Complexity Note on Mixed Hypergraphs

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Abstract

A mixed hypergraph H is a triple $(V, \mathcal{C}, \mathcal{D})$ where V is its vertex set and $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V, $\mathcal{C}$–edges and $\mathcal{D}$–edges. The degree of a vertex is the number of edges in which it is contained. A vertex coloring of H is proper if each $\mathcal{C}$–edge contains two vertices with the same color and each $\mathcal{D}$–edge contains two vertices with different colors. The feasible set of H is the set of all k's such that there exists a proper coloring using exactly k colors. The lower (upper) chromatic number of H is the minimum (maximum) number in the feasible set.

We prove that it is NP–complete to decide whether the upper chromatic number of mixed hypergraphs with maximum degree two is at least a given k. We present polynomial time algorithms for mixed hypergraphs with maximum degree two to decide their colorability, to find a coloring using the number of colors equal to the lower chromatic number and we present a 3/2–aproximation algorithm for the upper chromatic number. We further prove that it is coNP-hard to decide whether the feasible set of a given general mixed hypergraph is an interval of integers.

1 Introduction

Hypergraphs are well established combinatorial objects, see [1]. A hypergraph is a pair $(V, \mathcal{E})$ where $\mathcal{E}$ is a family of subsets of V of size at least 2; the members of V are called vertices and the members of $\mathcal{E}$ are called edges. A mixed hypergraph G is a triple $(V, \mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V; the members of $\mathcal{C}$ are called $\mathcal{C}$–edges and the members of $\mathcal{D}$ are called $\mathcal{D}$–edges. A mixed hypergraph is *reduced* if it does not contain a $\mathcal{C}$–edge of size two. A

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proper k–coloring c of G is a mapping $c: V \to \{1, \ldots, k\}$ such that there are two vertices with $\mathcal{D}$ifferent colors in each $\mathcal{D}$–edge and there are two vertices with a $\mathcal{C}$ommon color in each $\mathcal{C}$–edge. The coloring c is a *strict k–coloring* if it uses all k colors. The number of edges which contain a vertex v is called the *degree* of v; each edge which is both a $\mathcal{C}$–edge and a $\mathcal{D}$–edge is counted twice. The mixed hypergraph is colorable iff G has a proper coloring. Mixed hypergraphs were introduced in [13]; they find their application in different areas: coloring block designs (see [2, 3, 4, 10, 11, 12]), list–coloring of graphs and others ([9]).

The *feasible set* $\mathrm{F}(G)$ of a mixed hypergraph G is the set of all k's such that there exists a strict k–coloring of G. The *(lower) chromatic number* $\chi(G)$ of G is the minimum number in $\mathrm{F}(G)$ and the *upper chromatic number* $\bar{\chi}(G)$ of G is the maximum number in $\mathrm{F}(G)$. The feasible set of G is *gap–free* (unbroken) iff $\mathrm{F}(G) = [\chi(G), \bar{\chi}(G)]$; we write $[a, b]$ for all the integers between a and b (inclusively). If the feasible set of G contains a gap, we say it is broken. An example of a mixed hypergraph with a broken feasible set was first given in [5]. On the other hand, it was proved in [8] that feasible sets of mixed hypertrees, special class of mixed hypergraphs, are gap–free. Other definitions and basic properties of mixed hypergraphs, previously used also in [6, 8, 13], are in Section 2.

We focus our attention on complexity of coloring problems for mixed hypergraphs and in particular of those with maximum vertex degree two; mixed hypergraphs with maximum vertex degree two are in nice correspondence with mixed multigraphs as described in Observation 2. Mixed hypergraphs with vertex degree bounded by three (or more) are not interesting from the complexity point of view: The coloring problems for them are equally difficult as the coloring problems for general mixed hypergrahs due to Observation 1.

Properties of mixed hypergraphs with maximum degree two are studied and mixed multigraphs, natural representation of such mixed hypergraphs, are introduced in Section 3. We can often restrict our attention to reduced mixed hypergraphs (cf. Lemma 1) when we deal with mixed hypergraphs. We prove that the feasible sets of mixed hypergraphs with maximum degree two are gap–free and their lower chromatic numbers are at most three (Theorem 1). We state several properties of colorings using the maximum possible number of colors in Lemma 4. We conclude Section 3 with the proof that it is NP–complete to decide whether the upper chromatic number of a given mixed hypergraph with maximum degree two is at least a given k (Theorem 2).

In Section 4, we present some polynomial algorithms for mixed hypergraphs with maximum degree two. We find a polynomial algorithm to decide their colorability; this problem is NP–complete for general mixed hypergraphs. We find a polynomial algorithm to determine the lower chromatic number of a mixed hypergraph with maximum degree two which even finds a coloring using that number of colors. We find a 3/2–aproximation algorithm for the upper chromatic number; the NP–completeness of this problem is established in Theorem 2 in Section 3 and there is no aproximation algorithm (unless $P = NP$)

with a constant aproximation ratio for general mixed hypergraphs (thus also for those with maximum degree three) due to the results of [7].

