A note on injective spaces

In honour of Bernhard Banaschewski on the occassion of his 75th birthday

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Abstract

The structure of retracts of powers $\mathbb{S}^X$ of Sierpiński space gives rise to a representation of continuous lattices similar to the representation of (algebraic) Scott domains as information systems. Also, the case of such retracts that are topologies on X is discussed.

The injective spaces, that is, the spaces J such that for every embedding $X \subseteq Y$ and every $f : X \to J$ there is an extension $g : Y \to J$, were shown in [14] to coincide with continuous lattices endowed with the natural (Scott) topologies.

The category of continuous lattices is one of the important categories of theoretical computer science and related algebra. Another such category is that of algebraic domains. The latter one is known to be equivalent with the category of information systems (see [15]). Analyzing the structure of the retracts of the powers $\mathbb{S}^X$ of Sierpiński space (another characteristics of injectivity) we obtain a similar equivalence for the category of continuous lattices, in which the notion of information system is modified (the inductive system below: roughly speaking, the distinction between consistent and inconsistent sets of data is abandoned, but a set does not automatically entail its constituents). This is done in Section 3 after a brief analysis of the retracts of $\mathbb{S}^X$ in Section 2.

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If we view a retract τ of $\mathbb{S}^X$ as a set of subsets of X, the question naturally arises as to when τ is a topology on X, and what is the nature of the space (X, τ) if it is. This is discussed in Section 4 where such spaces are characterized. A class of thus obtained Scott topologies and a few related questions are discussed in some more detail.

1 Preliminaries

1.1 The set of all subsets (resp. of all finite subsets) of a set X will be denoted by $\mathfrak{P}(X)$ (resp. $\mathfrak{P}_{\mathrm{fin}}(X)$).

1.2 Let $\mathcal{C}$ be a class of monomorphisms of a category $\mathcal{A}$. An object A of $\mathcal{A}$ is said to be *injective* with respect to $\mathcal{C}$ if for every $m : B \to C$ in $\mathcal{C}$ and every $f : B \to A$ there is a $g : C \to A$ such that $g \cdot m = f$. Dually a projective object with respect to a class of epimorphisms is defined.

Unless otherwise stated, speaking of injective objects in the category $\mathbf{Top}_0$ of T_0-topological spaces we have in mind the injectivity with respect to all embeddings.

It is a well-known fact that

(inj) *The injective spaces (injective objects in* $\mathbf{Top}_0$*) are precisely the retracts of the powers of* $\mathbb{S}^X$ *where*

$$\mathbb{S} = (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\})$$

is the Sierpiński space. (See [14].)

1.3 A subset D of a poset $(X, \leq)$ is *directed* if it is non-empty and if for any $d_1, d_2 \in D$ there is a $d \in D$ such that $d_1, d_2 \leq d$.

In a poset $(X, \leq)$ one writes

$$x \ll y$$

if for each directed D such that $y \leq \sup D$ there is a $d \in D$ such that $x \leq d$.

In a complete lattice we will denote the suprema, as a rule, by $\bigvee M$, $a \vee b$, etc.

A *continuous lattice* L is a complete lattice such that

$$\text{for all } a \in L, \quad a = \bigvee\{b \mid b \ll a\}.$$

(For more on continuous lattices see, e.g., [10].)

Let $(X, \leq)$ be a poset. A subset $U \subseteq X$ is said to be *Scott open* if

(1) $U = \uparrow U = \{x \mid \exists y \in U,\ x \geq y\}$. and

(2) whenever $D \subseteq X$ is directed and $\sup D \in U$ then $U \cap D \neq \emptyset$.

The system of all Scott open subsets of $(X, \leq)$ is called the Scott topology (see, e.g., [10]; in [14] the author speaks of the *induced topology*).

In [14] D. Scott proved that

> *injective spaces are precisely the continuous lattices (with Scott topologies).*

Remark: Another interesting result of this nature is that the spaces injective with respect to dense embeddings are precisely the continuous Scott domains (see [10], [8], [9]; in [8] and [9] also other injectivities are treated).

1.4 The following is a well-known fact (see. e.g., [5]).

Proposition *Let $L : \mathcal{A} \to \mathcal{B}$ be a left adjoint to $R : \mathcal{B} \to \mathcal{A}$. If B is injective in $\mathcal{B}$ with respect to $\mathcal{C}$ then $R(B)$ is injective in $\mathcal{A}$ with respect to any $\mathcal{C}'$ such that $L[\mathcal{C}'] \subseteq \mathcal{C}$. Dually, if A is projective in $\mathcal{A}$ with respect to $\mathcal{C}$ then $L(A)$ is injective in $\mathcal{B}$ with respect to any $\mathcal{C}'$ such that $R[\mathcal{C}'] \subseteq \mathcal{C}$.*

Note. From this fact one can obtain very easily the statement (inj) above, and a little more. Applying the projectivity part to the forgetful functor **Frm**$\to$**Set** (**Frm** is the category of frames; for frames and locales see, e.g., [12]) and the associated left adjoint (free functor) **Set**$\to$**Frm** we immediately see that the projective frames with respect to surjections, are precisely the retracts of the free frames $F(X)$. It is easy to see that $F(X)$ is the topology of $\mathbb{S}^X$. Thus, the injective objects (with respect to embeddings) in the category of locales are the retracts of $\mathbb{S}^X$, and this can be restricted to $\mathbf{Top}_0$ since $\mathbb{S}^X$ is a sober space.

