Note

A Bijective Proof of an Identity for Noncrossing Graphs

Pavel Podbrdský

Department of Applied Mathematics, Charles University
Malostranské náměstí 25, 118 00 Prague
Czech Republic
e-mail: `podbrdsk@karlin.mff.cuni.cz`

By a *noncrossing graph* G we mean an undirected graph (it may have multiple edges) on the vertex set $[n] = \{1,\ 2, \ldots, n\}$ which contains no couple of edges $\{i, j\}$ and $\{k, l\}$ such that $i < k < j < l$. In other words, when we represent the vertices $1, 2, \ldots, n$ of G by the points $(1, 0), (2, 0), \ldots, (n, 0)$, respectively, of the x-axis and embed the edges as halfcircles in the upper halfplane, no edges of G intersect.

Let A be the set of all noncrossing graphs with no multiple edges (isolated vertices are allowed) and B be the set of all noncrossing graphs with no isolated vertices (multiple edges are allowed). Let a_n be the number of graphs of A with n vertices, let b_n be the number of graphs of B with n edges. Then the following identity holds:

$$a_{n+2} = 8b_n.$$

This identity was proved by Klazar in [2] by generating functions. He derived formula

$$\sum_{n=0}^{\infty} b_n x^n = 1 + x + 6x^2 + 44x^3 + \cdots = \frac{1 + 10x - \sqrt{1 - 12x + 4x^2}}{16x}.$$

Thus $b_n = 2^{n-1} S_n$ $(n \geq 1)$ where S_n are the well known Schröder numbers which have generating function

$$\sum_{n=0}^{\infty} S_n x^n = 1 + x + 3x^2 + 11x^3 + \cdots = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x}.$$

Flajolet and Noy proved in [1], also by generating functions, that $a_n = 2^n S_{n-2}$ ($n \geq 2$). Comparision of both expressions for a_n and b_n gives the identity. Our aim in the present note is to give a bijective proof without use of generating functions.

Theorem 1 *For all* $n \in \mathbb{N}$
$$a_{n+2} = 8b_n.$$

Proof: The idea of the proof is this: We find a certain recursive structures in the sets A and B which have the same behaviour. Then we match together graphs of A and B with the same recursive structure (which is done by induction). The scheme of this process is drawn in Figure 1. The precise proof follows.

Let $B_n = \{G \in B : |E(G)| = n\}$ and $A_n = \{G \in A : |V(G)| = n\}$. We construct a bijection φ_n between B_n and certain 8-element subsets of A_{n+2} which we call groups. Each group in A_k ($k \geq 3$) contains graphs which differ only in the edges $\{1,2\}$, $\{k-1,k\}$ or $\{1,k\}$ (presence or absence of any of these 3 edges does not affect whether G is noncrossing). We represent every group by the graph that contains none of these 3 edges. Clearly, the groups partition A_k. Let $A'_n \subset A_n$ be the set of graphs which do not contain edges $\{1,2\}$, $\{n-1,n\}$, and $\{1,n\}$. We construct the bijection φ_n between B_n and A'_{n+2} recursively, using previously constructed bijections φ_k between B_k and A'_{k+2} ($k < n$).

For $n = 1$ (the first row of Figure 1) we define φ_1 in the only possible way by matching the graph containing one edge (the only element of B_1) with the empty graph on three vertices (the only element of A'_3).

Let $n > 1$, let φ_k ($k < n$) be the (already constructed) bijections between B_k and A'_{k+2} and let $H \in B_n$. We define $\varphi_n(H)$. Let $e = \{1,l\}$ be the longest edge of H incident with the vertex 1 (if e is a multiple edge, we take e as the topmost embedded one). The edge e exists, because graph H contains no isolated vertices. We distinguish three cases:

