A new small emendation of Gödel's ontological proof

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1 Introduction

Gödel's ontological proof of necessary existence of a godlike being was finally published in the third volume of Gödel's collected works [7]; but it became known since 1970 when Gödel showed the proof to Dana Scott and Scott presented it (in fact a variant of it) at a seminar at Princeton. Detailed history is found in Adam's introductory remarks to the ontological proof in [7]. The proof uses modal logic and its analysis is an exciting exercise in systems of formal modal logic. Needles to say, formal modal logic has found several uses e.g. in the theory of consistency proofs (provability logic) as well as in foundations of computer science and artificial intelligence (dynamic logic, logics of belief, modal reasoning under uncertainty).

From the literature analyzing and modifying Gödel's proof let me first cite Sobel [11] observing that (with enough comprehension) the system suffers by collapse of modalities (proving $\varphi \rightarrow \Box\varphi$ for each φ); Anderson's emendation [1] having no collapse, my two papers [8, 9] in which I offer another analysis of the collapse and show that (under an appropriate modal predicate logic) Anderson's axiom system is redundant; further [2] where the authors point out that this is not the case in a weaker modal logic.[1] The original Gödel's proof as well as the emendations can be understood in a formalism having variables for objects, variables for (unary) properties of objects, a binary application predicate (object x has the property X) and a unary second order predicate (a property X is positive). Gödel (also in Scott's and Anderson's modification) has 5 axioms; below

[1] I also mention the paper [6], presently in final stages of preparation.

the reader will encounter Anderson's variant which will be our starting point. What makes the proof interesting is the fact that it depends on several choices: (1) the underlying modal propositional logic (S5 or less?), (2) the asssumed comprehension (which formulas define properties?), (3) the underlying behaviour of quantifiers with respect to modalities. To illustrate the last point the question is if we think of Kripke models with a set W of possible worlds, a fixed domain M of objects (not depending on W) and properties interpreted as mappings of $W \times M$ into $\{0,1\}$ (plus an interpretation of positiveness) *or* if we assume also the domain to be dependent on possible worlds, each world bearing its own domain. The latter semantics is reducible to the former by introducing a special property E describing for each possible worlds the subset of objects of the general domain existing in the given world. The details of these semantics and Gödel's proof are discussed in depth in the forthcoming Fitting's monograph [5] where the author also explains Anderson's emendation and mentions mine.

The emendations have concerned mainly Gödel's first axiom (saying that a property is positive iff its negation is not positive). The axiom (A2), i.e. $((P(X) \& \Box(X \subseteq Y)) \rightarrow P(Y))$ can be read as saying that if a property X is positive and Y is its necessary consequence then Y is also positive. It has not been questioned by any of the above authors but I find it difficult to motivate: If H stands for godlike and D for "devillike" (a predicate not occuring in Gödel) then assuming H to be positive we would have to assume that the disjunction of H and D (being godlike or being devillike) is also positive, which appears to be counter-intuitive. (This example also occurs in [4].) In the present paper we show how to weaken the axiom to get rid of this unpleasant feature, preserving the desired consequence.

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2 Preliminaries.

Let us first be explicit on the underlying logic. We shall keep the logic first-order, but two-sorted; we shall have a sort of individuals and sort of properties. (Note that the logic is described in my [9]). Variables for individuals are lower case $x, y, z, \ldots$; variables for properties are upper case $X, Y, Z, \ldots$ There will be one property-constant H for "godlike", one binary predicate $\in$ of application ($x \in X$ will be abbreviated as $X(x)$), one unary predicate P applied to properties ($P(X)$ is read "X is positive") and finally equality predicate $=$. There may be futher constants. Formulas are built from atoms of the form $X(x)$, $P(X)$, $x = y$, $X = Y$ using connectives, quantifiers $\forall, \exists$ and modalities $\Box, \Diamond$ ($\exists$ and $\Diamond$ being defined from $\forall, \Box$).