1.5 Let (X,τ), (Y,θ) be topological spaces. For $A\subseteq X$ and $B\subseteq Y$ set

$$\mathcal{W}(A,B)=\{f:(X,\tau)\to(Y,\theta)\mid f \text{ continuous, } f[A]\subseteq B\}.$$

We will consider two types of spaces of continuous mappings $f:(X,\tau)\to(Y,\theta)$. In

$$\mathsf{P}((X,\tau),(Y,\theta))$$

the topology is generated by all $\mathcal{W}(u,U)$ with $u\in\mathfrak{P}_{\text{fin}}(X)$ and U open in (Y,θ) (this is the *topology of pointwise convergence* and coincides with what we obtain from the embedding of the set of continuous maps into $(Y,\theta)^X$). In

$$\mathsf{C}((X,\tau),(Y,\theta))$$

the topology is generated by the $\mathcal{W}(K,U)$ with K compact and U open (this is the *compact-open topology*, and will be considered for locally compact (X,τ) only.

Both the constructions are functorial (with $\mathsf{P}(f,g)(\varphi)=\mathsf{C}(f,g)(\varphi)=g\varphi f$). It is a standard fact that $\mathsf{C}((X,\tau),-)$ with locally compact (X,τ) is a right adjoint to $(X,\tau)\times -$. Since the product preserves embeddings we obtain from 1.4 that

> *each* $\mathsf{C}(X,\tau),(Y,\theta))$ *with injective* (Y,θ) *(in particular, each* $\mathsf{C}((X,\tau),\mathbb{S})$*) is injective.*

(See, e.g. [11]. Also $\mathsf{P}((X,\tau),-)$ is a right adjoint – see, e.g., [4](theorem 7.1.6) –, but the associated left adjoint preserves closed embeddings only. Thus we can infer the injectivity of (e.g.) $\mathsf{P}((X,\tau),\mathbb{S})$ just with respect to closed embeddings.)

1.6 We will use the following standard (and straightforward) fact.

If L is a complete lattice and $h:L\to L$ a monotone map such that $h\cdot h=h$ then $h[L]$ is a complete lattice with the suprema given by the formula $\bigvee' M=h(\bigvee M)$.

2 Retracts of a power of Sierpiński space

2.1 Conventions. 1. Throughout this section, X will be a fixed set. The elements of X will be denoted by x, y, z (with indices, primes, etc.). The elements of $\mathfrak{P}_{\mathrm{fin}}(X)$ will be indicated by u, v, w and the general subsets of X by upper case letters (typically, U, V, W).

2. The power $\mathbb{S}^X$ of Sierpiński space will be viewed as $\mathfrak{P}(X)$ endowed with the topology generated by the basis

$$\{\mathcal{W}(u) \mid u \in \mathfrak{P}_{\mathrm{fin}}(X)\}, \quad \mathcal{W}(u) = \{U \mid u \subseteq U \subseteq X\} \ (= \mathcal{W}(u, \{1\}) \text{ from 1.5}).$$

For a continuous mapping $\rho : \mathbb{S}^X \to \mathbb{S}^X$ define a relation $\vdash \, = \, \vdash_\rho \, \subseteq \mathfrak{P}_{\mathrm{fin}}(X) \times X$ by setting

$$u \vdash x \quad \text{iff} \quad \mathcal{W}(u) \subseteq \rho^{-1}(\mathcal{W}(\{x\})).$$

Proposition 2.2 *The formula $\rho \mapsto \vdash_\rho$ constitutes an invertible correspondence between continuous mappings $\rho : \mathbb{S}^X \to \mathbb{S}^X$ and relations $R \subseteq \mathfrak{P}_{\mathit{fin}}(X) \times X$ such that*

(1) *$u \subseteq v$ and uRx implies vRx.*

The inverse is given by

$$R \mapsto r_R, \quad r_R(U) = \{x \mid \exists u, \ u \subseteq U, \ uRx\}.$$

Proof. First, r_R is continuous since

$$r_R^{-1}(\mathcal{W}(\{x\})) = \{U \mid \exists u, \ u \subseteq U, \ uRx\} = \bigcup \{\mathcal{W}(u) \mid uRx\} \qquad (*)$$

is open.

We have $v \vdash_{r_R} x$ iff $\mathcal{W}(v) \subseteq \bigcup\{\mathcal{W}(u) \mid uRx\}$ by $(*)$. This inclusion holds for vRx and on the other hand if it holds we have in particular $u \subseteq v$ with uRx and hence vRx by (1). Thus, $v \vdash_{r_R} x$ iff vRx.

We have $x \in r_{\vdash_\rho}(U)$ iff there is a $u \subseteq U$ such that $\mathcal{W}(u) \subseteq \rho^{-1}(\mathcal{W}(x))$. In particular, $U \in \rho^{-1}(\mathcal{W}(\{x\}))$ and $x \in \rho(U)$. On the other hand, if $x \in \rho(U)$, that is, $\rho(U) \in \mathcal{W}(\{x\})$, we have by continuity a u such that $U \in \mathcal{W}(u)$ and $\rho(\mathcal{W}(u)) \subseteq \mathcal{W}(\{x\})$. Thus, $x \in r_{\vdash_\rho}(U)$. $\square$

In particular we obtain that for each continuous $\rho : \mathbb{S}^X \to \mathbb{S}^X$

$$U \subseteq V \quad \Rightarrow \quad \rho(U) \subseteq \rho(V).$$

This, of course, is also seen directly from the obvious fact that in $\mathbb{S}^X$, $U \subseteq V$ iff $U \in \overline{\{V\}}$.

