbb{TOP}$ is not alg-universal and therefore nor universal, by Theorem 1. On the other hand, the author of this paper showed that there exists a class $\mathcal{C}$ of paracompact connected spaces such that the class $\mathcal{C}$ and all non-constant continuous mappings between the spaces from $\mathcal{C}$ form a universal category (thus non-constant continuous maps between them are closed under composition), see [20]. Analogous results were proved by V. Trnková; for example, she proved that there exists a class $\mathcal{C}$ of connected compact Hausdorff spaces such that $\mathcal{C}$ and all non-constant continuous mappings between the spaces from $\mathcal{C}$ form a dually alg-universal category [31], and that there exists a class $\mathcal{C}$ of connected metric spaces such that $\mathcal{C}$ and all non-constant continuous mappings between the spaces from $\mathcal{C}$ form an alg-universal category. These facts motivated V. Trnková to ask whether topological varieties can be universal. First we give several notions to formalize this question. We say that $\mathbf{A} = (A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\}, \tau)$ is a <u>topological</u> <u>algebra</u> of type Δ if $(A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ is an algebra of type Δ, (A, τ) is a topological space and all operations $\omega_{\mathbf{A}} : A^\alpha \to A$ with $\omega \in \Delta$ are continuous with respect to the product topology on A^α (α is the arity of the operation symbol $\omega \in \Delta$). Let $\mathbb{V}$ be a variety of algebras of a given type Δ. Then $\mathbb{TOP}(\mathbb{V})$ is a category formed by all topological algebras $\mathbf{A} = (A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\}, \tau)$ of type Δ where $(A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ belongs to $\mathbb{V}$ and all continuous homomorphisms between them. The category $\mathbb{TOP}(\mathbb{V})$ is called a <u>topological</u> <u>variety</u>, and we say that $\mathbb{TOP}(\mathbb{V})$ is a <u>topologization</u> of $\mathbb{V}$. Trnková's question may then be formalized as follows:

Characterize those varieties $\mathbb{V}$ of algebras for which $\mathbb{TOP}(\mathbb{V})$ is universal.

In [33], V. Trnková proved that $\mathbb{TOP}(\mathbb{SEM})$ is universal for the variety $\mathbb{SEM}$ of all semigroups.

An endomorphism f of an algebra $\mathbf{A}$ is called <u>internal</u> if f is constant or f is a derived operation of $\mathbf{A}$ other that the identity. We say that a variety $\mathbb{V}$ of algebras <u>has</u> <u>internal</u> <u>endomorphisms</u> if there exists a cardinal α such that every algebra $\mathbf{A}$ from $\mathbb{V}$ of cardinality at least α has an internal endomorphism. Then, by Theorem 1, $\mathbb{TOP}(\mathbb{V})$ can be neither universal nor alg-universal because any rigid topological algebra from $\mathbb{TOP}(\mathbb{V})$ has a size at most α. We prove that for topological regular varieties of unary algebras this poses the only obstacle to universality.

To formulate the main result, we need to give several notions. For a type Δ, let $\mathbb{A}(\Delta)$ denote the category of all algebras of type Δ and all their homomorphisms, and let Δ_0 and Δ_1 denote the respective collections of nullary and unary operation symbols $\omega \in \Delta$. Any type $\Delta = \Delta_0 \cup \Delta_1$ will be called <u>unary</u>, and any algebra $\mathbf{A}$ of type Δ a <u>unary</u> <u>algebra</u>.

Let $\Delta = \Delta_0 \cup \Delta_1$ be a unary type. Then the monoid freely generated by Δ_1 consists of all derived unary operations in $\mathbb{A}(\Delta)$ (and also in $\mathbb{A}(\Delta_1)$). Any variety of algebras is described by the identities satisfied in all algebras in the variety. All identities of type Δ can be divided into these four types:

(i) $\gamma_1(\omega_1) = \gamma_2(\omega_2)$ where γ_1 and γ_2 are derived unary operations and $\omega_1, \omega_2 \in \Delta_0$;
(ii) $\gamma_1(\omega_1) = \gamma_2(x)$ where γ_1 and γ_2 are derived unary operations, $\omega_1 \in \Delta_0$ and x is a variable;
(iii) $\gamma_1(x) = \gamma_2(x)$ where γ_1 and γ_2 are derived unary operations and x is a variable;

(iv) $\gamma_1(x) = \gamma_2(y)$ where γ_1 and γ_2 are derived unary operations and x and y are distinct variables.

A variety $\mathbb{V} \subseteq \mathbb{A}(\Delta)$ of unary algebras is regular if all identities satisfied in $\mathbb{V}$ are of type (i) or (iii). If $\mathbb{V}$ is a variety of unary algebras, then the relation $\sim$ on the monoid freely generated by Δ_1 defined by

$$\alpha_1 \sim \alpha_2 \text{ iff the identity } \alpha_1(x) = \alpha_2(x) \text{ holds in } \mathbb{V}$$

is a monoid congruence. Let $\mathrm{Mon}(\mathbb{V})$ be the quotient monoid of the free monoid over Δ_1 by the congruence $\sim$ and let $\mathrm{UOp}(\mathbb{V})$ be the underlying set of $\mathrm{Mon}(\mathbb{V})$. Then $\mathrm{UOp}(\mathbb{V})$ is the set of all distinct derived unary operations of $\mathbb{V}$, and the monoid operation of $\mathrm{Mon}(\mathbb{V})$ is their composition. For a variety $\mathbb{V}$ of unary algebras over Δ, let $\mathfrak{R}(\mathbb{V})$ denote the variety over Δ_1 satisfying exactly the identities of type (iii) that are satisfied in $\mathbb{V}$. Then $\mathrm{Mon}(\mathbb{V}) = \mathrm{Mon}(\mathfrak{R}(\mathbb{V}))$.

Let $\mathcal{K}$ be a full subcategory of $\mathbb{TOP}$ and let $\mathbb{V}$ be a variety of algebras. Then $\mathcal{K}(\mathbb{V})$ is a full subcategory of $\mathbb{TOP}(\mathbb{V})$ formed by all topological algebras $\mathbf{A} = (A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\}, \tau)$ from $\mathbb{TOP}(\mathbb{V})$ with $(A,\tau) \in \mathcal{K}$. Let $\mathbb{PAR}$ denote the full subcategory of $\mathbb{TOP}$ formed by all paracompact connected spaces. The aim of this paper is to prove Theorem 2 below.

An algebra is nontrivial if it has at least two elements.

Theorem 2. *For a regular variety $\mathbb{V}$ of unary algebras the following are equivalent:*

(1) $\mathbb{PAR}(\mathbb{V})$ *is universal;*
(2) $\mathbb{TOP}(\mathbb{V})$ *is universal;*
(3) $\mathbb{PAR}(\mathbb{V})$ *is alg-universal;*
(4) $\mathbb{TOP}(\mathbb{V})$ *is alg-universal;*
(5) $\mathbb{PAR}(\mathbb{V})$ *has a proper class of non-isomorphic rigid topological algebras;*
(6) $\mathbb{TOP}(\mathbb{V})$ *has a proper class of non-isomorphic rigid topological algebras;*
(7) $\mathbb{PAR}(\mathbb{V})$ *has an arbitrarily large rigid topological algebra;*
(8) $\mathbb{TOP}(\mathbb{V})$ *has an arbitrarily large rigid topological algebra;*
(9) $\mathbb{V}$ *does not have internal endomorphisms;*
(10) *either $\Delta_0 \neq \emptyset$ and $\mathbb{V}$ contains a nontrivial algebra $\mathbf{A}$ generated by $\emptyset$, or else $\Delta_0 = \emptyset$ and $\mathbb{V}$ contains a nontrivial algebra $\mathbf{A}$ with no internal endomorphisms.*

This result motivates the following three open problems (here $\mathbb{COMP}$ is the full subcategory of $\mathbb{TOP}$ formed by all compact Hausdorff spaces)

Problem 1. Characterize universal topological varieties of unary algebras.

Problem 2. Is there a topological variety $\mathbb{TOP}(\mathbb{V})$ that is not universal and $\mathbb{V}$ does not have internal endomorphisms?

Problem 3. Is there a variety $\mathbb{V}$ such that $\mathbb{COMP}(\mathbb{V})$ is universal?

We believe that the statements (2) and (9) from Theorem 2 are equivalent for any variety of algebras, that Theorem 2 is true for all unary varieties, and that the second problem has negative solution.

We give several examples illustrating Theorem 2. Let Δ be a unary type with exactly one operation symbol then $\mathbb{TOP}(\mathbb{V})$ is not universal for any subvariety $\mathbb{V}$ (not only regular) of $\mathbb{A}(\Delta)$. Indeed, if $\Delta = \Delta_0 = \{\omega\}$, then $\mathbb{V}$ has an internal endomorphism (any constant mapping to the value of ω is a homomorphism). If $\Delta = \Delta_1 = \{\omega\}$, then $\mathbb{V}$ has internal endomorphisms because $\omega_{\mathbf{A}}$ is an endomorphism of any algebra $\mathbf{A}$ of type Δ. On the other hand, if Δ consists of two operation symbols, then $\mathbb{TOP}(\mathbb{A}(\Delta))$ is universal regardless of the arities of the operational symbols. In addition, $\mathbb{TOP}(\mathbb{V})$ remains universal if $\mathbb{V}$ is the variety of type Δ consisting of one nullary operation symbol and one unary operation symbol ω that is an involution or an idempotent (both varieties are regular, the first variety is determined by the identity $\omega^2(x) = x$, the second one by the identity $\omega^2(x) = \omega(x)$, and the algebra freely generated by $\emptyset$ in either variety is nontrivial).

By Theorem 1 and from definitions, we immediately obtain that (1) $\implies$ (2) $\implies$ (4) $\implies$ (6) $\Leftrightarrow$ (8) $\implies$ (9) $\implies$ (10) and (1) $\implies$ (3) $\implies$ (5) $\Leftrightarrow$ (7) $\implies$ (9), and thus it remains to prove (10) $\implies$ (1). In fact, we prove the stronger version of the implication (10) $\implies$ (1). For this purpose we generalize the notion of topologization of a variety. Let $\mathcal{K}$ be a concrete category with a forgetful functor $\Lambda : \mathcal{K} \longrightarrow \mathbb{SET}$ such that $\mathcal{K}$ has products and Λ preserves them. A $\mathcal{K}$-algebra of type Δ is a pair $(A, \mathbf{B} = (U, \{\omega_{\mathbf{B}} \mid \omega \in \Delta\}))$ such that A is a $\mathcal{K}$-object with $\Lambda A = U$, $\mathbf{B}$ is an algebra of type Δ and for every $\omega \in \Delta$ there exists a $\mathcal{K}$-morphism $\omega_A : A^\alpha \longrightarrow A$ where α is the arity of ω with $\Lambda\omega_A = \omega_{\mathbf{B}}$. We say that a $\mathcal{K}$-morphism $f : A \longrightarrow A'$ is a $\mathcal{K}$-homomorphism from a $\mathcal{K}$-algebra $(A, \mathbf{B})$ to a $\mathcal{K}$-algebra $(A', \mathbf{B}')$ whenever Λf is a homomorphism from $\mathbf{B}$ to $\mathbf{B}'$. Then all $\mathcal{K}$-algebras and all $\mathcal{K}$-homomorphisms form a category $\mathcal{K}(\mathbb{A}(\Delta))$. If $\mathbb{V}$ is a variety of algebras of type Δ, then $\mathcal{K}(\mathbb{V})$ is a full subcategory of $\mathcal{K}(\mathbb{A}(\Delta))$ formed by all $\mathcal{K}$-algebras $(A, \mathbf{B})$ with $\mathbf{B} \in \mathbb{V}$. We then say that $\mathcal{K}(\mathbb{V})$ is a $\mathcal{K}$-variety of $\mathbb{V}$. Let $\mathbb{CL}$ be a category of all connected paracompact topological spaces and continuous closed mappings. Clearly $\mathbb{CL}$ is a subcategory of $\mathbb{TOP}$ and also of $\mathbb{PAR}$ (but not full subcategory). We prove

Theorem 3. *Let $\mathbb{V}$ be a regular unary variety of type Δ such that either $\Delta_0 \neq \emptyset$ and $\mathbb{V}$ contains a nontrivial algebra generated by $\emptyset$, or else $\Delta_0 = \emptyset$ and $\mathbb{V}$ contains a nontrivial algebra with no internal endomorphism. Then any subcategory $\mathcal{K}$ of $\mathbb{PAR}(\mathbb{V})$ containing $\mathbb{CL}(\mathbb{V})$ is universal.*

The remainder of the paper is devoted to the proof of Theorem 3 because the implication (10) $\implies$ (1) follows from Theorem 3.