Logical axioms are those of classical two-sorted predicate calculus (for example $(\forall X)(\psi \rightarrow \varphi(X)) \rightarrow (\psi \rightarrow (\forall X)\varphi(X)$ is X is not free in ψ), equality axioms, i.e.

reflexivity, symmetry, transitivity and the following:
$x = y \to X(x) \equiv X(y)$,
$X = Y \equiv (\forall x)(X(x) \equiv Y(x))$ (extensionality),
$\Box(X = Y) \to (P(X) \equiv P(Y))$.
Furthermore, we have usual axioms of the modal predicate logic S5, formulated as follows: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, $\Box\varphi \to \varphi$, $\varphi \equiv \Box\varphi$ for boxed φ (resulting from formulas of the form $\Box\psi$ using connectives and quantifiers). Finally, we have necessary equality for objects: $x = y \to \Box(x = y)$.

Semantics: Kripke models of the form $K = (W, M, Prop, \mathcal{P})$ where $W \neq \emptyset$ is the set of possible worlds, $M \neq \emptyset$ is the set of individuals, $Prop$ is a set of mappings $F : (W \times M) \to \{0, 1\}$ ($F(w, m) = 1$ meaning that in the world w, the individual m has the property coded by F) and $\mathcal{P} : (W \times Prop) \to \{0, 1\}$.

A (global) theory is a set Q of axioms. $Q \vdash \varphi$ means that φ is provable from Q in our logic using deduction rules modus ponens, generalization and necessitation. (Local theory is as above but without the rule of necessitation.)

Satisfaction of a formula in a possible world by an evaluation of free variables ($W, w \Vdash \varphi[e]$) and truth of a formula in a world ($K, w \Vdash \varphi$) is defined in the obvious way, in particular: e evaluates variables for individuals by elements in M and variables for properties by elements of $Prop$; $K, w \vdash X(x)[e]$ if $e(X)(w, e(x)) = 1$; equality is interpreted absolutely; $K, w \Vdash \Box\varphi[e]$ if for each $w' \in W, K, w' \Vdash \varphi[e]$ etc. $K \Vdash \varphi$ iff $(K, w) \Vdash \varphi$ for all possible worlds of K. K is a *model* of a theory Q of $K \Vdash \varphi$ for all $\varphi \in Q$.

Theorem *(Completeness).* Let Q be a theory and φ a formula. $Q \vdash \varphi$ iff $K, w \Vdash \varphi$ for each Kripke model $K = (W, M, Prop, \mathcal{P})$ of Q and each $w \in W$.

(See [9] for a proof.)

The following is the schema of *full comprehension* (C_{full}) : for each formula $\varphi(x)$ not containing the variable Y,

$$(\exists Y)\Box(\forall x)(Y(x) \equiv \varphi(x))$$

is the comprehension axiom for φ (stating that the formula $\varphi(x)$ defines a property). Note that φ may have free variables other than x, but different from Y.

If a theory contains the full comprehension schema then, in particular, properties can be combined using Boolean connectives, e.g. we can prove

$$(\exists Y)\Box(\forall x)(Y(x) \equiv \neg X(x)),$$

$$(\exists Z)\Box(\forall x)(Z(x) \equiv U(x)\&V(x))$$

etc.

Then we may safely denote the Y above by $\neg X$ and the Z above by $U\&V$. (Note that our axioms guarantee uniqueness.)

We also write $X \subseteq Y$ for $(\forall u)(X(u) \to Y(u))$. (Thus $X = Y$ is equivalent to $\Box((X \subseteq Y)\&(Y \subseteq X))$.

Lemma. ([9]). Our logic proves $\Box(\forall x)\varphi \equiv (\forall x)\Box\varphi$ and $(\exists x)\Box\varphi \rightarrow \Box(\exists x)\varphi$.

Proof. We prove the latter formula.

$\vdash \varphi \rightarrow (\exists x)\varphi$,
$\vdash \Box\varphi \rightarrow \Box(\exists x)\varphi$,
$\vdash (\forall x)[\Box\varphi \rightarrow \Box(\exists x)\varphi]$,
$\vdash (\exists x)\Box\varphi \rightarrow \Box(\exists x)\varphi$.

