COLORINGS AND HOMOMORPHISMS OF MINOR CLOSED CLASSES

JAROSLAV NEŠETŘIL AND PATRICE OSSONA DE MENDEZ

ABSTRACT. We relate acyclic (and star) chromatic number of a graph to the chromatic number of its minors and as a consequence we show that the set of all triangle free planar graphs is is homomorphism bounded by a triangle free graph. This solves a problem posed in [14] and completes results of [13] where a similar result has been proved for K_k-free graphs for $k = 4, 5$. It also improves the best known bound for the star chromatic number of planar graphs from 80 to 30. Our method generalize to all minor closed classes and puts Hadwiger conjecture in yet another context.

1. INTRODUCTION

Denote by $\chi_a(G)$ the acyclic chromatic number of a graph G, i.e. the minimal number of colors which are sufficient for a (proper) coloring of the vertices of G so that every cycle in G gets at least 3 colors. It is known that χ_a is bounded for graphs of bounded genus and also for bounded degree graphs, see [3, 2] for the best known bounds. Similarly, let $\chi_{\rm st}(G)$ denotes the *star chromatic* number of a graph G, i.e. the minimal number of colors which are sufficient for a (proper) coloring of the vertices of G so that 4 vertices of every path of length 3 in G get at least 3 colors. Clearly $\chi_{\rm st}(G) \geq \chi_a(G)$. It is known that $\chi_{\rm st}$ is bounded whenever χ_a is bounded (folklore, see e.g.[6]).

It is also well known that χ_a differs arbitrarily from χ as it is unbounded for bipartite graphs and even for bipartite 2-degenerated graphs: It suffices to consider the graph $K_n^\star$ which we get from K_n by subdividing every edge by a single vertex. We complement these results by the following result. (We denote by $\prec$ the minor relation for graphs: $G \prec H$ if G can be obtained from a subgraph of H by a sequence of edge-contractions. We consider only loopless simple graphs.)

Theorem 1.1. *There exists a function $f : \mathbf{N} \to \mathbf{N}$ such that for any graph G holds*

$$\chi_a(G) \leq \chi_{\rm st}(G) \leq f(\max \chi(H))$$

where maximum is taken over all minors H of G. In fact $f(n) \leq cn^2$.

It follows that for any minor closed class $\mathcal{C}$, $\{\chi_a(G), G \in \mathcal{C}\}$ is bounded iff $\{\chi(G), G \in \mathcal{C}\}$ is bounded. By the results of [11, 18, 1] we know that this is equivalent to bounded

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oriented chromatic number $\vec{\chi}$ and colorings of graphs with colored and mixed edges, see Theorem 6.1.

We shall make use of the following two (we believe) well known results. We sketch a proof of Lemma 2 for completeness.

Lemma 1 (Mader, 1967). *For each k, there exists some natural number $h(k) < ck\sqrt{\log k}$ (for some constant c) such that every graph of minimum degree at least $h(k)$ contains K_k as a minor.*

Lemma 2. *For any minor closed graph $\mathcal{C}$, $\{\chi(G), G \in \mathcal{C}\}$ is bounded if and only if $\mathcal{C}$ is different from the set of all graphs.*

Proof. If $\{\chi(G), G \in \mathcal{C}\}$ is unbounded then the vertex critical graphs in $\mathcal{C}$ have unbounded minimal degree. Mader's theorem (i.e. previous Lemma 1) applies. □

We discovered Theorem 1.1 in the context of graphs and their homomorphisms. Recall: A *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ satisfying $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. We also write $G \leq H$ if there exists a homomorphism $G \to H$. This quasiorder is called *homomorphism* or *coloring order*. It is well known *Grötsch' s Theorem* (see e.g. [10] and [19] for the best proof) that every triangle free planar graph is 3-colorable. Using homomorphism order this means that $G \leq K_3$ for any triangle free planar graph G. In yet another way we can also say that the class $\mathcal{P}_3$ of all K_3-free graphs is bounded by K_3. The following problem has been formulated in [14]:

Problem 1. *Does there exists a triangle free graph H such that $G \leq H$ for any triangle free planar graph? In other words, is the class $\mathcal{P}_3$ bounded by a triangle free graph H?*