We show in Section 5 that it is NP-hard to decide whether the feasible set of a given mixed hypergraph contains a gap (Theorem 4). (We also know that this problem is coNP–hard but we do not provide the proof of this fact and we refer the reader to [7].)

2 Basic Properties of Mixed Hypergraphs

First, we state a lemma about reducing mixed hypergraphs with maximum degree 2:

Lemma 1 *There is a polynomial time algorithm which for a given mixed hypergraph with maximum degree two either outputs that it is uncolorable or finds a reduced mixed hypergraph with maximum degree two such that its proper colorings one–to–one correspond to proper colorings of the given mixed hypergraph.*

Proof: Let H be the given mixed hypergraph. If H does not contain a $\mathcal{C}$–edge of size two, then we are done. Otherwise let $\{u, v\}$ be such a $\mathcal{C}$–edge. If H contains a $\mathcal{D}$–edge $\{u, v\}$, then H is uncolorable and we stop. If $\{u, v\}$ is a subset of another $\mathcal{C}$–edge of H, we remove the super–$\mathcal{C}$–edge from H (this clearly does not affect proper colorings of H). Next, we proceed as follows: Let w be a new vertex; we remove the $\mathcal{C}$–edge $\{u, v\}$ from H, replace (at most one other) occurrence of u by w and another (at most one) occurrence of v by w. The vertex w is contained in at most two edges of the new mixed hypergraph and thus the new mixed hypergraph has the maximum degree at most two. Moreover, its proper colorings one–to–one correspond to proper colorings of the original one (the colors of u and v have to be the same, and thus we can color w with their common color and vice versa). We stop when either we find that the mixed hypergraph is uncolorable or we obtain a reduced one. $\Box$

We made use of the fact that the maximum degree of the original mixed hypergraph was at most two. If we slighty weaken this condition, we can perform the following construction: Let H be any mixed hypergraph with n vertices $v_1, \ldots, v_n$ and m edges $E_1, \ldots, E_m$. We create the following mixed hypergraph H' with nm vertices v_{ij} where $1 \le i \le n$ and $1 \le j \le m$. We create $\mathcal{C}$–edges $\{v_{ij}, v_{ij+1}\}$ for $1 \le i \le n$ and $1 \le j \le m-1$; these $\mathcal{C}$–edges assure that the vertices v_{ij} and $v_{ij'}$ are colored with the same color. We create new edges E'_k such that $E'_k = \{v_{ik} | v_i \in E_k\}$ and we assign the same type to E'_k as E_k has. It is clear that proper colorings of the constructed mixed hypergraph one–to–one correspond to the proper colorings of the original one and moreover, its maximum degree is at most three; thus we can conclude:

Observation 1 *For each mixed hypergraph H, one can construct in polynomial time a mixed hypergraph with maximum degree three whose proper colorings one–to–one correspond to the proper colorings of H.*

Let H be a mixed hypergraph with maximum degree two. We create a multigraph G as follows: Each edge of H corresponds a vertex of G; two vertices of G are joined by as many edges as they share vertices in H and these edges correspond to the shared vertices. A vertex of H contained only in one (no) edge of H is represented by an edge of G whose one end–vertex (both of them) do not correspond to an edge of H. G contains vertices of three types: *$\mathcal{C}$–vertices*, *$\mathcal{D}$–vertices* and other vertices. We color edges of G and we demand that there are two edges with the same color incident to each $\mathcal{C}$–vertex and two edges with different colors incident to each $\mathcal{D}$–vertex. This allows us to introduce the notion of *mixed multigraphs* which contain $\mathcal{C}$ and $\mathcal{D}$–vertices (all these special vertices must have degree at least two) and their edges are to be colored as described above. All the definitions for mixed hypergraphs translate to mixed multigraphs, in particular, a mixed multigraph is reduced iff it does not contain a $\mathcal{C}$–vertex of degree two. We conclude this paragraph with the following observation:

Observation 2 *For each mixed hypergraph H with maximum degree two, there is the mixed multigraph G whose proper edge–colorings one–to–one correspond to the proper vertex–colorings of H. On the other hand, for each mixed multigraph G, there is a mixed hypergraph H with maximum degree two, such that their proper colorings are also in one–to–one correspondence.*

3 Colouring Bounded Degree Mixed Hypergraphs

We use the correspondence between mixed hypergraphs of maximum degree two and mixed multigraphs established in Observation 2; any statement for mixed hypergraphs holds also for mixed multigraphs and vice versa.

Lemma 2 *Let H be a mixed hypergraph with maximum degree two. Let c be a proper coloring of H using $k \geq 4$ colors. Then there exists a proper coloring c' of H using exactly $k-1$ colors.*

Proof: Assume that c uses colors $1, \ldots, k$. We replace each $\mathcal{C}$–edge of H containing two or more vertices colored with the color k with a $\mathcal{C}$–edge containing just two of its vertices colored by the color k and containing no other vertices. Let H' be this new mixed hypergraph. Clearly c is a proper coloring of H' and the maximum degree of H' is two. We contract the $\mathcal{C}$–edges of H' of size two as in Lemma 1. Let $\tilde{H}$ be the mixed hypergraph after performing the contractions and let $\tilde{c}$ be its coloring using k colors corresponding to the coloring c of H. It is clear that $\mathrm{F}(H') = \mathrm{F}(\tilde{H}) \subseteq \mathrm{F}(H)$ and thus it is enough to construct a strict $(k-1)$–coloring of $\tilde{H}$.