To close this section let us recall Anderson's emendation of Gödel's original axioms as presented in [1]. Note that his logic different from ours; this will be discussed in Section 4, where we present all five his axioms. Here we only introduce the first three and the definition of godlikeness. We use (A1)–(A5) to denote Anderson's axioms.

(A1) $P(X) \rightarrow \neg P(\neg X)$

(A2) $(P(X)\&\Box(X \subseteq Y)) \rightarrow P(Y)$

(Definition.) $H(x) \equiv (\forall Y)(P(Y) \equiv \Box\varphi(x))$

(x is godlike iff properties it necessarily has are exactly all positive properties).

(A3) $P(H)$.

In [9] (and before it in [8]) I proved that in our logic, Anderson's axioms (A1–A3) together with full comprehension prove $\Box(\exists x)H(x)$ (necessary existence of a godlike individual). We shall analyze this proof in the next section. Note also that in [8] several models of Anderson's system are constructed. A short example: $W \neq \emptyset \neq M$ arbitrary, $g \in M$ a fixed individual, *Prop* is the set of *all* mappings $F : W \times M \rightarrow \{0,1\}; \mathcal{P}(F) = 1$ (independently from $w \in W$) iff for all $w, F(w,g) = 1$. Anderson's (Gödels) axioms (A4), (A5) are formulated in Section 4.

3 The emendation

Gödel's explanation for (A2) is ([7] p. 434): "It is possible to interpret positive as perfective...The chief axiom runs then (essentially): A property is perfective iff it implies no negation of a perfective". And in the footnote (same page) says "it need not be assumed always either φ or $\neg\varphi$ is positive". This leads to the following suggestion of mine: *replace* the first two axioms by the following axiom (A12):

$$(P(X)\&\Box(X \subseteq Y)) \rightarrow \neg P(\neg Y).$$

(Here clearly $X \subseteq Y$ stands for $(\forall u)(X(u) \rightarrow Y(u)$.) Evidently then we can prove the following:

Lemma. (A12) proves $P(X) \rightarrow \neg P(\neg X)$ (which is Anderson's (A1)) and $P(X) \rightarrow \Diamond(\exists u)X(u)$.

Proof. Observe $\Box(X \subseteq X)$ and $\Box(X = \emptyset) \rightarrow \Box(X \subseteq \neg X)$ hence $((P(X)\&\Box(X = \emptyset)) \rightarrow \neg P(\neg\neg X)$, thus $P(X)\&\Box(X = \emptyset) \rightarrow \neg P(X)$, contradiction.

Let H be defined as follows:

$$H(u) \equiv (\forall X)(\Box X(u) \equiv (\exists Y)(\Box(Y \subseteq X) \& P(Y)))$$

u is godlike iff it necessarily has all positive properties and each property that u necessarily has is necessarily implied by a positive property.

(A3) is $P(H)$, as before.

Lemma. (A12)+(A3) proves $H(u) \to \Box H(u)$.

Proof. Clearly, $H(u) \& P(H)$ implies $\Box H(u)$.

Theorem. Over our logic, (A12)+(A3) proves $\Box(\exists u)H(u)$.

Proof. Conclude $(\exists u)H(u) \to (\exists u)\Box H(u)$, $\Diamond(\exists u)H(u) \to \Diamond(\exists u)\Box H(u) \to (\exists u)\Diamond\Box H(u) \to (\exists u)\Box H(u)$. In our logic $(\exists u)\Box H(u) \to \Box(\exists u)H(u)$ is a valid implication, which gives the desired conclusion

$$\Box(\exists u)H(u).$$

I find the axiom (A12) much more acceptable than original (A1), (A2) and even as Anderson's (A1), (A2); on the other hand, in [8] (cf. [9]) I have proved that Gödel's original system (with a "cautious" comprehension, not leading to the collapse of modalities) has a faithful interpretation in Anderson's system (A1–A3) (with full comprehension, over our logic). And similarly, we have the following easy result:

Theorem. Anderson's (A1)+(A2)+(A3) has a faithful interpretation in (A12)+(A3) both with full comprehension. Thus the original Gödel's system (with cautious comprehension) has a faithful interpretation in (A12)+(A3) (with full comprehension).