Next we recall several familiar properties of regular varieties of unary algebras and graphs. Graphs play an important role in the theory of embeddings and in a description of unary algebras. A graph is a pair (X, R) where X is a set and $R \subseteq X \times X$ (in literature, a graph is often called a directed graph or a digraph), elements of X are called nodes, elements of R are called arcs. The arc (x, x) for some $x \in X$ is called a loop. We say that a graph (X, R) is

connected whenever for any pair of nodes x and y of X there exists a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in R$ or $(x_{i+1}, x_i) \in R$ for all $i = 0, 1, \ldots, n-1$.
antireflexive whenever $(x, x) \notin R$ for all $x \in X$;
reflexive whenever $(x, x) \in R$ for all $x \in X$;
antisymmetric whenever $(x, y) \in R$ and $(y, x) \in R$ imply $x = y$.

For a graph $G = (X, R)$, let $G^0 = (X, R^0)$ denote the reflexive closure of G, i.e., $R^0 =$

$R \cup \{(x,x) \mid x \in X\}$. We say that a unary algebra $\mathbf{A} = (Z, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ over a type Δ is connected if the graph $(Z, \{(x, \omega_{\mathbf{A}}(x)) \mid x \in X,\ \omega \in \Delta$ is a unary operational symbol$\})$ is connected.

Let $M = (M, \cdot)$ be a monoid. Then a mapping $f : M \longrightarrow M$ is called an inner left translation (or an inner right translation) of $x \in M$ whenever $f(m) = x \cdot m$ (or $f(m) = m \cdot x$, respectively) for all $m \in M$. For a variety $\mathbb{V}$ of unary algebras over Δ, let $M_{\mathbb{V}}$ denote the set of all inner left translations of $\mathrm{Mon}(\mathbb{V})$, and for $\gamma \in \mathrm{UOp}(\mathbb{V})$, let $\phi_\gamma^{\mathbb{V}}$ (or $\psi_\gamma^{\mathbb{V}}$) denote the inner left translation (or the inner right translation) of γ in $\mathrm{Mon}(\mathbb{V})$, thus $\phi_\gamma^{\mathbb{V}}(\beta) = \gamma \circ \beta$ (or $\psi_\gamma^{\mathbb{V}}(\beta) = \beta \circ \gamma$, respectively) for all $\beta \in \mathrm{UOp}(\mathbb{V})$. For a variety $\mathbb{V}$ and a set X, let $\mathcal{F}_{\mathbb{V}}(X)$ denote the algebra freely generated in $\mathbb{V}$ by X. The following statement is folklore.

Theorem 4. *Let $\mathbb{V}$ be a regular unary variety of type Δ, and let X be a set. Then:*

(1) *if $X = \{x\}$ is a singleton, then $\mathcal{F}_{\mathfrak{R}(\mathbb{V})}(X) = \mathbf{A} = (\mathrm{UOp}(\mathbb{V}), \{\omega_{\mathbf{A}} = \phi_\omega^{\mathbb{V}} \mid \omega \in \Delta_1\})$ where we identify x with the identity derived operation from $\mathrm{Mon}(\mathbb{V})$;*
(2) *if $X \neq \emptyset$ then $\mathcal{F}_{\mathfrak{R}(\mathbb{V})}(X)$ is the disjoint union of $\mathcal{F}_{\mathfrak{R}(\mathbb{V})}(\{x\})$ for $x \in X$;*
(3) *if $X \neq \emptyset$, $\mathcal{F}_{\mathbb{V}}(\emptyset) = \mathbf{B} = (Z, \{\omega_{\mathbf{B}} \mid \omega \in \Delta\})$ and $\mathcal{F}_{\mathfrak{R}(\mathbb{V})}(X) = \mathbf{C} = (Y, \{\omega_{\mathbf{C}} \mid \omega \in \Delta_1\})$ with $Y \cap Z = \emptyset$, then $\mathcal{F}_{\mathbb{V}}(X) = \mathbf{D} = (Y \cup Z, \{\omega_{\mathbf{D}} \mid \omega \in \Delta\})$, where*

$$\omega_{\mathbf{D}}(x) = \begin{cases} \omega_{\mathbf{B}}(x) & \text{if } x \in Z, \\ \omega_{\mathbf{C}}(x) & \text{if } x \in Y \end{cases}$$

for every $\omega \in \Delta_1$ and all $x \in Y \cup Z$, and $\omega_{\mathbf{D}}(0) = \omega_{\mathbf{B}}(0)$ for every $\omega \in \Delta_0$. $\square$

Remark. If $\mathbf{A} = (A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ is an algebra of type Δ and if α is a derived operation of type Δ, then $\alpha_{\mathbf{A}}$ denotes a realization of α in the algebra $\mathbf{A}$.

Now we recall several facts from the theory of transformation monoids. We say that a pair (X, M) is a transformation monoid if X is a set and M is a set of mappings of X into itself closed under the composition and containing the identity mapping of X. An element $x \in X$ is called a source of (X, M) if for every $y \in X$ there exists some $f \in M$ with $f(x) = y$. If, moreover, for every $y \in X$ there exists exactly one $f \in M$ with $f(x) = y$ then we say that x is an exact source. Define $\mathrm{Com}(M) = \{g : X \longrightarrow X \mid \text{ for every } f \in M,\ f \circ g = g \circ f\}$. We recall the following folklore statement, proved in [12], for instance.

Theorem 5. *For a transformation monoid (X, M) and $x \in X$ the following are equivalent:*

(1) *$x \in X$ is an exact source of (X, M);*
(2) *$x \in X$ is a source of both (X, M) and $(X, \mathrm{Com}(M))$;*
(3) *there exists a binary operation $\cdot$ on X such that $(X, \cdot)$ is a monoid with the unit element x and M is the set of all inner left translations of $(X, \cdot)$;*
(4) *there exists a binary operation $\cdot$ on X such that $(X, \cdot)$ is a monoid with the unit element x, M is the set of all inner left translations of $(X, \cdot)$ and $\mathrm{Com}(M)$ is the set of all inner right translations of $(X, \cdot)$.* $\square$

Next we give some properties of regular varieties of unary algebras satisfying (10) of Theorem 2.

Proposition 6. *Let $\mathbb{V}$ be a regular unary variety of type Δ containing a nontrivial algebra generated by $\emptyset$. Then the algebra $\mathcal{F}_{\mathbb{V}}(\emptyset) = (Z, \{\omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)} \mid \omega \in \Delta\})$ is nontrivial, and it is either connected, or else has a quotient algebra $\mathbf{A} = (\{0,1\}, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ in which $\omega_{\mathbf{A}}(i) = i$ for all $\omega \in \Delta_1$ and $i = 0, 1$, and $\{0,1\} = \{\omega_{\mathbf{A}}(0) \mid \omega \in \Delta_0\}$.*

Let X be a set and $0, 1 \notin X$. If $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is connected, then $\mathbb{V}$ contains an algebra $\mathbf{B} = (Z \times (X \cup \{0\}), \{\omega_{\mathbf{B}} \mid \omega \in \Delta\})$ such that

$$\begin{aligned} \omega_{\mathbf{B}}(z,x) &= (\omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z), x) \qquad \text{for all } \omega \in \Delta_1 \\ &\qquad\qquad \text{and all } z \in Z, \text{ and } x \in X \cup \{0\}, \\ \omega_{\mathbf{B}}(0) &= (\omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(0), 0) \qquad \text{for all } \omega \in \Delta_0. \end{aligned}$$

If $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is not connected, then $\mathbb{V}$ contains an algebra $\mathbf{C} = (X \cup \{0,1\}, \{\omega_{\mathbf{C}} \mid \omega \in \Delta\})$ such that

$$\begin{aligned} \omega_{\mathbf{C}}(z) &= z \qquad \text{for all } \omega \in \Delta_1 \text{ and all } z \in X \cup \{0,1\}, \\ \omega_{\mathbf{C}}(0) &= \omega_{\mathbf{A}}(0) \qquad \text{for all } \omega \in \Delta_0 \end{aligned}$$

belongs to $\mathbb{V}$.

Proof. The algebra $\mathcal{F}_{\mathbb{V}}(\emptyset) = (Z, \{\omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)} \mid \omega \in \Delta\})$ is nontrivial because $\mathbb{V}$ contains a nontrivial $\emptyset$-generated algebra. Consider the relation

$$R = \{(z, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z)) \mid \omega \in \Delta_1,\ z \in Z\} \subseteq Z \times Z.$$

Choose $z \in Z$ and let U be the subset of Z consisting of all elements $u \in Z$ such that there exists a sequence $u = x_0, x_1, \dots, x_n = z$ of Z with $(x_i, x_{i+1}) \in R$ or $(x_{i+1}, x_i) \in R$ for all $i = 0, 1, \dots, n-1$. Then $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is connected if and only if $U = Z$. The equivalence $\sim$ on Z defined by $z \sim z'$ just when $z, z' \in U$ or $z, z' \in Z \setminus U$ is a congruence. If $U \neq Z$, then $\mathcal{F}_{\mathbb{V}}(\emptyset)/\sim$ is the desired algebra $\mathbf{A}$, and the first statement is proved. Since $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is isomorphic to a quotient algebra of $\mathcal{F}_{\mathbb{V}}(\Delta_0)$, the second statement follows by Theorem 4. □

Proposition 7. *Let $\mathbb{V}$ be a regular variety of a unary type $\Delta = \Delta_1$ containing a nontrivial algebra $\mathbf{A} = (A, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ with no internal endomorphisms. Then there exists a regular subvariety $\mathbb{W}$ of $\mathbb{V}$ such that $\mathcal{F}_{\mathbb{W}}(\{x\})$ is nontrivial and has no internal endomorphisms.*

Proof. Let $\mathbb{W}$ be the regular subvariety of $\mathbb{V}$ generated by $\mathbf{A}$. Choose a singleton set $X = \{x\}$. Then $\mathcal{F}_{\mathbb{W}}(X)$ is isomorphic to a quotient subalgebra of $\mathcal{F}_{\mathbb{V}}(X)$ modulo a fully invariant congruence $\sim$. By Theorem 4, the congruence $\sim$ is also a monoid congruence on $\mathrm{Mon}(\mathbb{V})$, and $\mathrm{Mon}(\mathbb{W})$ is isomorphic to $\mathrm{Mon}(\mathbb{V})/\sim$. If $\mathcal{F}_{\mathbb{W}}(X)$ has a constant endomorphism, then it has a singleton subalgebra, and hence the algebra $\mathbf{A}$ also has a singleton subalgebra, and therefore a constant endomorphism – this is a contradiction. Therefore for every $\gamma \in \mathrm{UOp}(\mathbb{W})$ there exists some $\omega \in \Delta$ with $\omega \circ \gamma \not\sim \gamma$.

