Extremal problems (and a bit of enumeration) for hypergraphs with linearly ordered vertex sets

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Abstract

A hypergraph $\mathcal{H} = (H_i : i \in I)$ with the vertex set $\bigcup_{i \in I} H_i = [n] = \{1, 2, \ldots, n\}$ contains another hypergraph $\mathcal{H}' = (H_i' : i \in I')$ with the vertex set $[m]$ ($m \le n$) if there is a subsequence $1 \le v_1 < v_2 < \cdots < v_m \le n$ of $[n]$ and an injection $f : I' \to I$ such that, for every $r \in [m]$ and $i \in I'$, $r \in H_i'$ implies that $v_r \in H_{f(i)}$. We investigate the extremal functions $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ defined as the maximum size $e(\mathcal{H}) = |\mathcal{H}| = |I|$, resp. weight $i(\mathcal{H}) = \sum_{i \in I} |H_i|$, of a simple $\mathcal{H}$ with n vertices if $\mathcal{H}$ does not contain $\mathcal{F}$. We determine both functions exactly if $\mathcal{F}$ has only disjoint singleton edges or if $i(\mathcal{F}) \le 4$ (there are 55 such $\mathcal{F}$). We give enumerative formulas for the numbers of both simple and all $\mathcal{H}$ with $i(\mathcal{H}) = n$ and derive two identities analogous to Dobiňski's formula for Bell numbers. In the extremal problem we derive, by means of Davenport–Schinzel sequences, two general almost linear bounds. We consider the forbidden 4-path $\mathcal{F}_{42} = 13, 15, 23, 24$ introduced by Füredi and prove that $H_e(\mathcal{F}_{42}, n)$ and $H_i(\mathcal{F}_{42}, n)$ are $O(n(\log n)^2(\log\log n)^3)$. (Füredi proved in the case of ordered bipartite graphs the $O(n \log n)$ bound.)

1 Introduction and motivation

Let us begin by stating a typical example of the extremal problems which we shall investigate in our article. If $\mathcal{H}$ is a simple hypergraph with the vertex set

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$[n] = \{1, 2, \ldots, n\}$ and such that for no four vertices $1 \le a < b < c < d \le n$ and for no two distinct edges $A, B \in \mathcal{H}$ the four incidences $a, c \in A$ and $b, d \in B$ occur, what is the maximum number of edges $|\mathcal{H}|$ and what is the maximum weight $\sum_{H \in \mathcal{H}} |H|$. Among other results we prove that the former maximum is $4n-5$ and that the latter is $8n-12$ ($n > 1$). Actually we proved it already in Klazar [15].

A *hypergraph* $\mathcal{H} = (H_i : i \in I)$ is a finite list of finite nonempty subsets H_i of $\mathbf{N} = \{1, 2, \ldots\}$, called *edges*. *Simple* hypergraphs have no repeated edges. The elements of $\bigcup \mathcal{H} = \bigcup_{i \in I} H_i \subset \mathbf{N}$ are called *vertices*. Our hypergraphs have no isolated vertices. Let $\mathcal{H} = (H_i : i \in I)$ and $\mathcal{H}' = (H_i' : i \in I')$ be two hypergraphs. If there exists an *increasing* (with respect to the standard linear ordering of $\mathbf{N}$) injection $F : \bigcup \mathcal{H}' \to \bigcup \mathcal{H}$ and an injection $f : I' \to I$ such that the implication $v \in H_i' \Longrightarrow F(v) \in H_{f(i)}$ holds for every $v \in \bigcup \mathcal{H}'$ and $i \in I'$, we say that $\mathcal{H}$ *contains* $\mathcal{H}'$ and write $\mathcal{H} \supset \mathcal{H}'$. Else we say that $\mathcal{H}$ is $\mathcal{H}'$-*free* and write $\mathcal{H} \not\supset \mathcal{H}'$. The subsets $F(\bigcup \mathcal{H}')$ and $f(I')$ form the $\mathcal{H}'$-*copy* in $\mathcal{H}$. For example, $\mathcal{H}$ is $(\{1\}_1, \{1\}_2)$-free iff its edges are pairwise disjoint, that is, $\mathcal{H}$ is a set partition. The initial example corresponds to $\mathcal{H}'$-freeness for $\mathcal{H}' = (\{1, 3\}, \{2, 4\})$. If F and f are bijections (remember that F is increasing) and the equivalence $v \in H_i' \Longleftrightarrow F(v) \in H_{f(i)}$ holds for every $v \in \bigcup \mathcal{H}'$ and $i \in I'$, we say that $\mathcal{H}'$ and $\mathcal{H}$ are *isomorphic*.

The *order* $v(\mathcal{H})$ of $\mathcal{H} = (H_i : i \in I)$ is the number of vertices $v(\mathcal{H}) = |\bigcup \mathcal{H}|$, the *size* $e(\mathcal{H})$ is the number of edges $e(\mathcal{H}) = |\mathcal{H}| = |I|$, and the *weight* $i(\mathcal{H})$ is the number of incidences between the vertices and the edges $i(\mathcal{H}) = \sum_{i \in I} |H_i|$. Trivially, $v(\mathcal{H}) \le i(\mathcal{H})$ and $e(\mathcal{H}) \le i(\mathcal{H})$.

We associate with every hypergraph $\mathcal{F}$ (the letter $\mathcal{F}$ is for "forbidden") two (extremal) functions $H_e(\mathcal{F}), H_i(\mathcal{F}) : \mathbf{N} \to \mathbf{N}$ defined by

$$\begin{aligned} H_e(\mathcal{F}, n) &= \max\{e(\mathcal{H}) : \mathcal{H} \not\supset \mathcal{F} \ \&\ \mathcal{H} \text{ is simple } \&\ v(\mathcal{H}) = n\} \\ H_i(\mathcal{F}, n) &= \max\{i(\mathcal{H}) : \mathcal{H} \not\supset \mathcal{F} \ \&\ \mathcal{H} \text{ is simple } \&\ v(\mathcal{H}) = n\}. \end{aligned}$$

It is clear that in the definition $\mathcal{H}$ must be simple. (For $\mathcal{F}$ of the form $(\{1\}_1, \{1\}_2, \ldots, \{1\}_k)$ the simplicity may be dropped but not for any other $\mathcal{F}$.) On the other hand, $\mathcal{F}$ may be any hypergraph, not necessarily simple. In Sections 5 and 6 we work also with the graph version $G_e(\mathcal{F}, n)$ of $H_e(\mathcal{F}, n)$ in which $e(\mathcal{H})$ is maximized only over graphs $\mathcal{H}$ and with the unordered versions $H_e^u(\mathcal{F}, n)$ and $G_e^u(\mathcal{F}, n)$ in which the vertex injection F is not required to be increasing. We also consider the slightly more general extremal function $G_e(B, n)$ where B is a set of forbidden graphs. Thus for a graph $\mathcal{F}$ the function $G_e^u(\mathcal{F}, n)$ equals to the classical graph extremal function $\mathrm{ex}(\mathcal{F}, n)$. The *reversal* $\overline{\mathcal{F}}$ is obtained from $\mathcal{F}$ by reverting the linear order of $\bigcup \mathcal{F}$. Obviously, $H_e(\overline{\mathcal{F}}, n) = H_e(\mathcal{F}, n)$ and $H_i(\overline{\mathcal{F}}, n) = H_i(\mathcal{F}, n)$. It is also obvious that, for every $n \in \mathbf{N}$ and $\mathcal{F}$, $H_e(\mathcal{F}, n) \le 2^n - 1$ and $H_i(\mathcal{F}, n) \le n2^{n-1}$ but much better bounds can be given. In the forthcoming sections we investigate the behaviour of $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ for various fixed $\mathcal{F}$ and n running through $\mathbf{N} = \{1, 2, \ldots\}$. We considered $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ implicitly already in [15]. Except this article, as far as we know, our extremal setting is new and was not investigated before. We stress again its two not so usual features: the containment is an ordered one (the vertex injection F is increasing) and $\mathcal{H}$ may have edges of any sizes (even if the forbidden $\mathcal{F}$ is a mere graph).

Before summarizing our results we say few words about our motivation and about connections to other results in extremal set systems theory. In this branch of combinatorics (see, for example, surveys of Frankl [10], Füredi [8], and Tuza [23, 24] or the collection [11]) one is interested in the maximum number of edges in set systems subject to some restrictions. These may restrict intersections of edges or they may exclude some forbidden (sub)configurations. Almost always the underlying universum of vertices is supposed to be unordered. We know of only one systematic study of a class of "ordered" extremal problems (for set systems; we are not speaking here of posets, words etc.), the work of Füredi and Hajnal [9] that deals with simple bipartite graphs with ordered parts. (In [9] the equivalent language of 0-1 matrices is used. So is in Anstee, Ferguson and Sali [1], see also further references thereof, but their extremal problems are "unordered".) One our aim is just to explore the properties of $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ and the possibilities which open here. Other aim is to apply results and techniques from the theory of *Davenport–Schinzel sequences* which deals with extremal problems for words; necessary definitions and references will be given in Section 5. Also, we want to extend some results of [9] from 2-element edges to edges of arbitrary cardinality.

In Section 2 we consider hypergraphs $\mathcal{S}_k$ which consist of k disjoint singleton edges. (For the containment of $\mathcal{S}_k$ the order of vertices is irrelevant.) Theorems 2.1 and 2.3 determine $H_e(\mathcal{S}_k, n)$ and $H_i(\mathcal{S}_k, n)$ exactly. We describe all extremal hypergraphs as well. In Theorem 2.2 we prove that if $\mathcal{F} \ne \mathcal{S}_k$ then $H_e(\mathcal{F}, n)$ is a strictly increasing function. Trivially, $H_e(\mathcal{F}, n) \le H_i(\mathcal{F}, n)$ for every $n \in \mathbf{N}$ and $\mathcal{F}$. Theorem 2.4 states that if $\mathcal{F}$ has no two edges of which one completely precedes the other, then for every $n \in \mathbf{N}$ also $H_i(\mathcal{F}, n) \leq cH_e(\mathcal{F}, n)$ where $c > 0$ depends only on $\mathcal{F}$. Theorem 2.5 gives a trivial polynomial upper bound on $H_e(\mathcal{F}, n)$. In Section 3 we precisely determine $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ for each of the 55 $\mathcal{F}$ with $1 \leq i(\mathcal{F}) \leq 4$. Section 4 is enumerative. In Theorem 4.1 we give formulas for the number of hypergraphs, both simple and all, with prescribed numbers of edges of a given cardinality. We use the formulas to calculate the total numbers of hypergraphs, both simple and all, of weight n for $n \leq 10$. In Corollary 4.3 two identities similar to Dobiǹski's formula are given. (Dobiǹski's formula deals with set partitions and our identities deal with hypergraphs.) In Section 5 we apply generalized Davenport–Schinzel sequences to obtain two almost linear bounds which involve $\alpha(n)$, the inverse Ackermann function. In Theorem 5.1 we prove that for every fixed set partition $\mathcal{F}$ for every $\mathcal{F}$-free $\mathcal{H}$ the weight $i(\mathcal{H})$ is bounded almost linearly in $e(\mathcal{H})$. An example is given showing that the superlinearity is inevitable. Theorem 5.2 gives an almost linear upper bound on $G_e(\mathcal{F}, n)$ ($\mathcal{H}$ has only two-element edges) in the case that $\mathcal{F}$ is a forest whose components are stars which have all centers smaller than all leaves. An example shows that the superlinearity is again genuine. In Section 6 we investigate the case when $\mathcal{F}$ is a forest such that one part of the bipartition of $\mathcal{F}$ is smaller than the other (it is easy to see that for other $\mathcal{F}$ we have $H_e(\mathcal{F}, n) \gg n^\gamma$, $\gamma > 1$). Theorem 6.2 gives a method for deriving good upper bounds on $H_e(\mathcal{F}, n)$ from those on $G_e(\mathcal{F}, n)$. We give three applications. Theorem 6.3 extends the classical (easy) unordered graph result $\mathrm{ex}(\mathcal{F}, n) \ll n$ if $\mathcal{F}$ is a forest to hypergraphs: $H_e^u(\mathcal{F}, n) \ll n$ for every forest $\mathcal{F}$. Theorem 6.4 extends the almost linear graph bound of Theorem 5.2 to hypergraphs: $H_e(\mathcal{F}, n)$ is almost linear whenever $\mathcal{F}$ is a star forest. In the last Theorem 6.6 we prove the bound $H_e(\mathcal{F}, n) \ll n(\log n)^2(\log\log n)^3$ if $\mathcal{F} = (\{1,3\}, \{1,5\}, \{2,3\}, \{2,4\})$. This forbidden path was investigated first by Füredi who in [7] and [9] proved graph bounds $n \log n \ll G_e(\mathcal{F}, n) \ll n \log n$ (for ordered bipartite graphs). Section 7 contains some open problems.

We need few more definitions. Notation $f(n) \ll g(n)$ is synonymous to the $f(n) = O(g(n))$ notation. If $m, n \in \mathbf{N}$ and $m \leq n$, then $[n] = \{1, 2, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$. The *degree* $\deg(v) = \deg_{\mathcal{H}}(v)$ of v in $\mathcal{H} = (H_i : \ i \in I)$ is the number of the edges H_i containing v. The *simplification* of $\mathcal{H}$ is a simple hypergraph $\mathcal{H}'$ obtained by keeping from each family of repeated edges of $\mathcal{H}$ just one member. The *deletion* of H_j from $\mathcal{H}$ gives the hypergraph $(H_i : \ i \in I')$ where $I' = I \backslash \{j\}$. The *deletion* of $a \in \bigcup \mathcal{H}$ from $\mathcal{H}$ gives the hypergraph $(H_i \backslash \{a\} : \ i \in I)$ where we omit $\emptyset$ if

$H_i = \{a\}$ (this operation in general destroys simplicity). We may also delete a only from some specified edges. A *(connected) component* $\mathcal{H}_1$ of $\mathcal{H}$ is the minimal subhypergraph $\mathcal{H}_1$ of $\mathcal{H}$ such that every $H \in \mathcal{H}\backslash\mathcal{H}_1$ is disjoint with every $H_1 \in \mathcal{H}_1$.

2 Singleton hypergraph $\mathcal{S}_k$

In this section $\mathcal{F} = \mathcal{S}_k = (\{1\}, \{2\}, \ldots, \{k\})$. We give exact formulas for $H_e(\mathcal{S}_k, n)$ and $H_i(\mathcal{S}_k, n)$. For $k = 1$ both extremal functions are undefined.

Theorem 2.1 *Let $k \ge 2$ and $\mathcal{S}_k = (\{1\}, \{2\}, \ldots, \{k\})$. Then*

$$H_e(\mathcal{S}_k, n) = \left\{ \begin{array}{lll} 2^n - 1 & \ldots & 1 \le n < k \\ 2^{k-2} & \ldots & n \ge k. \end{array} \right.$$

In particular, for $k \ge 3$ the function $H_e(\mathcal{S}_k, n)$ has the global maximum $H_e(\mathcal{S}_k, k-1) = 2^{k-1} - 1$.

Proof. The first formula is clear. For $k \ge 2$ and $n \ge k$ we have $H_e(\mathcal{S}_k, n) \ge 2^{k-2}$, because of the hypergraph $([n], X:\ \emptyset \ne X \subset [k-2])$. We prove by induction on k that for $n \ge k$ also $H_e(\mathcal{S}_k, n) \le 2^{k-2}$. For $k = 2$ this holds because $H_e(\mathcal{S}_2, n) = 1$ for every $n \in \mathbf{N}$. Let $n \ge k \ge 3$ and $\mathcal{H}$ be simple, $\mathcal{S}_k$-free, and $\bigcup \mathcal{H} = [n]$. We can suppose that (i) $\deg(v) \ge 2$ for every $v \in \bigcup \mathcal{H}$ and (ii) there is an $H \in \mathcal{H}$ with $|H| \ge 2$ and an $a \in H$ such that $H\backslash\{a\} \notin \mathcal{H}$.

If (i) is false, there is a vertex a and an edge H such that $a \in H$ and a lies in no other edge of $\mathcal{H}$. We delete H from $\mathcal{H}$ and obtain a simple hypergraph $\mathcal{H}'$ that must be $\mathcal{S}_{k-1}$-free because any $\mathcal{S}_{k-1}$-copy in $\mathcal{H}'$ can be extended by H and a to $\mathcal{S}_k$-copy in $\mathcal{H}$. By induction, $e(\mathcal{H}) = e(\mathcal{H}') + 1 \le (2^{(k-1)-1} - 1) + 1 = 2^{k-2}$. Suppose that (ii) is false. Let $a \in \bigcup \mathcal{H}$ be arbitrary and $H \in \mathcal{H}$, $a \in H$, be such that $|H|$ is as small as possible. If $|H| > 1$, we take $b \in H$, $b \ne a$, and the negation of (ii) gives $H\backslash\{b\} \in \mathcal{H}$, contradicting the minimality of $|H|$. Thus $|H| = 1$ and $\{a\} \in \mathcal{H}$. We obtain that $\{a\} \in \mathcal{H}$ for every vertex a of $\mathcal{H}$ but this implies the contradiction $\mathcal{H} \supset \mathcal{S}_k$ $(n \ge k)$.

