Distance constrained labeling of precolored trees*

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Abstract

Graph colorings with distance constraints are motivated by the frequency assignment problem. The so called $\lambda_{(p,q)}$-labeling problem asks for coloring the vertices of a given graph with integers from the range $\{0, 1, \ldots, \lambda\}$ so that labels of adjacent vertices differ by at least p and labels of vertices at distance 2 differ by at least q, where p, q

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are fixed integers and integer λ is part of the input. It is known that this problem is *NP*-complete for general graphs, even when λ is fixed, i.e., not part of the input , but polynomially solvable for trees for $(p,q) = (2,1)$. It was conjectured that the general case is also polynomial for trees. We consider the precoloring extension version of the problem (i.e., when some vertices of the input tree are already precolored) and show that in this setting the cases $q = 1$ and $q > 1$ behave differently: the problem is polynomial for $q = 1$ and any p, and it is *NP*-complete for any $p > q > 1$.

1 Introduction

The radio frequency (or channel) assignment problem stems from important practical applications. Its graph theoretical model [9] asks for a labeling of the vertices of an input graph by nonnegative integers so that labels of vertices at distance at most i differ by at least p_i, for every $i \leq k$, where k and $p_1, \ldots, p_k$ are parameters of the problem. As a particular subproblem, Roberts proposed the problem of assigning integers (frequencies) to vertices (transmitters) such that vertices that are "fairly close" to each other (at distance two) receive different labels and vertices that are very close (adjacent) receive labels that are at least two apart. This corresponds to the case of $k = 2$ and $(p_1, p_2) = (2, 1)$, referred to as $(2,1)$-labelings of graphs [2, 4, 7, 8, 10, 13, 14]. The more general two-parameter problem with $(p_1, p_2) = (p, q)$, $p \geq q > 1$, was considered in [1, 4, 5].

The minimum λ such that a graph G allows a (p,q)-labeling by integers from the range $\{0, 1, \ldots, \lambda\}$ is denoted by $\lambda_{(p,q)}(G)$. It was shown in [7, 14] that determining $\lambda_{(2,1)}(G)$ is an *NP*-complete problem even for graphs G with diameter two. The complexity of deciding $\lambda_{(2,1)}(G) \leq \lambda$ for fixed λ was shown *NP*-complete for every $\lambda \geq 4$ in [4] (answering a problem asked in [14]). It was also shown in [4] that for every $p \geq q \geq 1$, there is a λ (dependent on p,q) such that deciding $\lambda_{(p,q)}(G) \leq \lambda$ is *NP*-complete.

As concerns special graph classes, $\lambda_{(2,1)}(G)$ can be determined efficiently for paths, cycles and wheels [7], and for cographs and trees [2] (thus disproving the conjecture of [7] that the problem is *NP*-complete for trees). D. Welsh suggested [personal communication, 1999] that, by an algorithm similar to Chang and Kuo's, it should be possible to determine $\lambda_{(p,q)}(T)$ for a tree T for arbitrary p, q. We will review the algorithm of [2] in Section 2. The crucial step of the algorithm uses bipartite matchings (or systems of distinct representatives, SDR), and an analogous algorithm for $q > 1$ would need to be able to decide existence of "systems of *distant* representatives". This problem is, however, *NP*-complete in general, as shown in Section 3. It is therefore plausible to conjecture that determining $\lambda_{(p,q)}(T)$ is *NP*-hard for trees, when $q > 1$. Note also that the complexity of determining $\lambda_{(2,1)}(G)$ for graphs of bounded tree-width is not known.

For graph coloring problems, it is natural to consider the *precoloring extension* variants of the problems where some vertices of the input graph are given as already (pre)colored, and the question is if this precoloring can be extended to a proper coloring of the entire graph using a given number of colors. For several results on the complexity of precoloring extension for special graph classes see e.g. [11, 12]. The aim of this paper is to consider the precoloring extension variant of (p, q)-labelings of trees. We show that at least in this setting the cases $q = 1$ and $q > 1$ behave quite differently. We will show in Section 2 that Chang and Kuo's algorithm can be easily extended to precolored trees and parameters $(p, 1)$, for any p. On the other hand, as we will show in Section 4, the problem is *NP*-complete for every $p > q > 1$.