Suppose that α is a derived operation of $\mathbb{W}$ which is also an endomorphism of $\mathcal{F}_{\mathbb{W}}(X)$. Then $\alpha \in \mathrm{Mon}(\mathbb{W}) \cap \mathrm{Com}(\mathrm{Mon}(\mathbb{W}))$, that is, $\alpha \circ \gamma = \gamma \circ \alpha$ for every $\gamma \in \mathrm{UOp}(\mathbb{W})$, by Theorem 5, or, equivalently, $\phi_\alpha^{\mathbb{W}} = \psi_\alpha^{\mathbb{W}}$. Since $\mathbf{A} \in \mathbb{W}$, we have $\alpha_{\mathbf{A}}\gamma_{\mathbf{A}}(a) = \gamma_{\mathbf{A}}\alpha_{\mathbf{A}}(a)$ for every $a \in A$ and $\gamma \in \mathrm{UOp}(\mathbb{W})$, and therefore $\alpha_{\mathbf{A}} = 1_A$ by the hypothesis. Since $\mathbf{A}$ generates $\mathbb{W}$, it follows that the operation $\phi_\alpha^{\mathbb{W}}$ is the identity map, and hence the endomorphism $\psi_\alpha^{\mathbb{W}}$ is the identity map as well. The proof is complete. $\square$

Let $\mathbb{V}$ be a regular unary variety of type Δ fulfilling (10) of Theorem 2. Then, by Propositions 6 and 7, the variety $\mathbb{V}$ satisfies exactly one of these statements:

(d1) $\mathbb{V}$ contains an algebra $\mathbf{A} = (\{0,1\}, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ such that

$$\omega_{\mathbf{A}}(i) = i \qquad \text{for all } \omega \in \Delta_1 \text{ and } i = 0, 1,$$
$$\text{and} \qquad \{0,1\} = \{\omega_{\mathbf{A}}(0) \mid \omega \in \Delta_0\};$$

(d2) $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is a nontrivial connected algebra;

(d3) $\Delta = \Delta_1$ and there exists a regular subvariety $\mathbb{W}$ of $\mathbb{V}$ such that $M_{\mathbb{W}} \cap \mathrm{Com}(M_{\mathbb{W}})$ consists of the identity mapping alone, and for every $\gamma \in \mathrm{UOp}(\mathbb{W})$ there exists $\omega \in \Delta$ with $\omega \circ \gamma \neq \gamma$.

For a given set X, let $P_X : \mathrm{Set} \longrightarrow \mathrm{Set}$ denote the contravariant hom-functor $\hom(-, X)$. Thus $P_X Y = X^Y$ consists of all mappings from Y to X for every set Y, and $P_X f(g) = g \circ f$ for any given mapping $f : Z \longrightarrow Y$ and for all mappings $g : Y \longrightarrow X$. Let $S(P_X)$ denote the category whose objects are pairs (Y, V) where Y is a set and $V \subseteq P_X Y$, and the morphisms from (Y, V) to (Y', V') are all mappings $f : Y \longrightarrow Y'$ satisfying $P_X f(V') \subseteq V$. Theorem 8 below combines results by L. Kučera, Z. Hedrlín and A. Pultr.

Theorem 8. [27] *If X has at least two elements, then $S(P_X)$ is a universal category.* $\square$

When we combine the result of Z. Hedrlín [11] and Theorem 1, we obtain the following statement.

Theorem 9. [11] *and* [17] *For every set X there exists a family $\{G_x \mid x \in X\}$ of pairwise disjoint antireflexive antisymmetric connected graphs such that any graph homomorphism $f : G_x^0 \longrightarrow G_y^0$ with $x, y \in X$ is constant, or else $x = y$ and f is the identity.* $\square$

Now we turn to a simple but useful construction of connected paracompact spaces.

Proposition 10. *Let $\{A_i = (A_i, \tau_i) \mid i \in I\}$ be a family of pairwise disjoint connected compact Hausdorff spaces and let $\sim$ be an equivalence on $A' = \bigcup_{i\in I} A_i$ such that*

(a) *$|C \cap A_i| \leq 1$ for all classes C of $\sim$ and all $i \in I$;*

(b) *if $x \in A_i$ and $y \in A_j$ for distinct $i, j \in I$ with $x \sim y$, then the set $\{z \in A_i \mid z \sim v \text{ for a } v \in A' \text{ with } v \neq z\}$ is finite or the set $\{z \in A_j \mid z \sim v \text{ for a } v \in A' \text{ with } v \neq z\}$ is finite;*

(c) *the graph (I, R) is connected, where*

$$R = \{(i,j) \mid i, j \in I, \text{ there exists a class } C \text{ of } \sim \text{ with } C \cap A_i \neq \emptyset \neq C \cap A_j\}.$$

Then (A, τ) is a connected paracompact space for which $A = A'/\sim$ and $U \subseteq A$ is τ-open exactly when $\{c \in A_i \mid \text{there exists a class } C \text{ of } \sim \text{ with } c \in C \in U\}$ is τ_i-open for all $i \in I$.

Proof. The statement is clear for $|I| = 1$, and thus we may assume that $|I| > 1$. For an element $a \in \bigcup_{i\in I} A_i$, let $[a]$ denote the class of $\sim$ containing a. Clearly, τ is a topology on $A = A'/\sim = \bigcup_{i\in I} A_i$. A straightforward verification shows that (A,τ) is a Hausdorff space, because (A_i,τ_i) are Hausdorff spaces for all $i \in I$. Since (A_i,τ_i) is connected for all $i \in I$ we conclude, using (c), that (A,τ) is connected. Thus it remains to prove that any τ-open covering has a locally finite refinement. Let $\mathcal{O}$ be a τ-open covering of A. Let $\mathcal{C}$ be the set of all non-singleton classes of $\sim$ and let I' consist of all $i \in I$ such that $A_i \cap C \neq \emptyset$ for infinitely many $C \in \mathcal{C}$. Since (A_i,τ_i) is compact, for every $i \in I'$ there exists a finite set $\mathcal{O}_i \subseteq \mathcal{O}$ with $\{[a] \mid a \in A_i\} \subseteq \cup_{U\in\mathcal{O}_i} U$. By (b), for every $U \in \mathcal{O}_i$ with $i \in I'$ there exists a τ-open set $V_U \subseteq U$ such that $\{[a] \in U \mid a \in A_i\} = \{[a] \in V_U \mid a \in A_i\}$ and if $C \in V_U$ for some $C \in \mathcal{C}$ then $C \cap A_i \neq \emptyset$ because (A_i,τ_i) is a Hausdorff space. Then $\{[a] \mid a \in \bigcup_{i\in I'} A_i\} \subseteq \cup\{V_U \mid U \in \bigcup_{i\in I'} \mathcal{O}_i\}$. For every $C \in \mathcal{C} \setminus \{[a] \mid a \in \bigcup_{i\in I'} A_i\}$ choose a τ-open set $Q_C \in \mathcal{O}$ with $C \in Q_C$. Since (A_i,τ_i) is a Hausdorff space, for every $C \in \mathcal{C}\setminus\{[a] \mid a \in \bigcup_{i\in I'} A_i\}$ we may choose, by (b), a τ-open set $V_C \subseteq Q_C$ such that $C \in V_C$, $V_C \cap V_U = \emptyset$ for all $U \in \bigcup_{i\in I'} \mathcal{O}_i$ and $V_C \cap V_{C'} = \emptyset$ for distinct $C, C' \in \mathcal{C} \setminus \{[a] \mid \bigcup_{i\in I'} A_i\}$. Set $\mathcal{P}' = \{V_U \mid U \in \bigcup_{i\in I'} \mathcal{O}_i\} \cup \{V_C \mid C \in \mathcal{C}\setminus\{[a] \mid a \in \bigcup_{i\in I'} A_i\}\}$. Then for every $i \in I\setminus I'$, the set $B_i = \{a \in A_i \mid [a] \notin (\bigcup_{U\in\mathcal{P}'} U)\}$ is compact because it is τ_i-closed and $[b] \notin \mathcal{C}$ for all $b \in B_i$. Since the set $\{[a] \in \mathcal{C} \mid a \in A_i\}$ is finite for every $i \in I \setminus I'$, there exist τ_i-open sets B'_i for $i \in I\setminus I'$ such that $B_i \subseteq B'_i \subseteq A_i$ and $[b] \notin \mathcal{C}$ for all $b \in B'_i$. Hence for every $i \in I\setminus I'$, by the definition of τ and by the compactness of B_i, there exists a finite set $\mathcal{O}_i \subseteq \mathcal{O}$ such that $\{[b] \mid b \in B_i\} \subseteq \bigcup_{U\in\mathcal{O}_i} U$. Set $\mathcal{P} = \mathcal{P}' \cup \{U \cap B'_i \mid U \in \mathcal{O}_i, i \in I \setminus I'\}$. Then $\mathcal{P}$ is a desired locally finite refinement of $\mathcal{O}$, and thus (A,τ) is a paracompact space. $\square$

We exploit the following Cook's result.

Theorem 11. [4] or [27] *There exists an infinite metric continuuum K such that*

(1) *for every subcontinuum K_1 of K, if $f : K_1 \longrightarrow K$ is a continuous mapping, then f is a constant mapping or f is the inclusion mapping;*
(2) *there exist infinitely many pairwise disjoint infinite subcontinua of K.* $\square$

Choose pairwise disjoint infinite subcontinua A, B, C, D, E, and F of K and distinct points $a_0, a_1 \in A$, $b_0, b_1, b_2 \in B$, $c_0, c_1, c_2 \in C$, $d_0, d_1 \in D$, $e_0, e_1 \in E$, and $f_0, f_1 \in F$.

For a set X, let α_i^X denote the constant mapping from X to A with the value a_i for $i = 0, 1$. Assume that $\mathcal{X} = (X, V)$ is an object of $S(P_A)$. We then define $K(\mathcal{X})$ as the quotient space of the disjoint union of A^X, B, C, and $D \times V$ (where V is endowed with discrete topology) by the least equivalence $\sim$ such that

- ($\bullet$) $\alpha_0^X \sim b_0$, $\alpha_1^X \sim c_0$;
- ($\bullet$) $b_1 \sim (d_0, \nu)$ and $\nu \sim (d_1, \nu)$ for all $\nu \in V \subseteq A^X$.

Observe that if $x \sim y$ for $x, y \in A^X$, or $x, y \in B$, or $x, y \in C$, or $x, y \in D \times \{\nu\}$ for $\nu \in V$, then $x = y$. Therefore, for the sake of simplicity, we identify any point with the class of $\sim$ containing it. Thus, in particular, $\alpha_0^X = b_0$, $\alpha_1^X = c_0$, $\nu = (d_1, \nu)$ and $(d_0, \nu) = (d_0, \nu') = b_1$ for all $\nu, \nu' \in V$.

For an antireflexive graph $G = (X, R)$, we define $F(G)$ as the quotient space of the product space $F \times R$ (where R is endowed by the discrete topology) by the least equivalence $\sim$ such that

- ($\bullet$) $(f_0, (x, y)) \sim (f_0, (x, z))$ for all $(x, y), (x, z) \in R$;
- ($\bullet$) $(f_0, (x, y)) \sim (f_1, (z, x))$ for all $(x, y), (z, x) \in R$;

(•) $(f_1,(y,x)) \sim (f_1,(z,x))$ for all $(y,x),(z,x) \in R$.