We can assume that (i) and (ii) hold. Let a and H be as in (ii). Let $H' \in \mathcal{H}$ be such that $a \in H'$, $H' \ne H$, and, if possible, $|H'| = 1$. We define $\mathcal{H}'$ by deleting H' from $\mathcal{H}$ and then a from $\mathcal{H}\backslash\{H'\}$. Some edges may get duplicated and we set $\mathcal{H}''$ to be the simplification of $\mathcal{H}'$. By (i), $v(\mathcal{H}'') = v(\mathcal{H}) - 1 = n - 1 \ge k - 1$. Since any $\mathcal{S}_{k-1}$-copy in $\mathcal{H}''$ can be extended by H' and a to an $\mathcal{S}_k$-copy in $\mathcal{H}$, $\mathcal{H}''$ is $\mathcal{S}_{k-1}$-free. Also, $e(\mathcal{H}') \le 2e(\mathcal{H}'') - 1$ because, by (ii), $H\backslash\{a\}$ is not duplicated in $\mathcal{H}'$. Notice that $\emptyset \notin \mathcal{H}''$ because we have deleted $\{a\}$ as H'. By induction (now we use the stronger upper bound on $e(\mathcal{H}'')$),

$$e(\mathcal{H}) = e(\mathcal{H}') + 1 \le (2e(\mathcal{H}'') - 1) + 1 = 2e(\mathcal{H}'') \le 2 \cdot 2^{(k-1)-2} = 2^{k-2}.$$

□

$H_e(\mathcal{S}_k, n)$ has the strange feature of being independent of n. We show that the other functions $H_e(\mathcal{F}, n)$ are increasing, as one expects.

Theorem 2.2 *If $\mathcal{F} \neq \mathcal{S}_k$ then $H_e(\mathcal{F}, n) < H_e(\mathcal{F}, n+1)$ for every $n \in \mathbf{N}$.*

Proof. Let $\mathcal{F} \neq \mathcal{S}_k$ and $\bigcup \mathcal{F} = [m]$. We say that $\{u\} \in \mathcal{F}$ is an *isolated singleton* of $\mathcal{F}$ if $\deg(u) = 1$. Let l be the maximum number such that $\{1\}, \{2\}, \ldots, \{l\}$ are isolated singletons of $\mathcal{F}$. Since $\mathcal{F} \neq \mathcal{S}_k$, $0 \le l < m$. Clearly, any other isolated singleton of $\mathcal{F}$ is preceded by at least $l+1$ vertices.

We proceed by induction on n. The inequality holds for every $n < m-1$ because then $H_e(\mathcal{F}, n) = 2^n - 1$. Let $n \ge m-1$ and let $\mathcal{H}$ attain the value $H_e(\mathcal{F}, n)$. If $a \in H \in \mathcal{H}$ and $\{a\} \notin \mathcal{H}$, we replace H by $\{a\}$. The new hypergraph is simple, $\mathcal{F}$-free, and it has the same number of edges and vertices as $\mathcal{H}$; order does not decrease because else we would have contradiction with the inductive assumption. Repeating the replacements we obtain a simple $\mathcal{F}$-free hypergraph $\mathcal{H}'$ such that $e(\mathcal{H}') = e(\mathcal{H}) = H_e(\mathcal{F}, n)$, $\bigcup \mathcal{H}' = \bigcup \mathcal{H} = [n]$, and $\{a\} \in \mathcal{H}'$ for every $a \in [n]$. We define $\mathcal{H}''$ by inserting in $\mathcal{H}'$, between l and $l+1$, a new singleton edge $\{u\}$. $\mathcal{H}''$ is simple and satisfies $v(\mathcal{H}'') = n+1$ and $e(\mathcal{H}'') = e(\mathcal{H}') + 1 = H_e(\mathcal{F}, n) + 1$. We show that $\mathcal{H}''$ is $\mathcal{F}$-free. This gives $H_e(\mathcal{F}, n+1) \ge e(\mathcal{H}'') > H_e(\mathcal{F}, n)$. The new edge $\{u\}$ would have to participate in every $\mathcal{F}$-copy in $\mathcal{H}''$. It cannot play the role of any of the initial l isolated singletons of $\mathcal{F}$ because $\{1\}, \{2\}, \ldots, \{l\} \in \mathcal{H}'$ and we would have already $\mathcal{F} \subset \mathcal{H}'$. It cannot play the role of any other isolated singleton of $\mathcal{F}$ either because those are preceded in $\mathcal{F}$ by at least $l+1$ vertices but $\{u\}$ is preceded in $\mathcal{H}''$ by only l vertices. Thus $\mathcal{H}'' \not\supset \mathcal{F}$. □

Theorem 2.3 *Let $k \ge 2$ and $\mathcal{S}_k = (\{1\}, \{2\}, \ldots, \{k\})$. Then*

$$H_i(\mathcal{S}_k, n) = \begin{cases} n2^{n-1} & \ldots \quad 1 \le n < k \\ n + (k-2)2^{k-3} & \ldots \quad k \le n \le 2^{k-3} + 1 \\ (k-1)n - (k-2) & \ldots \quad n \ge \max(k, 2^{k-3} + 1). \end{cases}$$

Note that $H_i(\mathcal{S}_k, k-1) > H_i(\mathcal{S}_k, n)$ for $k \le n \le \max(k, 2^{k-2})$ ($k \ge 3$).

Proof. The formula is clear for $1 \le n < k$. We suppose that $n \ge k \ge 2$ and that $\mathcal{H}$ is a simple hypergraph with $\bigcup \mathcal{H} = [n]$. Its dual $\mathcal{H}^*$ is defined by

$$\mathcal{H}^* = (H_i^* : \ i \in [n]) \text{ where } H_i^* = \{H \in \mathcal{H} : \ i \in H\}.$$

Thus $e(\mathcal{H}^*) = v(\mathcal{H}) = n$. Let $\Gamma(X) = \Gamma_{\mathcal{H}}(X)$ be for $X \subset [n]$ defined by

$$\Gamma(X) = \left| \bigcup_{i \in X} H_i^* \right| = |\{H \in \mathcal{H} : \ H \cap X \neq \emptyset\}|.$$

By the defect form of P. Hall's theorem (Lovász [16, Problems 7.5 and 13.5]) applied on $\mathcal{H}^*$, $\mathcal{H}$ is $\mathcal{S}_k$-free if and only if

$$\max_{X \subset [n]} |X| - \Gamma(X) \ge n - k + 1.$$

Thus if $\mathcal{H}$ is $\mathcal{S}_k$-free, there exists a set $X \subset [n]$ of cardinality l, $n - k + 2 \le l \le n$ ($\Gamma(X) \ge 1$), intersected by only at most $l - n + k - 1$ edges of $\mathcal{H}$. And contrarywise, every such a hypergraph is (trivially) $\mathcal{S}_k$-free. Hence

$$i(\mathcal{H}) \le (l - n + k - 1)n - (l - n + k - 2) + (n - l)2^{n-l-1} = f(l, k, n)$$

and this bound is attained.

The first difference of $f(l, k, n)$ with respect to l is the increasing function

$$f(l + 1, k, n) - f(l, k, n) = n - 1 - (n - l + 1)2^{n-l-2}.$$

Therefore $f(l, k, n)$ attains its maximum in one of the endpoints $l = n - k + 2$ and $l = n$ (or in both). The corresponding values are $f(n - k + 2, k, n) = n + (k - 2)2^{k-3}$ and $f(n, k, n) = (k - 1)n - (k - 2)$. These values are equal for $n = 2^{k-3} + 1$. For $n < 2^{k-3} + 1$ the former value dominates and for $n > 2^{k-3} + 1$ the latter. We obtain the other two formulas. Maximum weights are attained by $\mathcal{H}_1$ or by $\mathcal{H}_2$ where the edges of $\mathcal{H}_1$, respectively of $\mathcal{H}_2$, are $[n]$ together with all nonempty subsets of some $(k-2)$-element set $Y \subset [n]$, respectively $[n]$ together with some $k - 2$ distinct $(n - 1)$-element subsets of $[n]$. $\square$

It follows from the proof that $\mathcal{H}_1$ and $\mathcal{H}_2$ are the only types of extremal hypergraphs for $n \ge k$. For $1 \le n < k$ the maximum weight is attained only by the complete hypergraph. We conclude that the number of simple $\mathcal{S}_k$-free hypergraphs having order n and the maximum weight is 1 if $1 \le n < k$ and $\eta_{k,n}\binom{n}{k-2}$ if $n \ge k$, where for $k = 2, 3, 4$ always $\eta_{k,n} = 1$ and for $k \ge 5$ we have $\eta_{k,n} = 1$ if $n \ne 2^{k-3} + 1$ and $\eta_{k,2^{k-3}+1} = 2$.

By means of P. Hall's theorem one can give a quick proof of Theorem 2.1 as well. The number of $\mathcal{H}$ attaining $H_e(\mathcal{S}_k, n)$ is seen to be 1 for $n < k$ and $2^{k-2}\binom{n}{k-2}$ for $n \ge k$.

Two subsets X and Y of $\mathbf{N}$ are *separated* if $\max X < \min Y$ or $\max Y < \min X$. Below we can assume that $e(\mathcal{F}) > 1$ because for $\mathcal{F}$ with just one edge $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ are easy to determine exactly.

Theorem 2.4 *Suppose that $\mathcal{F}$ has no two separated edges, $p = v(\mathcal{F})$, and $q = e(\mathcal{F}) > 1$. Then for every $n \in \mathbf{N}$*

$$H_i(\mathcal{F}, n) \le (2p-1)(q-1)H_e(\mathcal{F}, n).$$

Proof. Let $\mathcal{H}$ attain $H_i(\mathcal{F}, n)$. We transform $\mathcal{H}$ in a new hypergraph $\mathcal{H}'$ by keeping all edges with less than p vertices and replacing every edge $H = \{v_1, v_2, \ldots, v_s\}$ of $\mathcal{H}$ with $s \ge p$, where $v_1 < v_2 < \cdots < v_s$, by $t = \lfloor |H|/p \rfloor$ new p-element edges $\{v_1, \ldots, v_p\}$, $\{v_{p+1}, \ldots, v_{2p}\}, \ldots, \{v_{(t-1)p+1}, \ldots, v_{tp}\}$. $\mathcal{H}'$ may not be simple and we set $\mathcal{H}''$ to be the simplification of $\mathcal{H}'$. Two observations: (i) no edge of $\mathcal{H}'$ repeats q or more times and (ii) $\mathcal{H}''$ is $\mathcal{F}$-free. If (i) were false, there would be q distinct edges $H_1, \ldots, H_q$ in $\mathcal{H}$ such that $|\bigcap_{i=1}^q H_i| \ge p$. But this implies the contradiction $\mathcal{F} \subset \mathcal{H}$. As for (ii), note that any $\mathcal{F}$-copy in $\mathcal{H}''$ may use from every $H \in \mathcal{H}$ only at most one new edge and so it is an $\mathcal{F}$-copy in $\mathcal{H}$ as well. The observations and the definitions of $\mathcal{H}'$ and $\mathcal{H}''$ imply

$$\begin{aligned} H_i(\mathcal{F}, n) = i(\mathcal{H}) &\le \frac{(2p-1)i(\mathcal{H}')}{p} \le \frac{(2p-1)(q-1)i(\mathcal{H}'')}{p} \\ &\le (2p-1)(q-1)e(\mathcal{H}'') \\ &\le (2p-1)(q-1)H_e(\mathcal{F}, n). \end{aligned}$$

In the last innocently looking inequality we use Theorem 2.2. $\square$

The same idea gives for $H_e(\mathcal{F}, N)$ a trivial polynomial bound.

Theorem 2.5 *If $\mathcal{F}$ is a hypergraph with $p = v(\mathcal{F})$ and $q = e(\mathcal{F})$, then for every $n \in \mathbf{N}$*

$$H_e(\mathcal{F}, n) \le (q-1)\binom{n}{p} + \binom{n}{p-1} + \cdots + \binom{n}{1}.$$

Proof. Let $\mathcal{H}$ attain $H_e(\mathcal{F}, n)$. We put in $\mathcal{H}'$ every $H \in \mathcal{H}$ with $|H| < p$ and, for every $H \in \mathcal{H}$ with $|H| \ge p$, a p-element subset $H' \subset H$. Since no p-element edge of $\mathcal{H}'$ repeats more than $q-1$ times (else $\mathcal{H} \supset \mathcal{F}$) and other edges do not repeat at all, we have

$$H_e(\mathcal{F}, n) = e(\mathcal{H}) = e(\mathcal{H}') \le (q-1)\binom{n}{p} + \binom{n}{p-1} + \cdots + \binom{n}{1}.$$

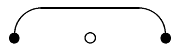
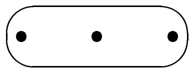
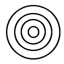
$\square$

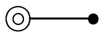
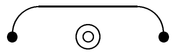
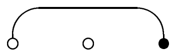
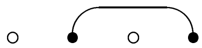
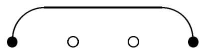
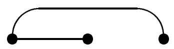
For $\mathcal{F} = ([p]_1, [p]_2, \ldots, [p]_q)$ this bound is best possible.

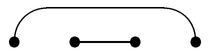
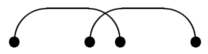
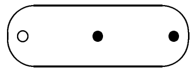
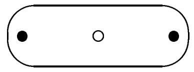
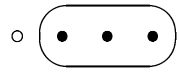
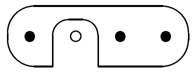
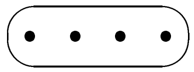
3 One hundred and ten extremal functions

The table below lists extremal functions of the 55 nonempty forbidden $\mathcal{F}$ with $i(\mathcal{F}) \le 4$. In the proofs we refer to $\mathcal{F}$ according to the numbers in column 1. Star indicates that the reversal $\overline{\mathcal{F}}$ is nonisomorphic to $\mathcal{F}$ and is not listed, because it has the same extremal functions. $\mathcal{F}$ are visualized in column 2. Hypergraphs $\mathcal{F}$ with $i(\mathcal{F}) = 1, 2, 3$, and 4 occupy lines 1, 2–4, 5–11, and 12–39, respectively. Empty circle $\circ$ denotes a vertex that is a singleton edge and full circle $\bullet$ a vertex that is not a singleton edge. Two-element edges are indicated by arcs and larger edges by ovals. Concentric circles or arcs sharing both endpoints ($\mathcal{F}_{31}$) indicate edge multiplicities. For example, $\mathcal{F}_{18} = (\{1\}_1, \{1\}_2, \{1,2\})$ and $\mathcal{F}_{36} = (\{1,2,3\}, \{2\})$. Columns 3 and 4 list functions $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$. The formulas given hold for all $n \in \mathbf{N}$ if not said otherwise. The extremal functions for hypergraphs $\mathcal{F}_{33}$ and $\mathcal{F}_{34}$ were determined already in [15] but we give the arguments here again for the sake of completeness.