Here we give an affirmative answer to this problem. In fact one can prove that the class $\mathcal{P}_3$ is bounded by a 3-colorable triangle free graph and we also prove an analogous result for any minor closed class $\mathcal{C}$ with bounded chromatic number:

Theorem 1.2. *Let $\mathcal{C}$ be a minor closed class of graphs all of which are k-colorable. Then the class $\mathcal{C}_3$ of all triangle free graphs in $\mathcal{C}$ is bounded by a k-colorable triangle free graph.*

Similar results (for planar graphs) were obtained recently in [13] for the class $\mathcal{P}_4$ (of all K_4-free planar graphs) and for the class $\mathcal{P}$ of all planar graphs. However the case of planar triangle-free graphs was left open. Here we treat the problem in a more general context. All results may be seen as an evidence for the following general conjecture.

Let $\mathcal{A}, B$ be classes of graphs, $\mathcal{A} \subset \mathcal{B}$. We say that the class $\mathcal{A}$ is *bounded in* $\mathcal{B}$ if there exists a graph $H \in \mathcal{B}$ such th at $G \leq H$ for any $G \in \mathcal{A}$. Thus $\mathcal{A}$ is bounded in $\mathcal{A}$ iff $\mathcal{A}$ has the greatest element (with respect to $\leq$). The study of boundednes phenomena is one of the basic problems and we are pleased that in our setting it relates questions like Hadwiger conjecture to the mainstream mathematics (see Remarks).

Given a finite set $\mathcal{F}$ of graphs we denote by $\mathrm{Forb}_h(\mathcal{F})$ the class of all graphs G with no homomorphism $F \to G$ for an $F \in \mathcal{F}$. Equivalently and more formally, $\mathrm{Forb}_h(\mathcal{F}) = \{G; F \in \mathcal{F} \Rightarrow F \not\leq G\}$. As an example, note that $\mathrm{Forb}_h\{K_3\}$ is the class of all triangle free graphs.

Conjecture 1. *Let $\mathcal{F}$ be any finite set of graphs. Then for any minor closed class of graphs $\mathcal{C}$ the class $\mathcal{C} \cap \mathrm{Forb}_h(\mathcal{F})$ is bounded in $\mathrm{Forb}_h(\mathcal{F})$.*

It has been proved in [9] (see also [4] and [13] for a different proof) that for the class $\mathcal{C}_d$ of all graphs with all their vertices bounded by d and for any finite set of graphs $\mathcal{F}$ the analogous conjecture holds. The results of this paper and [13] verify the conjecture for $\mathcal{F} = \{K_k\}$ (for $k \geq 4$ for planar graphs only). Note that for graphs in general and even degenerated classes of graphs the analogous statement fails to be true: For $\mathcal{F} = \{K_3\}$ consider graphs $K_n^{\star\star}$ formed from K_n by subdividing each edge by two new vertices. All the graphs $K_n^{\star\star}$ are 2-degenerated yet they are not bounded by a finite triangle-free graph. Note also that this cannot be saved by (large) girth: Let G be a graph of girth ℓ with chromatic number k. Then the graph $G^{\star\star}$ has girth 3ℓ and there is no homomorphism of G into a triangle free graph with at most k vertices.

This paper is organized as follows: In Section 1 we prove Theorem 1.1. In Section 2 we prove the following result which is perhaps of an independent interest:

Theorem 1.3. *For any minor closed class $\mathcal{C}$ with bounded chromatic number there exists $k = k(\mathcal{C})$ such that any graph $G \in \mathcal{C}$ has a proper k-coloring with the property that any cycle of length ≥ 5 get at least 4 different colors.*

Note that, again, an analogous statement fails to be true in general: In any k-coloring of the graph $K_n^{\star\star}$, n sufficiently large, there exists cycles of length ≥ 6 which are colored by at most 3 colors. Also the graphs $G^{\star\star}$ (see above) have girth 3ℓ and in any k-coloring contain cycles colored by at most 3 colors.(More complicated examples are provided by Ramsey theory.)

