On a Set of 2-colorings

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Abstract

We are interested in examining sets of 2-colorings of the set of positive integers that avoid long monochromatic arithmetic progressions having odd common difference. Here we give a lower bound for the number of 2-colorings of the interval $[1, m]$ of positive integers that avoid monochromatic arithmetic progressions of certain length having odd common difference. Also, we give a non-periodic 2-coloring of the set of positive integers that avoids long monochromatic arithmetic progressions having odd common difference.

Key words: 2-coloring, van der Waerden's theorem.

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1 Introduction

The questions we address are of the following type: If a set of finite colorings with a certain property is given, what can we say about its size and its intersections with other sets of finite colorings?

Let $\mathbb{N}$ be the set of positive integers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $[,]$ form closed intervals in $\mathbb{N}_0$. For $r \in \mathbb{N}$, an r-coloring of $\mathbb{N}$ is a map $f : \mathbb{N} \to A$, with $|A| = r$. A coloring is an r-coloring for some r. If f is a coloring of $\mathbb{N}$ and if $B \subseteq \mathbb{N}$ satisfies $|f(B)| = 1$, we say that B is f-monochromatic. An arithmetic progression of length k and common difference d, $k, d \in \mathbb{N}$, is a set of the form $\{a + (i-1)d : i \in [1,k]\}$, for some $a \in \mathbb{N}$.

Van der Waerden's theorem [3] on arithmetic progressions says that for any coloring f and any $k \in \mathbb{N}$ there is an f-monochromatic arithmetic progression of length k. Brown, Graham, and Landman in [1] study subsets L of $\mathbb{N}$ such that van der Waerden's theorem can be strengthened to guarantee the existence of arbitrarily long f-monochromatic progressions having common difference in L.

For $r \in \mathbb{N} \backslash \{1\}$ we say that L, $L \subseteq \mathbb{N}$, is r-large if every r-coloring yields arbitrarily long monochromatic progressions having common differences in L. We say that L is large if it is r-large for every r. Perhaps surprisingly, there are many large sets; for example, for any $m \in \mathbb{N}$, the set $m\mathbb{N}$ is large.

On the other hand, it is known that $2\mathbb{N} - 1$ is not 2-large. One can see that, for $n \in \mathbb{N} \backslash \{1\}$, the coloring

$$f_n : \mathbb{N} \to \{0, 1\}$$

defined by

$$f_n(i) = 0 \Leftrightarrow ((\exists t \in \mathbb{N}_0)\ i \in [2(n-1)t+1, (2t+1)(n-1)])$$

avoids n-term f_n-monochromatic arithmetic progressions having odd common difference. Clearly, f_n is periodic with the period $2(n-1)$.

Actually, $2\mathbb{N}-1$ fails the condition that any r-large set contains an infinite number of multiplies of any integer.

Another example of a set that is not 2-large is $\mathbb{N}! = \{n! : n \in \mathbb{N}\}$. This set fails the condition that, for any r-large set $L = \{a_n\}_{n\in\mathbb{N}}$,

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Brown, Graham, and Landman have conjectured that any 2-large set is large. It seems that this simply stated conjecture is difficult to prove or disprove and that the answer can go either way.

The only distinction between the known properties of the family of large sets and the family of 2-large sets is the "Ramsey property." It is known that the family of large sets has the Ramsey property, i.e., if $L_1 \cup L_2$ is large then at least one of L_1 and L_2 is large. It is an open question if the family of 2-large sets has the Ramsey property.

Thus, one way to approach the conjecture is to try to find two sets that are not 2-large and to show that their union is 2-large. Brown, Graham, and Landman suggest that $2\mathbb{N}-1$ and $\mathbb{N}!$ could be such sets. It is not difficult to see that $(2\mathbb{N}-1) \cup \mathbb{N}!$ satisfies both of the necessary conditions mentioned above.

It is known that $(\mathbb{N}!-1) \cup \mathbb{N}! \cup (\mathbb{N}!+1)$ is not 2-large.

We note that if χ is a periodic 2-coloring of $\mathbb{N}$ with a period T, then the set $\{1 + i \cdot T! : i \in \mathbb{N}\}$ is χ-monochromatic. Regarding to the problem if $(2\mathbb{N} - 1) \cup \mathbb{N}!$ is 2-large, this observation naturally leads to the following two questions.

1. Is there a non-periodic 2-coloring that avoids long monochromatic arithmetic progressions having their common differences in $2\mathbb{N} - 1$?

2. Generally, for a given interval of positive integers, how many 2-colorings that avoid monochromatic arithmetic progressions of a given length and having their common differences in $2\mathbb{N} - 1$ are there?