Theorem 3.1

no.	picture of $\mathcal{F}$	$H_e(\mathcal{F}, n)$	$H_i(\mathcal{F}, n)$
1		not defined	not defined
2		n	n
3		$1, 1, \ldots$	n
4		n	n
5		$\left\lfloor \frac{3n}{2} \right\rfloor$	$2n$ $(n > 1)$
6*		n	$2n - 1$
7		$1, 3, 2, 2, \ldots$	$2n - 1$ $(n \neq 2)$
8*		n	$2n - 1$
9*		$2n - 1$	$3n - 2$
10		$2n - 1$	$3n - 2$
11		$\binom{n+1}{2}$	n^2
12		$2n$ $(n > 2)$	$3n$ $(n > 2)$
13*		$2n - 1$	$\left\lfloor \frac{7(n-1)}{2} \right\rfloor + 1$
14		$n + 1$ $(n > 1)$	$3n - 2$
15*		$n + 1$ $(n > 1)$	$3n - 2$

no.	picture of $\mathcal{F}$	$H_e(\mathcal{F}, n)$	$H_i(\mathcal{F}, n)$
16		$n+1\ (n>1)$	$3n-2$
17		$1, 3, 7, 4, 4, \ldots$	$3n-2\ (n \neq 3)$
18*		$2n-1$	$4n-6\ (n>5)$
19*		$2n-1$	$3n-2$
20		$2n-1$	$3n-2$
21*		$2n-1$	$3n-2$
22*		$2n-1$	$3n-2$
23*		$2n-1$	$3n-2$
24		n	$2n-1$
25*		$4n-5\ (n>1)$	$8n-12\ (n>1)$
26*		$4n-5\ (n>1)$	$8n-12\ (n>1)$
27		$4n-5\ (n>1)$	$8n-12\ (n>1)$
28		$4n-5\ (n>1)$	$8n-12\ (n>1)$
29*		$2n-1$	$3n-2$
30		$\left\lfloor \frac{n^2}{4} \right\rfloor + n$	$2\left\lfloor \frac{n^2}{4} \right\rfloor + n\ (n \neq 3)$
31		$\binom{n+1}{2}$	n^2

no.	picture of $\mathcal{F}$	$H_e(\mathcal{F}, n)$	$H_i(\mathcal{F}, n)$
32		$2\left\lfloor\frac{(n+1)^2}{4}\right\rfloor - 1$	$5\left\lfloor\frac{(n+1)^2}{4}\right\rfloor - 2n - 2$
33		$4n-5$ $(n>1)$	$8n-12$ $(n>1)$
34		$4n-5$ $(n>1)$	$8n-12$ $(n>1)$
35*		$\binom{n+1}{2}$	n^2
36		$\binom{n+1}{2}$	n^2
37*		n^2-n+1	$\frac{5n^2-9n+6}{2}$
38*		n^2-n+1	$\frac{5n^2-9n+6}{2}$
39		$\frac{n^3+5n}{6}$	$\frac{n^3-n^2+2n}{2}$

Proof. $\mathcal{H}$ is a generic simple $\mathcal{F}$-free hypergraph with $\bigcup\mathcal{H} = [n]$, $n \in \mathbf{N}$. $\mathcal{H}$ is *full* if $\{a\} \in \mathcal{H}$ for every $a \in \bigcup\mathcal{H}$. If $a \in H \in \mathcal{H}$ and $\{a\} \notin \mathcal{H}$, we can replace H by $\{a\}$. (We used this replacement in the proof of Theorem 2.2.) Our hypergraph remains simple and $\mathcal{F}$-free, and its size has not changed (but order might decrease and weight decreases). Repeating this operation, we replace $\mathcal{H}$ by a full $\mathcal{H}'$ with $e(\mathcal{H}') = e(\mathcal{H})$ and $v(\mathcal{H}') = n' \le v(\mathcal{H}) = n$. If $e(\mathcal{H}') \le f(n')$ for a nondecreasing function f, we have also $e(\mathcal{H}) = e(\mathcal{H}') \le f(n') \le f(n)$. This little trick helps us to obtain upper bounds on $H_e(\mathcal{F}, n)$ (it does not help for no. 4, 11, 29–34, and 39 where $\mathcal{F}$ has no singleton) but it does not work for $H_i(\mathcal{F}, n)$. When determining $H_e(\mathcal{F}, n)$ we assume, without repeating it every time, that $\mathcal{H}$ is replaced by a full $\mathcal{H}'$ with $\bigcup\mathcal{H}' = [n']$.

For obtaining upper bounds on $H_i(\mathcal{F}, n)$ we use induction and/or other replacement arguments. To simplify the situation, we get rid of a large edge $H \in \mathcal{H}$ by replacing H by some sets H_i, usually (but not always, see $\mathcal{F}_{30}$) subsets of H. For the resulting hypergraph $\mathcal{H}_0$ one has to check three things: $\mathcal{H}_0$ remains simple ($H_i \notin \mathcal{H}$ for every i), $i(\mathcal{H}_0) \ge i(\mathcal{H})$ ($\sum_i |H_i| \ge |H|$), and $\mathcal{H}_0$ remains $\mathcal{F}$-free (the reason is usually that any $\mathcal{F}$-copy may use at most one of the new edges H_i and therefore, since $H_i \subset H$, $\mathcal{H}_0 \supset \mathcal{F}$ implies the contradiction $\mathcal{H} \supset \mathcal{F}$). For each particular replacement these three conditions are easy to check and we leave it to the reader. Repeating the replacements, we eliminate all large edges.

1. No such $\mathcal{H}$ exists. 2. $\mathcal{H}$ is a set partition. 3. $\mathcal{H}$ has one edge. 4. $\mathcal{H}$ has only singleton edges.

5. Recall that $\mathcal{H}'$ is full. Therefore edges with $|H| \geq 2$ must be mutually disjoint and $H_e(\mathcal{F}, n) \leq n + \lfloor n/2 \rfloor$, which is easy to attain. The value $H_i(\mathcal{F}, n) = 2n$ for $n > 1$ is clear; $H_i(\mathcal{F}, 1) = 1$.

6. $\mathcal{H}'$ besides singletons has no other edges and $H_e(\mathcal{F}, n) \leq n$, which is easy to attain. As for the weight, we have (in $\mathcal{H}$) $\deg(a) \leq 2$ for every $a \in [n-1]$, and equality for an a implies $\deg(n) \leq 2$. Hence $\deg(a) = 2$ for an $a < n$ implies $i(\mathcal{H}) \leq 2n$. Even $i(\mathcal{H}) \leq 2n - 1$ because $\deg(a) = 2$ for every $a \in [n]$ is impossible ($\mathcal{H}$ is simple). In the other case when $\deg(a) = 1$ for every $a < n$ again $i(\mathcal{H}) \leq 2n - 1$ because then $\deg(n) \leq n$. In both cases $i(\mathcal{H}) \leq 2n - 1$, attained by $\mathcal{H} = ([n], [n-1])$ and $\mathcal{H} = (\{i, n\}, \{n\} : \ i \in [n-1])$.

7. Particular case of Theorems 2.1 and 2.3; $H_i(\mathcal{F}, 2) = 4$.

8. $H_e(\mathcal{F}, n) = n$ for the same reason as in 6. As for $H_i(\mathcal{F}, n)$, every component $\mathcal{H}_1$ of $\mathcal{H}$ consists of several edges which pairwise intersect in one common vertex that is their largest vertex. Thus $i(\mathcal{H}_1) \leq 2v(\mathcal{H}_1) - 1$ and $H_i(\mathcal{F}, n) \leq 2n - 1$, attained by $\mathcal{H} = (\{i, n\}, \{n\} : i \in [n-1])$.

9. In $\mathcal{H}'$, $|H| \leq 2$ for every edge and $|H| = 2$ implies $1 \in H$. Hence $H_e(\mathcal{F}, n) \leq n + (n-1)$, attained by $\mathcal{H} = (\{1\}, \{1, i\}, \{i\} : i \in [2, n])$. As for $H_i(\mathcal{F}, n)$, we eliminate all edges with $|H| \geq 3$ by replacing H by two-element sets $\{a, b\}$ where $a = \min H$ and $a < b \in H$. Since $1 \in H$ for every two-element edge, $H_i(\mathcal{F}, n) \leq n + 2(n-1)$, attained by the already mentioned $\mathcal{H}$.

10. Use arguments similar to 9. Allowed two-element edges are now $\{i, i+1\}$.

11. $\mathcal{H}$ has only edges of cardinalities 1 and 2. Thus $H_e(\mathcal{F}, n) = \binom{n}{1} + \binom{n}{2}$ and $H_i(\mathcal{F}, n) = \binom{n}{1} + 2\binom{n}{2}$.

12. If $|H| \geq 3$ for an $H \in \mathcal{H}'$, then $H_1 \notin \mathcal{H}'$ for some $H_1 \subset H$ with $|H_1| = 2$. Replacing H by H_1, we get rid of all edges with three and more vertices. Every vertex is then contained in at most two two-element edges. Therefore $H_e(\mathcal{F}, n) \leq n + n$, attained for $n > 2$ by $\mathcal{H} = (\{i\}, \{i, i+1\} : i \in [n])$ (taken

modulo n). For $n = 1, 2$ we have $H_e(\mathcal{F}, n) = 1, 3$. The value $H_i(\mathcal{F}, n) = 3n$ for $n > 2$ is clear; for $n = 1, 2$ we have $H_i(\mathcal{F}, n) = 1, 4$.

13. We eliminate from $\mathcal{H}'$ all edges with three and more vertice as in 12. Two-element edges may intersect only in the very last vertex n'. Thus $H_e(\mathcal{F}, n) \leq n + (n-1)$, attained by $\mathcal{H} = (\{i\}, \{n\}, \{i, n\} : i \in [n-1])$. As for $H_i(\mathcal{F}, n)$, let $n \geq 3$ and v be the first vertex with $\deg(v) \geq 3$ (if v does not exist, $i(\mathcal{H}) \leq 2n$). If $\deg(v) = 3$, $i(\mathcal{H}) \leq 3n - 1$ because $\deg(w) \leq 3$ for every $w > v$ and $3n$ cannot be attained. If $\deg(v) > 3$, necessarily $v = n$ and $\deg(w) \leq 2$ for $w < n$. Hence $H_i(\mathcal{F}, n) \leq 2(n-1) + 1 + (n-1) + \lfloor \frac{n-1}{2} \rfloor$, attained by $\mathcal{H} = (\{i, n\}, \{2j-1, 2j, n\}, \{n\} : \ i \in [n-1], j \in [\lfloor \frac{n-1}{2} \rfloor])$ for odd $n \geq 3$ and the same $\mathcal{H}$ plus $\{n-1\}$ for even $n \geq 4$.

14. Recall that $\mathcal{H}'$ is full. Besides singletons it may have at most one other edge and $H_e(\mathcal{F}, n) \leq n + 1$, which is easy to attain; $H_e(\mathcal{F}, 1) = 1$. To determine $H_i(\mathcal{F}, n)$, notice that, in $\mathcal{H}$, $\deg(w) \geq 4$ implies that $\deg(v) = 1$ for every other vertex v. Then $i(\mathcal{H}) \leq 2n - 1$. Otherwise $\deg(w) \leq 3$ for every w and $i(\mathcal{H}) \leq 3n$. Since $\deg(w) = \deg(v) = 3$ implies that w and v lie in the same three edges, weights $3n$ and $3n - 1$ cannot be attained but $3n - 2$ can, by $\mathcal{H} = ([n], [n-1], [2, n])$.

15. $\mathcal{H}'$ has no edges H with $|H| > 2$ and may have only one two-element edge, $\{n' - 1, n'\}$. Thus, for $n > 1$, $H_e(\mathcal{F}, n) \leq n + 1$, which is easy to attain; $H_e(\mathcal{F}, 1) = 1$. $\mathcal{H} = ([n], [n-1], [2, n])$ shows that $H_i(\mathcal{F}, n) \geq 3n - 2$. We prove the opposite inequality by considering $\deg(1)$ in a general $\mathcal{H}$. Case $\deg(1) \geq 4$ is impossible because it implies that $\mathcal{H} \supset \mathcal{F}$. So does $\deg(1) = 3$ if an $H \in \mathcal{H}$ exists with $1 \notin H$. Thus $\deg(1) = 3$ implies that $e(\mathcal{H}) = 3$ and $i(\mathcal{H}) \leq 3n - 2$. If $\deg(1) = 2$, we delete the two edges containing 1 from $\mathcal{H}$ and obtain $i(\mathcal{H}) \leq n + (n-1) + (n-1) = 3n - 2$ because the resulting hypergraph does not contain $\mathcal{F}_3$. If $\deg(1) = 1$, we proceed by induction on $v(\mathcal{H})$. Let $H \in \mathcal{H}$ with $1 \in H$. If $H_1 = H \backslash \{1\} \notin \mathcal{H}$, we delete 1 (simplicity is preserved) and use induction. If $H_1 \in \mathcal{H}$ and $|H_1| \leq 2$, we delete 1 and H_1 and use induction. If $H_1 \in \mathcal{H}$ and $|H_1| \geq 3$, let $1, u, v$, and w be the first four vertices of H (in this order). If both sets $H_2 = H_1 \backslash \{u\}$ and $H_3 = H_1 \backslash \{v\}$ are edges of $\mathcal{H}$, we have $\mathcal{H} \supset \mathcal{F}$ since $u \in H \cap H_1$, $v \in H_2$, and $w \in H_3$. Hence one of the sets, say H_2, is not an edge and we can again use induction, deleting 1 from $\mathcal{H}$ and u from H_1.

16. $H_e(\mathcal{F}, n)$ is handled similarly to 15. $\mathcal{H} = ([n], [n-1], [2, n])$ shows that $H_i(\mathcal{F}, n) \geq 3n - 2$. We prove the opposite inequality. Let $n \geq 3$ and let $u \in [2, n-1]$ have the maximum degree in $\mathcal{H}$ among the vertices in $[2, n-2]$. If $\deg(u) = 1$, then $i(\mathcal{H}) \leq \deg(1) + \deg(n) + n - 2 \leq n + n + n - 2 = 3n - 2$.

If $\deg(u) \geq 4$, then $\mathcal{H} \supset \mathcal{F}$, which is a contradiction. The same holds if $\deg(u) = 3$ and an edge H exists with $u \notin H$. Thus $\deg(u) = 3$ implies $e(\mathcal{H}) = 3$ and $i(\mathcal{H}) \leq 3n-2$. Let $\deg(u) = 2$. If $\deg(1) \geq 3$ and $\deg(n) \geq 3$, we have again $\mathcal{H} \supset \mathcal{F}$ or $e(\mathcal{H}) = 3$. Thus, say $\deg(1) \leq 2$. If $\deg(1) = 1$, we delete the edge containing 1 and obtain $i(\mathcal{H}) \leq n + 2(n-1) - 1 = 3n-3$ because the rest of $\mathcal{H}$ does not contain $\mathcal{F}_6$. If $\deg(1) = 2$, we delete $H \in \mathcal{H}$ such that $1 \in H$ and $|H| \leq n-1$. The rest again does not contain $\mathcal{F}_6$ and thus $i(\mathcal{H}) \leq n-1+2n-1 = 3n-2$.

17. Particular case of Theorems 2.1 and 2.3; $H_i(\mathcal{F}, 3) = 12$.

18. In $\mathcal{H}'$, two nonsingleton edges may intersect only in the common last vertex, which implies that $e(\mathcal{H}_1) \leq 2v(\mathcal{H}_1) - 1$ holds for every component $\mathcal{H}_1$ of $\mathcal{H}'$. Hence $H_e(\mathcal{F}, n) \leq 2n-1$, attained by $\mathcal{H} = (\{i\}, \{i,n\}, \{n\} : \ i \in [n-1])$.

As for $H_i(\mathcal{F}, n)$, consider an $\mathcal{H}$ with $\bigcup \mathcal{H} = [n]$. Since $\mathcal{H} \not\supset \mathcal{F}$, $\deg(1) \leq 2$. We delete 1 from $\mathcal{H}$ and obtain $\mathcal{H}_1$. $\mathcal{H}_1$ has at most two duplicated edges. Let $H_1 = H_2$ be one of the duplications. If $|H_1| = 1$, we delete H_1 from $\mathcal{H}_1$. If $|H_1| \geq 2$, we delete from H_1 its last vertex. This creates no new duplication (else $\mathcal{H} \supset \mathcal{F}$). In this way we remove from $\mathcal{H}_1$ both possible duplications and obtain a simple $\mathcal{H}_2$ with $\bigcup \mathcal{H}_2 = [2,n]$ and $i(\mathcal{H}) \leq 4 + i(\mathcal{H}_2)$. We have the inductive inequality $i(\mathcal{H}) \leq 4 + H_i(\mathcal{F}, n-1)$. Note that $\deg(2) \leq 2$ and thus for induction we may as well delete 2 instead of 1 and that if one of $\{1\}$, $\{2\}$, and $\{1,2\}$ is an edge of $\mathcal{H}$, we obtain the streghtening $i(\mathcal{H}) \leq 3 + H_i(\mathcal{F}, n-1)$. Note also that $\deg(v) \geq 3$ implies that v is the last vertex of every $H \in \mathcal{H}, v \in H$.

We prove that for $n = 1,2,3,4,5,6$ we have $H_i(\mathcal{F}, n) = 1,4,8,11,15,18$ and that $H_i(\mathcal{F}, n) = 4n-6$ for $n \geq 6$. The first two values are trivial. By the inductive inequality, $H_i(\mathcal{F}, 3) \leq 4+4 = 8$. Weight 8 is attained by $\mathcal{H} = (\{3\}, \{1,3\}, \{2,3\}, [3])$. Let $n = 4$ and $\bigcup \mathcal{H} = [4]$. Clearly, $\deg(1), \deg(2) \leq 2$. Let first $\deg(3) \geq 3$ and p be the number of edges intersecting both $[2]$ and $[3,4]$. Clearly, $p \leq 2 \cdot 2$. Since no edge can contain both 3 and 4, $\deg(3) + \deg(4) \leq p+2 \leq 6$ and $i(\mathcal{H}) = \sum_1^4 \deg(i) \leq 2 \cdot 2 + 6 = 10$. If $\deg(3) \leq 2$, let p be the number of edges $H \in \mathcal{H}$ such that $4 \in H$ and $H \cap [3] \neq \emptyset$. Then $p \leq H_e(\mathcal{F}_5, 3) = 4$, $\deg(4) \leq 1+p \leq 5$, and $i(\mathcal{H}) = \sum_1^4 \deg(i) \leq 3 \cdot 2 + 5 = 11$. Weight 11 is attained by $\mathcal{H} = (\{4\}, \{i,4\}, [4] : \ i \in [3])$. Thus $H_i(\mathcal{F}, 4) = 11$. By the inductive inequality, $H_i(\mathcal{F}, 5) \leq 4+11 = 15$ and weight 15 is attained by $\mathcal{H} = (\{5\}, \{i,5\}, \{2j-1, 2j, 5\} : \ i \in [4], j \in [2])$.