Observe that if $x \sim y$ for some $x,y \in F \times \{r\}$ with $r \in R$, then $x = y$. For simplicity, we thus identify any point of $F \times R$ with the class of $\sim$ containing it.

First we prove two auxiliary statements about $F(G)$.

Lemma 12. *Let $G = (X,R)$ be an antireflexive, antisymmetric connected graph with $|X| > 1$. If $\phi : F \to F(G)$ is a continuous mapping, then either ϕ is a constant mapping or there exists an arc $r \in R$ such that $\phi(x) = (x,r)$ for every $x \in F$.*

Proof. First assume that there exist distinct $x,y \in F \setminus \{f_0,f_1\}$ such that $\phi(x) = (y,r)$ for some $r \in R$. Let S be the component of the set $\phi^{-1}(\{(z,r) \mid z \in F\})$ containing x. Then S is a continuum because $\{(z,r) \mid z \in F\})$ is a closed subset of $F(G)$. By Theorem 11, we deduce that $\phi(S) = \{(y,r)\}$. Assume that $S \neq F$. Let W be the closure of $F \setminus S$ in the continuum F. Since F is connected, there exists a component S' of W with $S \cap S' \neq \emptyset$ and $S' \cap (F \setminus S) \neq \emptyset$. Then $(y,r) \in \phi(S')$, and $\phi(S')$ is connected because ϕ is continuous. But then there exists a component S'' of $W \cap \phi^{-1}(\{(z,r) \mid z \in F\})$ contained in S' with $S'' \setminus S \neq \emptyset \neq S'' \cap S$. This is a contradiction because $S \cup S'' \neq S$ is a connected subset of $\phi^{-1}(\{(z,r) \mid z \in F\})$. Thus if there exist distinct $x,y \in F \setminus \{f_0,f_1\}$ with $\phi(x) = (y,r)$ for some $r \in R$ then ϕ is a constant mapping. Next we assume that for every $x \in F \setminus \{f_0,f_1\}$ there exists $r_x \in R$ with $\phi(x) = (x,r_x)$. Since F is a metric connected space, there exist two one-to-one sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ of points from $F \setminus \{f_0,f_1\}$ such that $\{x_i\}_{i=1}^{\infty}$ converges to f_0 and $\{y_i\}_{i=1}^{\infty}$ converges to f_1. Then $\{\phi(x_i) = (x_i,r_{x_i})\}_{i=1}^{\infty}$ converges to $\phi(f_0)$ and $\{\phi(y_i) = (y_i,r_{y_i})\}_{i=1}^{\infty}$ converges to $\phi(f_1)$. Hence there exist $r_{f_0},r_{f_1} \in R$ with $\phi(f_0) = (f_0,r_{f_0})$ and $\phi(f_1) = (f_1,r_{f_1})$. From the connectedness of $\phi(F)$ it follows that we can assume $r_{f_0} = r_{f_1}$ and that the component S_x of $\phi^{-1}(\{(z,r_x) \mid z \in F\})$ containing x contains f_0 or f_1 for all $x \in F \setminus \{f_0,f_1\}$. Observe that if $r_{f_0} = (y_0,y_1)$, then the component S_x for $x \in F \setminus \{f_0,f_1\}$ contains f_i if and only if y_i is a node of r_x. Thus if $r_x \neq r_{f_0}$ for some $x \in F \setminus \{f_0,f_1\}$, then $f_0 \in S_x$ just when $f_1 \notin S_x$. Define $\phi' : F \to F(G)$ so that

$$\phi'(x) = \begin{cases} (x,r_x) & \text{if } r_x = r_{f_0}, \\ (f_i,r_{f_0}) & \text{if } r_x \neq r_{f_0} \text{ and } f_i \in S_x \text{ for } i = 0,1. \end{cases}$$

From the above it follows that the definition of ϕ' is correct. From the definition of $F(G)$ and from the continuity of ϕ it follows that $(\phi')^{-1}(Y)$ is closed for every closed set $Y \subseteq F(G)$. Hence ϕ' is continuous and, by Theorem 11, we obtain that $\phi'(x) = (x,r_{f_0})$ for all $x \in F$. Therefore $\phi = \phi'$, and the proof is complete. □

Proposition 13. *Let $G = (X,R)$ and $G' = (X',R')$ be nonvoid connected antireflexive antisymmetric graphs. If $G \neq G'$ and any graph homomorphism from G^0 to $(G')^0$ is constant, then any continuous mapping from $F(G)$ into $F(G')$ is constant. If $G = G'$ and any graph endomorphism of G^0 is either the identity or a constant, then any continuous mapping from $F(G)$ to $F(G)$ is either the identity or a constant.*

Proof. For every $r \in R$, let $\mu_r : F \to F(G)$ be a mapping such that $\mu_r(x) = (x,r)$ for all $x \in F$. Clearly, μ_r is continuous and therefore $\phi \circ \mu_r$ is continuous for all $r \in R$. By Lemma 12, either $\phi \circ \mu_r$ is constant or there exists $s_r \in R'$ such that $\phi \circ \mu_r(x) = (x,s_r)$ for all $x \in F$. If $\phi \circ \mu_r$ is a constant mapping to (x,s) for some $x \in F \setminus \{f_0,f_1\}$, $r \in R$ and $s \in R'$, then

ϕ is the constant mapping with the value (x, s) because G is a nonvoid connected graph. Thus assume that either $\phi \circ \mu_r = \mu_{s_r}$ for some $s_r \in R'$ or $\phi \circ \mu_r$ is a constant mapping with a value (f_i, s) for some $i \in \{0, 1\}$ and some $s \in R'$. If we identify x with (f_0, r) and y with (f_1, r) for all $r = (x, y) \in R$ or $r = (x, y) \in R'$, then we obtain that $\phi(X) \subseteq X'$. Let ψ be the domain-range restriction of ϕ to X and X'. Then for every arc $(x, y) = r$ either $(\psi(x), \psi(y)) = s_r \in R'$ or $\psi(x) = \psi(y)$. Thus ψ is a graph homomorphism from G^0 to $(G')^0$. The statements immediately follow from the assumptions. $\square$

We return to the properties of algebras. For a regular variety $\mathbb{V}$ of unary algebras of type Δ satisfying the condition (di) for $i = 1, 2, 3$, we shall construct a contravariant functor $\Gamma_i : S(P_A) \longrightarrow \mathbb{CL}(\mathbb{V})$.

First assume that $\mathbb{V}$ satisfies the condition (d1) and the quotient algebra $\mathbf{A} = (Z, \{\omega_{\mathbf{A}} \mid \omega \in \Delta\})$ of $\mathcal{F}_{\mathbb{V}}(\emptyset)$ witnesses (d1). For an object $\mathcal{X} = (X, V)$ of $S(P_A)$, we define $\Gamma_1 \mathcal{X} = (\mathcal{U}_1 \mathcal{X}, \{\omega_{\Gamma_1 \mathcal{X}} \mid \omega \in \Delta\}, \tau)$ as follows:

(i) $(\mathcal{U}_1 \mathcal{X}, \tau)$ is the topological space obtained from the disjoint union of $K(\mathcal{X})$ and E by identifying b_1 with e_0 and c_1 with e_1;

(ii) $\omega_{\Gamma_1 \mathcal{X}}(u) = u$ for all $u \in \mathcal{U}_1 \mathcal{X}$ and all $\omega \in \Delta_1$, and $\omega_{\Gamma_1 \mathcal{X}}(0) = \alpha_i^X$ exactly when $\omega_{\mathbf{A}}(0) = i$ for $i = 0, 1$ and $\omega \in \Delta_0$.

For any morphism $\phi : (X, V) \longrightarrow (X', V')$ of $S(P_A)$ define $\Gamma_1 \phi : \Gamma_1(X', V') \longrightarrow \Gamma_1(X, V)$ by

$$\Gamma_1 \phi(x) = \begin{cases} x \circ \phi & \text{if } x \in A^{X'}, \\ x & \text{if } x \in B \cup C \cup E, \\ (d, \nu \circ \phi) & \text{if } x = (d, \nu) \in D \times V'. \end{cases}$$

Then

Proposition 14. *If a regular variety $\mathbb{V}$ of unary algebras satisfies the condition* (d1), *then $\Gamma_1 : S(P_A) \longrightarrow \mathbb{CL}(\mathbb{V})$ is a faithful contravariant functor.*

Proof. By Proposition 10, we obtain that $(\mathcal{U}_1 \mathcal{X}, \tau)$ is a connected paracompact space. By Proposition 6, $(\mathcal{U}_1 \mathcal{X}, \{\omega_{\Gamma_1 \mathcal{X}} \mid \omega \in \Delta\})$ is an algebra from $\mathbb{V}$. Since any mapping from a singleton topological space into a T_1-space is closed and continuous, and so is the identity mapping on any topological space, we conclude that $\Gamma_1 \mathcal{X} \in \mathbb{CL}(\mathbb{V})$. Let $\phi : (X, V) \longrightarrow (X', V')$ be a morphism of $S(P_A)$. First observe that $\Gamma_1 \phi(\alpha_i^{X'}) = \alpha_i^X$ for $i = 0, 1$ because $\Gamma_1 \phi(\nu) = \nu \circ \phi$ for all $\nu \in A^{X'}$ and because α_i^X or $\alpha_i^{X'}$ is the constant mapping to a_i for $i = 0, 1$. From $P_A \phi(V') \subseteq V$ it follws that $\nu \circ \phi \in V$ for all $\nu \in V'$; hence $\Gamma_1 \phi(D \times V') \subseteq D \times V$. Since $\Gamma_1 \phi(u) = u$ for all $u \in B \cup C \cup E$ and $\Gamma_1 \phi(d, \nu) = (d, \nu \circ \phi)$ for all $(d, \nu) \in D \times V'$, we conclude that $\Gamma_1 \phi$ is correctly defined, and since V and V' are endowed with the discrete topology we obtain that $\Gamma_1 \phi$ is closed and continuous exactly when the domain-range restriction of $\Gamma_1 \phi$ to $A^{X'}$ and A^X is closed and continuous. Since the continuity of the restriction is well-known, it suffices to show the closedness. It is well-known that the family $\{\{\nu \in A^{X'} \mid \nu(x) \in U\} \mid x \in X', U \subseteq A \text{ is closed}\}$ forms a basis of closed sets in $A^{X'}$. Since $\Gamma_1 \phi(\{\nu \in A^{X'} \mid \nu(x) \in U\}) = \{\nu \in A^X \mid \nu(y) \in U, y \in \phi^{-1}(x)\}$ is closed in A^X for all $x \in X'$ and all closed sets $U \subseteq A$, we obtain that $\Gamma_1 \phi$ is closed and continuous. Clearly, $\Gamma_1 \phi$ is the identity mapping whenever ϕ is the identity mapping and if $\phi : (X,V) \longrightarrow (X',V')$ and $\phi' : (X',V') \longrightarrow (X'',V'')$ are morphisms of $S(P_A)$, then $\Gamma_1(\phi' \circ \phi) = \Gamma_1\phi \circ \Gamma_1\phi'$. Hence $\Gamma_1 : S(P_A) \longrightarrow \mathbb{CL}(\mathbb{V})$ is a contravariant functor. The faithfulness of Γ_1 follows from the faithfulness of P_A. $\square$

Secondly, assume that $\mathbb{V}$ satisifes the condition (d2). The algebra $\mathcal{F}_{\mathbb{V}}(\emptyset) = (Z, \{\omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)} \mid \omega \in \Delta\})$ is thus connected and nontrivial. For any object $\mathcal{X} = (X,V)$ of $S(P_A)$, we define $\Gamma_2\mathcal{X} = (\mathcal{U}_2\mathcal{X}, \{\omega_{\Gamma_2\mathcal{X}} \mid \omega \in \Delta\}, \tau)$ as follows:

(i) $(\mathcal{U}_2\mathcal{X}, \tau)$ is the quotient space of the disjoint union of $K(\mathcal{X}) \times Z$ and $E \times Z \times Z \times \{1,2\}$ (where Z and $\{1,2\}$ are endowed with the discrete topology) by the least equivalence $\sim$ such that
- ($\bullet$) $(b_i, z) \sim (e_0, z, z', i)$ for all $z, z' \in Z$ and $i = 1, 2$;
- ($\bullet$) $(c_i, z) \sim (e_1, z', z, i)$ for all $z, z' \in Z$ and $i = 1, 2$.