The key notion for the proof of Theorem 1.3 is the notion of *folding* and using that we prove in Section 4 Theorem 1.2 by an universal construction similar to those given in [13, 18, 1]. Section 5 contains concluding remarks and open problems.

2. Star chromatic number bounded by density

Theorem 2.1. *Let G be a simple graph with density α, and let $\beta \geq \alpha$ be the maximum density over the minors of G which are simple (including G itself). Then, the star chromatic number $\chi_{\mathrm{st}}(G)$ is bounded by:*

$$\chi_{\mathrm{st}}(G) \leq \lceil\alpha\rceil(2\lceil\beta\rceil + \lceil\alpha\rceil - 1) + \lfloor 2\alpha \rfloor + 1$$

Proof. First, we shall find an acyclic orientation $\vec{G}$ in such a way that each vertex has indegree at most $\lceil\beta\rceil$. (It is well known that a graph G with density $\beta = \max_{H\subseteq G}\frac{|E(H)|}{|V(H)|}$ may be oriented in such a way that any vertex v of G has indegree $d^-(v)$ at most $\lceil\beta\rceil$ (see for instance, Hakimi [8].) Furthermore, we color each arc e of $\vec{G}$ with a color $c(e)\in\{1,\ldots,\lceil\beta\rceil\}$, in such a way that all the arcs incoming a same vertex have different colors.

Then, let $\widetilde{G}$ be the graph $(V(G), E(G)\cup E_1\cup E_2)$, where (denoting (x,y) an arc and $\{x,y\}$ an edge):

- $(x,y)\in E_1$ if $\{x,y\}\notin E(G)$ and $\exists z\in V(G), \{(x,z),(z,y)\}\subseteq E(\vec{G})$,
- $(x,y)\in E_2$ if $\{x,y\}\notin E(G)$ and $\exists z\in V(G), \{(x,z),(y,z)\}\subseteq E(\vec{G})$ with $c((x,z)) < c((y,z))$.

Thus the graph $\widetilde{G}$ is the graph G together with potentially conflicting pairs of vertices for a star coloring. We now estimate the density of $\widetilde{G}$. Let A be any subset of $V(G)$ and let B be the set of neighbors of A. Formally, B is the subset of $V(G)\setminus A$ defined by:

$$B = \{x\in V(G)\setminus A;\quad \exists v\in A, \{v,x\}\in E(G)\}$$

The edges of the subgraph $\widetilde{G}_A$ of $\widetilde{G}$ induced by A may be partitioned as follows: $E(\widetilde{G}_A) = E(G_A)\cup E_1^{\mathrm{i}}\cup E_1^{\mathrm{e}}\cup E_2^{\mathrm{i}}\cup E_2^{\mathrm{e}}$, where:

- $(x,y)\in E_1^{\mathrm{i}}$ if $\{x,y\}\notin E(G_A)$ and $\exists z\in A, \{(x,z),(z,y)\}\subseteq E(\vec{G})$,
- $(x,y)\in E_1^{\mathrm{e}}$ if $\{x,y\}\notin E(G_A)$ and $\exists z\in B, \{(x,z),(z,y)\}\subseteq E(\vec{G})$,
- $(x,y)\in E_2^{\mathrm{i}}$ if $\{x,y\}\notin E(G_A)$ and $\exists z\in A, \{(x,z),(y,z)\}\subseteq E(\vec{G})$ with $c((x,z)) < c((y,z))$.
- $(x,y)\in E_2^{\mathrm{e}}$ if $\{x,y\}\notin E(G_A)$ and $\exists z\in B, \{(x,z),(y,z)\}\subseteq E(\vec{G})$ with $c((x,z)) < c((y,z))$.