It remains to show that $H(\mathcal{F}, 6) = 18$ and not $4+15 = 19$. $H(\mathcal{F}, 6) \geq 18$ due to $\mathcal{H} = (\{6\}, \{i,6\}, \{1,2,6\}, \{3,4,5,6\} : \ i \in [5])$. We elaborate the argument for $n=4$. Let $\bigcup\mathcal{H}=[6]$. Clearly, $\deg(1),\deg(2)\le 2$ and $\deg(3)\le 4$. If $\deg(3)=4$, no edge intersects both $[3]$ and $[4,6]$ and $i(\mathcal{H})\le 2H_i(\mathcal{F},3)=16$. If $\deg(3)=3$, we delete 3 from $\mathcal{H}$. If this creates a duplication, one of $\{1\}$, $\{2\}$, and $\{1,2\}$ is an edge of $\mathcal{H}$ and by the above remark $i(\mathcal{H})\le 3+H_i(\mathcal{F},5)=18$. If no duplication arises, again $i(\mathcal{H})\le\deg(3)+H_i(\mathcal{F},5)=18$. So $\deg(3)\le 2$. Let $k=\deg(4)$. Let first $k\ge 3$ and p be the number of edges intersecting both $[4]$ and $[5,6]$ (none of them contains 4). The edges for which 4 is the last vertex contribute by at least $k-1$ to $\deg(1)+\deg(2)+\deg(3)\le 6$ and thus $k\le 7$ and $p\le 6-(k-1)=7-k$. If $\deg(5)\ge 3$, $\deg(5)+\deg(6)\le p+2\le 9-k$ (no edge contains both 5 and 6) and $i(\mathcal{H})=\sum_1^6\deg(i)\le 3\cdot 2+k+9-k=15$. If $\deg(5)\le 2$, we have $\deg(6)\le 2+p\le 9-k$ and $i(\mathcal{H})\le 4\cdot 2+k+9-k=17$. Thus $k=\deg(4)\le 2$ and we have $\deg(i)\le 2$ for every $i\in[4]$. If $\deg(5)\ge 3$ we set again p to be the number of $H\in\mathcal{H}$ intersecting both $[4]$ and $[5,6]$. We have $p\le 4\cdot 2=8$ and $\deg(5)+\deg(6)\le p+2\le 10$. Thus $i(\mathcal{H})=\sum_1^6\deg(i)\le 4\cdot 2+10=18$. If $\deg(5)\le 2$, let p be the number of $H\in\mathcal{H}$ intersecting $[5]$ and containing 6. Then $p\le H_e(\mathcal{F}_5,5)=7$ and $\deg(6)\le 1+p\le 8$. We have again $i(\mathcal{H})\le 5\cdot 2+8=18$. Thus $H_i(\mathcal{F},6)=18$.

Finally, using induction starting at $n=6$ and the inductive inequality we see that for $n\ge 6$ we have $H_i(\mathcal{F},n)\le 4n-6$. The opposite inequality is proved by the hypergraph $\mathcal{H}=(\{i,n-1\},\{i,n\},\{n-1\},\{n\}:\ i\in[n-2])$.

19. Let v be the first vertex in $\mathcal{H}'$ with $\deg(v)\ge 2$. If $\deg(v)=2$, $\mathcal{H}'$ has at most one nonsingleton edge and $e(\mathcal{H}')\le n+1$. If $\deg(v)>2$, every nonsingleton edge has two vertices and starts in v. Thus $H_e(\mathcal{F},n)\le 2n-1$, attained by $\mathcal{H}=(\{1\},\{1,i\},\{i\}:\ i\in[2,n])$. This hypergraph shows that $H_i(\mathcal{F},n)\ge 3n-2$. To prove the opposite inequality, we take a general $\mathcal{H}$ and argue as in 15. If $\deg(1)\ge 3$, $|H|\le 2$ for every edge of $\mathcal{H}$ and $|H|=2$ implies $1\in H$. Thus $i(\mathcal{H})\le 3n-2$. If $\deg(1)=2$, we delete the two edges containing 1. Since the rest does not contain $\mathcal{F}_4$, we have $i(\mathcal{H})\le n+(n-1)+(n-1)=3n-2$. If $\deg(1)=1$, let H and H_1 be given by $1\in H\in\mathcal{H}$, and $H_1=H\backslash\{1\}$. If $H_1\notin\mathcal{H}$ or $|H_1|\le 2$, we delete 1 and, if necessary, H_1, and use induction. If H_1 is an edge and $|H_1|\ge 3$, then $H_2=H_1\backslash\{u\}$, where $u=\min H_1$, is not an edge (else $\mathcal{H}\supset\mathcal{F}$). We delete 1 from $\mathcal{H}$ and u from H_1 and use induction.

20. In $\mathcal{H}'$, for every two nonsingleton edges $H_1\ne H_2$ we have $H_1\le H_2$ or $H_1\ge H_2$. ($H_1\le H_2$ means that $x\le y$ for every $x\in H_1, y\in H_2$.) Therefore $\mathcal{H}'$ has at most $n-1$ such edges. $H_e(\mathcal{F},n)\le 2n-1$, attained by $(\{i,i+1\},\{i\},\{n\}\ :\ i\in[n-1])$. This hypergraph shows also that

$H_i(\mathcal{F},n) \geq 3n-2$. We prove the opposite inequality by induction. Let $\mathcal{H}$ have $\bigcup\mathcal{H} = [n]$ with $n \geq 3$. If $\deg(v) = 1$ for every $v \in [2, n-1]$ then $i(\mathcal{H}) = \deg(1) + \deg(n) + n - 2 \leq 3n - 2$. If $\deg(v) \geq 3$ for some $v \in [2, n-1]$, we split $\mathcal{H}$ into $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ takes the edges of $\mathcal{H}$ lying to the left of v, $\mathcal{H}_2$ takes those lying to the right, and if $\{v\} \in \mathcal{H}$ then $\{v\} \in \mathcal{H}_1$; no edge lies on both sides of v because $\mathcal{H} \not\supset \mathcal{F}$. We have $v(\mathcal{H}_1) + v(\mathcal{H}_2) \leq n+1$. If $v(\mathcal{H}_1) + v(\mathcal{H}_2) \leq n$, then by induction $i(\mathcal{H}) = i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq 3v(\mathcal{H}_1) - 2 + 3v(\mathcal{H}_2) - 2 \leq 3n - 4$. If $v(\mathcal{H}_1) + v(\mathcal{H}_2) = n+1$, we note that $i(\mathcal{H}_2) \leq 3v(\mathcal{H}_2) - 3$ because now $v = \min\bigcup\mathcal{H}_2$ and $\{v\} \notin \mathcal{H}_2$. Again by induction $i(\mathcal{H}) = i(\mathcal{H}_1) + i(\mathcal{H}_2) \leq 3n - 2$. The last case is if $\deg(v) = 2$ for some $v \in [2, n-1]$. Let H_1 and H_2 be the edges containing v. If no edge jumps over v we split $\mathcal{H}$ and proceed as before. Else we have, say, $\min H_1 < v < \max H_1$. We delete v from $\mathcal{H}$. Since $H_1\backslash\{v\} \notin \mathcal{H}$, the only duplication that may arise is when $v = \min H_2$ (case $v = \max H_2$ is similar) and $H_3 = H_2\backslash\{v\} \in \mathcal{H}$. We cancel this duplication by deleting from H_3 its last vertex. No new duplication then arises ($\mathcal{H} \not\supset \mathcal{F}$) and we have by induction that $i(\mathcal{H}) \leq 3 + 3(n-1) - 2 = 3n - 2$.

21. $H_e(\mathcal{F},n) \leq 2n-1$ follows from the fact that, in $\mathcal{H}'$, $|H| \leq 2$ for every edge and $|H| = 2$ implies $1 \in H$. The bound is attained by $\mathcal{H} = (\{1\}, \{i\}, \{1,i\} : i \in [2,n])$. This hypergraph shows also that $H_i(\mathcal{F},n) \geq 3n-2$. To prove the opposite inequality, consider $\deg(1)$ in a general $\mathcal{H}$. If $\deg(1) = 1$, delete the edge containing 1. The rest does not contain $\mathcal{F}_8$ and thus $i(\mathcal{H}) \leq n + 2(n-1) - 1 = 3n - 3$. If $\deg(1) = 2$, delete an edge H such that $|H| \leq n-1$ and $1 \in H$. The rest does not contain $\mathcal{F}_{24}$ ($\mathcal{F}_8$ would do here but not in the next argument 22), so $i(\mathcal{H}) \leq n - 1 + 2n - 1 = 3n - 2$. If $\deg(1) \geq 3$, we delete 1 from $\mathcal{H}$. In the resulting hypergraph $\mathcal{H}_0$ only singletons may be duplicated and every component $\mathcal{H}_1$ of $\mathcal{H}_0$ satisfies $i(\mathcal{H}_1) \leq 2v(\mathcal{H}_1)$ since the only intersection in $\mathcal{H}_1$ is the common last vertex v (and $\{v\}$ may be duplicated). Thus $i(\mathcal{H}_0) \leq 2(n-1)$. $(H\backslash\{1\} : 1 \in H \in \mathcal{H}, H \neq \{1\})$ is a simple and $\mathcal{F}_{24}$-free, even $\mathcal{F}_8$-free, hypergraph. Hence $\deg(1) \leq 1 + H_e(\mathcal{F}_{24}, n-1) = n$. In total, $i(\mathcal{H}) \leq n + 2(n-1) = 3n - 2$.

22. The arguments are very similar to those in 21.

23. $\mathcal{H}'$ has no edge H with $|H| \geq 3$ and no two-element edge skipping one or more vertices. Again $H_e(\mathcal{F},n) \leq 2n-1$, attained by the same hypergraph as in 20. This hypergraph shows also that $H_i(\mathcal{F},n) \geq 3n-2$. We prove the opposite inequality by induction on $v(\mathcal{H}) = n$. It is easy to check that $\deg(1) \geq 3$ implies $\mathcal{H} \supset \mathcal{F}$. Let $\deg(1) = 2$. The first case is when $|H| \neq 2$ for both edges containing 1. Deletion of 1 from $\mathcal{H}$ gives then a simple hypergraph

and we have $i(\mathcal{H}) \le 2 + 3(n-1) - 2 = 3n - 3$. If $|H| = 2$ for exactly one of them, we set $H_1 = H$, and if both have two elements, we set H_1 to be the longer one. Deletion of H_1 from $\mathcal{H}$ and 1 from the rest gives a simple hypergraph and $i(\mathcal{H}) \le 2 + 1 + 3(n-1) - 2 = 3n - 2$. Let now $\deg(1) = 1$ and $1 \in H \in \mathcal{H}$. If $|H| \le 3$, we delete H and use induction. Let $|H| \ge 4$. If $H_1 = H\backslash\{1\}$ is not an edge, we delete 1 from $\mathcal{H}$ and use induction. If $H_1 \in \mathcal{H}$, let $u = \min H_1$. Clearly, $H_1\backslash\{u\}$ is not an edge (else $\mathcal{H} \supset \mathcal{F}$). We delete 1 from $\mathcal{H}$ and u from H_1 and use induction.

24. As in 6, $\mathcal{H}'$ has no nonsingleton edge and thus $H_e(\mathcal{F}, n) = n$. As for weights, notice that every component $\mathcal{H}_1$ of $\mathcal{H}$ either consists of at most two edges or the only intersection in $\mathcal{H}_1$ is one vertex common to all edges. Both cases give bound $i(\mathcal{H}_1) \le 2v(\mathcal{H}_1) - 1$ and thus $H_i(\mathcal{F}, n) \le 2n - 1$, attained by $\mathcal{H} = (\{i, n\}, \{n\} : i \in [n-1])$.

25. We remark that in 25–28 $H_e(\mathcal{F}, 1) = H_i(\mathcal{F}, 1) = 1$. $\mathcal{H}'$ has no edge H with $|H| \ge 4$, every three-element edge must contain 1 and 2, and two-element edges must start in 1 or in 2. Thus, for $n > 1$, $H_e(\mathcal{F}, n) \le n+(n-1)+2(n-2)$, attained by the hypergraph $\mathcal{H}^* = (\{1\}, \{i\}, \{1, i\}, \{2, j\}, \{1, 2, j\} : i \in [2, n], j \in [3, n])$. $\mathcal{H}^*$ shows that, for $n > 1$, $H_i(\mathcal{F}, n) \ge 8n - 12$. To prove the opposite inequality, we consider a general $\mathcal{H}$ with $v(\mathcal{H}) \ge 3$. If $|H \cap [3, n]| \le 1$ for every $H \in \mathcal{H}$, $i(\mathcal{H}) \le i(H^*) = 8n-12$. Let $|H \cap [3, n]| \ge 2$ for an edge H. If $\deg(1), \deg(2) \ge 3$ then $\mathcal{H} \supset \mathcal{F}$. So, say, $\deg(2) \le 2$ (case $\deg(1) \le 2$ is similar). We delete from $\mathcal{H}$ the edges containing 2 and observe that the rest avoids $\mathcal{F}_9$. Hence $i(\mathcal{H}) \le n+n-1+3(n-1)-2 = 5n-6 \le 8n-12$ $(n \ge 2)$.

26. $|H| \le 3$ for every edge of $\mathcal{H}'$, allowed three-element edges are $\{1, b, b+1\}$ $(n-2$ edges) and allowed two-element edges are $\{1, b\}$ and $\{b, b+1\}$ $(2n-3$ edges). Thus $H_e(\mathcal{F}, n) \le 4n - 5$ $(n > 1)$ and it is clear which hypergraph attains this value. We show that the same hypergraph attains also the maximum weight $8n-12$. If $\deg(1) \le 2$, we delete from $\mathcal{H}$ the edges containing 1 and conclude, since the rest avoids $\mathcal{F}_{10}$, that $i(\mathcal{H}) \le n+n-1+3(n-1)-2 = 5n-6 \le 8n-12$ $(n \ge 2)$. Let $\deg(1) \ge 3$. We delete 1 from $\mathcal{H}$. Consider two edges H_1 and H_2 of the resulting $\mathcal{H}_1$. $H_1 = H_2$ implies $|H_1| \le 2$ (else $\mathcal{H} \supset \mathcal{F}$) and no edge of $\mathcal{H}_1$ has higher multiplicity than 2. If $H_1 \ne H_2$ and neither H_i is a singleton, then $H_1 \le H_2$ or $H_2 \le H_1$ (else $\mathcal{H} \supset \mathcal{F}$). Thus $i(\mathcal{H}) = \deg(1) + i(\mathcal{H}_1) \le (1+n-2+n-1) + 2(n-1+2(n-2)) = 8n-12$.

27. Similar to 25. Only the interval $[3, n]$ is replaced by $[2, n-1]$.

28. In $\mathcal{H}'$ no edge has more than three elements, three-element edges must consist of consecutive vertices, and two-element edges must be of the

form $\{a, a+1\}$ and $\{a, a+2\}$. Again $H_e(\mathcal{F}, n) \leq n+2(n-2)+n-1 = 4n-5$, which is attained if we take all decribed edges (and singletons). To prove $H_i(\mathcal{F}, n) \leq 8n-12$, which is attained by the same hypergraph, we show that other edges can be eliminated using induction on n. If an $H \in \mathcal{H}$ exists with $|H| \geq 4$, let $u, v \in H$ be two distinct vertices, none of them the end of H. If $\deg(u) = \deg(v) = 1$, u or v may be deleted from $\mathcal{H}$ (one of these deletions does not create duplication) and induction applies. If $\deg(u) \geq 2$ and $\deg(v) = 1$, we can delete v from $\mathcal{H}$ unless $\deg(u) = 2$ and $H \backslash \{v\} \in \mathcal{H}$. But then we can delete u from $\mathcal{H}$. Similarly if $\deg(u) = 1$ and $\deg(v) \geq 2$. If $\deg(u) \geq 2$ and $\deg(v) \geq 2$, both inequalities must be equalities and u, v lie in the same two edges. Either of u and v can be deleted and induction applies. Thus we can assume that $|H| \leq 3$ for every $H \in \mathcal{H}$. If $H = \{a, b, c\}_< \in \mathcal{H}$ and $c > b+1$ (case $a < b-1$ is similar), then $\deg(b) \leq 2$ and $\deg(b) = 2$ implies that b and $b+1$ lie in the same edge. It is easy to see that b can be deleted. We may assume that every edge H with $|H| = 3$ is of the form $H = \{a, a+1, a+2\}$. Finally, if $\{a, b\} \in \mathcal{H}$ and $a < b-2$, it is clear that $b-2$ and $b-1$ have degree 1 and lie in the same edge. Either one of them can be deleted. We can assume that $\{a, b\}_< \in \mathcal{H}$ implies $b \leq a+2$.