In this paper we discuss the questions above.

Let $\mathcal{K}_{2\mathbb{N}-1}$ be the set of all 2-colorings of $\mathbb{N}$ with the property that there are no arbitrarily long monochromatic progressions having odd common difference.

In Section 2 we construct a family of subsets of $\mathcal{K}_{2\mathbb{N}-1}$. Also, we show that there are non-periodic elements of $\mathcal{K}_{2\mathbb{N}-1}$.

In Section 3, for certain values of m, we give a lower bound for the number of 2-colorings of the interval $[1, m]$ that avoid monochromatic arithmetic progressions of a given length having odd common difference.

2 A set of 2-colorings

We start with a lemma.

Let $p \in \mathbb{N} \setminus \{1\}$ be given.

Lemma 1 *Let* $a \in [1, 2(2p-1)]$ *and let* $l \in [0, 2(p-1)] \setminus \{p-1\}$. *There are* $i \in [0, 2p-1]$, $j \in [1, p]$, *and* $k \in [2p, 2(2p-1)]$ *so that*

$$a + i(2l+1) \equiv 2j \pmod{2p}$$

and

$$a + i(2l+1) \equiv k \pmod{2(2p-1)}.$$

Proof. Let

$$\{a, a+(2l+1)\} \cap 2\mathbb{N} = \{\alpha\}$$

Suppose that there is $q \in [1, p-1]$ so that

$$(\forall i \in [0, q])\,(\exists k_i \in [1, 2p-1])\ \ \alpha + 2i(2l+1) \equiv k_i \pmod{2(2p-1)}.$$

Since $2(2l+1) \not\equiv 0 \pmod{2(2p-1)}$ we have that

$$k_1 \neq k_0.$$

From, for all $i, i' \in [0, q]$

$$i \neq i' \ \Rightarrow\ k_i \neq k_{i'}$$

and the fact that k_i is even it follows that

$$q < p-1.$$

Therefore, there are $i \in [0, p-1]$ and $k \in [2p, 2(2p-1)]$ so that

$$\alpha + 2i(2l+1) \equiv k \pmod{2(2p-1)}. \ \blacksquare$$

Let M' be the set of two colorings such that $f \in M'_p$ if and only if the following three conditions are satisfied.

1. $f(2k) = f_{2p}(2k)$, for all k.

2. There is an odd number $N = N(f)$, not greater than $2p-1$, so that

$$f(2kp + N) = f_{2p}(2kp + N) \text{ for all } k.$$

3. For all j not greater than $2p-1$

$$f(2k(2p-1)+j) = f_{2p}(2k(2p-1)+j), \text{ for all } k.$$

For example, for f_6 and $N = 5$ we have that $f \in M'$ for $i \in [7, 19]$ is given by

i	7	8	9	10	11	12	13	14	15	16	17	18	19
$f_6(i)$	1	1	1	1	0	0	0	0	0	1	1	1	1
$f(i)$	$*$	1	$*$	1	0	0	0	0	0	1	1	1	$*$

Theorem 2 $M_p' \subseteq \mathcal{K}_{2\mathbb{N}-1}$.

Proof. Let $f \in M_p'$ and let a and l be any nonnegative integers. We prove that

$$\{a + i\,(2l+1) : i \in [0, 2p-1]\}$$

is not f monochromatic.

Let $a' \in [1, 2\,(2p-1)]$, and let $l' \in [0, 2\,(p-1)]$ be so that

$$a \equiv a' \pmod{2\,(2p-1)} \text{ and } l \equiv l' \pmod{(2p-1)}.$$

If $l' \neq p-1$, by Lemma 1 there is $i \in [0, 2p-1]$ so that

$$a' + i\,(2l'+1) \in 2\mathbb{N} \text{ and } f_{2p}\,(a' + i\,(2l'+1)) = 1.$$

This means that

$$f\left(a+i\left(2l+1\right)\right)=1.$$

On the other hand

$$\begin{aligned}1 &\leq \left|\left\{i\in[0,2p-1]:f_{2p}\left(a+i\left(2l+1\right)\right)=0\right\}\right|\leq\\ &\quad \left|\left\{i\in[0,2p-1]:f\left(a+i\left(2l+1\right)\right)=0\right\}\right|.\end{aligned}$$

Therefore, K is not f-monochromatic.