Let H be the graph obtained from $G_{A\cup B}$ by deleting any edges having both incidences in B, and let H_i be the graph obtained from H by deleting all the vertices of B having no incoming edge of color i and contracting all the arcs $(x,y), y\in B$ of color i. Moreover, $V(H_i)$ may be identified with A and the edge set of H_i is given by:

$$\begin{aligned} E(H_i) = E(G_A) &\cup \{(x,y)\in E_1^{\mathrm{e}}, \exists z\in B, \{(x,z),(z,y)\}\subseteq E(H) \text{ and } c((x,z))=i\}\\ &\cup \{(x,y)\in E_2^{\mathrm{e}}, \exists z\in B, \{(x,z),(y,z)\}\subseteq E(H) \text{ and } c((x,z))=i\}\end{aligned}$$

Hence, we have, as H_i being a minor of H an thus a minor of G, has density at most β (and order $|A|$):

$$\lceil\alpha\rceil\lceil\beta\rceil|A| \geq \sum_{i=1}^{\lceil\alpha\rceil}|E(H_i)| \geq \lceil\alpha\rceil|E(G_A)| + |E_1^{\mathrm{e}}| + |E_2^{\mathrm{e}}|$$

It follows that

$$\lceil\alpha\rceil\lceil\beta\rceil|A| - \lceil\alpha\rceil|E(G_A)| \geq |E_1^{\mathrm{e}}| + |E_2^{\mathrm{e}}|$$

Moreover, we obviously have $|E_1^{\mathrm{i}}| \leq \lceil\alpha\rceil |E(G_A)|$ and $|E_2^{\mathrm{i}}| \leq \binom{\lceil\alpha\rceil}{2}|A|$.
Summarizing, we get:

$$|E(\widetilde{G}_A)| \leq |E(G_A)| + \left(\lceil\alpha\rceil\lceil\beta\rceil + \binom{\lceil\alpha\rceil}{2}\right)|A|$$

and thus, the density of $\widetilde{G}$ is at most:

$$\max_{A\subseteq V(G)} \frac{|E(\widetilde{G}_A)|}{|A|} \leq \lceil\alpha\rceil\lceil\beta\rceil + \binom{\lceil\alpha\rceil}{2} + \alpha$$

It follows that $\chi(\widetilde{G}) \leq 2\lceil\alpha\rceil\lceil\beta\rceil + \lceil\alpha\rceil(\lceil\alpha\rceil - 1) + \lfloor 2\alpha\rfloor + 1$.

Now, consider a coloring of $\widetilde{G}$ with $\chi(\widetilde{G})$ colors and think of this as a coloring of G. Then, for any path P of length 3 in $\vec{G}$, P either contains a subpath of length 2 which internal vertex is a sink or it contains a directed subpath of length 2. In both cases, according to the definition of $\widetilde{G}$, this path induces a triangle in $\widetilde{G}$ and thus is 3-colored. As a consequence, no bi-colored subgraph of G may include a path of length 3 and, as the coloring is obviously a proper coloring of G (G is a partial graph of $\widetilde{G}$), we get $\chi_{\mathrm{st}}(G) \leq \chi(\widetilde{G})$, which concludes the proof. □

Corollary 1. *The star chromatic number of a planar graph is at most* 30*.*

This improves bounds given in [6].

Proof. For any planar graph G of order $n \geq 3$, we get $\alpha \leq \beta \leq 3 - \frac{6}{n} < 3$. □

As an example where $\alpha \neq \beta$:

Corollary 2. *The star chromatic number of a bipartite planar graph is at most* 18*, the one of a* 4*-regular planar graph is at most* 19*.*

Proof. For any bipartite planar graph G of order $n \geq 3$, we get $\alpha \leq 2 - \frac{4}{n} < 2$ and $\beta < 3$ (as any minor of a bipartite planar graph is planar). For a 4-regular planar graph, $\alpha = 2$ and $\beta < 3$ (as previously). □

Let us remark that the above proof of Theorem 2.1 gives a structural characterization of star chromatic number in terms of chromatic number. Given an orientation $\vec{G}$ of G we define graph $\widetilde{G}$ similarly as in the above proof: $V(G) = V(\widetilde{G})$ with the edges of $E(\widetilde{G})$ being all edges of G together with all pairs $\{x, y\}$ for which there exists a vertex $z \in V(G)$ such that either $(x, z), (z, y) \in E(\vec{G})$ or $(x, z), (y, z) \in E(\vec{G})$.