29. We delete the last vertex from every $H \in \mathcal{H}$, $|H| \geq 2$. The resulting sets are mutually disjoint and lie in $[n-1]$. Thus $H_e(\mathcal{F}, n) \leq n+(n-1)$ and $H_i(\mathcal{F}, n) \leq n+(n-1)+(n-1)$, attained by $\mathcal{H} = (\{i\}, \{n\}, \{i, n\} : i \in [n-1])$.

30. If $H \in \mathcal{H}$ with $|H| \geq 3$, we replace H by the two-element set of the first two vertices of H. Thus, for bounding $H_e(\mathcal{F}, n)$ from above, we may assume that $|H| \leq 2$ for every edge. It is clear that two-element edges form a triangle-free graph on at most n vertices. By a special case of Turán's theorem (see [16, Problem 10.30]), it has at most $\lfloor \frac{n^2}{4} \rfloor$ edges. The value of $H_e(\mathcal{F}, n)$ is attained by $(\{i\}, \{j, k\} : i \in [n], j \in [\lfloor n/2 \rfloor], k \in [\lfloor n/2 \rfloor + 1, n])$. We show that the maximum weight is attained by the same hypergraph with the exception $n = 3$ when $H_i(\mathcal{F}, 3) = 8$ (and not 7). Large edges $H = \{a_1, a_2, \ldots, a_t\}_<$ with $t \geq 4$ are eliminated by the replacement $H \to \{a_1, a_{t-1}\}, \{a_2, a_{t-1}\}, \ldots, \{a_{t-2}, a_{t-1}\}$. If $t = 3$ and $a_3 < n$, we eliminate H by $H \to \{a_2, a_3\}, \{a_2, n\}$. Similarly if $1 < a_1$. Let k be the number of the troublesome edges $\{1, a, n\}$. No two-element edge is incident with any of the as and they form a triangle-free graph on at most $n-k$ vertices. By Turán's theorem, $H_i(\mathcal{F}, n) \leq n+2\lfloor \frac{(n-k)^2}{4} \rfloor + 3k$ where the bound is attained. For $n \geq 4$ this is maximized for $k = 0$ (and $k = 2$ for $n = 4$; $H_i(\mathcal{F}, 4) = 12$ is attained by $(\{i\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\})$ and $(\{i\}, \{1,4\}, \{1,2,4\}, \{1,3,4\})$ where

$i \in [4]$) and for $n = 3$ by $k = 1$. Indeed, $(\{i\}, \{1, 3\}, \{1, 2, 3\})$ is better than $(\{i\}, \{1, 2\}, \{1, 3\})$ where $i \in [3]$.

31. No two distinct edges of $\mathcal{H}$ intersect in two or more vertices. Hence every $H \in \mathcal{H}$ with $|H| \geq 3$ may be replaced by its two-element subsets; this works for both size and weight. Therefore $H_e(\mathcal{F}, n) = \binom{n}{1} + \binom{n}{2}$ and $H_i(\mathcal{F}, n) = \binom{n}{1} + 2\binom{n}{2}$, as in 11.

32. As for $H_e(\mathcal{F}, n)$, we eliminate from $\mathcal{H}$ all edges with $|H| \geq 4$ by replacing H by the two-set of its first two elements. So $|H| \leq 3$ for every $H \in \mathcal{H}$. Let $a+1$ be the first vertex that is the last point of a two-element edge or the middle point of a three-element edge. $\mathcal{H}$ consists of singletons, of a bipartite graph with parts $[a]$ and $[a+2, n]$, and of edges of the form $\{b, a+1\}$, $\{a+1, c\}$, and $\{b, a+1, c\}$ where $b \in [a], c \in [a+2, n]$; other edges would create $\mathcal{F}$ or they would contradict the minimality of $a+1$. We see that $H_e(\mathcal{F}, n) \leq n + 2a(n-1-a) + (n-1)$, which is attained and maximized by $a = \lfloor (n-1)/2 \rfloor$. The same hypergraph attains the maximum weight because large edges $H = \{a_1, a_2, \dots, a_t\}_<$ can be eliminated by the replacement $H \to \{a_1, a_2\}, \{a_1, a_2, a_3\}$ if $t = 4, 5$ and by $H \to \{a_1, a_2\}, \{a_1, a_3\}, \dots, \{a_1, a_{t-2}\}$ if $t \geq 6$. Counting the weight instead of size, we obtain the second formula.

33. First, we bound the number of two-element edges in $\mathcal{H}$. Let $\mathcal{H} = \mathcal{G}$ be a ($\mathcal{F}$-free) graph with the vertex set $[n]$. The sets $X_i = \{x \in [i+1, n] : \{i, x\} \in \mathcal{G}\}$, $i \in [n-1]$, are subsets of $[2, n]$ and ($\mathcal{G} \not\supset \mathcal{F}$) $\max X_i \leq \min X_{i+1}$. Thus $e(\mathcal{G}) = \sum_{i=1}^{n-1} |X_i| \leq n - 1 + n - 2 = 2n - 3$. Hence $\mathcal{H}$ has at most $2n-3$ two-element edges. We delete from every $H \in \mathcal{H}$, $|H| \geq 3$, its first and last vertex. If two of the resulting sets intersect, we have two distinct edges $H_1, H_2 \in \mathcal{H}$ and five not necessarily distinct vertices $u_1, u_2 < v < w_1, w_2$ such that $\{u_1, v, w_1\} \subset H_1$ and $\{u_2, v, w_2\} \subset H_2$. Moreover, we can assume that $u_1 \neq u_2$ or $w_1 \neq w_2$ because $H_1 \neq H_2$. But this implies $\mathcal{H} \supset \mathcal{F}$. Thus the resulting sets, subsets of $[2, n-1]$, are mutually disjoint. We conclude that $e(\mathcal{H}) \leq n + 2n - 3 + n - 2 = 4n - 5$ and $i(\mathcal{H}) \leq n + 2(2n-3) + 3(n-2) = 8n - 12$. These bounds are attained by $\mathcal{H} = (\{1\}, \{n\}, \{1, n\}, \{1, i\}, \{i, n\}, \{1, i, n\} : i \in [2, n-1])$.

34. It is easily checked that the argument bounding the number of edges with more than 2 elements works here as well. We prove by induction on n that the number of two-element edges is again at most $2n-3$. Let $\mathcal{H} = \mathcal{G}$ be a ($\mathcal{F}$-free) graph with the vertex set $[n]$. If $\deg(1) = 1$, we have by induction that $e(\mathcal{G}) \leq 1 + 2n - 5 = 2n - 4$. For $\deg(1) > 1$, if $\{1, n\} \in \mathcal{G}$ let m be the second largest neighbour of 1 and if $\{1, n\} \notin \mathcal{G}$ let m be the largest neighbour

of 1. Clearly, $m < n$ and every edge of $\mathcal{G}$, except possibly only $\{1, n\}$, lies either in $[m]$ or in $[m+1, n]$. By induction, $e(\mathcal{G}) \leq 1 + 2m - 3 + 2(n - m + 1) - 3 = 2n - 3$. Thus again $e(\mathcal{H}) \leq 4n - 5$ and $i(\mathcal{H}) \leq 8n - 12$. The extremal hypergraph is, for example, $\mathcal{H} = (\{1\}, \{n\}, \{1, n\}, \{i\}, \{1, i\}, \{i, i+1\}, \{1, i, i+1\} : i \in [2, n-1])$.

35. We have $|H| \leq 2$ for every $H \in \mathcal{H}'$. Thus $H_e(\mathcal{F}, n) = \binom{n}{1} + \binom{n}{2}$. As for the weight, if $H \in \mathcal{H}$ with $|H| \geq 3$, we replace H by the two-element sets $\{a, b\}$ where $a = \min H$ and $a < b \in H$. Thus we may suppose that $|H| \leq 2$ for every $H \in \mathcal{H}$ and we conclude that $H_i(\mathcal{F}, n) = \binom{n}{1} + 2\binom{n}{2}$.

36. Same argument as in 35.

37. In $\mathcal{H}'$, $|H| \leq 3$ for every edge and $|H| = 3$ implies $1 \in H$. Thus $H_e(\mathcal{F}, n) = \binom{n}{1} + \binom{n}{2} + \binom{n-1}{2}$. As for the weight, we get rid of all H with $|H| \geq 4$ by the same replacements as in 35. If H with $|H| = 3$ is present, again $1 \in H$. Thus $H_i(\mathcal{F}, n) = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n-1}{2}$.

38. Same argument as in 37. Allowed three-element edges are now $H = \{a, a+1, b\}_<$ and we have again $1 + 2 + \cdots + (n-2) = \binom{n-1}{2}$ of these.

39. Clearly, $H_e(\mathcal{F}, n) = \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$ and $H_i(\mathcal{F}, n) = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3}$. $\square$

We do not have 110 distinct extremal functions and not even close to 78. Hypergraphs $\mathcal{F}$ with $1 \leq i(\mathcal{F}) \leq 4$ have 28 distinct extremal functions $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ (included the "undefined function"). Of these 25 differ for infinitely many arguments. The formulas for $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$ hold for $n \geq v(\mathcal{F})$ with the exception of $\mathcal{F}_5, \mathcal{F}_{12}$, $\mathcal{F}_{18}$, and $\mathcal{F}_{30}$ but only the initial values of $H_i(\mathcal{F}_{18}, n)$ caused some troubles. We conclude this section by a nice geometric derivation of the formula for $H_e(\mathcal{F}_{34}, n)$ (crossing pattern) due to Attila Pór. Put the vertices $1, 2, \ldots, n$ in this order clockwise on a circle in the plane and consider the convex hulls $C_i = \mathrm{conv}(H_i)$, $H_i \in \mathcal{H}$. The condition $\mathcal{H} \not\supset \mathcal{F}_{34}$ is equivalent to the condition that the relative interiors of C_i do not intersect. So it is clear that we may have at most $n-2$ edges H with $|H| \geq 3$, maximized by the triangulations, and at most $3n - 6 - (n-3) = 2n - 3$ two-element edges because these form a planar graph with a big outer face. Thus $H_e(\mathcal{F}_{34}, n) \leq n - 2 + 2n - 3 + n = 4n - 5$.

4 Enumerative intermezzo

Besides the extremal problems for $\mathcal{F}$-free hypergraphs there is also the enumerative problem to count them. Let

$$h_n^{(v)}(\mathcal{F}) = |\{\mathcal{H}:\ \mathcal{H} \text{ is simple } \&\ \mathcal{H} \not\succ \mathcal{F}\ \&\ \bigcup\mathcal{H} = [n]\}|$$

be the number of simple nonisomorphic $\mathcal{F}$-free hypergraphs $\mathcal{H}$ with $v(\mathcal{H}) = n$. Let $h_n^{(i,s)}(\mathcal{F})$ and $h_n^{(i)}(\mathcal{F})$ be the analogous counting functions with $i(\mathcal{H}) = n$ in the place of $v(\mathcal{H}) = n$ and with the simplicity of $\mathcal{H}$ dropped in $h_n^{(i)}(\mathcal{F})$. For example, for $\mathcal{F}_2 = (\{1\}_1, \{1\}_2)$ all three counting functions equal to the nth Bell number B_n that counts partitions of $[n]$.

The enumerative problem to determine or to bound, for $\mathcal{F}$ fixed and $n \to \infty$, the three counting functions is already for $i(\mathcal{F}) \le 4$ much more difficult than the extremal problem. In Klazar [15] we found the ordinary generating functions $F_1(x)$, $F_2(x)$, and $F_3(x)$ of $h_n^{(v)}(\mathcal{F})$, $h_n^{(i,s)}(\mathcal{F})$, and $h_n^{(i)}(\mathcal{F})$, respectively, for the crossing pattern $\mathcal{F}_{34} = (\{1,3\}, \{2,4\})$. F_1, F_2, and F_3 are algebraic over $\mathbf{Z}(x)$ of degrees 3, 4, and 4, respectively, and their coefficients grow roughly like $(63.97055\ldots)^n$, $(5.79950\ldots)^n$, and $(6.06688\ldots)^n$ where the bases of the exponentials are algebraic numbers of degrees 4, 15, and 23, respectively. We did not succeed in enumerating $\mathcal{F}_{33}$-free hypergraphs where $\mathcal{F}_{33} = (\{1,4\}, \{2,3\})$ and we believe it is a problem that deserves interest.

In this article we drop the condition of $\mathcal{F}$-freeness and we determine the total numbers $h_n^{(v)}$, $h_n^{(i,s)}$, and $h_n^{(i)}$, that is, the number of simple $\mathcal{H}$ with $v(\mathcal{H}) = n$, the number of simple $\mathcal{H}$ with $i(\mathcal{H}) = n$, and the number of all $\mathcal{H}$ with $i(\mathcal{H}) = n$. The numbers $h_n^{(v)}$ have been already investigated before, in the slightly different terminology of set covers, but the remaining two problems seem new. We review the known formulas for $h_n^{(v)}$, derive a new recurrence for them, and then we proceed to $h_n^{(i,s)}$ and $h_n^{(i)}$.

Write s_n for the number of simple set systems on $[n]$, which are (possibly empty) sets of nonempty subsets of $[n]$. Clearly, $s_n = 2^{2^n-1}$ and

$$s_n = 2^{2^n-1} = \sum_{j=0}^{n} \binom{n}{j} h_j^{(v)} \tag{1}$$

because set systems 1-1 correspond to simple $\mathcal{H}$ with $\bigcup\mathcal{H} \subset [n]$. Hence we can easily calculate $h_n^{(v)}$ starting by $h_0^{(v)} = 1$ and continuing by the recurrence

$$h_n^{(v)} = 2^{2^n-1} - \sum_{j=0}^{n-1} \binom{n}{j} h_j^{(v)} \tag{2}$$

given in Hearne and Wagner [13]. Using exponential generating functions $F(x)=\sum_{n\ge 0}s_n x^n/n!$ and $H(x)=\sum_{n\ge 0}h_n^{(v)}x^n/n!$ we invert relation (1) by noting that it amounts to $F(x)=\mathrm{e}^x H(x)$. Thus $H(x)=\mathrm{e}^{-x}F(x)$ and we have the explicit formula

$$h_n^{(v)}=\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}2^{2^j-1} \tag{3}$$

that can be found in Comtet [4, p. 165] and that was derived independently by Macula [17].

We show that for $n\ge 0$ also

$$h_{n+1}^{(v)}=2\sum_{0\le k,l\le n}\frac{h_k^{(v)}h_l^{(v)}n!}{(k+l-n)!(n-k)!(n-l)!}-h_n^{(v)}. \tag{4}$$

(The actual summation range is $\max(k,l)\le n\le k+l$.) We take a simple $\mathcal{H}$, $\bigcup\mathcal{H}=[n+1]$, and decompose it into $\mathcal{H}_1$ and $\mathcal{H}_2$ where $\mathcal{H}_1$ consists of the sets $H\backslash\{1\}$ such that $1\in H\in\mathcal{H}$ (we omit the $\emptyset$ if $\{1\}\in\mathcal{H}$) and $\mathcal{H}_2$ consists of the remaining edges of $\mathcal{H}$. We relabel the vertices so that $\bigcup\mathcal{H}_1=[k]$ and $\bigcup\mathcal{H}_2=[l]$. It is clear that $\mathcal{H}_1$ and $\mathcal{H}_2$ are simple and that $k,l\le n$. To invert the decomposition, we first select two simple $\mathcal{H}_1$ and $\mathcal{H}_2$ of order k and l, which can be done in $h_k^{(v)}h_l^{(v)}$ ways. We unite their vertex sets so that n vertices arise. This can be done in $\binom{n}{k+l-n,n-k,n-l}$ ways by choosing, from n vertices, $k+l-n, n-k$, and $n-l$ vertices lying in $\bigcup\mathcal{H}_1\cap\bigcup\mathcal{H}_2$, only in $\bigcup\mathcal{H}_2$, and only in $\bigcup\mathcal{H}_1$, respectively. We append to every edge in $\mathcal{H}_1$ a new least vertex $1'$ and obtain a simple $\mathcal{H}$ with $n+1$ vertices. Finally, the possible addition of $\{1'\}$ to $\mathcal{H}$ (we always loose edge $\{1\}$ when decomposing) gives two further options, except for $\mathcal{H}_1=\emptyset$ when $\{1'\}$ must be always added. This explains the factor 2 and the subtraction of $h_n^{(v)}$ in (4).