2.3 The formulas

$$R \mapsto \widetilde{R} = \{(u,v) \mid \forall x \in v,\ uRx\}, \quad S \mapsto S' = \{(u,x) \mid uS\{x\}\}$$

constitute a one-one correspondence between

- the $R \subseteq \mathfrak{P}_{\text{fin}}(X) \times X$ such that $u' \supseteq u$ and uRx implies $u'Rx$, and
- the $S \subseteq \mathfrak{P}_{\text{fin}}(X) \times \mathfrak{P}_{\text{fin}}(X)$ such that
  (1) $u' \supseteq u$, uSv and $v \supseteq v'$ implies $u'Sv'$, and
  (2) if uSv_i, $i = 1, 2$, then $uS(v_1 \cup v_2)$.

It is easy to check that

> $\widetilde{R} \circ \widetilde{R} = \widetilde{R}$ iff (a) uRy for all $y \in v$ and vRx implies uRx, and (b) if uRx the there is a v such that uRy for all $y \in v$ and vRx.

Lemma 2.4 *$\rho\rho(U) = \rho(U)$ for all U iff* $\widetilde{\vdash} \circ \widetilde{\vdash} = \widetilde{\vdash}$.

Proof. $\Rightarrow$: We have in particular $\rho\rho(u) = \rho(u)$ for $u \in \mathfrak{P}_{\text{fin}}(X)$. Since (recall 2.2) $\rho(U) = \{x \mid \exists u,\ u \subseteq U,\ u \vdash x\}$ we obtain that

$$\{x \mid u \vdash x\} = \{x \mid \exists v \subseteq \{y \mid u \vdash y\},\ v \vdash x\}.$$

$\Leftarrow$: First observe that

$$u \subseteq \rho(U) \quad \text{iff} \quad \exists v \subseteq U,\ v \widetilde{\vdash} u.$$

(Indeed, if $u \subseteq \rho(U)$ we have for each $x \in u$ a $v_x \subseteq U$ such that $v_x \vdash x$; set $v = \bigcup v_x$. The other implication is trivial.)

Thus,

$$\rho\rho(U) = \{x \mid \exists u \subseteq \rho(U),\ u \widetilde{\vdash} \{x\}\} = \{x \mid \exists v \subseteq U, \exists u,\ v \widetilde{\vdash} u \widetilde{\vdash} \{x\}\} =$$
$$= \{x \mid \exists v \subseteq U,\ v \vdash x\} = \rho(U). \quad \square$$

Lemma 2.5 *$\rho(U) = U$ if and only if*

(S1) *for every $x \in U$ there is a $u \subseteq U$ such that $u \vdash x$, and*

(S2) *if $u \subseteq U$ and $u \vdash x$ then $x \in U$.*

Proof. We obviously have (S1) iff $U \subseteq \rho(U)$ and (S2) iff $\rho(U) \subseteq U$. $\square$

2.6 The subsets $U \subseteq X$ satisfying (S1) and (S2) will be said to be *stable.* Thus,

each retract of a power of the Sierpiński space is constituted by the stable sets of a relation $\vdash \subseteq \mathfrak{P}_{fin}(X) \times X$ such that $u' \supseteq u \vdash x$ implies $u' \vdash x$, satisfying (a) and (b) from 2.3.

2.7 The relational case. We will say that a retraction $\rho : \mathbb{S}^X \to \mathbb{S}^X$ (or, the retract $\rho[\mathbb{S}^X]$) is *relational* if for each instance $u \vdash x$ there is a $y \in u$ such that $\{y\} \vdash x$. Write

$$yRx \quad \text{iff} \quad \{y\} \vdash x. \tag{$*$}$$

It is easy to check that

- the formula $(*)$ together with

  $$u \vdash x \quad \text{iff} \quad \exists y \in u,\ yRx$$

  constitutes a one-one correspondence between the relational retracts and the relations $R \subseteq X \times X$ such that $R \circ R = R$.

- we have

  $$\rho(U) = \{x \mid \exists y \in U,\ yRx\}, \quad \text{and}$$

- the stable subsets $U \subseteq X$ are characterized by

  (S1') for every $y \in U$ there is an $x \in U$ such that xRy, and

  (S2') if $x \in U$ and xRy then $y \in U$.

3 Inductive systems and continuous lattices

Recall the equivalence between Scott's information systems and (algebraic) domains ([15], for a more explicit proof see e.g. [1] where the case of the more general continuous domains is also discussed; for the relation between (algebraic) domains and frames see also [3]). In this section we will establish a very similar equivalence concerning what we call "inductive systems" and continuous lattices.

3.1 An *inductive system* is a couple $S = (X_S, \vdash_S)$ where X_S is a set and $\vdash_S \subseteq \mathfrak{P}_{\text{fin}}(X_S) \times X_S$ is a relation such that

(1) if $u \supseteq v$ and $v \vdash x$ then $u \vdash x$,

(2) if $v \vdash x$ and $u \vdash y$ for all $y \in v$ then $u \vdash x$,

(3) if $u \vdash x$ then there is a v such that $v \vdash x$ and $u \vdash y$ for all $y \in v$.