If $l'=p-1$ then $2l'+1=2p-1$ and

$$f_{2p}\left(a'+i\left(2p-1\right)\right)\neq f_{2p}\left(a'+\left(i+1\right)\left(2p-1\right)\right),\text{ for all } i\in\left[0,2p-2\right].$$

Note that

$$a'+i\left(2p-1\right)\equiv N \pmod{2p} \text{ and } i\in\left[0,2p-1\right]$$

implies

$$\left\{i-1,i+1\right\}\cap\left[0,2p-2\right]\neq\phi.$$

Thus, $\left\{a'+i\left(2p-1\right):i\in\left[0,2p-1\right]\right\}$ is not f-monochromatic. ∎

The following corollary gives a way to construct a non-periodic elements of $\mathcal{K}_{2\mathbb{N}-1}$.

Corollary 3 *Let g be a non-periodic 2-coloring of $\mathbb{N}_0$ and let*

$$f:\mathbb{N}\rightarrow\{0,1\}$$

be defined by

$$f\left(n\right)=\begin{cases} g\left(\frac{n-2p-1}{2p(2p-1)}\right) & \textit{if} \qquad n\equiv\left(2p+1\right)\pmod{2p\left(2p-1\right)} \\ f_{2p}\left(n\right) & \textit{otherwise}\end{cases}.$$

Then f is non-periodic and $f\in M_p'$.

An example of a non-periodic 2-coloring of $\mathbb{N}$ is the Morse sequence. See, [2].

Also, we note that for any f obtained in this way, there are arbitrarily long monochromatic progressions having their common differences in $\mathbb{N}!$.

It is not difficult to see that, if M''_p is the set of all 2-colorings so that $f \in M''_p$ if and only if

1. $f(2k-1) = f_{2p}(2k-1)$, for all k;

2. there is an even number $N = N(f)$, not greater than $2p$, so that

$$f(2kp+N) = f_{2p}(2kp+N) \text{ for all } k;$$

3. for all $j \in [2p, 2(2p-1)]$

$$f(2k(2p-1)+j) = f_{2p}(2k(2p-1)+j), \text{ for all } k;$$

then

$$M''_p \subseteq \mathcal{K}_{2\mathbb{N}-1}.$$

Therefore,

$$M_p = \left\{\chi : \{\chi, 1-\chi\} \cap \left(M'_p \cup M''_p\right) \neq \phi\right\} \subseteq \mathcal{K}_{2\mathbb{N}-1}.$$

We note that the elements of M_p **permit** monochromatic $(2p-1)$-term arithmetic progressions having common difference 1.

3 The Lower Bound

One can see the facts given in the previous section in the following way. Let us represent the coloring f_{2p} as an array $A_p = \{a_{i,j}\}_{(i,j) \in \mathbb{N}_0 \times [1,2p]}$ with $2p$

columns and an infinite number of rows and with $a_{i,j} \in \{0,1\}$ as its entries. We put

$$a_{i,j} = 1 \;\Leftrightarrow\; f_{2p}(2ip + j) = 1.$$

The fact that $M_p' \subseteq \mathcal{K}_{2\mathbb{N}-1}$ means that any 2-coloring, represented by an array that is obtained from A_p by changing any number of 1's into 0's in all odd columns except at least one, belongs to M_p'. Similarly, a 2-coloring, represented by an array that is obtained from A_p by changing any number of 0's into 1's in all even columns except at least one, belongs to M_p''.

This "visualization" of the elements of $M_p' \cup M_p''$ leads to the following fact.

Theorem 4 *For any non-negative integers k and p, the number of 2-colorings of the interval $[1, 2kp(2p-1)]$ that avoid monochromatic $2p$-term arithmetic progressions having common differences in $2\mathbb{N}-1$ and that permit monochromatic $(2p-1)$-term arithmetic progressions having common difference 1 is not less than*

$$4\left(2^{kp(p-1)} - \left(2^{k(p-1)} - 1\right)^p\right) - 2.$$

Proof. The claim is obviously true if $p = 1$.

Let $p \geq 2$ and let us consider an array A_p^k obtained from the first $k(2p-1)$ rows of A_p. In each odd column of A_p^k there are kp entries equal to 0. We note the fact that if exactly i, $i \in [1, p-1]$, odd columns are unchanged then there are exactly

$$\left(2^{k(p-1)} - 1\right)^{p-i}$$

different ways to change 1's into 0's in the rest $p - i$ odd columns. Thus, we have that there are

$$\sum_{i=1}^{p-1} \binom{p}{i} \left(2^{k(p-1)} - 1\right)^{p-i} = 2^{kp(p-1)} - \left(2^{k(p-1)} - 1\right)^{p} - 1$$

different $k\,(2p-1) \times 2p$ arrays that can be obtained by changing 1's into 0's in odd columns of A_p^k and leaving at least one odd column unchanged.