Corollary 3. $\chi_{st}(G) = \min \chi(\widetilde{G})$ *where minimum is taken over all orientations* $\vec{G}$ *of* G*.*

Proof. Clearly any orientation $\vec{G}$ of G and any coloration of $\widetilde{G}$ gives a star coloring of G. Conversely, given any star coloring c of G define orientation $\vec{G}$ as follows: for an edge $e = \{x, y\}$ of G with colors $c(x) = i, c(y) = j$ the subgraph induced by all the vertices with colors i and j is a star forest and we orient e from the center of the star (in the case that the star has 2 vertices we take any orientation). One can check that c is a proper coloring of the graph $\widetilde{G}$ defined using this orientation. □

3. Acyclic chromatic number bounded by chromatic number

In this section we prove Theorem 1.1. First, we shall relate the density of a graph with the maximum value of the chromatic numbers of its minors.

Lemma 3. *There exists a function $g : \mathbb{N} \to \mathbb{N}$, such that any graph G has a density bounded as follows:*

$$\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|} \leq g(\max_{H \prec G} \chi(H)) \tag{1}$$

Proof. According to Lemma 1, there exists a natural number $h(k)$, such that every graph of minimum degree at least $h(k)$ contains K_k as a minor.

Thus, assume G has density $\alpha = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|}$. Then, it includes as a subgraph a graph H with minimum degree at least α and thus G has K_p as a minor with $h(p) \geq \alpha$. Thus,

$$\max_{H \subseteq G} \frac{|E(H)|}{|V(H)|} \leq h^{-1}(\max_{H \prec G} \chi(H)) \tag{2}$$

□

Now we can apply

$$\chi_a \leq \chi_{st} \leq \lfloor 2\alpha \rfloor + 3\lceil \alpha \rceil^2 - \lceil \alpha \rceil + 1$$

and we get Theorem 1.1.

4. Foldings

Definition 1. *Let G and H be two graphs and let $f : V(G) \to V(H)$ be a homomorphism from G to H. Then f is a* folding *of G in H if*

$$\forall x, y, z \in V(G), \quad \big((x, z) \in E(G) \wedge (y, z) \in E(G)\big) \implies f(x) \neq f(y) \tag{3}$$

The following result generalizes [17] where a similar result was obtained for (no partially colored) degree 3 planar graphs.

Proposition 1. *Let G be an undirected graph with acyclic chromatic number $\chi_a(G)$. Assume G is partially oriented and let Δ^- be the maximum indegree of the partial orientation. Then, there exists an extension of the partial orientation of G into a full orientation and a folding, with respect to this orientation, from G to K_p (with some orientation), where $p \leq \chi_a(G)(\Delta^- + 2)^{\chi_a(G)-1}$.*

Proof. Consider an acyclic coloring of G with $\chi_a(G)$ colors $\{1, \ldots, \chi_a(G)\}$. Let $1 \leq i < j \leq \chi_a(G)$. According to the definition of an acyclic coloring, the subgraph $G_{i,j}$ induced by the vertices colored i or j is a forest. Consider an extension of the orientation of the edges of $G_{i,j}$ such that the originally non-oriented edges of any tree not reduced to an isolated vertex is oriented from a root. Then, each vertex has indegree at most Δ^-+1. Then, compute a function $\mu_{i,j} : V(G_{i,j}) \to \{0, 1, \ldots, \Delta^-+1\}$ as follows: For any isolated vertex v, $\mu_{i,j}(v) = 0$. For any non trivial tree Y, let r be a vertex of the tree of color i and let $X = \{r\}$. While $X \neq Y$, we consider a vertex v in $Y \setminus X$ having a neighbor u in X. Then, we compute $\mu(v)$ as follows:

- if $\{u, v\}$ is oriented from u to v and u has color i, then $\mu_{i,j}(v) = \mu_{i,j}(u)$;
- if $\{u, v\}$ is oriented from u to v and u has color j, then $\mu_{i,j}(v) \equiv \mu_{i,j}(u) + 1 \pmod{\Delta^- + 2}$;
- if $\{u, v\}$ is oriented from v to u and u has color i, then choose $\mu_{i,j}(v) \not\equiv \mu_{i,j}(u) \pmod{\Delta^- + 2}$ while avoiding the values given to the predecessors of u (that is: the neighbors z of u, such that $\{z, u\}$ is oriented from z to u) already in X (at most Δ^- such vertices exist);
- if $\{u, v\}$ is oriented from v to u and u has color j, then choose $\mu_{i,j}(v) \not\equiv \mu_{i,j}(u)+1 \pmod{\Delta^- + 2}$ while avoiding the values given to the predecessors of u already in X (at most Δ^- such vertices exist).