- (The last two rows of Figure 1.) No edge lies under e. Let H_2 be the graph consisting of the edges $E(H) \setminus \{e\}$ (H_2 lies to the right of e). We say that H has type $[x]$ where $x = 1$ if H_2 is "tangent" to e (i.e. some edge in H_2 is incident to the vertex $l = 2$) and $x = 0$ otherwise. Let $G_2 = \varphi_{n-1}(H_2)$ have the vertex set $\{1,\ 2, \ldots, n+1\}$. We define $\varphi_n(H) = G$ where G is constructed from G_2 by adding a new vertex $n+2$ and the edge $\{1, n+1\}$. When $x = 1$ (where $[x]$ is the type of H) we also add the edge $\{n, n+1\}$.
- (Rows 10–13 of Figure 1.) There is no edge to the right of $e = \{1,l\}$. Let H_1 be the graph consisting of the edges $E(H) \setminus \{e\}$ (H_1 lies under the edge e). We say that the graph H has type $[x,y]$ where $x = 1$ if H_1 contains an edge incident to the vertex 1 (otherwise $x = 0$) and $y = 1$ if H_1 contains an edge incident to the vertex l (otherwise $y = 0$). Let $G_1 = \varphi_{n-1}(H_1)$ have the vertex set $\{2,\ 3, \ldots, n+2\}$. We define $\varphi_n(H) = G$ where G is

constructed from G_1 by adding vertex 1, when $x = 1$ we also add the edge $\{2, n+2\}$ and when $y = 1$ we also add the edge $\{2, 3\}$.

- (Rows 2–9 of Figure 1.) There are edges both under and to the right of $e = \{1, l\}$. Let H_1 be the graph consisting of b_1 edges lying under e in H and let H_2 be the graph consisting of b_2 edges lying to the right of e in H ($b_1 + b_2 + 1 = n$). We say that H has type $[x, y, z]$ where $x = 1$ if H_1 contains an edge adjacent to the vertex 1 (otherwise $x = 0$), $y = 1$ if H_1 contains an edge adjacent to the vertex l (otherwise $y = 0$) and $z = 1$ if H_2 contains an edge adjacent to the vertex l (otherwise $z = 0$). Let $G_1 = \varphi_{b_1}(H_1)$ and $G_2 = \varphi_{b_2}(H_2)$ have the vertex sets $\{1,\ 2, \ldots, b_1+2\}$ and $\{b_1+2,\ b_1+3, \ldots, b_1+b_2+3\}$ (clearly $b_1+b_2+3 = n+2$). Then we define $\varphi_n(H) = G$ where G is constructed by putting graphs G_1 and G_2 together (the last vertex of G_1 is the same as the first vertex of G_2), adding the edge $(1, b_1+2)$ always and adding the edges $\{b_1+1, b_1+2\}$, $\{b_1+2, b_1+3\}$, $\{b_1+2, n+2\}$ when $x = 1$, $y = 1$, $z = 1$, respectively.

It is clear from the definitions that always $\varphi_n(H) = G \in A'_{n+2}$. We have to prove that the mappings $\varphi_n : B_n \to A'_{n+2}, n = 1, 2, \ldots,$ are bijections. We do this recursively as well.

It it trivial for $n = 1$.

Let $n > 1$ and let all φ_k $(k < n)$ be bijections. We observe few facts:

- φ_n is correctly defined. Every graph $H \in B_n$ has a unique type and the mappings φ_k which we applied on the subghraphs H_1 and H_2 are bijections. Thus we have defined for every $H \in B_n$ exactly one image $\varphi_n(H)$.

- φ_n is surjective. Having $G \in A'_{n+2}$, we take $f = \{1, l\}$ to be the longest edge adjacent to vertex 1 in G (if there exist any) and find a good type for the preimage H:

  - If f does not exist then H has type $[x, y]$ where the values of x and y depend on presence or absence of the edges $\{2, n+2\}$ and $\{2, 3\}$ in G. We take $H_1 = \varphi_{n-1}^{-1}(G_1)$ where G_1 is the subgraph of G with the vertex set $\{2,\ 3,\ \ldots,\ n+2\}$ containing all the edges of G except for $\{2, n+2\}$ and $\{2, 3\}$. We construct H by adding one edge to H_1 (according to its type) so that $\varphi_n(H) = G$.
  - If $f = \{1, n+1\}$ then H has type $[x]$ where the value of x depends on presence or absence of the edge $\{n, n+1\}$ in G. Similarly to the first case we take $H_2 = \varphi_{n-1}^{-1}(G_2)$ where G_2 is the subgraph of G with the vertex set $\{1,\ 2,\ \ldots,\ n+1\}$ containing all the edges of G except for $\{1, n+1\}$ and $\{n, n+1\}$. We construct H by adding one edge to H_2 (according to its type) so that $\varphi_n(H) = G$.