There is certainly no danger of confusion in writing simply $\vdash$ for $\widetilde{\vdash}$. By 2.3, the definition above is equivalent with requiring that

(I1) if $u' \supseteq u \vdash v \supseteq v'$ then $u' \vdash v'$,

(I2) if $u \vdash v_1, v_2$ then $u \vdash (v_1 \cup v_2)$, and

(I3) $\vdash$ is transitive and interpolative, that is, $u \vdash v$ iff there is a w such that $u \vdash w \vdash v$.

3.2 Notes. 1. By the observations in the previous section, the inductive systems $(X, \vdash)$ with X fixed are in a one-one correspondence with idempotent (retraction) maps $\rho : \mathbb{S}^X \to \mathbb{S}^X$ defined by

$$\rho(U) = \{X \mid \exists u \subseteq U,\ u \vdash x\}.$$

The symbol ρ will be henceforth used in this sense.

2. In a certain analogy with Scott's information systems we can think of an inductive system as follows. The elements of X_S represent predicates or observables, and $u \vdash x$ represent an inference, or induced knowledge, of x based on the data, or observations, in u. The difference is that here we do

not distinguish between consistent and inconsistent $u \in \mathfrak{P}_{\text{fin}}(X)$ and that we do not have in general $u \vdash x$ for all $x \in u$ ($\vdash$ is only interpolative, in general not reflexive; in the interpretation of knowledge induced by observables we may think of not necessarily taking an observation at the face value unless confirmed).

3.3 In analogy with [15] define an *approximable map* $f : S \to T$ as a relation $f \subseteq \mathfrak{P}_{\text{fin}}(X_S) \times \mathfrak{P}_{\text{fin}}(X_T)$ such that

(M1) $\emptyset f \emptyset$,

(M2) $u' \supseteq u f v \supseteq v' \quad \Rightarrow \quad u' f v'$,

(M3) $u f v_1, v_2 \quad \Rightarrow \quad u f (v_1 \cup v_2)$, and

(M4) $u f v$ iff there exist u', v' such that $u \vdash u' f v' \vdash v$

The following are straightforward

Observations *1. If $f : S \to T$, $g : T \to R$ are approximable maps then so is $g \cdot f : S \to R$ defined as the composition of relations (written backwards in accordance with the usual composition of mappings).*

2. Each $\vdash_S$ is an approximable map $S \to S$ and we have, for $f : S \to T$, $f \cdot \vdash_S = \vdash_T \cdot f = f$.

The resulting category will be denoted by

$$\mathbf{IndS}.$$

By the second observation, the $\vdash_S : S \to S$ are the identities in **IndS**.

The system of all subsets of X_S stable in $\vdash_S$ (recall 2.6), ordered by inclusion, will be denoted by

$$\mathcal{L}(S).$$

Thus, $\mathcal{L}(S) = \rho[\mathbb{S}^X]$ and hence (see 1.6) it is a complete lattice, with the suprema $\bigvee_{i \in I} U_i = \rho(\bigcup_{i \in I} U_i)$. If $\{U_i \mid i \in I\}$ is a directed system of stable sets then obviously $\bigcup_{i \in I} U_i$ is stable. Thus we have

Observation 3.4 *The supremum of a directed system in $\mathcal{L}(S)$ is the union.*

For $u \in \mathfrak{P}_{\mathrm{fin}}(X_S)$ set

$$(u\vdash) = \{X \mid u \vdash x\} \; (= \rho(u)).$$

Proposition 3.5 *1. Each $(u\vdash)$ is stable.*

2. $u \subseteq v \quad \Rightarrow \quad (u\vdash) \subseteq (v\vdash)$.

3. For each stable U the system $\{(u\vdash) \mid u \subseteq U\}$ is directed and

$$U = \bigcup\{(u\vdash) \mid u \subseteq U\} = \bigvee\{(u\vdash) \mid u \subseteq U\}.$$

4. $U \ll V$ in $\mathcal{L}(S)$ iff there is a $u \in \mathfrak{P}_{fin}(X_S)$ such that $u \subseteq (u\vdash)$ and $u \subseteq V$. Thus in particular, $u \vdash v \quad \Rightarrow \quad (u\vdash) \ll (v\vdash)$.

5. Consequently, $\mathcal{L}(S)$ is a continuous lattice.

Proof. 1 since $\rho\rho(u) = \rho(u)$ and 2 follows from (1) in 3.1.

3: If $u_1, u_2 \subseteq U$ we have $u_1 \cup u_1 \subseteq U$ and $(u_i \vdash) \subseteq ((u_1 \cup u_2)\vdash)$, By (S1) for each $x \in U$ there is a $u \subseteq U$ with $u \vdash x$, and hence, by (S2), $U = \bigcup\{(u\vdash) \mid u \subseteq U\}$.

4: If $U \ll V$ then by 3, $U \subseteq (u\vdash)$ for some $u \subseteq V$. If $U \subseteq (u\vdash)$, $u \subseteq V$ and $V = \bigcup_{i \in I} V_i$ with $\{V_i \mid i \in J\}$ directed then $u \subseteq V_i$ for some i, u being finite. By (S2), $U \subseteq (u\vdash) \subseteq V_i$.