As before, we identify any point of $K(\mathcal{X}) \times Z$ and $E \times Z \times Z \times \{1,2\}$ with the class of $\sim$ containing it – thus $(b_i, z) = (e_0, z, z', i)$ and $(c_i, z) = (e_1, z', z, i)$ for all $z, z' \in Z$ and $i = 1, 2$;

(ii)

$$\omega_{\Gamma_2\mathcal{X}}(u) = \begin{cases} (x, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z)) & \text{if } u = (x,z) \in K(\mathcal{X}) \times Z, \\ (x, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z), \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z'), i) & \text{if } u = (x, z, z', i) \in E \times Z \times Z \times \{1,2\} \end{cases}$$

for all $\omega \in \Delta_1$ and, for all $\omega \in \Delta_0$

$$\omega_{\Gamma_2\mathcal{X}}(0) = (\alpha_0^X, z) \qquad \text{whenever} \quad \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(0) = z.$$

For any morphism $\phi : (X,V) \longrightarrow (X',V')$ of $S(P_A)$, we define $\Gamma_2\phi : \Gamma_2(X',V') \longrightarrow \Gamma_2(X,V)$ by

$$\Gamma_2\phi(x) = \begin{cases} (\nu \circ \phi, z) & \text{if } x = (\nu, z) \in A^{X'} \times Z, \\ x & \text{if } x \in ((B \cup C) \times Z) \cup (E \times Z \times Z \times \{1,2\}), \\ (d, \nu \circ \phi, z) & \text{if } x = (d, \nu, z) \in D \times V' \times Z. \end{cases}$$

Then

Proposition 15. *If a regular variety $\mathbb{V}$ of unary algebras satisfies the condition* (d2), *then $\Gamma_2 : S(P_A) \longrightarrow \mathbb{CL}(\mathbb{V})$ is a faithful contravariant functor.*

Proof. Using Proposition 10 and the fact that $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is connected, we conclude that $(\mathcal{U}_2\mathcal{X}, \tau)$ is a connected paracompact space. By Theorem 4 and Proposition 6, $(\mathcal{U}_2\mathcal{X}, \{\omega_{\Gamma_2\mathcal{X}} \mid \omega \in \Delta\})$ is an algebra from $\mathbb{V}$ because any variety is closed under square powers. Observe that $(b_i, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z)) = (e_0, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z), \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z'), i)$, $(c_i, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z)) = (e_1, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z'), \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z), i)$ for all $z, z' \in Z$ and $i = 1, 2$. Hence, for any $\omega \in \Delta_1$, the mapping $\omega_{\Gamma_2\mathcal{X}} \restriction (E \times Z \times Z \times \{1,2\})$ is the product of the identity mapping on E with $\omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)} \times \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}$ and the identity mapping on $\{1,2\}$. Since any mapping from a singleton topological space into a T_1-space is closed and continuous, and so is the identity mapping of any space, and because Z and $\{1,2\}$ carry discrete topology, it follows that $\Gamma_2\mathcal{X} \in \mathbb{CL}(\mathbb{V})$. For the same reason as in the proof of Proposition 14, we obtain that $\Gamma_2\phi$ is continuous and closed for any morphism ϕ from $S(P_A)$. Since Γ_2 preserves the identity morphisms and composition we conclude that Γ_2 is a contravariant functor. The faithfulness of Γ_2 follows from the faithfulness of P_A. $\square$

Finally, assume that a regular variety $\mathbb{V}$ satisfies the condition (d3). Then $\Delta = \Delta_1$. Since our aim is to prove the universality of $\mathbb{V}$, we may assume that $\mathbb{V}$ has the properties of the variety $\mathbb{W}$ from (d3). By Theorem 9, there exists a family $\{G_\omega \mid \omega \in \Delta\}$ of pairwise disjoint antireflexive antisymmetric connected graphs such that if $\rho : G^0_\omega \to G^0_{\omega'}$ is a graph homomorphism for $\omega, \omega' \in \Delta$, then either ρ is a constant mapping or $\omega = \omega'$ and ρ is the identity mapping. For every $\omega \in \Delta$ select distinct arcs r^ω_0 and r^ω_1 of G_ω. For any object $\mathcal{X} = (X, V)$ of $S(P_A)$, define $\Gamma_3\mathcal{X} = (\mathcal{U}_3\mathcal{X}, \{\omega_{\Gamma_3\mathcal{X}} \mid \omega \in \Delta\}, \tau)$ as follows:

(i) $(\mathcal{U}_3\mathcal{X}, \tau)$ is the quotient space of the disjoint union of $K(\mathcal{X}) \times \mathrm{UOp}(\mathbb{V})$, $E \times \mathrm{UOp}(\mathbb{V}) \times \mathrm{UOp}(\mathbb{V}) \times \{1, 2\}$, and $F(G_\omega) \times \mathrm{UOp}(\mathbb{V})$ with $\omega \in \Delta$ (where $\mathrm{UOp}(\mathbb{V})$ and $\{1, 2\}$ are endowed with discrete topology) by the least equivalence $\sim$ such that
   - $(\bullet)$ $(b_i, \gamma) \sim (e_0, \gamma, \gamma', i)$ and $(c_i, \gamma) \sim (e_1, \gamma', \gamma, i)$ for all $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$ and $i = 1, 2$;
   - $(\bullet)$ $(b_1, \gamma) \sim (f_0, r^\omega_0, \gamma)$ and $(c_1, \gamma \circ \omega) \sim (f_1, r^\omega_1, \gamma)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$, and all $\omega \in \Delta$.

   As before, we identify any point of $K(\mathcal{X}) \times \mathrm{UOp}(\mathbb{V})$ or of $E \times \mathrm{UOp}(\mathbb{V}) \times \mathrm{UOp}(\mathbb{V})$ or of $F(G_\omega) \times \mathrm{UOp}(\mathbb{V})$ for $\omega \in \Delta$ with the class of $\sim$ containing it. Thus $(b_i, \gamma) = (e_0, \gamma, \gamma', i)$ and $(c_i, \gamma) = (e_1, \gamma', \gamma, i)$ for all $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$ and $i = 1, 2$, and $(b_1, \gamma) = (f_0, r^\omega_0, \gamma)$ and $(c_1, \omega \circ \gamma) = (f_1, r^\omega_1, \gamma)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$.

(ii)

$$\omega_{\Gamma_3\mathcal{X}}(u) = \begin{cases} (x, \omega \circ \gamma) & \text{if } u = (x, \gamma) \in K(\mathcal{X}) \times \mathrm{UOp}(\mathbb{V}), \\ (x, \omega \circ \gamma, \omega \circ \gamma', i) & \text{if } u = (x, \gamma, \gamma', i) \in E \times \mathrm{UOp}(\mathbb{V}) \times \mathrm{UOp}(\mathbb{V}) \times \{1, 2\}, \\ (x, \omega \circ \gamma) & \text{if } u = (x, \gamma) \in (\bigcup_{\omega \in \Delta} F(G_\omega)) \times \mathrm{UOp}(\mathbb{V}) \end{cases}$$

   for all $\omega \in \Delta$.

For any morphism $\phi : (X, V) \to (X', V')$ of $S(P_A)$, define $\Gamma_3\phi : \Gamma_3(X', V') \to \Gamma_3(X, V)$ by

$$\Gamma_3\phi(u) = \begin{cases} (\nu \circ \phi, \gamma) & \text{if } u = (\nu, \gamma) \in A^{X'} \times \mathrm{UOp}(\mathbb{V}), \\ (d, \nu \circ \phi, \gamma) & \text{if} u = (d, \nu, \gamma) \in D \times V' \times \mathrm{UOp}(\mathbb{V}), \\ u & \text{if} u \in ((B \cup C) \times \mathrm{UOp}(\mathbb{V})) \\ & \qquad \cup (E \times \mathrm{UOp}(\mathbb{V}) \times \mathrm{UOp}(\mathbb{V}) \times \{1, 2\}) \\ & \qquad \cup ((\bigcup_{\omega \in \Delta} F(G_\omega)) \times \mathrm{UOp}(\mathbb{V})). \end{cases}$$

Then

Proposition 16. *If a regular variety $\mathbb{V}$ of unary algebras satisfies the condition* (d3), *then $\Gamma_3 : S(P_A) \to \mathbb{CL}(\mathbb{V})$ is a faithful contravariant functor.*

Proof. By Proposition 10, we obtain that $(\mathcal{U}_3\mathcal{X}, \tau)$ is a connected paracompact space. Let X be a singleton set. Since $\{(x, \gamma) \mid \gamma \in \mathrm{UOp}(\mathbb{V})\}$ is a subalgebra of $\{\mathcal{U}_3\mathcal{X}, \{\omega_{\Gamma_3\mathcal{X}} \mid \omega \in \Delta\}\}$ isomorphic to $\mathcal{F}_\mathbb{V}(X)$ for all $x \in K(\mathcal{X}) \cup (\bigcup_{\omega \in \Delta} F(G_\omega))$ and because the set $\{(x, \gamma, \gamma', i) \mid \gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})\}$ is a subalgebra of $(\mathcal{U}_3\mathcal{X}, \{\omega_{\Gamma_3\mathcal{X}} \mid \omega \in \Delta\})$ isomorphic to $(\mathcal{F}_\mathbb{V}(X))^2$ for all $x \in E \setminus \{e_0, e_1\}$ and $i = 1, 2$, we deduce, by Theorem 4, that $(\mathcal{U}_3\mathcal{X}, \{\omega_{\Gamma_3\mathcal{X}} \mid \omega \in \Delta\})$ is an algebra from $\mathbb{V}$. Noting that $(b_i, \omega \circ \gamma) = (e_0, \omega \circ \gamma, \omega \circ \gamma', i)$ and $(c_i, \omega \circ \gamma) = (e_1, \omega \circ \gamma', \omega \circ \gamma, i)$

for all $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$ and $i = 1, 2$, we see that $\omega_{\Gamma_3\mathcal{X}} \restriction (E \times \mathrm{UOp}(\mathbb{V}) \times \mathrm{UOp}(\mathbb{V}) \times \{1,2\})$ is the product of the identity mapping on E with $\omega^2_{\mathcal{F}_{\mathbb{V}}(X)}$ and the identity mapping of $\{1,2\}$. Analogously, $(f_1, r_1^{\omega'}, \omega \circ \gamma) = (c_1, \omega \circ \gamma \circ \omega')$ for all $\omega, \omega' \in \Delta$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. Hence $\omega_{\Gamma_3\mathcal{X}} \restriction (F(G_\omega) \times \mathrm{UOp}(\mathbb{V}))$ is the product of the identity mapping and $\omega_{\mathcal{F}_{\mathbb{V}}(X)}$ for all $\omega \in \Delta$. Since the identity mapping of any space and any mapping from a singleton topological space into a T_1-space are closed and continuous and because $\mathrm{UOp}(\mathbb{V})$ carries discrete topology, we conclude that $\Gamma_3\mathcal{X} \in \mathbb{CL}(\mathbb{V})$. For the same reason as in the proof of Proposition 14, we obtain that $\Gamma_3\phi$ is continuous and closed for all morphisms ϕ of $S(P_A)$. Since Γ_3 preserves the identity morphisms and composition, we conclude that Γ_3 is a contravariant functor. The faithfulness of Γ_3 follows from the faithfulness of P_A. $\square$

Next we give several properties of $\Gamma_i\mathcal{X}$.