2 Algorithm for Trees and $q = 1$

We present a slightly modified version of the algorithm of [2].

The input consists of an integer value λ and a tree T with some vertices precolored by a function $g : U \longrightarrow \{0, 1, \ldots, \lambda\}$, where $U \subseteq V(T)$ is the set of precolored vertices. We first choose a leaf r as a root of T, which defines the parent-child relation between every pair of adjacent vertices. For any edge $xy \in E(T)$ such that x is a child of y (i.e., y is the neighbor of x on the path from x to the root), we denote by T_{xy} the subtree of T rooted in y and containing y, x and all descendants of x. For every such edge and for every pair of colors a, b, we introduce a boolean variable $\phi(x, y; a, b)$ valued true if and only if T_{xy} has a $(p, 1)$-coloring $f : V(T_{xy}) \longrightarrow \{0, 1, \ldots, \lambda\}$ which extends the precoloring and such that $f(x) = a, f(y) = b$. We show how to evaluate ϕ in polynomial time (using dynamic programming from the leaves of T towards the root r). This will constitute a polynomial time algorithm for T, since the tree T allows a precoloring extension if and only if there exist colors a, b such that $\phi(r', r; a, b) = \mathsf{true}$, where r' is the only child of r. The evaluation of ϕ goes as folows:

1. Set $\phi(x, y; a, b) = \mathsf{false}$ for every edge xy and any two colors $a, b \in \{0, 1, \ldots, \lambda\}$.

2. Consider an edge xy such that x is a leaf. Set $\phi(x, y; a, b) = \mathsf{true}$ for all pairs $a, b \in \{0, 1, \ldots, \lambda\}$ such that $|b - a| \geq p$ and, if applicable, such that $a = g(x)$ if $x \in U$ and/or $b = g(y)$ if $y \in U$.

3. Consider an edge xy such that $\phi(x', y'; a, b)$ was already evaluated for all edges $x'y'$ of T_{xy} except xy. Let $z_1, \ldots, z_k$ be the children of x.

   For every pair of colors $a, b \in \{0, 1, \ldots, \lambda\}$ such that $|b - a| \geq p$, and $a = g(x)$ if $x \in U$ and/or $b = g(y)$ if $y \in U$, define

   $$M_i = \{c | \ \phi(z_i, x; c, a) = \mathsf{true} \text{ and } b \neq c\}, \tag{$*$}$$

   and set

   $$\phi(x, y; a, b) = \mathsf{true}$$

   if and only if the set system $M_i, i = 1, 2, \ldots, k$, has a System of Distinct Representatives.

4. If there are colors $a, b \in \{0, 1, \ldots, \lambda\}$ such that $\phi(r', r; a, b) = \mathsf{true}$ then output "T allows a precoloring extension." else output "T does not allow a $(p, 1)$-labeling extending the precoloring."

The correctness of the algorithm follows by an easy inductive argument. For the time complexity, $\phi(x, y; a, b)$ is evaluated for $n-1$ edges and for $O(\lambda^2)$ pairs of labels for each edge. The recursive step (assuming x has k children) takes time $O(k\lambda)$ for constructing the sets M_i and then $O((k + \lambda)^2 k\lambda)$ for bipartite matching (SDR). Altogether the running time is $O(n^2\lambda^5)$. Thus we have proved:

Theorem 1 *Existence of a $(p, 1)$-labeling which extends a given precoloring of a given tree can be decided in polynomial time, for any integer p.*

A similar algorithm could be used for the general case of (p,q)-colorings. The only difference is that condition $(*)$ must now read

$$M_i = \{c |\ \phi(z_i, x; c, a) = \textsf{true} \text{ and } |b - c| \geq q\}$$

and we set $\phi(x, y; a, b) = \textsf{true}$ if and only if the set system $M_i, i = 1, 2, \ldots, k$, allows a System of q-Distant Representatives, i.e., a system of representatives $c_i, i = 1, 2, \ldots, k$ such that $c_i \in M_i$ for each i and $|c_i - c_j| \geq q$ for all $i \neq j$. We will show in the next section that this problem is in general *NP*-complete, and hence the algorithm is not straightforwardly polynomial for $q > 1$. It is, however, still possible that the existence of a system of q-distant representatives within the algorithm can be decided in polynomial time because of some special properties of the sets M_i. But most likely (unless *P*=*NP*) this can be the case only for the plain (p,q)-labeling problem (when T has no precolored vertices). We will show in Section 4 that for any $p \geq q > 1$, the precoloring extension (p,q)-labeling problem is *NP*-complete.