By means of any of (2), (3), and (4) one finds that

$$(h_n^{(v)})_{n\ge 1}=(1,5,109,32297,2147321017,9223372023970362989,\ldots).$$

This quite quickly growing sequence is entry A003465 of Sloane [22].

We turn to counting hypergraphs, both simple and all, by weight. Inspection of the long table in Section 3 reveals that $(h_n^{(i,s)})_{n\ge 1}=(1,2,7,28,\ldots)$ and $(h_n^{(i)})_{n\ge 1}=(1,3,10,41,\ldots)$. What comes next?

Recall that a partition $\lambda=1^{a_1}2^{a_2}\ldots l^{a_l}$ of $n\in\mathbf{N}$, where $a_i\ge 0$ are integers and usually $a_l>0$, is the decomposition $n=1+1+\cdots+1+2+$

$\cdots + 2 + \cdots + l + \cdots + l$ with the part i appearing a_i times (parts i with $a_i = 0$ may be ommited). Thus $\sum ia_i = n$. We write briefly $\lambda \vdash n$. If $\mathcal{H}$ has weight n and a_i edges of cardinality i, the maximum edge cardinality being l, then $\lambda = 1^{a_1}2^{a_2}\ldots l^{a_l} \vdash n$ and we say that $\mathcal{H}$ has *edge type* λ. We derive formulas for numbers of hypergraphs with a given edge type.

Theorem 4.1 *Let $\lambda = 1^{a_1}2^{a_2}\ldots l^{a_l} \vdash n$ where $a_l > 0$. The number of simple hypergraphs with weight n and edge type λ is*

$$\sum_{j=l}^{n} \binom{\binom{j}{1}}{a_1}\binom{\binom{j}{2}}{a_2}\cdots\binom{\binom{j}{l}}{a_l}\sum_{m=j}^{n}(-1)^{m-j}\binom{m}{j}$$

and the number of all hypergraphs with weight n and edge type λ is

$$\sum_{j=l}^{n} \binom{\binom{j}{1}+a_1-1}{a_1}\binom{\binom{j}{2}+a_2-1}{a_2}\cdots\binom{\binom{j}{l}+a_l-1}{a_l}\sum_{m=j}^{n}(-1)^{m-j}\binom{m}{j}.$$

Proof. Consider the polynomials

$$W_n = W_n(x_1, x_2, \ldots, x_n) = \sum_{\mathcal{H}}\prod_{i=1}^{n} x_i^{e(i,\mathcal{H})}$$

where we sum over all simple $\mathcal{H}$ with $\bigcup\mathcal{H} = [n]$ and $e(i, \mathcal{H})$ is the number of i-element edges in $\mathcal{H}$. Refining (1) we have

$$\prod_{i=1}^{n}(1+x_i)^{\binom{n}{i}} = \sum_{j=0}^{n}\binom{n}{j}W_j$$

where on the left is a polynomial (analogous to W_n) counting simple set systems on $[n]$ according to the edge cardinalities. In terms of exponential generating functions,

$$\sum_{n\geq 0}\prod_{i=1}^{n}(1+x_i)^{\binom{n}{i}}\cdot\frac{y^n}{n!} = \mathrm{e}^y\cdot\sum_{n\geq 0}\frac{W_n y^n}{n!}. \tag{5}$$

Thus, as in (3),

$$W_n(x_1, \ldots, x_n) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\prod_{i=1}^{j}(1+x_i)^{\binom{j}{i}}.$$

The number of simple $\mathcal{H}$ with $i(\mathcal{H})=n$ and edge type $\lambda = 1^{a_1}2^{a_2}\dots l^{a_l} \vdash n$ is the coefficient at $x_1^{a_1}\dots x_l^{a_l}$ in $W_l + W_{l+1} + \cdots + W_n$ which is

$$\sum_{m=l}^{n}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\prod_{i=1}^{l}\binom{\binom{j}{i}}{a_i} = \sum_{j=l}^{n}\prod_{i=1}^{l}\binom{\binom{j}{i}}{a_i}\sum_{m=j}^{n}(-1)^{m-j}\binom{m}{j}.$$

Derivation of the second formula is almost identical, only W_n becomes a power series and $1+x_i$ is replaced by $(1-x_i)^{-1}$ because now any i-element edge may come in arbitrary many copies. $\square$

We give for illustration the distribution of hypergraphs with weight $n=6$ according to their edge types (the first entry is the number of simple $\mathcal{H}$ and the second, given only if different, is the number of all $\mathcal{H}$):

6^1	$1^1 5^1$	$2^1 4^1$	$1^2 4^1$	3^2	$1^1 2^1 3^1$	$1^3 3^1$	2^3
1	11	41	41, 50	31, 32	239	63, 120	62, 75

$1^2 2^2$	$1^4 2^1$	1^6
198, 264	41, 160	1, 32

Collecting the numbers over all edge types we obtain formulas for $h_n^{(i,s)}$ and $h_n^{(i)}$.

Corollary 4.2 *The numbers of hypergraphs with weight n, simple and all, are ($\lambda = 1^{a_1}2^{a_2}\dots l^{a_l}$ with $a_l > 0$)*

$$h_n^{(i,s)} = \sum_{\lambda\vdash n}\sum_{j=l}^{n}\prod_{i=1}^{l}\binom{\binom{j}{i}}{a_i}\sum_{m=j}^{n}(-1)^{m-j}\binom{m}{j} \tag{6}$$

$$h_n^{(i)} = \sum_{\lambda\vdash n}\sum_{j=l}^{n}\prod_{i=1}^{l}\binom{\binom{j}{i}+a_i-1}{a_i}\sum_{m=j}^{n}(-1)^{m-j}\binom{m}{j}. \tag{7}$$

Using (6), (7), and computer algebra system MAPLE we have found that

$$(h_n^{(i,s)})_{n\ge 1} = (1, 2, 7, 28, 134, 729, 4408, 29256, 210710, 1633107, \dots)$$
$$(h_n^{(i)})_{n\ge 1} = (1, 3, 10, 41, 192, 1025, 6087, 39754, 282241, 2159916, \dots).$$

As of May 2001, these sequences were absent in [22].

From the point of view of complexity theory formulas (6) and (7) are inferior compared to those for $h_n^{(v)}$. While any of (2), (3), and (4) needs only polynomially many (in n) operations to turn the input n into the output $h_n^{(v)}$, (6) and (7) require roughly $n^c p(n)$ operations where $p(n) = |\{\lambda : \lambda \vdash n\}|$. Numbers $p(n)$ grow superpolynomially because by the famous Hardy–Ramanujan–Uspensky asymptotics $p(n) \sim (n \cdot 4\sqrt{3})^{-1} \cdot \exp(\pi\sqrt{2n/3})$ if $n \to \infty$. (An elementary proof was given by Erdős [5] who proved that $p(n) \sim cn^{-1} \cdot \exp(\pi\sqrt{2n/3})$ and by Newman [18] who showed that $c = (4\sqrt{3})^{-1}$. A simpler complex-analytical proof was given later by Newman [19]. See also Newman's book [20, chapter 2].) On the other hand, $p(n)$ is subexponential and thus formulas (6) and (7) are nontrivial in the sense that the numbers of operations which they require are substantially smaller than $h_n^{(i,s)}$ and $h_n^{(i)}$ themselves (obviously $h_n^{(i,s)}, h_n^{(i)} > 2^n$ for $n > 3$). A polynomial algorithm generating $h_n^{(i,s)}$ and $h_n^{(i)}$ can be given by means of the recurrence approach that we used to derive (4).

For any rational polynomial $P(m) \in \mathbf{Q}[m]$ we have $\sum_{m=0}^{\infty} P(m)/m! = \mathrm{e} \cdot q$ where $\mathrm{e} = 2.71828\ldots$ is Euler number and $q \in \mathbf{Q}$. This follows simply by expressing $P(m)$ as a $\mathbf{Q}$-linear combination in the basis $\{1, m, m(m-1), m(m-1)(m-2), \ldots\}$. Dobiǹski's formula ([16, Problems 1.9a and 1.13] and [4, p. 210]) belongs to this family of identities and has $P(m) = m^n$ and $q = B_n$ where B_n is the nth Bell number. Setting in (5), respectively in the analogous equation for all hypergraphs, $y = 1$ and $x_i = x^i$ and comparing coefficients at x^n we obtain two identities of this type.

Corollary 4.3 *For every $n \in \mathbf{N}$ we have the identities ($\lambda = 1^{a_1}2^{a_2}\ldots m^{a_m}$ with $a_m = 0$ allowed)*

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{1}{m!} \cdot \sum_{\lambda \vdash n} \prod_{i=1}^{m} \binom{\binom{m}{i}}{a_i} &= \mathrm{e} \cdot \sum_{i(\mathcal{H})=n}^{*} \frac{1}{v(\mathcal{H})!} \\ \sum_{m=0}^{\infty} \frac{1}{m!} \cdot \sum_{\lambda \vdash n} \prod_{i=1}^{m} \binom{\binom{m}{i} + a_i - 1}{a_i} &= \mathrm{e} \cdot \sum_{i(\mathcal{H})=n} \frac{1}{v(\mathcal{H})!} \end{aligned}$$

where $\mathrm{e} = 2.71828\ldots$ and the star indicates that the sum is over simple $\mathcal{H}$ only.

For $n = 1, 2, 3$, and 4 the factors at e in the first identity are $1, 1, \frac{11}{6}$, and $\frac{25}{8}$ and in the second identity $1, 2, \frac{23}{6}$, and $\frac{89}{8}$.

5 Two applications of Davenport–Schinzel sequences

We begin with reminding a bound from the theory of generalized Davenport–Schinzel sequences. A sequence $v = a_1 a_2 \dots a_l \in [n]^*$ over the alphabet $[n]$ is *k-sparse* if $a_i = a_j, i < j$, implies $j - i \geq k$. The length of v is denoted $|v|$. If $u, v \in [n]^*$ are two sequences and v has a subsequence that differs from u only by an injective renaming of symbols, we say that v *contains* u. For example, $v = 2131425$ contains $u = 4334$ but v does not contain $u = 2323$. We write $u(k,l)$ to denote the sequence

$$u(k,l) = 12\dots k12\dots k\dots 12\dots k \in [k]^*$$

with l segments $12\dots k$. In Klazar [14] we proved that if $v \in [n]^*$ is k-sparse and does not contain $u(k,l)$, where $k \geq 2$ and $l \geq 3$, then for every $n \in \mathbf{N}$

$$|v| \leq n \cdot 2k2^{kl-4}(10k)^{2\alpha(n)^{kl-4}+8\alpha(n)^{kl-5}} \tag{8}$$

where $\alpha(n)$ is the inverse Ackermann function. (If $k = 1$ or $l \leq 2$, it is not difficult to prove that $|v| = O(n)$.)

Recall that $\alpha(n) = \min\{m : \ A(m) \geq n\}$ where $A(n) = F_n(n)$, the Ackermann function, is the diagonal function of the hierarchy of functions $F_i : \mathbf{N} \to \mathbf{N}$, $i \in \mathbf{N}$, starting with $F_1(n) = 2n$ and continuing by the rule $F_{i+1}(n) = F_i(F_i(\dots F_i(1)\dots))$ with n iterations of F_i. Thus $F_2(n) = 2^n$ and $F_3(n)$ is the tower function

$$F_3(n) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\} n.$$

We write $\beta(k,l,n)$ to denote the factor at n in (8). Thus

$$\beta(k,l,n) = 2k2^{kl-4}(10k)^{2\alpha(n)^{kl-4}+8\alpha(n)^{kl-5}}. \tag{9}$$

We utilize (8) in another approach to bounding $H_i(\mathcal{F},n)$ from above in terms od $H_e(\mathcal{F},n)$. Recall that $\mathcal{H}$ is a set partition if it has disjoint edges.

Theorem 5.1 *Suppose that $\mathcal{F}$ is a set partition with $p = v(\mathcal{F}), q = e(\mathcal{F}) > 1$ and $\mathcal{H}$ is a $\mathcal{F}$-free hypergraph with $v(\mathcal{H}) = n$, not necessarily simple. Then*

$$i(\mathcal{H}) < (q-1)n + \beta(q, 2p, e(\mathcal{H})) \cdot e(\mathcal{H})$$

where $\beta(k,l,n)$ is defined in (9).

Proof. Let $\bigcup \mathcal{H} = [n]$ and the edges of $\mathcal{H}$ be $H_1, H_2, \dots, H_e$ where $e = e(\mathcal{H})$. We set for $i \in [n]$

$$S_i = \{j \in [e] : \; i \in H_j\}$$

and consider the sequence $v = I_1 I_2 \dots I_n$ where I_i is an arbitrary permutation of S_i. Clearly, $v \in [e]^*$ and $|v| = i(\mathcal{H})$. The sequence v may not be q-sparse, because of the transitions $I_i I_{i+1}$, but it is easy to see that by deleting at most $q-1$ terms from the beginning of every I_i, $i > 1$, one can obtain a q-sparse subsequence w with length $|w| \geq |v| - (q-1)(n-1)$. It is also easy to see that if w (or v) contains $u(q, 2p)$ then $\mathcal{H}$ contains $\mathcal{F}$, which is forbidden. (Note that the subsequence aab in v forces the first a and the b to appear in two distinct segments I_i and thus it gives incidences of H_a and H_b with two distinct vertices.) Hence w does not contain $u(q, 2p)$ and we can apply (8):

$$i(\mathcal{H}) = |v| < (q-1)n + |w| \leq (q-1)n + \beta(q, 2p, e) \cdot e.$$

$\square$

For fixed numbers k, l the function $\beta(k, l, n)$ grows to infinity extremly slowly and for all practical purposes it is bounded. We give an example showing that in the last theorem some unbounded factor at $e(\mathcal{H})$ is necessary.

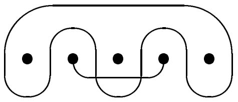
Hart and Sharir [12] constructed sequences $v \in [n]^*$ which are 2-sparse, do not contain sequence 12121, and have length $|v| \gg n\alpha(n)$. See Sharir and Agarwal [21] for more information. We take such a sequence v, consider the subsequence w of v consisting of the first and last appearances of symbols $i \in [n]$ in v, and decompose v into segments

$$v = I_1 I_2 \dots I_m$$

where every I_i ends by a term from w and contains no other term of w. Clearly, $n \leq m = |w| \leq 2n$ (we may assume that v uses every $i \in [n]$). Note that $|I_1| = |I_m| = 1$. If an I_j contains a symbol $a \in [n]$ twice, we have in I_j a subsequence aba, $b \neq a$, because v is 2-sparse. By the definition of segments, the first b appears in v before I_j and the last b after I_j or on its end and v is forced to have the forbidden $babab$ subsequence. Thus every I_j must be a permutation of a set $S_j \subset [n]$ and we can defined the hypergraph

$$\mathcal{H} = (H_i : \; i \in [n]) \quad \text{where} \quad H_i = \{j \in [m] : \; i \in S_j\}.$$

Clearly, $\bigcup \mathcal{H} \subset [m]$ and $n \le v(\mathcal{H}) \le 2n$, $e(\mathcal{H}) = n$, and $i(\mathcal{H}) = |v| \gg n\alpha(n)$. It is also clear that $\mathcal{H}$ is $\mathcal{F}_{40}$-free where $\mathcal{F}_{40}$ is the set partition

$$\mathcal{F}_{40} = (\{1,3,5\},\{2,4\}) =$$  .

For $\mathcal{F} = \mathcal{F}_{40}$ the factor at $e(\mathcal{H})$ in Theorem 5.1 must be $\gg \alpha(n)$.

Taking in Theorem 5.1 a simple $\mathcal{H}$ with the maximum weight, we obtain as a corollary for every set partition $\mathcal{F}$ ($p = v(\mathcal{F})$ and $q = e(\mathcal{F}) > 1$; case $q = 1$ is trivial) the inequality

$$H_i(\mathcal{F}, n) < (q-1)n + \beta(q, 2p, H_e(\mathcal{F}, n)) \cdot H_e(\mathcal{F}, n).$$

But here Theorem 2.4, when it applies, gives better bound.

In the second application of (8) we obtain an almost linear bound on $H_e(\mathcal{F}, n)$ in the case when $\mathcal{F}$ is a *star forest*. These are simple graphs $\mathcal{G}$ which have no two separated edges and such that $\deg(v) = 1$ whenever $v = \max H$, $H \in \mathcal{H}$. Thus every component of a star forest is a star and every centre of a star is smaller then every leaf. We begin with the graph case.