Lemma 17. *Let X and Y be sets. Then*

(1) *for every $\nu_0 \in A^X$ the set $S_{\nu_0} = \{\nu \in A^X \mid$ the set $\{x \in X \mid \nu(x) \neq \nu_0(x)\}$ is finite$\}$ is a dense subset of A^X;*

(2) *a mapping $\phi : A^X \longrightarrow A^Y$ is continuous if and only if there exist $\nu_0 \in A^Y$ and a partial mapping $\psi : Y \longrightarrow X$ such that*

$$(\phi(\nu))(y) = \begin{cases} \nu_0(y) & \text{if } \psi(y) \text{ is not defined,} \\ \nu(\psi(y)) & \text{if } \psi(y) \text{ is defined} \end{cases}$$

for all $\nu \in A^X$.

Proof. The first statement is folklore. Indeed, let $\mathcal{T}$ be a family of all finite sequences of non-empty open subsets of A indexed by a finite subsets of X. For $\mathcal{O} = \{O_x\}_{x \in X'} \in \mathcal{T}$ where $X' \subseteq X$ is finite define $O_{\mathcal{O}} = \{\nu \in A^X \mid \nu(x) \in O_x \text{ for all } x \in X'\}$. Then $O_{\mathcal{O}}$ is a non-empty open subset of A^X with $O_{\mathcal{O}} \cap S_\nu \neq \emptyset$ for all $\nu \in A^X$. Since $\{O_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{T}\}$ is a basis of open subsets of A^X the proof of (1) is complete. The statement (2) was proved by H. Herrlich in [18]. $\square$

Proposition 18. *Let Y be a continuum from the set $\{B, C, D, E, F\}$, and let $\mathcal{X} = (X, V)$ be an object of $S(P_A)$. If $\phi : Y \longrightarrow \Gamma_i\mathcal{X}$ is a continuous non-constant mapping and $i \in \{1, 2, 3\}$, then:*

(1) *$Y = B$ or $Y = C$ and*
- (•) *if $i = 1$ then $\phi(y) = y$ for all $y \in Y$,*
- (•) *if $i = 2$ then there exists $z \in Z$ such that $\phi(y) = (y, z)$ for all $y \in Y$,*
- (•) *if $i = 3$ then there exists $\gamma \in \mathrm{UOp}(\mathbb{V})$ such that $\phi(y) = (y, \gamma)$ for all $y \in Y$;*

(2) *$Y = D$ and*
- (•) *if $i = 1$ then there exists $\nu \in V$ such that $\phi(y) = (y, \nu)$ for all $y \in Y$,*
- (•) *if $i = 2$ then there exist $z \in Z$ and $\nu \in V$ such that $\phi(y) = (y, \nu, z)$ for all $y \in Y$,*
- (•) *if $i = 3$ then there exist $\gamma \in \mathrm{UOp}(\mathbb{V})$ and $\nu \in V$ such that $\phi(y) = (y, \nu, \gamma)$ for all $y \in Y$;*

(3) *$Y = E$ and*
- (•) *if $i = 1$ then $\phi(y) = y$ for all $y \in Y$,*

(•) *if $i = 2$ then there exist $z, z' \in Z$ and $j \in \{1,2\}$ such that $\phi(y) = (y, z, z', j)$ for all $y \in Y$,*

(•) *if $i = 3$ then there exist $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$ and $j \in \{1,2\}$ such that $\phi(y) = (y, \gamma, \gamma', j)$ for all $y \in Y$;*

(4) *$Y = F$, $i = 3$ and there exist $\gamma \in \mathrm{UOp}(\mathbb{V})$ and an arc r of G_ω for some $\omega \in \Delta$ such that $\phi(y) = (y, r, \gamma)$ for all $y \in Y$.*

Proof. Assume that $Y = B$. First we prove that $Im(\phi) \subseteq B$ when $i = 1$, $Im(\phi) \subseteq B \times Z$ when $i = 2$, $Im(\phi) \subseteq B \times \mathrm{UOp}(\mathbb{V})$ when $i = 3$. If contrary then there exists $y \in Y$ such that either $i = 1$ and $\phi(y) \neq y'$ for all $y' \in B$, or $i = 2$ and $\phi(y) \neq (y', z)$ for all $y' \in B$ and all $z \in Z$, or $i = 3$ and $\phi(y) \neq (y', \gamma)$ for all $y' \in B$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. Since $\phi(B)$ is connected, by Theorem 11 and the definition of $\Gamma_i \mathcal{X}$ (for $i = 1, 2, 3$), we deduce that ϕ is a constant mapping with the value $\phi(y)$ – a contradiction. If $i = 1$ then $Im(\phi) \subseteq B$ and Theorem 11 completes the proof. If $i = 2$ then a normality of $(\mathcal{U}_2 \mathcal{X}, \tau)$ implies that there exists $z \in Z$ with $Im(\phi) \subseteq B \times \{z\}$ because $B \times \{z'\}$ and $B \times \{z''\}$ are disjoint closed subsets of $\Gamma_2 \mathcal{X}$ for distinct $z', z'' \in Z$ and $Im(\phi)$ is connected. Now Theorem 11 completes the proof. For $i = 3$ the proof is the same as for $i = 2$. If $Y = C$ the proof is analogous.

Assume that $Y = D$. By the same reason as above we obtain that $Im(\phi) \subseteq D \times V/\sim$ when $i = 1$, $Im(\phi) \subseteq (D \times V/\sim) \times Z$ when $i = 2$, and $Im(\phi) \subseteq (D \times V/\sim) \times \mathrm{UOp}(\mathbb{V})$ when $i = 3$. First assume that $i = 1$. Let ψ be the range restriction of ϕ to $D \times V/\sim$ and let $\pi : (D \times V/\sim) \to D$ be a mapping given by $\pi(x, \nu) = x$ for all $x \in D$ and all $\nu \in V$. Then π is continuous. By Theorem 11, $\pi \circ \psi$ is either a constant or the identity. If $\pi \circ \psi$ is a constant mapping then from the connectedness of D it follows that ψ (and also ϕ) is constant. Thus for every $x \in D$ there exists $\nu_x \in V$ with $\phi(x) = \psi(x) = (x, \nu_x)$. Set $\nu = \nu_{d_1}$ and define a mapping $\eta : D \to D$ by

$$\eta(x) = \begin{cases} x & \text{if } \nu_x = \nu, \\ d_0 & \text{if } \nu_x \neq \nu. \end{cases}$$

From the continuity of ψ and the connectednes of D it follows that η is continuous. Since η is non-constant we conclude, by Theorem 11, that η is the identity mapping and hence $\phi(x) = (x, \nu)$ for all $x \in D$. If $i = 2$ then $Im(\phi) \subseteq (D \times V)/\sim) \times Z$. Since $\{(x, \nu, z) \mid x \in D,\ \nu \in V\}$ and $\{(x, \nu, z') \mid x \in D,\ \nu \in V\}$ are disjoint closed sets for distinct $z, z' \in Z$, since $\Gamma_2 \mathcal{X}$ is a normal space and D is connected, we conclude that there exists $z \in Z$ with $Im(\phi) \subseteq \{(x, \nu, z) \mid x \in D,\ \nu \in V\}$. The same argument as above shows that there exists $\nu \in V$ with $\phi(y) = (y, \nu, z)$ for all $y \in Y$. The proof for $i = 3$ is analogous. By the same argument we obtain the statement for $Y = E$. If $Y = F$ then, analogously as above, Theorem 11 implies that ϕ is constant whenever $i = 1$ or $i = 2$. Thus we can assume that $i = 3$, then, as before, we conclude that $Im(\phi) \subseteq (\bigcup_{\omega \in \Delta} F(G_\omega)) \times \mathrm{UOp}(\mathbb{V})$. Since $(\bigcup_{\omega \in \Delta} F(G_\omega)) \times \{\gamma\}$ and $(\bigcup_{\omega \in \Delta} F(G_\omega)) \times \{\gamma'\}$ are disjoint closed sets for distinct $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$, there exists $\gamma \in \mathrm{UOp}(\mathbb{V})$ with $Im(\phi) \subseteq (\bigcup_{\omega \in \Delta} F(G_\omega)) \times \{\gamma\}$, and Lemma 12 then completes the proof. □

Lemma 19. *Let $\mathcal{X} = (X, V)$ be an object of $S(P_A)$ and let Y be a set. If $\phi : A^Y \to \Gamma_i \mathcal{X}$ is a continuous non-constant mapping for some $i = 1, 2, 3$, then there exist $\nu_0 \in A^X$ and a* *partial mapping* $\psi : X \longrightarrow Y$ *such that*

(1) *if* $i = 1$ *then*

$$(\phi(\nu))(x) = \begin{cases} \nu(\psi(x)) & \text{if } \psi(x) \text{ is defined,} \\ \nu_0(x) & \text{if } \psi(x) \text{ is not defined} \end{cases}$$

for all $\nu \in A^Y$;

(2) *if* $i = 2$ *then for some* $z \in Z$

$$(\phi(\nu))(x) = \begin{cases} (\nu(\psi(x)), z) & \text{if } \psi(x) \text{ is defined,} \\ (\nu_0(x), z) & \text{if } \psi(x) \text{ is not defined} \end{cases}$$

for all $\nu \in A^Y$;

(3) *if* $i = 3$ *then for some* $\gamma \in \mathrm{UOp}(\mathbb{V})$

$$(\phi(\nu))(x) = \begin{cases} (\nu(\psi(x)), \gamma) & \text{if } \psi(x) \text{ is defined,} \\ (\nu_0(x), \gamma) & \text{if } \psi(x) \text{ is not defined} \end{cases}$$

for all $\nu \in A^Y$.