Proof. The interpretation is identical except for P; define $P^{\#}(X) \equiv (\exists Y)(P(Y) \& \Box(Y \subseteq X))$ (property is #-positive if there is a positive property Y necessarily implying X). Let g be such that $H(g)$; if $P^{\#}(X)$ then $\Box X(g)$, thus $\neg P^{\#}(\neg X)$ – this is (Anderson's) (A1) for $P^{\#}$. (A2) is evident from the definition of $P^{\#}$. Further we obviously have $H(u) \equiv (\forall X)(P^{\#}(X) \equiv \Box X(u))$ and $P^{\#}(H)$. Thus # interprets Anderson in our system. On the other hand, in Anderson we get $P(X) \equiv P^{\#}(X)$, thus # as an interpretation of Anderson in Anderson is identity. Thus # is a faithful interpretation of our system in Anderson.

4 A version with variable domains

In his analysis Anderson [1] uses a weaker (more general) logic (introduced in [3]) admitting that the set of (actually existing) individuals varies from one possible world to another. This possibility is studied in depth in [5]; we use the technique of this book and embed the variable domain logic into ours by introducing a new constant property E of actual existence (whose interpretation in a Kripke model $K = (W, M, Prop, \mathcal{P})$ is a particular member of $Prop$,) saying for each $w \in W$ and $m \in M$ m actually exists in the world w. We use Fitting's notation: $(\forall^E x)\varphi$ stands for $(\forall x)(E(x) \to \varphi)$ and $(\exists^E x)\varphi$ for $(\exists x)(E(x)\&\varphi)$. We also write $X \subseteq^E Y$ for $(\forall^E u)(X(u) \to Y(u))$. The only new axiom is that of non-emptiness: $\Box(\exists^E x)(x = x)$. Anderson's axioms (A1), (A3) (as well as his definition of H) are as before, (A2) becomes $(P(x)\&\Box(X \subseteq^E Y)) \to P(Y)$. As stated in [2], the above of $\Box(\exists x)H(x)$ proof does not modify to a (desired) proof of $\Box(\exists^E u)H(u)$ since $(\exists^E u)\Box\varphi \to \Box(\exists^E u)\varphi$ is not provable (being not a tautology). In this section we inspect Anderson's proof for the case of variable domains, modifying also here (A2). We replace (A1), (A2) by

(A12)^E $\quad (P(X)\&\Box(X \subseteq^E Y)) \to \neg P(\neg Y))$.

For this purpose we shall use the remaining two Anderson's (Gödel's) axioms (A4), (A5). Our underlying logic remains unchanged, only enriched by the properly E (and the axiom of non-emptiness).

Lemma. (A12)^E implies

(1) $\quad P(X) \to \neg P(\neg X)$,

(2) $\quad P(X) \to \Diamond(\exists^E u)X(u)$.

Proof. (1) is evident. We prove (2). Assume $\neg\Diamond(\exists^E u)X(u)$, i.e. $\Box(\forall^E u)\neg X(u)$. Then $\Box(X \subseteq^E \neg X)$, thus $P(X) \to \neg P(\neg\neg X)$, hence $P(X) \to \neg P(X)$, which gives $\neg P(X)$.

Definition. $H(u) \equiv (\forall Y)(\Box Y(u) \equiv (\exists Z)(P(Z)\&\Box(Z \subseteq^E Y))$.

(A3) is $P(H)$ as before; the next axiom is

(A4) $\quad P(X) \to \Box P(X)$.

The last Anderson's axiom will be presented below.

Lemma. (3) (A12)^E, (A3) implies $H(u) \to \Box H(u)$.

(4) (A12)^E, (A3), (A4) implies $P(Z) \to \Box(H \subseteq^E Z)$.

Proof. (3) Since $\Box(H \subseteq^E H)$ and $P(H)$ we get $H(u) \to \Box H(u)$ from the definition of H.

(4) We have $P(Z) \to (\forall^E u)(H(u) \to \Box Z(u))$; using (S5) delete the box and apply necessitation, getting $\Box P(Z) \to \Box(H \subseteq^E Z)$. Then (A4) gives the result.