Theorem 5.2 *Let $\mathcal{F}$ be a star forest with $r > 1$ components and p vertices and $G_e(\mathcal{F}, n)$ be the maximum number of edges in a simple graph $\mathcal{G}$ such that $\mathcal{G} \not\supset \mathcal{F}$ and $v(\mathcal{G}) = n$. Then*

$$G_e(\mathcal{F}, n) < (r-1)n + n \cdot \beta(r, 2(p-r+1), n)$$

where $\beta(k, l, n)$ is the almost constant function defined in (9).

Proof. Let $\mathcal{G}$ attain $G_e(\mathcal{F}, n)$ and $\bigcup \mathcal{G} = [n]$. We consider the sequence

$$v = I_1 I_2 \dots I_n \in [n]^*$$

where I_j is any permutation of the set $\{i \in [n] : \ \{i, j\} \in \mathcal{G}, i < j\}$. As in the previous proof, we select an r-sparse subsequence w of v with length $|w| \ge |v| - (r-1)(n-1)$. It is not hard to see that if w (or v) contains the sequence $u(r, 2(p-r+1))$ then $\mathcal{G} \supset \mathcal{F}$. Thus w does not contain $u(r, 2(p-r+1))$ and we can apply (8):

$$G_e(\mathcal{F}, n) = e(\mathcal{G}) = |v| < (r-1)n + n \cdot \beta(r, 2(p-r+1), n).$$

$\square$

For $r = 1$ component we have $G_e(\mathcal{F}, n) \ll n$. We extend the bound of Theorem 5.2 from graphs to hypergraphs by means of a more generally aplicable technique in the next section. We conclude the present section by an example showing that in general $G_e(\mathcal{F}, n)$ is superlinear for star forests.

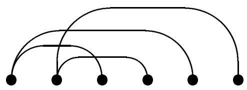
Let $v \in [n]^*$ be the same 12121-free sequence as in the previous example for $\mathcal{F}_{40}$ and let

$$v = I_1 I_2 \dots I_m$$

be the same decomposition into segments containing no repeated symbol, $n \le m \le 2n$. We rename the symbols in v so that if $i < j$ then the first appearance of j in v precedes that of i. (This does not affect the 12121-freeness.) We define the simple bipartite graph $\mathcal{G}$ with $\bigcup \mathcal{G} = [n+m]$ by

$$\{i, j\} \in \mathcal{G} \iff i \in [n] \ \&\ j \in [n+1, n+m] \ \&\ i \text{ appears in } I_{j-n}.$$

Then $e(\mathcal{G}) = |v| \gg n\alpha(n)$ and $2n \le v(\mathcal{G}) \le 3n$. We show that $\mathcal{G} \not\succ \mathcal{F}_{41}$ where $\mathcal{F}_{41}$ is the star forest

$$\mathcal{F}_{41} = (\{1,3\}, \{1,5\}, \{2,4\}, \{2,6\}) =$$  .

Suppose for the contrary that $\mathcal{F}_{41} \prec \mathcal{G}$ and $a_1 < a_2 < \dots < a_6$ are the vertices of a $\mathcal{F}_{41}$-copy in $\mathcal{G}$. By the definition of $\mathcal{G}$, $z = a_1 a_2 a_1 a_2$ is a subsequence of v, with terms appearing in $I_{a_3-n}, \dots, I_{a_6-n}$, respectively. But since $a_2 > a_1$, an a_2 must appear in v before z starts and v contains a subsequence of the type 12121, which is forbidden. So $\mathcal{G}$ is $\mathcal{F}_{41}$-free and shows that $G_e(\mathcal{F}_{41}, n) \gg n\alpha(n)$.

6 Orderly bipartite forests

$\mathcal{H}$ is an *orderly bipartite forest* (OBF) if it is a simple graph which has no cycle and such that $\min H < \max H'$ holds for every two edges $H, H' \in \mathcal{H}$. Star forests are OBF. Orderly bipartite forests with some singleton edges (which may repeate) form the largest class of $\mathcal{F}$ for which one can hope for linear or close to linear extremal functions. (Since every OBF with singletons is contained in an OBF without singletons, it is enough to consider only OBF.) We state this simple but important observation as a theorem.

Theorem 6.1 *If the hypergraph $\mathcal{F}$ is not an orderly bipartite forest with singletons, then there is a constant $\gamma > 1$ such that $H_e(\mathcal{F}, n) \gg n^{\gamma}$.*

Proof. If $\mathcal{F}$ is not an OBF with singletons, then $\mathcal{F}$ has (i) an edge with more than two elements or (ii) two separated two-element edges or (iii) a two-path isomorphic to $(\{1,2\},\{2,3\})$ or (iv) a repeated two-element edge or (v) an even cycle of two-element edges (odd cycles are subsumed in (iii)). In the cases (i)–(iv) it is easy to see that $H_e(\mathcal{F}, n) \gg n^2$ (cf. the results for $\mathcal{F}_{11}$, $\mathcal{F}_{32}$, $\mathcal{F}_{30}$, and $\mathcal{F}_{31}$ in Section 3). An application of the probabilistic method (Erdős [6]) provides an unordered graph that has n vertices, $\gg n^{1+1/k}$ edges, and no even cycle of length k. Thus $H_e(\mathcal{F}, n) \gg n^{1+1/k}$ in case (v) if $\mathcal{F}$ has an even cycle of length k. $\square$

In the unordered case it is well known that $G_e^u(\mathcal{F}, n) = \mathrm{ex}(\mathcal{F}, n) \ll n$ iff $\mathcal{F}$ is a forest, and if $\mathcal{F}$ is not a forest then $\mathrm{ex}(\mathcal{F}, n) \gg n^{\gamma}$ for some $\gamma > 1$ (by the aforementioned result). In the ordered case the class OBF enjoys much larger variety of linear and close to linear extremal functions.

We say, for $k \in \mathbf{N}$, that a graph $\mathcal{G}'$ is a *k-blowup* of a graph $\mathcal{G}$ if for every edge coloring $\chi : \mathcal{G}' \to \mathbf{N}$ that uses every color $i \in \mathbf{N}$ at most k times, there exists a subgraph in $\mathcal{G}'$ which is isomorphic to $\mathcal{G}$ and whose edges have totally different colors (no color is repeated on the subgraph). For example, it is not difficult to construct for every OBF $\mathcal{G}$ and $k \in \mathbf{N}$ an OBF $\mathcal{G}'$ that is a k-blowup of $\mathcal{G}$. For $k \in \mathbf{N}$ and a graph $\mathcal{G}$ we write $B(k, \mathcal{G})$ to denote the set of all k-blowups of $\mathcal{G}$. The following theorem shows how to derive bounds for hypergraphs from the graph case.

Theorem 6.2 *Suppose that $\mathcal{F}$ is a graph with $p = v(\mathcal{F})$ and $q = e(\mathcal{F}) > 1$. If $f : \mathbf{N} \to \mathbf{N}$ is a nondecreasing function such that*

$$G_e(B(\tbinom{p}{2}, \mathcal{F}), n) < n \cdot f(n)$$

holds for every $n \in \mathbf{N}$, then

$$H_e(\mathcal{F}, n) < q \cdot G_e(\mathcal{F}, n) \cdot H_e(\mathcal{F}, 2f(n) + 1) \tag{10}$$

holds for every $n \in \mathbf{N}$.

Proof. Let $\mathcal{H}$ attain $H_e(\mathcal{F}, n)$ and $\bigcup \mathcal{H} = [n]$. We put in $\mathcal{H}'$ every edge with more than 1 and less than p vertices and for every $H \in \mathcal{H}$ with $|H| \geq p$ we put in $\mathcal{H}'$ an arbitrary subset $H' \subset H$, $|H'| = p$. No edge of $\mathcal{H}'$ repeats more then $q-1$ times for else $H \supset \mathcal{F}$. Let $\mathcal{H}''$ be the simplification of $\mathcal{H}'$. So $e(\mathcal{H}) \leq n + (q-1)e(\mathcal{H}'')$. Let $\mathcal{G}$ be the simple graph consisting of all the edges E such that $E \subset H$ for some $H \in \mathcal{H}''$. Observe that if $\mathcal{F}' \in B(\binom{p}{2}, \mathcal{F})$, meaning that $\mathcal{F}'$ is a $\binom{p}{2}$-blowup of $\mathcal{F}$, and $\mathcal{F}' \subset \mathcal{G}$, then $\mathcal{F} \subset \mathcal{H}''$ and thus $\mathcal{F} \subset \mathcal{H}$. (For the edges $E \in \mathcal{G}$ lying in an $\mathcal{F}'$-copy consider the coloring $\chi(E) = H \in \mathcal{H}''$ where $E \subset H$.) Hence $\mathcal{F}' \subset \mathcal{G}$ for no $\mathcal{F}' \in B(\binom{p}{2}, \mathcal{F})$. Let $v(\mathcal{G}) = n'$; $n' \leq n$. We have

$$e(\mathcal{G}) \leq G_e(B(\tbinom{p}{2}, \mathcal{F}), n') < n' \cdot f(n').$$

There exists a vertex $v_0 \in \bigcup \mathcal{G}$ such that

$$d = \deg_{\mathcal{G}}(v_0) < 2f(n') \leq 2f(n).$$

Fix an arbitrary E_0, $v_0 \in E_0 \in \mathcal{G}$. Let $X \subset [n]$ be the union of all $H \in \mathcal{H}''$ with $E_0 \subset H$ and m be the number of such edges in $\mathcal{H}''$. We have the inequalities

$$m \leq H_e(\mathcal{F}, |X|) \quad \text{and} \quad |X| \leq d+1.$$

Thus ($H_e(\mathcal{F}, n)$ is increasing by Theorem 2.2)

$$m \leq H_e(\mathcal{F}, |X|) \leq H_e(\mathcal{F}, d+1) < H_e(\mathcal{F}, 2f(n)+1).$$

We see that the two-element set E_0 is contained in at least one but less than $H_e(\mathcal{F}, 2f(n)+1)$ edges of $\mathcal{H}''$. Deleting those edges we obtain a subhypergraph $\mathcal{H}''_1$ of $\mathcal{H}''$ on which the same argument can be applied. That is, a two-element set E_1 exists such that $E_1 \subset H$ for at least one but less than $H_e(\mathcal{F}, 2f(n)+1)$ edges $H \in \mathcal{H}''_1$ (clearly $E_1 \neq E_0$). Continuing this way until the whole $\mathcal{H}''$ is exhausted, we define a mapping

$$F : \mathcal{H}'' \to \{E : \; E \subset [n], |E| = 2\}$$

such that

$$F(H) \subset H \quad \text{and} \quad |F^{-1}(E)| < H_e(\mathcal{F}, 2f(n)+1)$$

holds for every $H \in \mathcal{H}''$ and every $E \subset [n], |E| = 2$. Let $\mathcal{G}'$ be the simple graph $\mathcal{G}' = F(\mathcal{H}'')$. Let $v(\mathcal{G}') = n'$; $n' \leq n$.

The containment $\mathcal{F} \subset \mathcal{G}'$ implies, by the definition of $\mathcal{G}'$, that $\mathcal{F} \subset \mathcal{H}''$ and thus $\mathcal{F} \subset \mathcal{H}$, which is forbidden. We have (it is easy to see that $G_e(\mathcal{F}, n)$ is increasing)

$$e(\mathcal{G}') \le G_e(\mathcal{F}, n') \le G_e(\mathcal{F}, n).$$

Putting it all together, we obtain ($G_e(\mathcal{F}, n) \ge n - 1$ if $q > 1$)

$$\begin{aligned} H_e(\mathcal{F}, n) = e(\mathcal{H}) &\le n + (q-1) \cdot e(\mathcal{H}'') \\ &< n + (q-1) \cdot H_e(\mathcal{F}, 2f(n)+1) \cdot e(\mathcal{G}') \\ &\le q \cdot H_e(\mathcal{F}, 2f(n)+1) \cdot G_e(\mathcal{F}, n). \end{aligned}$$

□

Recursive inequality (10) is nontrivial only if $f(n) = o(n)$ and thus it has any value only if $\mathcal{F}$ is an OBF (or perhaps if $\mathcal{F}$ is an even cycle). If $\mathcal{F}$ is an OBF and in Theorem 6.2 we replace $B(\binom{p}{2}, \mathcal{F})$ by some subclass $B \subset B(\binom{p}{2}, \mathcal{F}) \cap \mathrm{OBF}$, the number of colors $\binom{p}{2}$ can be replaced by $p - 1$. (Because for $|H| = p$ every p two-element edges $E \subset H$ contain a cycle but now no $\mathcal{F}' \in B$ has a cycle.) Note that the ordering of vertices was not used in the proof (it is crucial only for obtaining linear or close to linear bounds on $G_e(\mathcal{F}, n)$ and $G_e(B, n)$) and therefore Theorem 6.2 holds in the unordered case as well. We make use of this in the first of its three applications.

Theorem 6.3 *Let $\mathcal{F}$ be an unordered forest. Its unordered hypergraph extremal function satisfies*

$$H_e^u(\mathcal{F}, n) \ll n.$$

Proof. Let $v(\mathcal{F}) = p$ and $e(\mathcal{F}) = q > 1$ (case $q = 1$ is trivial). It is not hard to prove that $G_e^u(\mathcal{F}, n) = \mathrm{ex}(\mathcal{F}, n) \le (q-1)n$ (e.g. Bollobás [2, Exercise 24 in IV.7]). It is also easy to define a large forest $\mathcal{F}'$ with $Q = e(\mathcal{F}') = ((p-1)(q-1)+1)e(\mathcal{F}) = (pq - p - q + 2)q \le pq(q-1)$ edges that is a $(p-1)$-blowup of $\mathcal{F}$. We set $B = \{\mathcal{F}'\}$ and use (10) with the bounds $G_e^u(\mathcal{F}, n) \le (q-1)n$, $f(n) = Q - 1$ (since $G_e^u(B, n) = G_e^u(\mathcal{F}', n) \le (Q-1)n$), and $H_e^u(\mathcal{F}, n) < 2^n$ (trivial):

$$H_e^u(\mathcal{F}, n) < q(q-1) \cdot n \cdot 2^{2Q-1} = \tbinom{q}{2} 4^{pq(q-1)} \cdot n.$$

□

One can prove this bound also directly without using Theorem 6.2 by adapting the proof of $\mathrm{ex}(\mathcal{F}, n) \le (q-1)n$ to the hypergraph case.

In the second application of Theorem 6.2 we extend the bound of Theorem 5.2 to hypergraphs.

Theorem 6.4 *Let $\mathcal{F}$ be a star forest with $r > 1$ components, p vertices, and q edges. Let $t = (p-1)(q-1)+1$. Then*

$$H_e(\mathcal{F}, n) \ll n \cdot \beta(r, 2t(p-r)+2, n)^3$$

where $\beta(k, l, n)$ is the almost constant function defined in (9).

Proof. We replace $\mathcal{F}$ by the star forest $\mathcal{F}'$ in which every edge $\{i, j\} \in \mathcal{F}$, $i < j$, is replaced by t edges $\{i, j(1)\}, \ldots, \{i, j(t)\}$ where $i < j(1) < \cdots < j(t)$ and the set $\{j(1), \ldots, j(t)\}$ is slightly blowned up leaf j, that is, $j_1 < j_2$ implies $j_1(a) < j_2(b)$ for all $1 \le a, b \le t$ and all leaves j_1, j_2 of $\mathcal{F}$. It is easy to see that $\mathcal{F}' \in B(p-1, \mathcal{F})$.

We set $B = \{\mathcal{F}'\}$ and use (10) with the bounds $G_e(\mathcal{F}, n) \ll n \cdot \beta(r, 2(p-r)+2, n)$ (Theorem 5.2 for $\mathcal{F}$), $f(n) \ll \beta(r, 2t(p-r)+2, n)$ (Theorem 5.2 for $\mathcal{F}'$), and $H_e(\mathcal{F}, n) \ll n^p$ (Theorem 2.5):

$$H_e(\mathcal{F}, n) \ll n \cdot \beta(r, 2t(p-r)+2, n)^{p+1}.$$

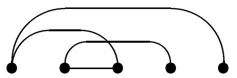
Feeding in (10) this improved upper bound on $H_e(\mathcal{F}, n)$, the second application of Theorem 6.2 gives

$$H_e(\mathcal{F}, n) \ll n \cdot \beta(r, 2t(p-r)+2, n)^2 \cdot \beta(r, 2t(p-r)+2, c\beta(\cdots))^{p+1}$$

where $c > 0$ is a constant. Since $\beta(r, 2t(p-r)+2, c\beta(r, 2t(p-r)+2, n)) < \beta(r, 2t(p-r)+2, n)^{1/(p+1)}$ for every sufficiently large n, we obtain the stated bound. $\square$

By Theorem 2.4, for $H_i(\mathcal{F}, n)$ we have the same bound.