Proof. Assume that $i = 1$ and that there exists $\nu \in A^Y$ such that $\phi(\nu) \notin A^X$. For $y \in Y$ define a mapping $\mu_y : A \longrightarrow A^Y$ such that

$$(\mu_y(x))(u) = \begin{cases} \nu(u) & \text{if } u \neq y, \\ x & \text{if } u = y \end{cases}$$

for all $x \in A$. Clearly, μ_y is a continuous mapping, and from Theorem 11 and the definition of $\Gamma_1\mathcal{X}$ we conclude that $\phi \circ \mu_y$ is a constant mapping with the value $\phi(\nu)$. Then, by a simple induction, we obtain that $\phi(S_\nu) = \{\phi(\nu)\}$, where $S_\nu = \{\nu' \in A^Y \mid \text{the set } \{y \in Y \mid \nu(y) \neq \nu'(y)\} \text{ is finite}\}$. By Lemma 17(1), S_ν is a dense subset of A^Y, and therefore ϕ is a constant mapping. Thus $Im(\phi) \subseteq A^X$, and Lemma 17(2) completes the proof. If $i = 2$ (or $i = 3$) then, by the same argument, we obtain that $Im(\phi) \subseteq A^X \times Z$ (or $Im(\phi) \subseteq A^X \times \mathrm{UOp}(\mathbb{V})$). Since A^Y is a connected space and since $A^X \times \{z\}$ and $A^X \times \{z'\}$ for distinct $z, z' \in Z$ are disjoint closed sets we obtain, by the normality of $\Gamma_2\mathcal{X}$, that $Im(\phi) \subseteq A^X \times \{z\}$ for some $z \in Z$. Analogously, if $i = 3$ then $Im(\phi) \subseteq A^X \times \{\nu\}$ for some $\nu \in \mathrm{UOp}(\mathbb{V})$. Lemma 17(2) completes the proof in both cases. $\square$

To finish the proof of Theorem 3, we need to prove that if $\mathcal{X}$ and $\mathcal{X}'$ are objects of $S(P_A)$ and $\phi : \Gamma_i\mathcal{X} \longrightarrow \Gamma_i\mathcal{X}'$ is a continuous homomorphism for some $i = 1, 2, 3$, then there exists a morphism $\psi : \mathcal{X}' \longrightarrow \mathcal{X}$ of $S(P_A)$ with $\Gamma_i\psi = \phi$. Therefore we shall assume that a continuous homomorphism $\phi : \Gamma_i\mathcal{X} \longrightarrow \Gamma_i\mathcal{X}'$ is given for some objects $\mathcal{X} = (X, V)$ and $\mathcal{X}' = (X', V')$ of $S(P_A)$ and for some $i = 1, 2, 3$.

Lemma 20. *Let* $\mathcal{X} = (X, V)$ *and* $\mathcal{X}' = (X', V')$ *be objects of* $S(P_A)$ *and let* $\phi : \Gamma_i\mathcal{X} \longrightarrow \Gamma_i\mathcal{X}'$ *be a continuous mapping for* $i = 1, 2, 3$. *If there exists a non-constant continuous mapping* μ *from* B *or* C *to* $\Gamma_i\mathcal{X}$ *such that* $\phi \circ \mu$ *is a constant mapping, then* ϕ *is a constant mapping.*

Proof. First assume that $i = 1$ and $\beta : B \longrightarrow \Gamma_1\mathcal{X}$. If $\phi \circ \beta$ is a constant mapping with the value b_1 then $\phi(B) = \{b_1\}$ and, by Lemma 19, $\phi(A^X) = \{b_1\}$ and, by Lemma 18,

$\phi(C) = \phi(D \times \{\nu\}) = \{b_1\} = \phi(E)$ for all $\nu \in V$. Thus ϕ is a constant mapping with the value b_1. If $\phi \circ \beta$ is a constant mapping with the value c_1 then $\phi(B) = \{c_1\}$ and, by Lemma 18, $\phi(E) = \phi(D \times \{\nu\}) = \{c_1\}$ for all $\nu \in V$. By Lemma 19, $\phi(A^X) = \{c_1\}$, and, by Lemma 18, we conclude that ϕ is a constant mapping with the value c_1 because $\phi(C) = \{c_1\}$. If $\phi \circ \beta$ is a constant mapping with the value $x \neq b_1, c_1$, then $\phi(B) = \{x\}$ and, by Lemma 18, $\phi(C) = \phi(E) = \phi(D \times \{\nu\}) = \{x\}$ for all $\nu \in V$. Hence $\phi(\alpha_0) = \phi(\alpha_1)$, and Lemma 19 implies $\phi(A^X) = \{x\}$ because α_i is a constant mapping from X to A with the value a_i. Thus ϕ is a constant mapping to x. Similarly we obtain that ϕ is a constant mapping if $\beta : C \longrightarrow \Gamma_1\mathcal{X}$.

Assume that $i = 2$. Let $\beta : B \longrightarrow \Gamma_2\mathcal{X}$ be such that there exists $z \in Z$ with $\beta(y) = (y, z)$ for all $y \in B$. Assume that $\phi \circ \beta$ is a constant mapping to (b_k, z') for some $z' \in Z$ and $k = 1, 2$. Then $\phi(\{(y, z) \mid y \in B\}) = (b_k, z')$ and, by Lemma 19, we conclude that $\phi(\nu, z) = (b_k, z')$ for all $\nu \in A^X$. By Lemma 18, we obtain that $\phi(x, z) = (b_k, z')$ for $x \in C \cup (\bigcup_{\nu \in V} D \times \{\nu\})$. In particular, $\phi(c_1, z) = \phi(c_2, z) = (b_k, z')$ and, by Lemma 18, $\phi(x, z'', z, j) = (b_k, z')$ for all $x \in E$, $z'' \in Z$ and $j = 1, 2$. Hence, by Lemma 18, $\phi(x, z'') = (b_k, z')$ for all $x \in B$ and all $z'' \in Z$. Therefore ϕ is a constant mapping to (b_k, z'). Assume that $\phi \circ \beta$ is a constant mapping to (c_k, z') for some $z' \in Z$ and $k = 1, 2$. Then $\phi(y, z) = (c_k, z')$ for all $y \in B$ and, by Lemma 18, $\phi(x, \nu, z) = (c_k, z')$ for all $x \in D$ and all $\nu \in V$ and $\phi(x, z, z'', j) = (c_k, z')$ for all $x \in E$, $z'' \in Z$ and $j = 1, 2$. Hence we deduce that $\phi(y, z) = (c_k, z')$ for all $y \in C$ and then $\phi(x, z'', z, 3-k) = (c_k, z')$ for all $x \in E$, $z'' \in Z$. By Lemma 19, $\phi(\nu, z) = (c_k, z')$ for all $\nu \in A^X$ and, by Lemma 18, $\phi(y, z'') = (c_k, z')$ for all $y \in B$ and all $z'' \in Z$. Thus ϕ is a constant mapping to (c_k, z'). Assume that $\phi \circ \beta$ is a constant mapping to x with $x \notin \{(b_k, z), (c_k, z) \mid k = 1, 2,\ z \in Z\}$. Then, by Lemma 18, we obtain that $\phi(y, z) = \phi(y', z, z', j) = x$ for all $y \in B \cup C \cup (\bigcup_{\nu \in V} D \times \{\nu\})$, $y' \in E$, $z, z' \in Z$, $j = 1, 2$. Thus $\phi(\alpha_0, z) = \phi(\alpha_1, z) = x$ for all $z \in Z$ and, by Lemma 19, we obtain that ϕ is a constant mapping to x. The proof is analogous for $\beta : C \longrightarrow \Gamma_2\mathcal{X}$.

Assume that $i = 3$. Then, by the same argument as in the case of $i = 2$, we obtain that if $\phi \circ \beta$ is a constant mapping to x then $\phi(y, z) = \phi(y, z, z', j) = x$ for all $y \in A^X \cup B \cup C \cup (\bigcup_{\nu \in V} D \times \{\nu\})$, $y' \in E$, $z, z' \in Z$, $j = 1, 2$. Since $(b_1, \gamma) = (f_0, r_0^\omega, \gamma)$ and $(c_1, \gamma \circ \omega) = (f_1, r_1^\omega, \gamma)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$, it follows that $\phi(f_0, r_0^\omega, \gamma) = \phi(f_1, r_1^\omega, \gamma) = x$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$. By the assumptions on the family $\{G_\omega \mid \omega \in \Delta\}$ and by Proposition 13, we obtain that $\phi(F(G_\omega) \times \{\gamma\}) = x$ for all $\omega \in \Delta$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. Thus ϕ is a constant mapping to x. $\square$

Corollary 21. *Let $\mathbb{V}$ be a regular variety of unary algebras of type Δ satisfying the condition* (d1). *If $\phi : \Gamma_1\mathcal{X} \longrightarrow \Gamma_1\mathcal{X}'$ is a continuous homomorphism for objects $\mathcal{X}$ and $\mathcal{X}'$ of $S(P_A)$, then there exists a morphism $\psi : \mathcal{X}' \longrightarrow \mathcal{X}$ of $S(P_A)$ with $\Gamma_1\psi = \phi$. Thus $\mathcal{K}(\mathbb{V})$ is universal for every subcategory $\mathcal{K}$ of $\mathbb{PAR}$ containing the category $\mathbb{CL}$.*

Proof. Let $\phi : \Gamma_1\mathcal{X} \longrightarrow \Gamma_1\mathcal{X}'$ be a continuous homomorphism for objects $\mathcal{X} = (X, V)$ and $\mathcal{X}' = (X', V')$ of $S(P_A)$. By the defintions of $\Gamma_1\mathcal{X}$ and $\Gamma_1\mathcal{X}'$ and by Proposition 6, $\{\alpha_i^X \mid i = 0, 1\} = \{\omega_{\Gamma_1\mathcal{X}}(0) \mid \omega \in \Delta_0\}$, $\{\alpha_i^{X'} \mid i = 0, 1\} = \{\omega_{\Gamma_1\mathcal{X}'}(0) \mid \omega \in \Delta_0\}$. Hence $\phi(\alpha_i^X) = \alpha_i^{X'}$ for $i = 0, 1$, and thus ϕ is not a constant mapping. By Lemmas 18 and 20, $\phi(y) = y$ for all $y \in B \cup C$. Hence $\phi(e_i) = e_i$ for $i = 0, 1$ and, by Lemma 18, $\phi(y) = y$ for all $y \in E$. Since $\phi(\alpha_i^X) = \alpha_i^{X'}$ for $i = 1, 2$ and since $\alpha_i^{X'}$ are distinct constant mappings, we deduce, by Lemma 19, that there exists a mapping $\psi : X' \to X$ such that $(\phi(\nu))(x) = \nu(\psi(x))$ for all $\nu \in A^X$ and all $x \in X'$. For $\nu \in V$ we have $\nu = (d_1, \nu)$ and $b_1 = (d_0, \nu)$ and, by Lemma 18, we conclude that $\phi(y, \nu) = (y, \phi(\nu))$ for all $y \in D$. Thus $\phi(\nu) \in V'$. Hence $(P_A(\psi))(V) = \phi(V) \subseteq V'$, and therefore $\psi : \mathcal{X}' \to \mathcal{X}$ is a morphism of $S(P_A)$ with $\Gamma_1\psi = \phi$. The rest follows from Theorem 8. $\square$

Theorem 22. *Let $\mathbb{V}$ be a regular variety of unary algebras of type Δ satisfying the condition* (d2). *If $\phi : \Gamma_2\mathcal{X} \to \Gamma_2\mathcal{X}'$ is a continuous homomorphism for objects $\mathcal{X}$ and $\mathcal{X}'$ of $S(P_A)$, then there exists a morphism $\psi : \mathcal{X}' \to \mathcal{X}$ of $S(P_A)$ with $\Gamma_2\psi = \phi$. Thus $\mathcal{K}(\mathbb{V})$ is universal for every subcategory $\mathcal{K}$ of $\mathbb{PAR}$ containing the category $\mathbb{CL}$.*