– If $f = \{1, l\}$, $l \leq n$ the graph H has type $[x, y, z]$ where the values of x, y and z depend on presence or absence of the edges $\{l-1, l\}$, $\{l, l+1\}$ and $\{l, n+2\}$ in G. We find G_1 and G_2 in a similar way as in the first two cases, take H_1 and H_2 to be the preimages of G_1 and G_2 and construct H (according to its type) so that $\varphi_n(H) = G$.

- It is also clear that the way of choosing the type of $\varphi^{-1}(G)$ and the graphs H_1 and H_2 above is unique. Then also the graph H such that $\varphi_n(H) = G$ is unique and φ_n is an injection.

We proved that φ_n is a bijection for every $n \in \mathbb{N}$. $\square$

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References

[1] P. Flajolet and M. Noy, Analytic combinatorics of non-crossing configurations, *Discrete Math.* **204** (1999), 203–229.

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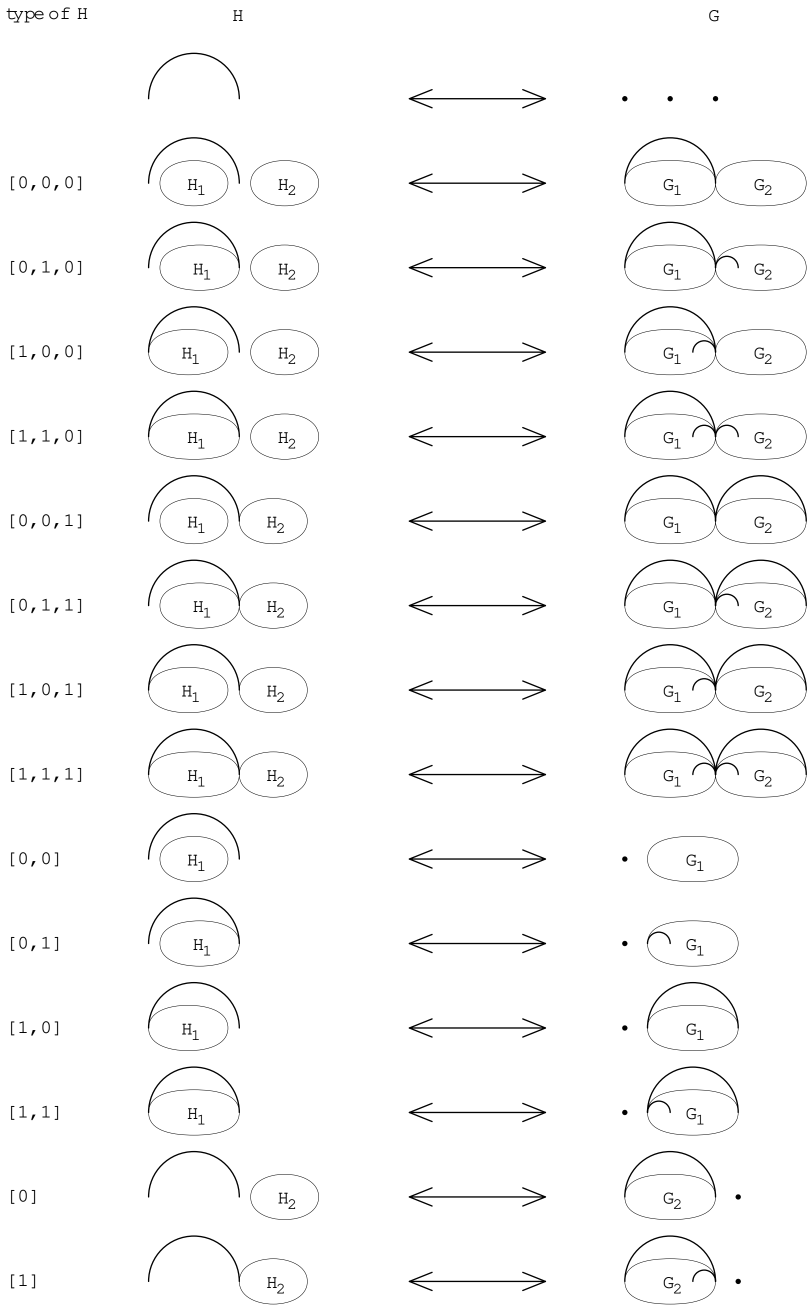

Figure 1: The recursive construction of bijection φ_n.