Our third and last application of Theorem 6.2 is to the graph

$$\mathcal{F}_{42} = (\{1,3\}, \{1,5\}, \{2,3\}, \{2,4\}) =$$  .

It arises from $\mathcal{F}_{41}$ by identifying 3 and 4 but it is more important to note that the starting points of the two edges ending in 3 emanate two noncrossing edges ending to the right of 3. $\mathcal{F}_{42}$ is an OBF but it is not a star forest. $\mathcal{F}_{42}$ in its matrix form

$$\begin{pmatrix} 1 & 1 & \\ 1 & & 1 \end{pmatrix}$$

(configuration C_2 of [9]) was introduced by Füredi [7] in order to prove that every convex n-gon has $O(n \log n)$ diagonals with unit length. (Recently simpler proof was given by Braß and Pach [3].) In [7] he proved that

$$n \log n \ll G_e(\mathcal{F}_{42}, n) \ll n \log n.$$

(The proof was given for ordered bipartite graphs but it works without changes for all ordered graphs.) For our purposes we need somewhat stronger version of the upper bound, which we prove (by the same argument of [7]) in Lemma 6.5. For completeness, we reproduce here the construction proving the lower bound, as given in [9, Construction 3.2].

We define inductively bipartite graphs $\mathcal{G}_n$, $n \in \mathbf{N}$, with parts $[2^n]$ and $[2^n+1, 2^{n+1}]$. $\mathcal{G}_1 = (\{1,3\}, \{2,3\}, \{2,4\})$. Let $A = [2^n]$, $B = [2^n+1, 2^{n+1}]$, $C = [2^{n+1}+1, 2^{n+1}+2^n]$, and $D = [2^{n+1}+2^n+1, 2^{n+2}]$. $\mathcal{G}_{n+1}$ consists of two copies of $\mathcal{G}_n$, one on the parts A, C and the other on B, D, and a matching between B and C. An easy induction shows that $e(\mathcal{G}_n) = (n+2)2^{n-1}$, $v(\mathcal{G}_n) = 2^{n+1}$, and $\mathcal{G}_n \not\succ \mathcal{F}_{42}$. Thus $G_e(\mathcal{F}_{42}, n) \gg n \log n$.

For $k \in \mathbf{N}$ consider graphs $\mathcal{G}$ with the following structure. $\bigcup \mathcal{G} = [k+1+a+b]$, $a, b \geq k$, and $\mathcal{G}$ has $2k^2+k$ edges: $\{i, k+1\} \in \mathcal{G}$ for $i \in [k]$ and every $i \in [k]$ is connected by k edges to $[k+2, k+1+a]$ and by k edges to $[k+2+a, k+1+a+b]$. Thus $\deg(k+1) = k$ and $\deg(i) = 2k+1$ for $i \in [k]$. We write $\mathcal{F}_{42}(k)$ to denote the set of all such graphs.

Lemma 6.5 *The sets of graphs $\mathcal{F}_{42}(k)$, $k \in \mathbf{N}$, are as defined above.*

1. $\mathcal{F}_{42}(3k+1) \subset B(k, \mathcal{F}_{42})$. *In particular,* $\mathcal{F}_{42}(31) \subset B(\binom{5}{2}, \mathcal{F}_{42})$.
2. $G_e(\mathcal{F}_{42}(k), n) \ll_k n \log n$.

Proof. Let $K = 3k+1$ and $\mathcal{G} \in \mathcal{F}_{42}(K)$ be edge colored so that each color appears at most k times. We select two edges $\{i, K+1\}$ and $\{j, K+1\}$, $i < j < K+1$, with different colors. There are K edges $\{i, l\}$ and K edges $\{j, l'\}$ such that $K+1 < l' < l$ holds for every two vertices l' and l. Because in each of the two K-tuples we have at least 4 different colors, we can select vertices l' and l so that the colors of $\{i, K+1\}, \{j, K+1\}, \{j, l'\}$, and $\{i, l\}$

are all different. Since $i < j < K+1 < l' < l$, this subgraph is isomorphic to $\mathcal{F}_{42}$.

Let $n \geq 2$ and $\mathcal{G}$ be a simple graph with $\bigcup \mathcal{G} = [n]$ which contains no $\mathcal{F} \in \mathcal{F}_{42}(k)$. For every fixed $i \in [n]$, we list the endpoints j of $\{i,j\} \in \mathcal{G}$, $i < j$: $i < j_0 < j_1 < \cdots < j_{t-1} \leq n$. Let $s = \lfloor t/(k+1) \rfloor$. We keep only the edges with endpoints $j_{(i-1)(k+1)}$, $i = 1, 2, \ldots, s$. (If $t < k+1$, we keep no edge $\{i,j\}$.) The graph $\mathcal{G}'$ obtained satisfies $e(\mathcal{G}') \geq e(\mathcal{G})/(2k+1) - kn$ and for every two edges $\{i,j\}, \{i,j'\} \in \mathcal{G}'$, $i < j < j'$, there are at least k edges $\{i,l\} \in \mathcal{G}, j < l < j'$, and at least k edges $\{i,l\} \in \mathcal{G}, l > j'$. Now we proceed as in [7]. We say that $\{i,j\} \in \mathcal{G}'$, $i < j$, has *type* (j,m) if there are two edges $\{i,l\}$ and $\{i,l'\}$ of $\mathcal{G}'$ such that $j < l < l'$ and $l - j \leq 2^m < l' - j$. Consider the partition $\mathcal{G}' = \mathcal{G}_1 \cup \mathcal{G}_2$ where $\mathcal{G}_1$ is formed by edges with at least one type and $\mathcal{G}_2$ by edges without type. It follows from the definitions that if k edges of $\mathcal{G}_1$ have the same type, then $\mathcal{F} \subset \mathcal{G}$ for some $\mathcal{F} \in \mathcal{F}_{42}(k)$. The number of types is less than $n(1 + \log_2 n)$. Thus $|\mathcal{G}_1| < kn + kn\log_2 n$. Let $i \in [n]$ and $i < j_0 < j_1 < \cdots < j_{t-1} \leq n$ be the endpoints j, $j > i$, of the edges incident with i which have no type. Let $d_r = j_r - j_{r-1}$, $r \in [t-1]$, and $D = d_1 + \cdots + d_{t-1} = j_{t-1} - j_0$. If $d_1 \leq D/2$, then $d_1 \leq 2^m < D$ for some $m \in \mathbf{N}_0$ and $\{i, j_0\}$ has type (j_0, m) because of $\{i, j_1\}$ and $\{i, j_{t-1}\}$. Thus $d_1 > D/2$ and $D - d_1 < D/2$. For similar reason $d_2 > (D - d_1)/2$ and $D - d_1 - d_2 < D/4$. In general $1 \leq D - d_1 - \cdots - d_r < D/2^r$ for $r \in [t-2]$. We obtain that $t \leq \lfloor \log_2 D \rfloor + 2 < 3 + \log_2 n$, $|\mathcal{G}_2| < 3n + n\log_2 n$, and $|\mathcal{G}'| = |\mathcal{G}_1| + |\mathcal{G}_2| < (k+3)n\log_2 n$. Therefore

$$e(\mathcal{G}) < kn + (2k+1)e(\mathcal{G}') < (2k+2)(k+3)n\log_2 n.$$

$\square$

Theorem 6.6 *Let* $\mathcal{F}_{42} = (\{1,3\}, \{1,5\}, \{2,3\}, \{2,4\})$. *Then*

$$n \cdot \log n \ll H_e(\mathcal{F}_{42}, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^3.$$

Proof. The lower bound holds already in the graph case and was proved above. To prove the upper bound, we set $B = \mathcal{F}_{42}(31)$ and apply (10) of Theorem 6.2 with the bounds $f(n) \ll \log n$ (because $G_e(B(\binom{5}{2}, \mathcal{F}_{42}), n) \leq G_e(B,n) \ll n\log n$ by Lemma 6.5), $G_e(\mathcal{F}_{42}, n) \ll n\log n$ ([7] or from the previous bound by $G_e(\mathcal{F}_{42}, n) \leq G_e(B, n)$), and $H_e(\mathcal{F}_{42}, n) \ll n^5$ (Theorem 2.5; we could as well start with the completely trivial bound $H_e(\mathcal{F}_{42}, n) < 2^n$):

$$H_e(\mathcal{F}_{42}, n) \ll n \cdot (\log n)^6.$$

Using this bound, the next application gives

$$H_e(\mathcal{F}_{42}, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^6.$$

The third application gives

$$\begin{aligned} H_e(\mathcal{F}_{42}, n) &\ll n \cdot (\log n)^2 \cdot (\log\log n)^2 \cdot (\log\log\log n)^6 \\ &\ll n \cdot (\log n)^2 \cdot (\log\log n)^3. \end{aligned}$$

□

By Theorem 2.4,

$$H_i(\mathcal{F}_{42}, n) \ll n \cdot (\log n)^2 \cdot (\log\log n)^3$$

as well.

7 Concluding remarks

We mention possible directions for further research. As for Theorem 2.4, singleton hypergraphs $\mathcal{S}_k$ show that $H_i(\mathcal{F}, n) \ll H_e(\mathcal{F}, n)$ is not always true. We conjecture that $\mathcal{S}_k$ are the only exceptions: if $\mathcal{F} \neq \mathcal{S}_k$, then $H_i(\mathcal{F}, n) < cH_e(\mathcal{F}, n)$ holds with a constant $c > 0$ depending only on $\mathcal{F}$ (but cf. Theorem 5.1 and the example with $\mathcal{F}_{40}$). One may try to extend the precise results of Section 3, say to the case when $\mathcal{F}$ is a graph with 3 or 4 edges. One may try to explain the repetitions of functions $H_e(\mathcal{F}, n)$ and $H_i(\mathcal{F}, n)$, such as for no. 9, 10, and 19–23 or for no. 25–28, 33, and 34.

An interesting question is about the order of magnitude of the hypergraph counting functions considered in Section 4. We can prove that $\log(h_n^{(i,s)})$ and $\log(h_n^{(i)})$ are equal to $n\log n - n\log\log n + O(n)$ but more precise asymptotics are desirable. The ratio $h_n^{(i)}/h_n^{(i,s)}$ seems to tend to a limit lying between 1.2 and 1.3. What is this limit? Which partition $\lambda \vdash n$ maximize the number of (simple or all) hypergraphs with weight n and edge type λ?

As for the class OBF (orderly bipartite forests), we define a hierarchy of three subclasses of OBF

$$\mathrm{LIN} \subset \mathrm{ALIN} \subset \mathrm{CLIN} \subset \mathrm{OBF}.$$

LIN contains $\mathcal{F}$ with linear extremal function: $H_e(\mathcal{F}, n) \ll n$. ALIN contains $\mathcal{F}$ with almost linear extremal function: $H_e(\mathcal{F}, n) \ll n \cdot f(\alpha(n))$ where $\alpha(n)$

is the inverse Ackermann function and $f(n)$ is primitive recursive. CLIN contains $\mathcal{F}$ with close to linear extremal function: $H_e(\mathcal{F}, n) \ll n \cdot (\log n)^c$ where $c > 0$ is a constant depending only on $\mathcal{F}$. (If $\mathcal{F}$ is a hypergraph and $H_e(\mathcal{F}, n) \ll n \cdot (\log n)^c$, then by Theorem 6.1 $\mathcal{F}$ must be an OBF.) The first two inclusions are sharp: we have seen that $\mathcal{F}_{41} \in$ ALIN$\backslash$LIN and that $\mathcal{F}_{42} \in$ CLIN$\backslash$ALIN. Is it true that CLIN=OBF? If not, what general upper bound can one give for $H_e(\mathcal{F}, n)$ if $\mathcal{F}$ is an OBF? As for Theorem 6.6, what is the exact asymptotics of $H_e(\mathcal{F}_{42}, n)$?

A basic but difficult question is to determine which OBF lie in LIN, which in ALIN, and which in CLIN. We summarize briefly our knowledge. Here we have proved that the four OBF with $e(\mathcal{F}) \le 2$, namely $\mathcal{F}_4, \mathcal{F}_{29}, \mathcal{F}_{33}$, and $\mathcal{F}_{34}$, are in LIN. A more general result is given in [15, Theorem 3.3]: for every $k \in \mathbf{N}$ the star forest

$$\mathcal{N}(k) = (\{i, 2k-i+1\}, \{i, 2k+i\} :\ i \in [k])$$

($[k]$ is matched with $[k+1, 2k]$ decreasingly and with $[2k+1, 3k]$ increasingly) is in LIN. (In [15] the linear bound is proved only for the graph case but blowing up the leaves of $\mathcal{N}(k)$ and using Theorem 6.2 we can extend it to hypergraphs.) We have proved here that every star forest is in ALIN; the containment of $\mathcal{F}_{41}$ forces it to be in ALIN$\backslash$LIN. As for CLIN$\backslash$ALIN, it contains $\mathcal{F}_{42}$ and some modifications of it but we do not know any large subfamily.

References

[1] R. Anstee, R. Ferguson and A. Sali, Small forbidden configurations II, *Electr. J. Comb.,* **8** (2001), R4, 24 pages.

[2] B. Bollobás, *Modern Graph Theory,* Springer, Berlin, 1998.

[3] P. Brass and J. Pach, The maximum number of times the same distance can occur among the vertices of a convex n-gon is $O(n \log n)$, *J. Comb. Theory, Ser. A,* **94** (2001), 178–179.

[4] L. Comtet, *Advanced Combinatorics,* D. Reidel Publ. Co., Boston, MA, 1974.

[5] P. Erdős, On an elementary proof of some asymptotic formulas in the theory of partitions, *Ann. Math.,* **43** (1942), 437–450.

[6] P. Erdős, Graph theory and probability, *Canadian J. Math.,* **11** (1959), 34–38.

[7] Z. Füredi, The maximum number of unit distances in a convex n-gon, *J. Comb. Theory, Ser. A,* **55** (1990), 316–320.

[8] Z. Füredi, Turán type problems. In: A. D. Keedwell (eds.), *Surveys in Combinatorics, 1991,* Cambridge University Press, Cambridge, UK, 1991; pp. 253–300.

[9] Z. Füredi and A. Hajnal, Davenport–Schinzel theory of matrices, *Discrete Math.,* **103** (1992), 233–251.

[10] P. Frankl, Extremal set systems. In: R. L. Graham, M. Grötschel and L. Lovász (eds.), *Handbook of Combinatorics, Volume 2,* Elsevier, Amsterdam, 1995; pp. 1293–1329.

[11] P. Frank, Z. Füredi, G. Katona, D. Miklós (editors), *Extremal Problems for Finite Sets,* János Bolyai Mathematical Society, Budapest, 1994.

[12] S. Hart and M. Sharir, Nonlinearity of Davenport–Schinzel sequences and of generalized path compression schemes, *Combinatorica,* **6** (1986), 151–177.

[13] T. Hearne and C. Wagner, Minimal covers of finite sets, *Discrete Math.,* **5** (1973), 247–251.

[14] M. Klazar, A general upper bound in extremal theory of sequences, *Commentat. Math. Univ. Carol,* **33** (1992), 737–746.

[15] M. Klazar, Counting pattern-free set partitions II. Noncrossing and other hypergraphs, *Electr. J. Comb.,* **7** (2000), R34, 25 pages.

[16] L. Lovász, *Combinatorial Problems and Exercises,* Akadémiai Kiadó, Budapest, 1993.

[17] A. J. Macula, Covers of a finite set, *Math. Mag.,* **67** (1994), 141–144.

[18] D. J. Newman, The evaluation of the constant in the formula for the number of partitions of n, *Amer. J. Math.,* **73** (1951), 599–601.

[19] D. J. Newman, A simplified proof of the partition formula, *Mich. Math. J.,* **9** (1962), 283–287.

[20] D. J. Newman, *Analytic Number Theory,* Springer, Berlin, 1998.

[21] M. Sharir and P. K. Agarwal, *Davenport–Schinzel Sequences and Their Geometric Applications,* Cambridge University Press, Cambridge, UK, 1995.

[22] N. J. A. Sloane (2000), The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/~njas/sequences/.

[23] Zs. Tuza, Applications of the set-pair method in extremal hypergraph theory. In: P. Frankl et al. (eds.), *Extremal Problems for Finite Sets,* János Bolyai Mathematical Society, Budapest, 1994; pp. 479–514.

[24] Zs. Tuza, Applications of the set-pair method in extremal problems, II. In: D. Miklós et al. (eds.), *Combinatorics, Paul Erdős is Eighty, Volume 2,* János Bolyai Mathematical Society, Budapest, 1996; pp. 459–490.

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