5 just summarizes 3 and 4. $\square$

Denote by

ContLat

the category of continuous lattices and the maps preserving suprema of directed subsets.

For an approximable map $f : S \to T$ and $u \in \mathcal{L}(S)$ set

$$\mathcal{L}(f)(U) = \bigcup\{(v\vdash) \mid \exists u \subseteq U,\ ufv\}.$$

By (M2) and (M3), if there are $u_i \subseteq U$ and v_i such that $u_i f v_i$ then we have $u_1 \cup u_2 \subseteq U$ with $(u_1 \cup u_2) f (v_1 \cup v_2)$ so that the union is directed. Thus, $\mathcal{L}(f)(U) \in \mathcal{L}(T)$. For a directed join $\bigcup_{d \in D} U_d$ we have

$$\begin{aligned}\mathcal{L}(f)(\bigcup_{d \in D} U_d) &= \bigcup\{(v\vdash) \mid \exists u \subseteq \bigcup_{d \in D} U_d,\ ufv\} = \\ &= \bigcup\{(v\vdash) \mid \exists d \in D, \exists u \subseteq U_d,\ ufv\} = \bigcup_{d \in D} \mathcal{L}(f)(U_d).\end{aligned}$$

Thus we have a morphism

$$\mathcal{L}(f) : \mathcal{L}(S) \to \mathcal{L}(T)$$

in **ContLat**.

Lemma 3.6 *$v \subseteq \mathcal{L}(f)(U)$ iff there is a $u \subseteq U$ such that ufv.*

Proof. $\Rightarrow$: For $y \in v$ choose $u_y \subseteq U$ and v_y such that $u_y f v_y \vdash y$. Set $u = \bigcup u_y$, $v' = \bigcup v_y$. Then by (M2) and (M3), $ufv' \vdash v$. Hence ufv.

$\Leftarrow$: Let $u \subseteq U$ and ufv. Choose v' such that $ufv' \vdash v$. Then $x \in \mathcal{L}(f)(U)$ for all $x \in v$ and hence $v \subseteq \mathcal{L}(f)(U)$. $\square$

Theorem 3.7 *The functor $\mathcal{L}$ constitutes an equivalence of the categories* **IndS** *and* **ContLat**.

Proof. We will prove that

I. $\mathcal{L}$ is a functor,

II. $\mathcal{L}$ is a full embedding, and

III. for each continuous lattice L there is an inductive system S such that $L \cong \mathcal{L}(S)$.

I : We have, by 3.6,

$$\begin{aligned}\mathcal{L}(g)(\mathcal{L}(f)(U)) &= \bigcup\{(w\vdash) \mid \exists v \subseteq \mathcal{L}(f)(U),\ vgw\} = \\ &= \bigcup\{(w\vdash) \mid \exists u \subseteq U, \exists v,\ ufvgw\} = \\ &= \bigcup\{(w\vdash) \mid \exists u \subseteq U,\ u(gf)w\} = \mathcal{L}(gf)(U).\end{aligned}$$

$\mathcal{L}(\vdash_S)(U) = \bigcup\{(v\vdash) \mid \exists u \subseteq U,\ u \vdash v\} = U$ since if $x \in U$ we have , by (S1) and (I3), a $u \subseteq U$ and a v such that $u \vdash v \vdash x$, and if $u \subseteq U$ and $u \vdash v \vdash x$ we have $x \in U$ by (I3) and (S2).

II : Let $h : \mathcal{L}(S) \to \mathcal{L}(T)$ preserve directed suprema. For $(u, v) \in \mathfrak{P}_{\text{fin}}(X_S) \times \mathfrak{P}_{\text{fin}}(X_T)$ put

$$u\tilde{h}v \quad \text{iff} \quad h((u\vdash)) \supseteq v.$$

$\widetilde{h}$ *is an approximable map* $S \to T$*:* (M1),(M2),(M3) and the fact that $u \vdash u'\widetilde{h}v' \vdash v$ implies $u\widetilde{h}v$ are immediate. Now let $u \vdash v$, that is, $h((u\vdash)) \supseteq v$. Recall 3.5. We have $h((u\vdash)) \gg (v\vdash)$. Interpolate $h((u\vdash)) \gg V \gg (v\vdash)$ to obtain a v' such that $h((u\vdash)) \gg (v'\vdash) \gg (v\vdash)$. We have the directed unions $(u\vdash) = \bigcup\{(u'\vdash) \mid u' \subseteq (u\vdash)\}$ and $h((u\vdash)) = \bigcup\{h((u'\vdash)) \mid u' \subseteq (u\vdash)\}$, and hence there is a u' such that $u \vdash u'$ and $h((u'\vdash)) \supseteq v' \supseteq v$.

$\widetilde{\mathcal{L}(f)} = f$: By 3.6 and (M1), $u\widetilde{\mathcal{L}(f)}u$ iff $v \subseteq \mathcal{L}(f)((u\vdash))$ iff there is a w such that $u \vdash w$ and wfv, and this holds iff ufv.