Similarly, there are

$$2^{kp(p-1)} - \left(2^{k(p-1)} - 1\right)^{p} - 1$$

different $k\,(2p-1) \times 2p$ arrays that can be obtained by changing 0's into 1's in even columns of A_p^k and leaving at least one even column unchanged. Taking into account the array A_p^k and the definition of M_p, we have that the number of 2-colorings of the interval $[1, 2kp\,(2p-1)]$ that avoid monochromatic $2p$-term arithmetic progressions having common differences in $2\mathbb{N}-1$ and that permit monochromatic $(2p-1)$-term arithmetic progressions having common difference 1 is at least

$$2\left(2\left(2^{kp(p-1)} - \left(2^{k(p-1)} - 1\right)^{p} - 1\right) + 1\right) = 4\left(2^{kp(p-1)} - \left(2^{k(p-1)} - 1\right)^{p}\right) - 2\,.\blacksquare$$

An immediate consequence of Theorem 4 follows.

Corollary 5 *If* $p,\, m \in \mathbb{N}$ *are so that*

$$\varepsilon_p \cdot p! \prod_{i=\lceil \frac{p+1}{2} \rceil}^{p} (2i-1) \,|\, m\,,$$

where $\varepsilon_2 = 2$ *and* $\varepsilon_p = 1$ *otherwise, then there are at least*

$$2\left(2\sum_{i=1}^{p}\left(2^{\frac{(i-1)m}{2(2i-1)}} - \left(2^{\frac{(i-1)m}{2i(2i-1)}} - 1\right)^{i}\right) - 1\right)$$

2-colorings of the interval of $[1, m]$ *that avoid monochromatic* $2p$*-term arithmetic progressions having common differences in* $2\mathbb{N}-1$.

Proof. The claim is obviously true if $p = 1$.

Let $p \geq 2$. Since

$$2i\,(2i-1)\,|m \ \text{ for all } I \in [1,p]$$

by Theorem 4, the number of 2-colorings that avoid $2i$-term monochromatic arithmetic progressions having common differences in $2\mathbb{N}-1$ and that permit monochromatic $(2i-1)$-term arithmetic progressions having common difference 1 is at least

$$4\left(2^{\frac{(i-1)m}{2(2i-1)}} - \left(2^{\frac{(i-1)m}{2i(2i-1)}} - 1\right)^{i}\right) - 2.$$

Also, for any $i \in [2,p]$, there are at least two 2-colorings of $[1,m]$ that avoid $(2i-1)$-term monochromatic arithmetic progressions having common difference in $2\mathbb{N}-1$ and that permit a monochromatic $2\,(i-1)$-term arithmetic progression having common difference 1.

Therefore, the number of 2-colorings of the interval of $[1,m]$ that avoid monochromatic $2p$-term arithmetic progressions having common differences in $2\mathbb{N}-1$ is at least

$$\begin{aligned} & 2\left(2\sum_{i=1}^{p}\left(2^{\frac{(i-1)m}{2(2i-1)}} - \left(2^{\frac{(i-1)m}{2i(2i-1)}} - 1\right)^{i}\right) - p\right) + 2\,(p-1) \\ = \ & 2\left(2\sum_{i=1}^{p}\left(2^{\frac{(i-1)m}{2(2i-1)}} - \left(2^{\frac{(i-1)m}{2i(2i-1)}} - 1\right)^{i}\right) - 1\right). \ \blacksquare \end{aligned}$$

It is not difficult to verify that, for p and m as in the corollary

$$(\forall i \in [1,p])\ 2^{\frac{(i-1)m}{2(2i-1)}} - \left(2^{\frac{(i-1)m}{2i(2i-1)}} - 1\right)^{i} \geq 2^{\frac{(i-1)^2 m}{2i(2i-1)}}.$$

From the fact that

$$i \geq 8 \;\Rightarrow\; \frac{(i-1)^2}{2i\,(2i-1)} > \frac{1}{5}$$

we have that for $p > 8$ and m as in the corollary, the number of 2-colorings of the interval $[1, m]$ that avoid $2p$-term monochromatic progressions with odd common difference is greater than

$$4\,(p-7)\,2^{\frac{m}{5}} + 26.$$

References

1. T. Brown, R. Graham, and B. Landman. On the set of Common Differences in van der Waerden's Theorem on Arithmetic Progressions. Canad. Math. Bull. Vol. **42** (1), 25-36, 1999.

2. H. Furstenberg. Recurrence in Ergodic Theory and Combinatorial Number Theory. Princeton University Press, Princenton, 1981.

3. B.L. van der Waerden. Beweis einer Baudetschen Vermutung.Nieuw Arch. Wisk. **15**, 212-216, 1927.