Let $v_1, \ldots, v_l$ be all the vertices of $\tilde{H}$ colored by $\tilde{c}$ with the color k; no two vertices of $v_1, \ldots, v_l$ are contained together in a $\mathcal{C}$–edge of $\tilde{H}$. We recolor these vertices one by one by colors different from the color k. We distinguish four cases:

- If v_i is contained in no $\mathcal{D}$–edges of $\tilde{H}$, then we can assign to v_i any color, e.g. the color 1.
- If v_i is contained in exactly one $\mathcal{D}$–edge D, we color v_i as follows: Let c_0 be a color of another vertex of D and we color v_i with a color different from c_0.
- If v_i is contained in two $\mathcal{D}$–edges D_1 and D_2 of $\tilde{H}$, let c_1 be a color used on some other vertex of D_1 and let c_2 be a color used on some other vertex of D_2. We color v_i with a color different from both c_1 and c_2 (note that $k - 1 \geq 3$).

This construction gives a proper coloring of $\tilde{H}$ using exactly $k-1$ colors and this finishes the proof due to the above mentioned inclusions of the feasible sets. $\Box$

The immediate corollary of this lemma is the following theorem:

Theorem 1 *The feasible set of any mixed hypergraph H with maximum degree two is gap–free. If H is colorable, then $[3, \overline{\chi}(H)] \subseteq \mathrm{F}(H)$. Moreover, the chromatic number of H is at most* 3*.*

Proof: If H is not colorable or $\chi(H) = 1$ (and thus H contains no $\mathcal{D}$–edges), then its feasible set is gap–free. In other cases, it holds that $[3, \overline{\chi}(H)] \subseteq \mathrm{F}(H)$ due to Lemma 2. This assures that $\mathrm{F}(H)$ is gap–free and $\chi(H) \leq 3$. $\Box$

Let G be a mixed multigraph and let c be a proper coloring of G. The edges colored with the same color form a *color–subgraph* of G. If it is a path/cycle, we call it a *color–path/color–cycle*. The length of a color–path/cycle is the number of edges it contains. The following two lemmas give some information about the structure of colorings using the maximal possible number of colors:

Lemma 3 *If c is a proper coloring of a mixed multigraph G using k colors and if there exists a vertex adjacent to three or more edges colored by c with the same color, then there exists a proper coloring c' using $k+1$ colors.*

Proof: Let C_0 be a color–subgraph containing a vertex of degree at least three and let c_0 be the color to which C_0 corresponds. We distinguish two cases:

- C_0 contains a cycle with a vertex of degree at least three.
  Let C' be a cycle which C_0 contains; let c' be the coloring obtained from c by recoloring the edges contained in C' with a new color. We claim that c' is a proper coloring. It is clear that all the $\mathcal{D}$–vertices are colored properly; let C be a $\mathcal{C}$–vertex. If C is not colored properly, it has to be adjacent to exactly two edges colored by c with the color c_0 (otherwise it would be still properly colored) — but either both these edges are recolored (if they are in C_0) or both of them keep the color c_0 and thus C is colored properly.

- C_0 does not contain a cycle with a vertex of degree at least three.
  Let v be a vertex of degree at least three in C_0. Since v is not contained in any cycle of C_0, it is an articulation with respect to C_0. Let C' be one of the components of $C_0 \setminus v$. Let c' be the coloring obtained from c by recoloring the edges of C' and the edge joining C' to v with a new color. We claim that c' is a proper coloring. The only affected vertex through this procedure is v (since the edges colored with c_0 incident to other vertices were either all recolored or none of them was recolored). But v still contains at least two vertices colored with c_0 and thus v is colored properly.

□

Lemma 4 *If c is a coloring of a mixed multigraph G using $\overline{\chi}(G)$ colors, then no vertex of G is incident to three or more edges colored with the same color. Moreover, the color–subgraphs are only color–paths and color–cycles.*

Proof: All the vertices of the color–subgraphs have degree at most two due to Lemma 3. If there is a disconnected color subgraph, recoloring its components with mutually different colors gives a proper coloring using more colors. Thus any color subgraph is connected and its vertices have degree at most two. □

We are ready to prove our NP–completeness result on the upper chromatic number of mixed hypergraphs with maximum degree two.

Theorem 2 *It is NP–complete to decide whether $\overline{\chi}(H) \geq k$ for a given mixed hypergraph H with maximum degree two and a given k.*

Proof: We present a reduction from a well–known NP–complete graph problem of triangle–covering. This problem is to decide whether a given graph G with $3n$ vertices can be covered by n triangles in such way that each vertex of G is contained in precisely one triangle.