$\mathcal{L}(\widetilde{h}) = h$: We have

$$\mathcal{L}(\widetilde{h})(U) = \bigcup\{(v\vdash) \mid \exists u \subseteq U,\ h((u\vdash)) \supseteq v\} = V.$$

If $x \in V$ there are u and v such that $u \subseteq U$ and $h((u\vdash)) \supseteq v \vdash x$. Since $(u\vdash) \subseteq U$ we have $x \in (v\vdash) \subseteq h((u\vdash)) \subseteq h(U)$ and $v \subseteq h(U)$. On the other hand, if $x \in h(U)$ then $x \in \bigvee\{h((u\vdash)) \mid (u\vdash) \subseteq U\}$ and there is a u such that $(u\vdash) \subseteq U$ and $x \in h((u\vdash))$. Since $h((u\vdash))$ is stable there is a v such that $v \subseteq h((u\vdash))$ and $v \vdash x$, and hence $x \in V$.

III : For a continuous lattice L put $X_L = \{x \in L \mid x \ll 1\}$ and for $(u, x) \in \mathfrak{P}_{\text{fin}}(X_L) \times X_L$ set

$$u \vdash_S x \quad \text{iff} \quad x \ll \bigvee u.$$

$S = (X_L, \vdash_L)$ *is an inductive system :* (I1) follows from the implication $a \ll b \leq c \;\Rightarrow\; a \ll c$, and we have (I2) since $a_i \ll b$ implies $a_1 \vee a_2 \ll b$. (I3) follows from the transitivity and interpolativity of $\ll$.

Define $\alpha : L \to \mathcal{L}(S)$ by setting $\alpha(a) = \{x \in X_L \mid x \ll a\}$. If $x \in \alpha(a)$ we can interpolate $x \ll y \ll a$ to obtain $\{y\} \subseteq \alpha(a)$ and $\{y\} \vdash x$. If $u \subseteq \alpha(a)$ then $\bigvee u \ll a$ and if $u \vdash x$, that is, $x \ll \bigvee u$ we conclude that $x \ll a$. Thus, $\alpha(a) \in \mathcal{L}(S)$.

For a directed supremum $a = \bigvee_{d \in D} a_d$ we have

$$\alpha(\bigvee_{d \in D} a_d) = \{x \mid x \ll \bigvee a_d\} = \{x \mid \exists y,\ x \ll y \ll \bigvee a_d\} =$$
$$= \{x \mid \exists d \in D,\ x \ll a_d\} = \bigcup \alpha(a_d).$$

Thus, α is a morphism.

Finally define $\beta : \mathcal{L}(S) \to L$ by setting $\beta(U) = \bigvee U$. This is obviously a morphism and we have $\beta\alpha(a) = a$. Now if $x \in \alpha\beta(U)$, that is, $x \ll \bigvee U$, interpolate $x \ll y \ll \bigvee U$. There are $x_1, \ldots, x_n \in U$ such that $x \ll y \leq x_1 \vee \cdots \vee x_n$ and hence $U \supseteq \{x_1, \ldots, x_n\} \vdash x$ and $x \in U$. If $x \in U$ we have, since U is stable, $x_1, \ldots, x_n \in U$ such that $\{x_1, \ldots, x_n\} \vdash x$. Then $x \ll x_1 \vee \cdots \vee x_n \leq \beta(U)$ and $x \in \alpha\beta(U)$. Thus we also have $\alpha\beta(U) = U$. $\square$

4 Retracts of a power of Sierpiński space that are topologies

Let (X, τ) be a topological space. The *specialization order* in X will be denoted by $\leq$; that is,

$$x \leq y \quad \text{iff} \quad x \in \overline{\{y\}}.$$

In this sense we will also use the symbol $\uparrow A$ for $\{x \in X \mid \exists a \in A,\ a \leq x\}$.

Observations 4.1 *1. If $A \subseteq U \in \tau$ then $\uparrow A \subseteq U$.*

2. If $u \subseteq X$ is finite then $\uparrow u$ is compact.

A topological space (X, τ) is said to be a *locally up-finite space* if for each $x \in X$ and each open $U \ni x$ there is a finite $u \subseteq U$ such that $\uparrow u$ is a neighbourhood of x.

By 4.1.2 we immediately obtain

Observation 4.2 *A locally up-finite space is locally compact.*

Lemma 4.3 *Let $K \subseteq U$ in a locally up-finite space, let K be compact and U open. Then there is a finite u such that $K \subseteq \uparrow u \subseteq U$. Moreover, u can be chosen such that $\uparrow u$ is a neighbourhood of K.*

Proof. For each $x \in K$ choose a finite $u(x) \subseteq U$ and an open $V(x) \subseteq \uparrow u(x)$ such that $x \in V(x)$. By compactness there are $x_1, \ldots, x_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} V(x_i) \subseteq \bigcup_{i=1}^{n} \uparrow u(x_i) = \uparrow\Big(\bigcup_{i=1}^{n} u(x_i)\Big).$$

Set $u = \bigcup_{i=1}^{n} u(x_i)$. $\square$

Proposition 4.4 *If (X, τ) is locally up-finite and if (Y, θ) is arbitrary then*

$$\mathsf{P}((X, \tau), (Y, \theta)) = \mathsf{C}((X, \tau), (Y, \theta)).$$

Proof. Recall 1.5. It suffices to prove that each $\mathcal{W}(K, V)$ with K compact in (X, τ) and V open in (Y, θ) is open in $\mathsf{P}((X, \tau), (Y, \theta))$. Fix an $f \in \mathcal{W}(K, V)$ and set $U = f^{-1}(V)$. By 4.3 there is a finite subset $u \subseteq U$ such that $K \subseteq\uparrow u \subseteq U$. We have $f[u] \subseteq ff^{-1}(V) \subseteq V$ so that $f \in \mathcal{W}(u, V)$, and $\mathcal{W}(u, V) \subseteq \mathcal{W}(K, V)$ since if $g[u] \subseteq V$ we have $g[\uparrow u] \subseteq\uparrow g[u] \subseteq V$ and hence $g[K] \subseteq V$ $\square$