Note that the hardness results apply only when the bound λ is part of the input:

Theorem 2 *For every fixed p, q and fixed λ, the precoloring extension (p,q)-labeling problem with labels in the range $\{0, 1, \ldots, \lambda\}$ is polynomially solvable for trees.*

Proof Since $\lambda_{(p,q)}(G) \geq q\Delta(G)$ (where $\Delta(G)$ denotes the maximum degree in G), the precoloring may only allow an extension to a (p,q)-labeling if the maximum degree of T is bounded. But then k, the number of children of x in the recursive step is bounded as well and the existence of a system of q-distant representatives can be decided in time $O(\lambda^k)$ by brute force. $\square$

3 Systems of Distant Representatives

The theory of Systems of Distinct Representatives (or, equivalently, matchings in bipartite graphs) is well known and very important for discrete optimization problems. Both for the elegant Hall theorem that describes necessary and sufficient conditions for the existence of an SDR, and for a polynomial time algorithm (augmenting paths or the matching algorithm of Edmonds [3]). Though several generalizations of the concept have been studied, we believe that the concept of *distant* representatives is new.

Definition 3.1 *Given integers q and m and a family $\mathcal{F} = \{M_i|\ i \in I\}$ of subsets of $X = \{1, 2, \ldots, m\}$, a mapping $f : I \longrightarrow X$ is called a* System of q-Distant Representatives *(shortly an Sq-DR) if*

(1) $f(i) \in M_i$ for every $i \in I$,

(2) $|f(i) - f(j)| \geq q$ for every $i, j \in I, i \neq j$.

Note that for $q = 1$ the condition (2) merely says that $f(i) \neq f(j)$, i.e., a system of 1-distant representatives is a system of distinct representatives. For this case the ordering of the elements of X becomes irrelevant (as long as any two elements of X are at distance at least 1). For $q > 1$, the linear ordering of the elements of X becomes a determining factor. The fact that deciding whether a given family has an Sq-DR is *NP*-complete for $q > 1$ is interesting on its own. For the purpose of λ-colorings we need a somewhat stronger result. We call a set of numbers *t-sparse* if $|x - y| \geq t$ for every two distinct members x, y of the set.

Theorem 3 *For every $q > 1$ and every t, it is NP-complete to decide if a family $\mathcal{F}$ of t-sparse sets of integers has a system of q-distant representatives.*

Proof We reduce from 3-Satisfiability of Boolean formulas in conjunctive normal form. This problem is known to be *NP*-complete even when restricted to formulas whose each clause contains 2 or 3 literals and every variable occurs in exactly 3 clauses – once positive and twice negated [6]. Suppose $\Phi = (V, C)$ is such a formula (where V is its variable set and C its clause set). We assume that the variables are numbered $x_1, x_2, \ldots, x_n$.

Fix a number $s > t + q \geq 2q$ (we may assume without loss of generality that $t \geq q$). For a variable x_i, let clause c contain the positive occurence of x_i and let clauses d, e contain $\neg x_i$. Denote $x_i(c) = (i-1)s+2$, $x_i(d) = (i-1)s+1$ and $x_i(e) = (i-1)s + q + 1$. For every clause $c \in C$ create a 3-element set $M_c = \{x_i(c)|x_i \in c \text{ or } \neg x_i \in c\}$.

Observe first that the sets M_c, $c \in C$ are t-sparse. This is because the smallest difference of any two numbers $x_i(c), x_j(d)$, $i \neq j$ is at least $s - q > t$.