Let G be an instance of the problem of triangle–covering; let $3n$ be the number of vertices of G and let m be the number of its edges. We construct a mixed multigraph from G by setting all its vertices to be $\mathcal{C}$–vertices. We claim that $\overline{\chi}(G) = m - 2n$ iff it admits a triangle–covering.

We first prove that if G admits a triangle–covering, then it can be colored with $m-2n$ colors: Let $T_1, \ldots, T_n$ be the triangles of the triangle–covering. We color the edges of T_i with the color i and the remaining $m-3n$ edges with colors $n+1, \ldots, m-2n$. This gives a proper coloring using exactly $m-2n$ colors.

Let c be a coloring using $\overline{\chi}(H)$ colors. The color subgraphs of G are only color–paths and color–cycles due to Lemma 4. Let P be the number of the color–paths and Q be the number of the color–cycles; let p_i be the lengths of the color–paths and q_j be the lengths of the color–cycles. The coloring of H uses exactly the following number of colors:

$$P+Q = m - \sum_i p_i - \sum_j q_j + P + Q =$$

$$m - (\sum_i p_i + \sum_j q_j - P)(1 - \frac{Q}{\sum_i p_i + \sum_j q_j - P})$$

Since each $\mathcal{C}$–vertex has to be either an inner vertex of a color–path or a vertex of a color–cycle, it has to hold that $\sum_i p_i + \sum_j q_j - P \geq 3n$. If the coloring c uses at least $m-2n$ colors, than it has to hold that $\frac{Q}{\sum_i p_i + \sum_j q_j - P} \geq \frac{1}{3}$. On the other hand, it holds that $P \leq \sum_i p_i$ and $Q \leq \frac{1}{3}\sum_j q_j$, since each color–path has length at least one and each color–cycle has length at least three; the equality is only in the case that all p_i's are ones and all q_j's are threes. Thus the inequality $\frac{Q}{\sum_i p_i + \sum_j q_j - P} \geq \frac{1}{3}$ implies that $p_i = 1$ for all i's (thus $\sum_i p_i = P$) and $q_j = 3$ for all j's. Since $\frac{Q}{\sum_i p_i + \sum_j q_j - P} = \frac{1}{3}$, it has to hold that $\sum_i p_i + \sum_j q_j - P = \sum_j q_j = 3n$. Thus each vertex is contained in exactly one color–cycle and all the color–cycles have length three; the color–cycles are triangles of a covering of G. This finishes the reduction. $\Box$

4 Algorithms

We focus our attention to algorithms for mixed hypergraphs with maximum degree two in this section. We first introduce several definitions.

We say that a vertex u of a mixed hypergraph H of maximum degree two is *doubled* if there is another vertex v such that u and v are in the same two $\mathcal{C}$–edges of H; the *doubling number* of H is the maximum number of its doubled vertices such that no two of them are in the same $\mathcal{C}$–edge. The doubling number of a mixed hypergraph can be easily obtained by a matching algorithm: This can be easily seen if we translate the notion of doubling number to mixed multigraphs: The doubling number of the original mixed hypergraph is equal to the maximum number of disjoint pairs of $\mathcal{C}$–vertices of the mixed multigraph such that the vertices of each pair are joined by at least two edges. We say that a triple of vertices u, v and w of a mixed hypergraph H of maximum degree two form a *$\mathcal{C}$–triangle* if each pair of them is contained in a $\mathcal{C}$–edge of H and they are not all contained in one $\mathcal{C}$–edge (thus the $\mathcal{C}$–edges of H for different pairs of them have to be different). Note that the edges corresponding to u, v and w in the mixed multigraph corresponding to the mixed hypergraph H form a triangle whose all three vertices are $\mathcal{C}$–vertices. A *triangular number* of H is the maximum possible number of triples of vertices such that the $\mathcal{C}$–edges containing pairs of their vertices are mutually distinct, i.e. the triangular number of H is equal to the maximum number of vertex disjoint triangles in the corresponding mixed multigraph such that the vertices of these triangles are only $\mathcal{C}$–vertices. Note that the determining of a triangular number of a mixed hypergraph of maximum vertex degree two is NP–complete since the problem of graph triangle–covering is a well–known NP–complete problem as mentioned in the proof of Theorem 2 in Section 3. On the other hand, it can be checked in polynomial time whether the triangular number is non–zero.

We first prove an upper bound on the upper chromatic number of mixed hypergraphs with maximum vertex degree two:

Lemma 5 *Let H be a reduced mixed hypergraph with maximum degree at most two. Then $\overline{\chi}(H) \le n - 3m/4 + d/2 + 5t/4$ where n is the number of the vertices of H, m is the number of the $\mathcal{C}$–edges of H, d is the doubling number of H and t is the triangular number of H.*

Proof: Let G be the mixed multigraph from Observation 2; G contains exactly m $\mathcal{C}$–vertices and n edges. Let c be its coloring using $\overline{\chi}(G) = \overline{\chi}(H)$ colors. The color subgraphs are only color–paths or color–cycles due to Lemma 4. Let P be the number of the color–paths and Q the number of the color–cycles; let p_i be the lengths of the color–paths and q_j the lengths of the color–cycles.