Definition (essence; necessary existence).

$Ess_X(u) \equiv (\forall Y)(\Box Y(u) \equiv \Box(X \subseteq^E Y))$,

$NE(u) \equiv (\forall Y)(Ess_Y(u) \to \Box(\exists^E x)Y(x))$.

(A5) $\qquad P(NE)$.

Lemma. (A12)^E, (A3), (A4) implies $H(u) \to Ess_H(u)$.

Proof. Assume $H(u)$ and $\Box Y(u)$; then for some Z, $P(Z)\&\Box(Z \subseteq^E Y)$. By the preceding lemma, $\Box(H \subseteq^E Z)$ and hence $\Box(H \subseteq^E Y)$. Consequently, if $H(u)$ and $\Box(H \subseteq^E Y)$ then (since $P(H)$,) $\Box Y(u)$ by the definition of H.

Theorem. (A12)^E (A3), (A4), (A5) imply $\Box(\exists^E u)H(u)$.

Proof. First assume $H(u)$; then $Ess^H(u)$ and $NE(u)$ (since NE is positive, we have $\Box NE(u)$). Thus, by the definition of NE, we get $H(u) \to \Box(\exists^E x)H(x)$, and therefore by generalizing by $(\forall^E u)$ and moving the quantifier $(\forall^E u)$ we get

$$(\exists^E u)H(u) \to \Box(\exists^E x)H(x),$$

$$\Diamond(\exists^E u)H(u) \to \Diamond\Box(\exists^E x)H(x),$$

$$\Diamond(\exists^E u)H(u) \to \Box(\exists^E x)H(x),$$

which gives $\Box(\exists^E x)H(x)$. This completes the proof; our emendation (use of $(\mathrm{A12})^E$) works.

5 Yet another emendation

Continuing the investigation of the last system, let us prove the following:

Lemma: $(\mathrm{A12})^E$, (A3)–(A5) prove $P(H\&E)$, i.e. being an actually existing godlike individuum is positive.

Proof. Note that Fitting [5] has this fact too. It is enough to prove $H(x) \to E(x)$, hence $\Box(\forall x)(H(x) \equiv (H(x)\&E(x)))$, (thus $\Box(H = (H\&E))$; then $P(H)$ implies $P(H\&E)$. Assume $H(x)$ and let $(\forall y)\Box(Y(y) \equiv y = x)$ (by comprehension). Since $\Box(x = x)$, we get $\Box Y(x)$ and by the above $H(x)$ implies $Ess_H(x)$, thus $\Box(H \subseteq^E Y)$, which gives $H(x) \to \Box(\forall u)((H(u)\&E(u)) \to u = x)$. But we know that $\Box(\exists^E u)H(u)$, hence $H(x) \to \Box(\exists u)(E(u)\&H(u)\&u = x)$, which gives $H(x) \to (\exists u)(E(u)\&u = x)$, i.e. $H(x) \to E(x)$.

This leads us to another emendation: instead of introducing Ess, NE, (A4) and (A5), just modify (A3) to the following:

$(\mathrm{A3})^E$ $\quad P(H\&E)$.

Theorem. $(\mathrm{A12})^E$ and $(\mathrm{A3})^E$ imply $\Box(\exists^E u)H(u)$.

Proof. Note the provability of $\Diamond(\exists^E u)H(u)$. Now $H(u)$ implies $\Box(H(u)\&E(u))$ (by $(\mathrm{A3})^E$ and the definition of H), hence $\Box(\exists x)(E(x)\&H(x))$, which is $\Box(\exists^E x)$ $H(x)$.

Theorem. Let $P^\#(X)$ stand for $(\exists Z)(\Box(Z \subseteq^E X)\&P(Z))$, let $(\mathrm{A4})^\#$ be $P(X) \to \Box P^\#(X)$ and let $(\mathrm{A5})^\#$ be $P^\#(NE)$. Then

$(\mathrm{A12})^E$, $(\mathrm{A3})^E$ proves $(\mathrm{A4})^\#$, $(\mathrm{A5})^\#$;

$(\mathrm{A12})^E$, (A3), $(\mathrm{A4})^\#$, $(\mathrm{A5})^\#$ proves $(\mathrm{A3})^E$.