Note. The spaces (X, τ) we are interested in are typically non-T_1. For T_1-spaces (X, τ) the situation is trivial. It is a well-known fact that then $\mathsf{P}((X, \tau), (Y, \theta)) = \mathsf{C}((X, \tau), (Y, \theta))$ (if and) only if all compact subspaces of (X, τ) are finite.

From now on, $\rho : \mathbb{S}^X \to \mathbb{S}^X$ will be an idempotent (retraction) mapping and $\vdash$ the associated relation. We will be interested in the case where the $\rho[\mathbb{S}^X]$ is a topology on X.

Thus, the open sets in our space will be the stable sets, and each neighbourhood U of x contains a $(u\vdash)$ such that $u \vdash x$.

We will write $\vdash x$ for $\{u \mid u \vdash x\}$.

Proposition 4.5 *$\rho[\mathbb{S}^X]$ is a topology iff*

($\bigcup$) *if $(v_1 \cup \cdots \cup v_n) \vdash x$ and $u_i \vdash v_i$ for $i = 1, \ldots, n$ then there is a k such that $u_k \vdash x$, and*

($\cap$) *for each $x \in X$, $\vdash x$ is directed by the relation $\vdash$.*

Proof. If U_i are stable then obviously $\bigcup U_i$ satisfies (S1); if ($\bigcup$) holds, the union also satisfies (S2). On the other hand, if unions of stable sets are stable then in particular $\bigcup_{i=1}^n (u_i\vdash)$ satisfies (S2) and ($\bigcup$) follows.

($\cap$) is easily seen to be equivalent with $\rho[\mathbb{S}^X]$ being a basis of a topology. $\square$

Lemma 4.6 *Let $\rho[\mathbb{S}^X]$ be a topology. Then*

(1) *for the specialization order we have*

$$x \leq y \quad \textit{iff} \quad \vdash x \subseteq \vdash y \ ;$$

(2) *for each* $u \in \mathfrak{P}_{fin}(X)$*,* $(u\vdash) \subseteq \uparrow u$*.*

Proof. (1): The $(u\vdash)$ with $u \vdash x$ constitute a neighbourhood basis of x.

(2): Let $x \in (u\vdash)$. Suppose $x \notin \uparrow u$. Thus, for each $y \in u$, $y \notin \overline{\{x\}}$ and hence there is an open U_y containing y but not x. Setting $U = \bigvee U_y$ we obtain an open set containing u but not x in contradiction with (S2). $\square$

Theorem 4.7 *Let* τ *be a topology on a set* X*. Then the following statements are equivalent.*

(1) τ *is a retract of* $\mathbb{S}^X$*.*

(2) (X, τ) *is locally up-finite.*

(3) $\mathsf{P}((X,\tau),(Y,\theta)) = \mathsf{C}((X,\tau),(Y,\theta))$ *for any space* (Y, θ)*.*

(4) $\mathsf{P}((X,\tau),\mathbb{S})$ *is a retract of* $\mathbb{S}^X$*.*

Proof. (1)$\Rightarrow$(2) : Let $x \in U$, U open. Then there is a $u \subseteq U$ such that $u \vdash x$. By 4.6.1, $x \in (u\vdash) \subseteq \uparrow u \subseteq U$.

(2)$\Rightarrow$(3) by 4.4.

(3)$\Rightarrow$(4) : $\mathsf{P}(X,\tau),\mathbb{S})$ can be viewed as the subspace $\tau \subseteq \mathbb{S}^X$. Since it coincides with $\mathsf{C}(X,\tau),\mathbb{S})$ it is injective by 1.5, and $\mathrm{id} : \mathsf{P}(X,\tau),\mathbb{S})) \to \mathsf{P}(X,\tau),\mathbb{S})$ can be extended to a retraction $\mathbb{S}^X \to \mathsf{P}(X,\tau),\mathbb{S})$.

(4)$\Rightarrow$(1) : View, again, $\mathsf{P}(X,\tau),\mathbb{S})$ as the subspace $\tau \subseteq \mathbb{S}^X$. Then (4) is just a reformulation of (1). $\square$

Proposition 4.8 *(Recall 2.7.) Relational retracts* τ *of* $\mathbb{S}^X$ *that are topologies are obtained by the formulas* (S1') *and* (S2') *from transitive interpolative relations* $R \subseteq X \times X$ *such that each* Rx *is directed. We have*

$$x \leq y \quad \textit{iff} \quad Rx \subseteq Ry$$

(in particular, $xRy \quad \Rightarrow \quad x \leq y$*).*

Proof. Recall 3.5 and realize that for a relational τ the condition $(\bigcup)$ is always satisfied. $\square$

Thus the (X,τ) with τ relational retracts are special cases of Erné's C-spaces (see [6]). In particular (see [6],[7]) they are co-frames, that is, besides the standard distributivity one also has the dual

$$(\bigwedge U_i) \cup V = \bigwedge(U_i \cup U).$$

Example. For instance, the Alexandroff topologies of partial orders are relational retracts: consider $R = \leq$.