We can assume w.l.o.g. that all the color–cycles are vertex–disjoint and all the vertices of each color–cycle are only $\mathcal{C}$–vertices: Otherwise, let C_1 and C_2 be two (non–disjoint) color–cycles, let v be one of their possibly more common vertices and let w be one of the neigbours of v in C_1. Or, let C_1 be a color–cycle containing a non–$\mathcal{C}$–vertex v and let w be one of the neighbours of v in C_1. We create a new vertex v' which is neither a $\mathcal{C}$–vertex nor a $\mathcal{D}$–vertex and we replace the edge vw by an edge $v'w$. The same coloring (which assigns the color of the replaced edge vw to the edge $v'w$) is a proper coloring of the new mixed multigraph and thus its upper chromatic number is at least the upper chromatic number of the original one. Neither n nor m is affected and the doubling number d and the triangular number t can only be decreased. Repeating this procedure yields a mixed multigraph such that one of its coloring using the upper chromatic number of colors contains only vertex–disjoint color–cycles.

The coloring c of G uses exactly the following number of colors:

$$P+Q=n-\sum_i p_i-\sum_{j,q_j=2} q_j-\sum_{j,q_j=3} -q_j\sum_{j,q_j>3} q_j+P+Q=$$

$$n-\sum_i p_i-\sum_{j,q_j=2} 2-\sum_{j,q_j=3} 3-\sum_{j,q_j>3} q_j+P+Q=$$

$$n-(\sum_i p_i+\sum_{j,q_j>3} q_j-P)(1-\frac{Q-\sum_{j,q_j=2}1-\sum_{j,q_j=3}1}{\sum_i p_i+\sum_{j,q_j>3} q_j-P})-\sum_{j,q_j=2} 1-\sum_{j,q_j=3} 2\leq$$

$$n-\frac{3}{4}(\sum_i p_i+\sum_{j,q_j>3} q_j-P)-\sum_{j,q_j=2} 1-\sum_{j,q_j=3} 2=$$

$$n-\frac{3}{4}(\sum_i p_i+\sum_j q_j-P)+\sum_{j,q_j=2} 1/2+\sum_{j,q_j=3} 1/4\leq n-\frac{3}{4}m+\frac{1}{2}d+\frac{1}{4}t$$

The first inequality is due to the trivial facts that $P\leq\sum_i p_i$ and $Q-\sum_{j,q_j=2}1-\sum_{j,q_j=3}1\leq 1/4\sum_{j,q_j>3}q_j$ (the length of each summed cycle on each side is at least four). The last inequality holds, since each $\mathcal{C}$–vertex is either an inner vertex of a color–path or a vertex of a color–cycle and thus $\sum_i p_i+\sum_j q_j-P\geq m$, and since the number of the disjoint color cycles of length two consisting only of $\mathcal{C}$–vertices in the coloring using the maximum possible number of colors is at most the doubling number and the number of the disjoint color cycles of length three consisting only of $\mathcal{C}$–vertices is at most the triangular number. $\Box$

We start with the promised algorithmic issues in the following lemmas:

Lemma 6 *Let H be a reduced mixed hypergraph with maximum degree two. There is a polynomial algorithm which outputs a coloring of H using at least $n-m+d+\max\{0,(t-2d)/3\}$ colors where n is the number of the vertices of H, m is the number of the $\mathcal{C}$–edges of H, d is the doubling number of H and t is the triangular number of H.*

Proof: Let us deal with the case that $d=t=0$, first. Let M be the number of edges of H. We first find a matching of size M between the vertices and the edges of H such that each edge is matched to one of its vertices and no two edges are matched to the same vertex. This matching exists, since each vertex of H is contained in at most two edges and each edge of H contains at least two vertices. Let $V_{\mathcal{C}}$ be the set of the vertices of H matched to the $\mathcal{C}$–edges of H. We color the remaining $n-m$ vertices of H with mutually distinct colors. We extend this coloring to the remaining m vertices of H. If there exists an edge such that the vertex matched to it is its only yet uncolored vertex, we color it as follows: Let C be such an edge and let v be its uncolored vertex. We distinguish two cases:

- The vertex v is contained only in $\mathcal{C}$–edges.
  We color v with any color used on some other vertex of C.
- The vertex v is contained in the $\mathcal{C}$–edge C and in a $\mathcal{D}$–edge D.
  Let c_0 be a color used on some vertex of D (D contains a colored vertex, since at least the vertex matched to D is colored). If the vertices of C different from v are all colored by c_0, we color v with any color different from c_0. If there are more colors used to color other vertices of C, we color v with any color used to color a vertex of C different from the color c_0.

We repeat this procedure as long as there exists an edge whose only uncolored vertex is the vertex matched to it. Note that when we finish coloring the vertices of the $\mathcal{C}$–edge C, the edge C is colored properly (recall that H is reduced and thus each C–edge of H contains at least three vertices). We don't violate coloring of $\mathcal{D}$–edges of H during the extension of the coloring.