Proof. We work in the theory with axioms $(\mathrm{A12})^E$, $(\mathrm{A3})^E$; recall that the theory proves $\Box(\exists^E x)H(x), H(u) \to \Box H(u)$. For simplicity, introduce a constant g and postulate $H(g)\&E(g)$. Also recall the definition
$Ess_X(u) \equiv (\forall Z)(\Box Z(u) \equiv \Box(Y \subseteq^E Z)$. The theorem is a consequence of the following lemmas.

Lemma. $P^\#(Y) \to \Box P^\#(Y)$ (which is $(\mathrm{A4})^\#$).

Proof. $H(u) \to (P^{\#}(Z) \equiv \Box Z(u))$, hence
$\Box H(u) \to \Box P^{\#}(Z) \equiv \Box Z(u)))$ thus
$H(u) \to \Box H(u), (\exists u)H(u) \to (P^{\#}(Z) \equiv \Box P^{\#}(Z))$ thus $P^{\#}(Z) \equiv \Box P^{\#}(Z)$.

Lemma. $Ess_H(g)$

Proof. Let $\Box Y(g)$, thus $P^{\#}(Y)$ and for some Z, $P(Z)\&\Box(Z \subseteq^E Y)$. From the definition of H we get
$H(u) \to (P(Z) \to \Box Z(u)) \to (P(Z) \to Z(u))$,
$P(Z) \to (\forall u)(H(u) \to Z(u)) \to (\forall^E u)(H(u) \to Z(u))$,
$P^{\#}(Y) \to (\forall^E u)(H(u) \to Y(u))$,
$\Box P^{\#}(Y) \to \Box(H \subseteq^E Y)$ and an appliction of $(A4)^{\#}$ gives $P^{\#}(Y) \to \Box(H \subseteq^E Y)$. Thus $(\forall Y)(\Box Y(y) \to \Box(H \subseteq^E Y))$; conversely, $H(g)\&E(g)$ and $\Box(H \subseteq^E Y)$ immediately gives $\Box Y(g)$. This proves $Ess_H(g)$.

Lemma. $Ess_Y(u)\&Ess_Z(u) \to \Box(Y \subseteq^E Z)$ (and $\Box(Z \subseteq^E Y)$).

Proof. $Ess_Z(u) \to (\Box Z(u) \equiv \Box(Z \subseteq^E Z))) \to \Box Z(u)$
$Ess_Y(u) \to (\Box Z(u) \equiv \Box(Y \subseteq^E Z))) \to \Box(Y \subseteq^E Z)$.

(Corollary.) $(\forall Y)(Ess_Y(y) \to \Box(H \subseteq^E Y))$

Lemma. $P^{\#}(NE)$

Proof. Fact 4 and $\Box(\exists^E u)H(u)$ gives $(\forall y)(Ess_Y(g) \to \Box(\exists^E U)Y(u))$, then $NE(g)$. Apply necessitation: $\Box NE(g)$, hence $H(g)$ gives $P^{\#}(NE)$.

This completes the proof of the first claim of our theorem. To prove the second, i.e. to show the provability of $(A3)^E$, in $(A12)^E$, (A3), $(A4)^{\#}$, $(A5)^{\#}$ just check the proof of the lemma (at the beginning of this section); this needs also checking the proofs from Sect. 4 with $(A4)^{\#}$, $(A5)^{\#}$ instead of (A4), (A5) (but it is routine).

Remark. In [9] I mention that the proof (from A1–A3) works even the belief logic KD45 (our S5 minus the schema $\Box\varphi \to \varphi$); one can prove $\Box(\exists x)H(x)$ (it is believed that there is a godlike being) but not $(\exists x)H(x)$. The same appears to work for our proof from (A12), (A3) and even for the proof of $\Box(\exists^E x)H(x)$ from $(A12)^E$, $(A3)^E$; but the proof from (A1) $(A2)^E$ (or $(A12^E)$, (A3)–(A5) does use the schema $\Box\varphi \to \varphi$ and I do not know if it can be eliminated. Thus some room remains for further analysis (including construction of some interesting models in the style of [8]).