Proof. Let $\phi : \Gamma_2\mathcal{X} \to \Gamma_2\mathcal{X}'$ be a continuous homomorphism for objects $\mathcal{X} = (X, V)$ and $\mathcal{X}' = (X', V')$ of $S(P_A)$. Since $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is a subset of $Im(\phi)$ and because $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is a nontrivial algebra, we deduce that the homomorphism ϕ is not constant. By Lemmas 18 and 20, for every $z \in Z$ there exist $t_z, u_z \in Z$ such that $\phi(y, z) = (y, t_z)$ for all $y \in B$ and $\phi(y, z) = (y, u_z)$ for all $y \in C$. Thus, by Lemma 19, $t_z = u_z$ and there exists a mapping $\psi_z : X' \to X$ such that $\phi(\nu, z) = (\nu \circ \psi_z, t_z)$ for all $\nu \in A^X$ because $\phi(\alpha_i^X, z) = (\alpha_i^{X'}, t_z)$ for $i = 1, 2$. By Lemma 18, $\phi(y, z, z', j) = (y, t_z, t_{z'}, j)$ for all $y \in E$, $z, z' \in Z$ and $j = 1, 2$. From $(P_A\psi_z(\nu), z) = \phi(\nu, z)$ for all $\nu \in V$ and all $z \in Z$ it follows that $P_A\psi_z(V) \subseteq V'$ for all $z \in Z$. Since $\{(\alpha_0, z) \mid z \in Z\}$ is a subalgebra of $\Gamma_2\mathcal{X}$ and $\Gamma_2\mathcal{X}'$ isomorphic to $\mathcal{F}_{\mathbb{V}}(\emptyset)$, we deduce that $\phi(\alpha_0, z) = (\alpha_0, z)$ and therefore $z = t_z$ for all $z \in Z$. Since $\omega_{\Gamma_2\mathcal{X}}(\nu, z) = (\nu, \omega_{\mathcal{F}_{\mathbb{V}}(\emptyset)}(z))$ and since $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is connected we conclude that $\psi_z = \psi_{z'}$ for all $z, z' \in Z$. Set $\psi = \psi_z$ for $z \in Z$, then $\Gamma_2\psi = \phi$. The rest follows from Theorem 8. $\square$

Theorem 23. *Let $\mathbb{V}$ be a regular variety of unary algebras of type Δ satisfying the condition* (d3). *If $\phi : \Gamma_3\mathcal{X} \to \Gamma_3\mathcal{X}'$ is a continuous homomorphism for objects $\mathcal{X}$ and $\mathcal{X}'$ of $S(P_A)$, then there exists a morphism $\psi : \mathcal{X}' \to \mathcal{X}$ of $S(P_A)$ with $\Gamma_3\psi = \phi$. Thus $\mathcal{K}(\mathbb{V})$ is universal for every subcategory $\mathcal{K}$ of $\mathbb{PAR}$ containing the category $\mathbb{CL}$.*

Proof. Let $\phi : \Gamma_3\mathcal{X} \to \Gamma_3\mathcal{X}'$ be a continuous homomorphism for objects $\mathcal{X} = (X, V)$ and $\mathcal{X}' = (X', V')$ of $S(P_A)$. If $\Gamma_3\mathcal{X}' = (\mathcal{U}_3\mathcal{X}', \{\omega_{\Gamma_3\mathcal{X}'} \mid \omega \in \Delta\}, \tau)$ then from the construction of $\Gamma_3\mathcal{X}'$ it follows that the algebra $(\mathcal{U}_3\mathcal{X}', \{\omega_{\Gamma_3\mathcal{X}'} \mid \omega \in \Delta\})$ is a disjoint union of algebras isomorphic to $\mathcal{F}_{\mathbb{V}}(\{x\})$ or to the second power of $\mathcal{F}_{\mathbb{V}}(\{x\})$ for some singleton set $\{x\}$. Hence, by (d3), $(\mathcal{U}_3\mathcal{X}', \{\omega_{\Gamma_3\mathcal{X}'} \mid \omega \in \Delta\})$ has no singleton subalgebra, and thus ϕ is a non-constant mapping, because $Im(\phi)$ is a subalgebra of $(\mathcal{U}_3\mathcal{X}', \{\omega_{\Gamma_3\mathcal{X}'} \mid \omega \in \Delta\})$. By Lemmas 18 and 20, there exist mappings $\lambda_\phi : \mathrm{UOp}(\mathbb{V}) \to \mathrm{UOp}(\mathbb{V})$ and $\lambda'_\phi : \mathrm{UOp}(\mathbb{V}) \to \mathrm{UOp}(\mathbb{V})$ such that $\phi(b, \gamma) = (b, \lambda_\phi(\gamma))$ for all $b \in B$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and $\phi(c, \gamma) = (c, \lambda'_\phi(\gamma))$ for all $c \in C$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. In particular, $\phi(\alpha_0^X, \gamma) = (\alpha_0^{X'}, \lambda_\phi(\gamma))$ and $\phi(\alpha_1^X, \gamma) = (\alpha_1^{X'}, \lambda'_\phi(\gamma))$. By Lemma 19, we conclude that $\lambda_\phi(\gamma) = \lambda'_\phi(\gamma)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and that there exists $\psi_\gamma : X' \to X$ such that $\phi(\nu, \gamma) = (\nu \circ \psi_x, \lambda_\phi(\gamma))$ for all $\nu \in A^X$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$, because $\alpha_0^{X'}$ and $\alpha_1^{X'}$ are distinct constant mappings. Lemma 18 implies that $\phi(d, \nu, \gamma) = (d, \nu \circ \psi_\gamma, \lambda_\phi(\gamma))$ for all $d \in D$, $\nu \in V$ and $\phi(e, \gamma, \gamma', j) = (e, \lambda_\phi(\gamma), \lambda_\phi(\gamma'), j)$ for all $e \in E$, $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$ and $j = 1, 2$. Hence $\psi_\gamma(V) \subseteq V'$, and therefore $\psi_\gamma : (X', V') \to (X, V)$ is a morphism of $S(P_A)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$. From Lemmas 12 and 18 it follows that if $\omega \in \Delta$ then $\phi(f, r, \gamma) = (f, r, \lambda_\phi(\gamma))$ for all $f \in F$, all arcs r of G_ω and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. Since

$(f_1, r_1^\omega, \gamma) = (c_1, \gamma \circ \omega)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$, we deduce that

$$(c_1, \lambda_\phi(\gamma) \circ \omega) = (f_1, r_1^\omega, \lambda_\phi(\gamma)) = \phi(f_1, r_1^\omega, \gamma) = \phi(c_1, \gamma \circ \omega) = (c_1, \lambda_\phi(\gamma \circ \omega))$$

and hence $\lambda_\phi(\gamma) \circ \omega = \lambda_\phi(\gamma \circ \omega)$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$. Thus λ_ϕ commutes with the inner right translation of ω in $\mathrm{Mon}(\mathbb{V})$ for every $\omega \in \Delta$. Since the set Δ generates the monoid $\mathrm{Mon}(\mathbb{V})$, we obtain that λ_ϕ commutes with all inner right translations of $\mathrm{Mon}(\mathbb{V})$ and, by Theorem 5, λ_ϕ is equal to an inner left translation of $\mathrm{Mon}(\mathbb{V})$, in particular $\lambda_\phi \in M_{\mathbb{V}}$. Since ϕ is a homomorphism we obtain that $\phi \circ \omega_{\Gamma_3 \mathcal{X}} = \omega_{\Gamma_3 \mathcal{X}'} \circ \phi$ for all $\omega \in \Delta$. Hence we deduce that

$$\begin{aligned}(b_1, \lambda_\phi(\omega \circ \gamma)) &= \phi(b_1, \omega \circ \gamma) = \phi \circ \omega_{\Gamma_3 \mathcal{X}}(b_1, \gamma) = \omega_{\Gamma_3 \mathcal{X}'} \circ \phi(b_1, \gamma) \\ &= \omega_{\Gamma_3 \mathcal{X}'}(b_1, \lambda_\phi(\gamma)) = (b_1, \omega \circ \lambda_\phi(\gamma))\end{aligned}$$

for all $\omega \in \Delta$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. Hence $\lambda_\phi(\omega \circ \gamma) = \omega \circ \lambda_\phi(\gamma)$ for all $\omega \in \Delta$ and all $\gamma \in \mathrm{UOp}(\mathbb{V})$. Thus λ_ϕ commutes with the inner left translation of ω in $\mathrm{Mon}(\mathbb{V})$ for all $\omega \in \Delta$. Since Δ generates $\mathrm{Mon}(\mathbb{V})$ we obtain that λ_ϕ commutes with all inner left translations of $\mathrm{Mon}(\mathbb{V})$ and thus $\lambda_\phi \in \mathrm{Com}(M_{\mathbb{V}})$. By Proposition 7, λ_ϕ is the identity mapping. From

$$\begin{aligned}(\nu \circ \psi_\gamma, \omega \circ \gamma) &= \omega_{\Gamma_3 \mathcal{X}'}(\nu \circ \psi_\gamma, \gamma) = \omega_{\Gamma_3 \mathcal{X}'} \circ \phi(\nu, \gamma) = \phi \circ \omega_{\Gamma_3 \mathcal{X}}(\nu, \gamma) \\ &= \phi(\nu, \omega \circ \gamma) = (\nu \circ \psi_{\omega \circ \gamma}, \omega \circ \gamma)\end{aligned}$$

for all $\nu \in A^X$, all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$ it follows that $\psi_\gamma = \psi_{\omega \circ \gamma}$ for all $\gamma \in \mathrm{UOp}(\mathbb{V})$ and all $\omega \in \Delta$. Since Δ generates $\mathrm{Mon}(\mathbb{V})$, we conclude, by Theorem 5, that $\psi_\gamma = \psi_{\gamma'}$ for all $\gamma, \gamma' \in \mathrm{UOp}(\mathbb{V})$. Set $\psi = \psi_\gamma$. Then $\psi : \mathcal{X}' \to \mathcal{X}$ is a morphism of $S(P_A)$ with $\Gamma_3 \psi = \phi$. The rest follows from Theorem 8. $\square$

Now Theorem 3 is a consequence of Corollary 21 and Theorems 22 and 23. Thus Theorems 2 and 3 are proved. $\square$

Remark. The characterization of all universal topological unary varieties remains open, but Theorem 21 can be generalized to certain non-regular varieties. Consider a variety $\mathbb{V}$ of unary algebras such that $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is a nontrivial algebra and for a singleton set $X = \{x\}$ and for $S(X) = \{\gamma_{\mathcal{F}_V(X)}(x) \mid \gamma \in \mathrm{UOp}(\mathbb{V})\}$, we have $\mathcal{F}_{\mathbb{V}}(\emptyset) \setminus S(X) \neq \emptyset$ and $\mathcal{F}_{\mathbb{V}}(\emptyset) \cap S(X) \neq \emptyset$ (we assume that $\mathcal{F}_{\mathbb{V}}(\emptyset)$ is the subalgebra of $\mathcal{F}_{\mathbb{V}}(X)$ generated by $\emptyset$). For any object $\mathcal{X} = (X, V)$, we let $\Gamma_4 \mathcal{X}$ be the quotient topological algebra of $\Gamma_2 \mathcal{X}$ by the least equivalence $\sim$ such that

- (•) $(x, z) \sim (x', z)$ for all $x, x' \in A^X \cup B \cup C \cup (D \times V)$ and all $z \in S(X) \cap \mathcal{F}_{\mathbb{V}}(\emptyset)$;
- (•) $(x, z, z, j) \sim (x', z, z, j')$ for all $x, x \in E$, $j, j' \in \{1, 2\}$ and all $z \in S(X) \cap \mathcal{F}_{\mathbb{V}}(\emptyset)$.

In a standard way we obtain that Γ_4 is a faithful contravariant functor from $S(P_A)$ into $\mathbb{CL}(\mathbb{V})$ and, analogously to the proof of Theorem 21, it can be shown that $\Gamma_4 : S(P_A) \to \mathcal{K}(\mathbb{V})$ is a contravariant full embedding for any subcategory $\mathcal{K}$ of $\mathbb{PAR}$ containing $\mathbb{CL}$.

The main obstacle of the generalization of Theorem 2 to all varieties of unary algebras is the fact that the construction of Γ_3 and hence the proof of Theorem 23 cannot be used for non-regular varieties of unary algebras.

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