We have extended the coloring to some vertices of H, but possibly not to all of them yet. Each $\mathcal{C}$–edge contains either no or at least two uncolored vertices. There may also be some $\mathcal{D}$–edges with uncolored vertices, but each $\mathcal{D}$–edge contains at least one colored vertex (that one which was matched to it). We color all the uncolored vertices with a completely new color — this finishes properly coloring of all the yet not fully colored $\mathcal{C}$–edges and $\mathcal{D}$–edges. Thus we have just obtained a proper coloring of H using at least $n-m$ colors.

We deal with the case that $d>0$ or $t>0$. Let $u_1,\ldots,u_d$ be a set of its doubled vertices such that no two of them are contained together in a $\mathcal{C}$–edge and let $v_1,\ldots,v_d$ be the copies of these vertices (i.e. u_i and v_i are contained in the same two $\mathcal{C}$–edges), obtained by a matching algorithm as described earlier in this section. We color each pair of vertices u_i and v_i with the same color and we remove these vertices together with the $\mathcal{C}$–edges, which they are contained in, from the mixed hypergraph. We obtain a reduced mixed hypergraph with $n-2d$ vertices, $m-2d$ $\mathcal{C}$–edges and with no doubled vertices. Let t' be the triangular number of the obtained mixed hypergraph; it is trivial to see that $t'\geq t-2d$. If it contains a $\mathcal{C}$–triangle, we colour all its vertices by the same colour and remove them together with the three $\mathcal{C}$–edges in which they are contained from the mixed hypergraph. We repat this step until the mixed hypergraph contains a $\mathcal{C}$–traingle. Let t'' be the number of the repetitions of this step. Note that $t''\geq t'/3$, since in each step the triangular number is decreased by at most 3. We obtain a reduced mixed hypergraph whose both the doubling number and the triangular number are equal to zero. It contains $n-2d-3t''$ vertices and $m-2d-3t''$ $\mathcal{C}$–edges. We proceed as described above and we obtain its coloring using at least $(n-2d-3t'')-(m-2d-3t'')=n-m$ colors. We extend this coloring to the removed vertices (using $d+t''$ new colors) and we get a proper coloring of the original mixed hypergraph using at least $n-m+d+t''$ colors. Since $t''\geq t'/3\geq(t-2d)/3$, the lemma is proven. $\Box$

Lemma 7 *Let H be a reduced mixed hypergraph with maximum degree at most two. H is two–colorable iff it does not contain an odd cycle consisting of vertices and $\mathcal{D}$–edges of size two. Moreover there exists a polynomial algorithm which finds the coloring using exactly $\chi(H)$ colors.*

Proof: If H contains an odd cycle consisting of vertices and $\mathcal{D}$–edges of size two, then it is clearly not two–colorable. Let us prove the opposite implication. Since H is reduced, all the $\mathcal{C}$–edges of H have size at least three and they are colored properly in any (not necessarily proper) two–coloring of H. Thus we can assume w.l.o.g. that H contains only $\mathcal{D}$–edges. We further replace any $\mathcal{D}$–edge D of H with four or more vertices by any three–vertex subset D' of D. This new mixed hypergraph H' has still maximum degree at most two, it does not contain an odd cycle consisting of $\mathcal{D}$–edges of size two and the sizes of its $\mathcal{D}$–edges are only two or three. We replace further its three–vertex edges with their two–vertex subsets, but this step needs some care. Let $D = \{u, v, w\}$ be a $\mathcal{D}$–edge of H' consisting of three vertices. If replacing D with $\{u, v\}$ introduces an odd cycle consisting of $\mathcal{D}$–edges of size two, then there is a path between u and v consisting of an even number $\mathcal{D}$–edges of size two; each vertex of this path is contained in exactly two $\mathcal{D}$–edges, those edges used by this path; it cannot be contained in more, since the maximum degree of H_0 is two. Thus there is not such a path between u and w and replacing D with $\{u, w\}$ will not introduce an odd cycle consisting of $\mathcal{D}$–edges of size two. This assures that we can replace D with $\{u, v\}$ or $\{u, w\}$ without introducing an odd cycle consisting of $\mathcal{D}$–edges of size two. We proceed in this manner as long as there are some edges of size three. Let H'' be the obtained mixed hypergraph. This is actually a simple bipartite graph which is clearly two–colorable. Its coloring is also a proper coloring of H' and thus a proper coloring of the original mixed hypergraph H.