Summary. Our main results are the following:
(1) In the logic with fixed domains, the system of axioms

(A12) $(P(X)\&\Box(X \subseteq Y)) \to \neg P(\neg Y)$
(def1) $(H(x) \equiv (\forall Y)(\Box Y(x) \equiv (\exists Z)(P(Z)\&\Box(Z \subseteq Y))$
(A3) $P(H)$

proves $\Box(\exists x)H(x)$.
(2) In the logic will the existence predicate E, the system of axioms

(A12)^E $(P(X)\&\Box(X \subseteq^E Y)) \to \neg P(\neg Y)$
(def1) as above
(A3)^E $\Box(\exists^E x)H(x)$.
proves $\Box(\exists^E x)H(x)$.

(3) Alternatively, you may take (A12)^E, (def1), (A3) as above together with Anderson's definitions and axioms (A4), (A5), i.e.

(def2) $Ess_X(u) \equiv (\forall Y)(\Box Y(u) \equiv \Box(X \subseteq^E Y)$
(def3) $NE(u) \equiv (\forall Y)(Ess_Y(u) \to \Box(\exists^E x)Y(x))$
(A4) $P(X) \to \Box P(X)$
(A5) $P(NE)$

This axiom system proves $\Box(\exists^E x)H(x)$, as well as it proves (A3)^E.

Our axiom (A12) (or (A12)^E) seems to correspond exactly to Gödel's ideas and is free from the unwanted consequence of the original axiom (A2).

This seems to show that (A12)^E, (A3)^E is a rather pleasant emendation of Gödel's proof.

6 Conclusion

Admittedly, our emendation is simple and small. Nevertheless, I hope it makes the assumptions more natural and acceptable. Needless to say, the relevance for theology and religion remains very limited, mainly as a sort of mathematical analysis or modelling of old attempts to prove existence of God. (See [10] for a fascinating discussion of the question of existence of God.) On the other hand, I hope that the reader will agree that the analysis of Gödels proof is an unusual exercise in modal logic.

References

[1] ANDERSON C. A.: *Some emendations of Gödel's ontological proof,* Faith and philosophy 7 (1990) 291–303.

[2] ANDERSON C. A., GETTINGS M.: *Gödel's ontological proof revisited,* in: Hájek, ed.: Gödel'96, Springer 1996, 167–172.

[3] COCCHIARELLA N. B.: *A completeness theorem in second order modal logic,* Theoria 35 (1969) 81–103.

[4] ESSLER W. K.: *Gödel's Beweis.* In: (Ricken F., ed.) Klassische Gottesbeweise in der Sicht der gegenwärtigen Logik und Wissenschaftstheorie. Verlag W. Kohlhammer 1991.

[5] FITTING M.: *Types, Tableaux and Gödel's God.* Draft of a book, August 2000, see http://comet.lehman.cuny.edu/fitting

[6] FUHRMANN A.: *'G' für Gödel: Kurt Gödels axiomatische Theologie.* In: (P. Schroeder-Haister et. al, ed.) Aus Logic in der Philosophie. Book in preparation.

[7] GÖDEL K.: *Collected works III,* ed. by S. Feferman et al., Oxford Univ. Press 1995.

[8] HÁJEK P.: *Der Mathematiker und die Frage des Existenz Gottes,* should appear in Buldt and Schimmanowich, ed.: Wahrkeit und Beweisbarkeit (the book was planned to appear in 1995 (!) but still has not appeared)

[9] HÁJEK P.: *Magari and others on Gödel's ontological proof,* in: Ursini et. al, ed. Logic and Algebra, Marcel Dekker 1996, pp. 125–136.

[10] KÜNG J.: *Existiert Gott?* Piper Verlag, Stuttgart 1978.

[11] SOBEL J.H.: *Gödel's ontological proof,* in (J.V.Thomson, ed.) On Being and Saying, Essays for Richard Cartwright, The MIT Press 1987.