A more interesting example is the following. Let $\leq$ be a partial order on X (it will coincide with the specialization order of the ensuing topology, but we start with it just as an order) such that

(cof) whenever $x = \sup(\downarrow x \setminus \{x\})$ then each directed $D \subseteq \downarrow x$ is cofinal in $\downarrow x$.

Then the Scott topology on $(\leq)$ is the relational retract associated with the R defined by

$$xRy \quad \text{iff} \quad \begin{cases} \text{either} \quad x < y, \\ \text{or} \quad x = y \neq \sup(\downarrow x \setminus \{x\}). \end{cases}$$

These Scott topologies form quite a rich class of spaces. We have

Proposition 4.9 *Each metric space (X,d) can be embedded into a poset $(Y,\leq)$ satisfying (cof) with the Scott topology.*

Proof. We can assume that the (X,d) has no isolated points. Since a metric space is paracompact, we can choose, for each n, a locally finite refinement $\mathcal{U}_n$ of $\{U \text{ open} \mid \text{diam}(U) < \frac{1}{n}\}$. Set $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ and $Y = X \cup \mathcal{U}$. On Y define a partial order by setting

$$\begin{aligned} u \leq v \quad &\text{iff} \quad v \subseteq u \text{ for } u, v \in \mathcal{U}, \\ u \leq x \quad &\text{iff} \quad x \in u \text{ for } u \in \mathcal{U} \text{ and } x \in X, \text{ and} \\ x \leq y \quad &\text{iff} \quad x = y \text{ for } x, y \in X. \end{aligned}$$

This order obviously satisfies (cof).

Claim *Let D be directed in Y and let* $\sup D = \xi \notin D$. *Then $D \subseteq \mathcal{U}$ and $\xi \in X$ (and $\{\xi\} = \bigcap D$).*

(Obviously $D \subseteq \mathcal{U}$. Suppose $\sup D = u \in \mathcal{U}$ Then $u \subseteq \bigcap\{v \mid v \in D\}$. Choose $x \in u$. As $x \in u \subseteq v$ for all $v \in D$, there are only finitely many v in $D \cap \mathcal{U}_n$. If D is infinite there is a $v \in D \cap \mathcal{U}_m$ with $m \geq n$ for any n. Since $\mathrm{diam}(u) > 0$ this cannot be; hence D is finite and u is its largest element.)

Now let U be open in X. Set $\widetilde{U} = \{V \in Y \mid V \subseteq U\} \cup U$. Using Claim we easily see that $\widetilde{U}$ is Scott open and $U = \widetilde{U} \cap X$.

On the other hand, let V be Scott open in Y and let $U = V \cap X$. For $x \in U$ consider $D = \{W \in \mathcal{U} \mid x \in W\}$. Then D is directed and $\sup D = x \in V$ Hence there is a $W \in D \cap V$. Since V is an increasing set, all $y \in W$ are in V and hence $W \subseteq V \cap X$ and U is open in X.

Thus, the topology on X induced by the Scott topology on Y coincides with the metric one. $\square$

Note The example of an embedding of a general T_0-space into a B-space in [7] can be viewed as an embedding into a (Y, τ) with τ a relational retract, albeit not a Scott one.

Remark 4.10 Using the more general $R \subseteq \mathfrak{P}_{\mathrm{fin}}(X) \times X$ we can represent a much wider variety of Scott spaces. The pattern will be apparent from the following trivial example.

Consider $(\mathbb{N} \times \{0,1\}) \cup \{\omega\}$ ($\mathbb{N}$ is the set of natural numbers) with the order given by

$$(n,i) \leq (n,j) \quad \text{iff} \quad m \leq n \text{ and } i = j,$$
$$\xi \leq \omega \text{ for all } \xi \in (\mathbb{N} \times \{0,1\}) \cup \{\omega\}.$$

Then the Scott topology is obviously locally up-finite and hence it is a retract of $\mathbb{S}^X$ (incidentally, this example shows that such a space is not necessarily a co-frame, in contrast with the more special relational case).

Characterizing the Scott topologies that are retracts of $\mathbb{S}^X$ may be of interest. There naturally arises the following

Problem: Is every locally compact Scott space locally up-finite?

(We should note that we thought of extending the condition (cof) by requiring that whenever $x = \sup({\downarrow}x \setminus \{x\})$ then ${\downarrow}x$ can be decomposed into finitely many subsets so that each directed $D \subseteq {\downarrow}x$ is cofinal in some of them. An easy counterexample is $X = \mathbb{N} \cup (\mathbb{N} \times \mathbb{N}) \cup \{\omega\}$ with $(n,k) \leq (m,l)$ iff $n \leq m$ and $k = l$, $(n,k) \leq m$ iff $k \leq m$, $\mathbb{N}$ in the standard order, and $\xi \leq \omega$ for all

ξ. Here we can have the required decompositions into at most two sets at each ξ, while the Scott topology is not even locally compact. – In fact, all examples of Scott topologies that were not retracts of the corresponding $\mathbb{S}^X$ we came across turned out not to be locally compact.)

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