The algorithm for finding the coloring with $\chi(H)$ colors proceeds as follows: If H contains no $\mathcal{D}$–edges, we color all the vertices of H with the same color. If H contains $\mathcal{D}$–edges, the algorithm checks whether it contains an odd cycle consisting of $\mathcal{D}$–edges of size two. If not, we use the procedure described in the previous paragraph to get a two–coloring. If it contains an odd cycle consisting of $\mathcal{D}$–edges of size two, we find a proper coloring of H as described in Lemma 6 and then we find a three–coloring as described in Lemma 2. □

We are now ready to state and prove the main theorem of this section:

Theorem 3 *There exist the following polynomial algorithms for mixed hypergraphs with maximum degree two:*

- *to decide colorability;*
- *to determine the lower chromatic number and to find a coloring using $\chi(H)$ colors;*

- *to find a coloring using at least $2\overline{\chi}(H)/3$ colors. This algorithm is a $3/2$–aproximation algorithm for the upper chromatic number of H.*

Proof: The first statement follows from Lemma 1. The second statement follows from Lemma 7. As to the last statement: We first reduce the given mixed hypergraph; this does not affect its feasible set, in particular it does not change its upper chromatic number, and the proper colorings of the reduced mixed hypergraph one–to–one correspond to the proper colorings of the original one (Lemma 1). Let H be the obtained mixed hypergraph. We can find a proper coloring using at least $n - m + d + \max\{0, (t - 2d)/3\}$ colors due to Lemma 6 where n is the number of the vertices of H, m is the number of its $\mathcal{C}$–edges, d is its doubling number and t is its triangular number; let k be the actual number of colors this coloring uses. Since each $\mathcal{C}$–edge contains at least three vertices, it holds that $m \le 2n/3$. On the other hand, we know that $\overline{\chi}(H) \le n - 3m/4 + d/2 + t/4$ due to Lemma 5. We distinguish two cases: $t \le 2d$ and $t > 2d$. In the first case ($t \le 2d$), easy calculation gives the desired estimate:

$$\frac{k}{\overline{\chi}(H)} \ge \frac{n-m+d}{n-3m/4+d/2+t/4} = 1 - \frac{m/4-d/2+t/4}{n-3m/4+d/2+t/4} \ge$$

$$1 - \frac{m/4}{n-3m/4+d/2+t/4} \ge 1 - \frac{m/4}{n-3m/4} \ge 1 - \frac{2n/12}{n-6n/12} =$$

$$1 - \frac{1/6}{1/2} = 1 - \frac{1}{3} = \frac{2}{3}$$

The case that $t > 2d$ is no harder:

$$\frac{k}{\overline{\chi}(H)} \ge \frac{n-m+d+(t-2d)/3}{n-3m/4+d/2+t/4} = \frac{n-m+d/3+t/3}{n-3m/4+d/2+t/4} =$$

$$1 - \frac{m/4+d/6-t/12}{n-3m/4+d/2+t/4} \ge 1 - \frac{m/4+d/6}{n-3m/4+d/2} \ge 1 - \max\left\{\frac{m/4}{n-3m/4}, \frac{d/6}{d/2}\right\} \ge$$

$$1 - \max\left\{\frac{2n/12}{n-6n/12}, \frac{1}{3}\right\} = 1 - \max\left\{\frac{1/6}{1-1/2}, \frac{1}{3}\right\} = 1 - \frac{1}{3} = \frac{2}{3}$$

We used the following trivial inequality for $\alpha, \beta, \gamma, \delta > 0$:

$$\frac{\alpha+\gamma}{\beta+\delta} \le \max\left\{\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right\}$$

Due to the compatition above, the algorithm really finds a coloring using at least $2\overline{\chi}(H)/3$ colors. $\square$

5 NP–hardness of Gaps in Feasible Sets

We focus our attention on the following algorithmic problem: "Is the feasible set of an input mixed hypergraph gap–free?" We first deal with NAE–satisfiability to prepare for the proof of the main theorem in this section.

A formula is called *necessarily unbalanced* if in every NAE–satisfying assignment at least 80% of variables have the same value.

Lemma 8 *Not-all-equal satisfiability of necessarily unbalanced formulae without negations is NP-complete.*

Proof: Not-all-equal satisfiability is known to be NP-complete for formulae without negations. Given such a formula Φ with n variables, construct Φ' by introducing $4n+5$ new variables $y_0, y_1, \ldots, y_{4n+4}$ and adding to Φ clauses (y_0, y_i), $i = 1, 2, \ldots, 4n+4$. Obviously, Φ' is NAE-satisfiable iff Φ is, and in any NAE-satisfying assignment, all variables y_i, $i = 1, 2, \ldots, 4n+4$ have the same value; i.e., at least $4n+4$ variables out of $5n+5$ variables of Φ' have the same value. $\Box$

Theorem 4 *It is NP-hard to decide whether the feasible set of a given mixed hypergraph contains a gap.*

Proof: For a necessarily unbalanced formula Φ on n variables without negations, we create a mixed hypergraph $H = (V, \mathcal{C}, \mathcal{D})$ with $v = 4n$ vertices such that:

1. H has nonempty feasible set iff Φ is NAE-satisfiable
2. $\mathrm{F}(H) \subset [1..\frac{11v}{20}] \cup [\frac{14v}{20}..v]$
3. $\mathrm{F}(H) \cap [1..\frac{11v}{20}] \neq \emptyset$ and $\mathrm{F}(H) \cap [\frac{14v}{20}..v] \neq \emptyset$ if Φ is NAE-satisfiable (and hence $\mathrm{F}(H)$ contains a gap in this case)