We claim that $\mathcal{M} = \{M_c\}_{c \in C}$ contains an Sq-DR if and only if Φ is satisfiable. Suppose first that $\mathcal{M}$ contains an Sq-DR f. We define a truth valuation ϕ of the variables V so that

- $\phi(x_i) = \textsf{true}$ if $f(c) = x_i(c)$ for some clause c such that $x_i \in c$,

- $\phi(x_i) = \mathsf{false}$ if $f(c) = x_i(c)$ for some clause c such that $\neg x_i \in c$,
- $\phi(x_i)$ is arbitrary if none of the above applies.

Obviously every clause is satisfied by this assignment. We have to show, though, that ϕ is defined correctly. Assume to the contrary that for some variable x_i, $\phi(x_i)$ should be both true and false. That means that $(i-1)s+2$ and at least one of $(i-1)s+1$ and $(i-1)s+q+1$ are both representatives of some sets. This is, however, impossible, since their difference $(i-1)s+2-((i-1)s+1) = 1$ or $(i-1)s+q+1-((i-1)s+2) = q-1$ would be less than q.

On the other hand, suppose that Φ is satisfied by a truth valuation ϕ. For every clause c, pick a variable (say x_i) which satisfies c and set $f(c) = x_i(c)$. This is clearly a set of representatives of $\mathcal{M}$. To see that the system is q-distant, note that if different clauses c and d are satisfied by different variables then the difference of $f(c)$ and $f(d)$ is at least $s-q > t \geq q$, while if c and d are satisfied by the same variable, say x_i, then this variable must occur negated in both clauses, and hence $f(c) = (i-1)s+1$ and $f(d) = (i-1)s+q+1$ (or vice versa), and thus their difference is q. $\square$

4 *NP*-completeness for $q > 1$

We prove the main result in this section:

Theorem 4 *For every fixed $p \geq q > 1$, it is NP-complete to decide if a precoloring of a tree can be extended to a (p,q)-labeling whose labels do not exceed a given λ.*

Proof We show a reduction from Systems of q-Distant Representatives. Let $t \geq q^2 + 2p$ and let a family $\mathcal{M} = \{M_i\}_{i=1}^n$ of t-sparse sets be given. We may assume that $\min(\bigcup_{i=1}^n M_i) \geq t$ and we set $\lambda = t + \max(\bigcup_{i=1}^n M_i)$.

We build a tree T with a root v_0 and children of the root $v_i, i = 1, 2, \ldots, n$ (and another level of nodes $v_{i,j,m}$, as defined later). The root will be precolored by 0, vertices v_i will not be precolored, but each of them will have a certain number of children precolored so that the only admissible colors for v_i will be exactly the elements of M_i. This can be attained as follows.

For a vertex v_i, let $M_i = \{a_{i,1} < a_{i,2} < \ldots < a_{i,k_i}\}$. Now consider the interval $[a_{i,j}+1, a_{i,j+1}-1]$. Since M_i is t-sparse, $a_{i,j+1} - a_{i,j} \geq q^2 + 2p$ and one can choose numbers $c_{i,j,1} = a_{i,j} + p < c_{i,j,2} < \ldots < c_{i,j,l_j} = a_{i,j+1} - p$ so that $q \leq c_{i,j,m+1} - c_{i,j,m} \leq 2p - 1$ for every $m = 1, 2, \ldots, l_j - 1$. (In more detail, if $a_{i,j+1} - a_{i,j} - 2p = \alpha q + \beta$ then $\alpha \geq q$ and we set $l_j = \alpha + 1$. Then take $c_{i,j,m+1} = c_{i,j,m} + q + 1$ for $m = 1, 2, \ldots, \beta$ and take $c_{i,j,m+1} = c_{i,j,m} + q$ for $m = \beta + 1, \ldots, \alpha$. Note that $2p - 1 \geq q + 1 > q$.) Finally for all $m = 1, 2, \ldots, l_j$ add leaves $v_{i,j,m}$ precolored by $c_{i,j,m}$ pending on the vertex v_i. In every (p,q)-labeling f, it must be $f(v_i) \notin [a_{i,j}+1, a_{i,j+1}-1]$, since any integer from this interval differs by at most $p-1$ from some $c_{i,j,m}$.