Let X be the set of variables of Φ and let C be the set of its clauses. For every variable $x \in X$, we introduce four vertices x_1, x_2, x_3, x_4, and we set $V = \{x_1, x_2, x_3, x_4 | x \in X\}$. On the four vertices arising from a variable x we create the following edges:

$$\mathcal{C}_x = \{(x_1, x_3, x_4), (x_2, x_3, x_4)\} \text{ and } \mathcal{D}_x = \{(x_1, x_2), (x_1, x_3, x_4), (x_2, x_3, x_4)\}$$

and we call $H_x = (\{x_1, x_2, x_3, x_4\}, \mathcal{C}_x, \mathcal{D}_x)$ the *variable gadget.*

For any two distinct variables $x, y \in X$, we introduce the following $\mathcal{D}$-edges:

$$\mathcal{D}_{xy} = \{(x_i, y_j) | 1 \leq i, j \leq 4\}$$

Thus the vertices of different variable gadgets must be colored by different colors.

Finally, for every clause $\phi \in C$, we introduce the following $\mathcal{C}$-edges:

$$\mathcal{C}_\phi = \{(x_3, x_4 | x \in \phi), (x_1, x_3 | x \in \phi)\}$$

Our hypergraph H is then constructed as follows:

$$H = (V, \mathcal{C} = \bigcup_{x \in X} \mathcal{C}_x \cup \bigcup_{\phi \in C} \mathcal{C}_\phi, \mathcal{D} = \bigcup_{x \in X} \mathcal{D}_x \cup \bigcup_{x \neq y \in X} \mathcal{D}_{xy})$$

Suppose H admits a proper coloring c. Since colors used on different variable gadgets are distinct, we may assume that colors used on vertices of H_x are x_1^*, x_2^*, x_3^* (since each H_x has at least one $\mathcal{C}$-edge, the four vertices x_1, x_2, x_3, x_4 of H_x cannot be all colored by distinct colors). We may further assume without loss of generality that $c(x_1) = x_1^*, c(x_2) = x_2^*$. If $c(x_3) \notin \{x_1^*, x_2^*\}$ then $c(x_3) = c(x_4) = x_3^*$ (since both (x_1, x_3, x_4) and (x_2, x_3, x_4) must contain two vertices colored by the same color). We call this coloring of Type 1.

On the other hand, if $c(x_3) \in \{x_1^*, x_2^*\}$ (and thus $c(x_4) \in \{x_1^*, x_2^*\}$ due to symmetry of H_x), the $\mathcal{D}$-edges $(x_1, x_3, x_4), (x_2, x_3, x_4)$ imply that either $c(x_3) = x_1^*, c(x_4) = x_2^*$ or $c(x_3) = x_2^*, c(x_4) = x_1^*$. We call such a coloring of Type 2.

We define a truth assignment $\varphi : X \longrightarrow \{\mathrm{T}, \mathrm{F}\}$ so that $\varphi(x) = \mathrm{T}$ iff $c[H_x]$ is of Type 1. We claim that φ NAE-satisfies Φ. Consider a clause ϕ. Since $(x_3, x_4 | x \in \phi) \in \mathcal{C}$, for some variable occuring in ϕ, say y, it must be $c(y_3) = c(y_4)$, i.e. $\varphi(y) = T$. Further, since $(x_1, x_3 | x \in \phi) \in \mathcal{C}$, for some variable occuring in ϕ, say z, it must be $c(z_1) = c(z_3)$, i.e. $\varphi(z) = F$. Hence each clause has at least one True and at least one False variable.

Suppose on the other hand that $\varphi : X \longrightarrow \{\mathrm{T}, \mathrm{F}\}$ NAE-satisfies Φ. Define a coloring c of H as follows:

$$c(x_1) = x_1^*, c(x_2) = x_2^*$$

$$c(x_3) = c(x_4) = x_3^* \text{ if } \varphi(x) = \mathrm{T}$$

$$c(x_3) = x_1^*, c(x_4) = x_2^* \text{ if } \varphi(x) = \mathrm{F}$$

for every variable $x \in X$. This coloring is proper on every variable gadget, it uses different colors for vertices of different gadgets and so it is proper in the hyperedges $\mathcal{D}_{xy}, x, y \in X$. Also the $\mathcal{C}$-edges arising from the clause gadgets are colored properly, because all clauses are NAE-satisfied.

Let c be a proper coloring and φ the corresponding NAE-satisfying truth assignment. The actual number of colors used by c is $k = 2n + |\{x : \varphi(x) = \mathrm{T}\}|$. Since Φ was necessarily unbalanced, either $k \geq 2n + \frac{4}{5}n$ or $k \leq 2n + \frac{n}{5}$, and substituting $n = \frac{v}{4}$ gets the desired bounds. The claim that $\mathrm{F}(H)$ has nonempty intersection with both $[1..\frac{11}{20}v]$ and $[\frac{14}{20}v..v]$ follows from the fact that the negation of a NAE-satisfying truth assignment is a NAE-satisfying assignment as well. $\square$

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