We perform this construction for every vertex v_i and for each interval $[a_{i,j}+1, a_{i,j+1}-1]$. Because we have assumed that $a_{i,1} \geq t$ and $\lambda - a_{i,k_i} \geq t$, the initial and terminal intervals $[0, a_{i,1}-1]$ and $[a_{i,k_i}+1, \lambda]$ are handled in the same way, with dummy border values $a_{i,0} = -1$ and $a_{i,k_i+1} = \lambda + 1$.

It follows that T allows a (p,q)-labeling which extends the precoloring (and uses labels from $\{0, 1, \ldots, \lambda\}$) if and only if $\mathcal{M}$ has a system of q-distant representatives.

Suppose that $f : V(T) \longrightarrow \{0, 1, \ldots, \lambda\}$ is a (p,q)-labeling which extends the precoloring. For each i, $f(v_i) \in M_i$, since (by the argument above) for every j, $f(v_i) \notin [a_{i,j}+1, a_{i,j+1}-1]$. Since for $i \neq i'$, v_i and $v_{i'}$ have a common neighbor (the root v_0), we have $|f(v_i) - f(v_{i'})| \geq q$ and f restricted to $\{v_1, v_2, \ldots, v_n\}$ yields a system of q-distant representatives for $\mathcal{M}$.

The converse, i.e., showing that T allows a (p,q)-labeling extending the precoloring, provided $\mathcal{M}$ allows a system of q-distant representatives, is straightforward. Note here that the precoloring of T constructed from $\mathcal{M}$ itself satisfies the (p,q)-constraints, since for each i the colors used on $v_{i,j,m}$ are at least q apart, they are at least p (and thus at least q away from the label 0 of the root), and they do not interfere with other $v_{i'}$ or $v_{i',j',m'}$ since such vertices are at distance at least three. $\square$

5 Conclusion and open problems

We have shown a significant difference in the computational complexity of (p,q)-labeling of trees for $q = 1$ and $q > 1$, when some vertices are precolored. This has been achieved by introducing the notion of q-distant representatives for set systems, which generalizes the well known distinct representatives (bipartite matching). We believe that the result that Sq-DR is *NP*-complete for $q > 1$ (while S1-DR, identical with SDR, is polynomial) is interesting on its own.

However, for the (p, q)-labeling problem of trees without precolored vertices, the complexity is still open when $q > 1$. It is tempting to try to prove *NP*-completeness along the lines above. One possibility would be to replace precolored vertices by trees that allow only certain labels at the root. Of course one cannot ask for trees that would allow only one possible label, since the set of admissible labels for a particular vertex is always symmetric with respect to the interval $[0, \lambda]$ (if f is a (p, q)-labeling then so is $f' = \lambda - f$). This does not cause problems for the desired reduction since one can show that Systems of q-Distant Representatives are *NP*-complete even when all sets $M_i \in \mathcal{M}$ are symmetric. This observation leads to the following open problems (affirmative solution to the first one would imply *NP*-completeness of (p, q)-labelings of trees):

Problem 1 Does there exist, for any relatively prime $p > q > 1$, any sufficiently large λ and any $(q^2 + 2p)$-sparse set M, a construction of trees $T_{x,\lambda}$, $x \in [p, \lambda - p]$, such that

(1) the size of $T_{x,\lambda}$ is polynomial in λ,

(2) $T_{x,\lambda}$ allows a (p, q)-labeling in which the root is labeled by x and all labels of children of the root are at distance at least q from the set M,

(3) in any (p, q)-labeling of $T_{x,\lambda}$, the root is labeled either by x or by $\lambda - x$?

The condition (1) is imposed to guarantee polynomiality of the desired reduction. However, we do not even know of the existence of any $T_{x,\lambda}$ satisfying at least (2-3), and therefore we deem the following problem interesting on its own:

Problem 2 Prove that for any relatively prime $p > q > 1$, any large enough λ and any $x \in [p, \lambda - p]$, there exists a tree $T_{x,\lambda}$ such that in any (p, q)-labeling of $T_{x,\lambda}$ the root is labeled either by x or by $\lambda - x$, and $T_{x,\lambda}$ has such a labeling.

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