If we consider the union of the preceding partial orientations for all the possibles values of i and j ($1 \leq i < j \leq \chi_a(G)$) and if we recolor any vertex v of color c with the t-uple $(c, \mu_{1,c}(v), \ldots \mu_{c-1,c}(v), \mu_{c,c+1}(v), \ldots, \mu_{c,\chi_a(G)}(v))$, we obtain a natural folding of G in $K_{\chi_a(G)(\Delta^-+2)^{\chi_a(G)-1}}$. □

We prove Theorem 1.3 in the following more technical form. Together with the previous Proposition and Theorem 1.1 this implies Theorem 1.3.

Theorem 4.1. *Let G be a graph and f a folding of G in a complete graph K_p. Then, there exists a coloring of G with at most q colors, such that any cycle of G of length at least 5 has vertices of at least 4 colors, where $q \leq \left(\max_{H<G} \chi(H)\right)^{\binom{p}{3}}$, where $H < G$ stands for "H is a loop-less minor of G".*

Proof. Consider any numbering of the vertices of K_p with integers $1, \ldots, p$ and let $i, j, k \in \{1, \ldots, p\}$ be three distinct integers. Then, f induces a folding of the subgraph $G_{i,j,k}$ of G induced by the vertices mapped by f into one of i, j, k in a directed triangle, which vertices i, j, k. Two case may then occur:

- The triangle (i, j, k) is directed as a circuit.

  In such a case, any vertex in $G_{i,j,k}$ has at most one incoming edge and any cycle of $G_{i,j,k}$ is thus oriented as a circuit. Moreover, no vertex may belong to more than one circuit and every circuit has a length which is a multiple of 3, with vertices successively colored i, j, k. Then, we assign a mark from $\{0, 1\}$ to the vertices: 0 for all the vertices, but one in each circuit. This way, in each cycle of length bigger than 3 there exists at least 3 vertices of different colors with mark 0 and one vertex with mark 1.

- The triangle (i, j, k) is acyclically oriented.

  Then, we may also assume it is directed as $i \to j \to k$. In such a case, no cycle of $G_{i,j,k}$ may be directed as a circuit and hence every cycle γ of $G_{i,j,k}$ includes at least one sink v. As the sink has to have at least two incoming edges, it is colored k and its neighbors have respectively color i and j. Let H be the minor of G obtained by contracting all the edges incident to a vertex colored i and call $g : V(G) \to V(H)$ the identification mapping of the contraction. The loops in H then correspond to initial cycles of length 3. Let $\hat{H}$ be the graph obtained from G by removing these loops (we identify $V(\hat{H})$ with $V(H)$) and let $\chi(\hat{H})$ be its chromatic number. Color $\hat{H}$ with $\chi(\hat{H})$ colors and call $\mu : V(H) \to \{1, \dots, \chi(\hat{H})\}$ the coloring. Then, it is easily checked that any cycle of length at least 5 of G contains at least four vertices v_1, v_2, v_3, v_4 with distinct $(f(v), \mu \circ g(v))$ pairs.

□

5. Triangle free bounds

In this section we prove Theorem 1.2. We proceed analogously as in [13]. Let $\mathcal{C}$ be a minor closed class of graphs. We assume that $\chi(G), G \in \mathcal{C}$ is a bounded set. By Theorems 1.1 and 1.3 we know that there exists a positive q such that any graph $G \in \mathcal{C}$ may be proper colored by q colors in such a way that any cycle of G of length ≥ 5 gets at least 4 colors. We construct graph $H = (V, E)$ as follows: The vertices V are all pairs of the form (i, ϕ) where $1 \leq i \leq q$ and ϕ is a function which assigns to every triple $T = \{i, j, k\}; 1 \leq j < k < l \leq q$ value $\phi(T) \in \{0, 1\}$ with $\phi(T) = 1$ whenever $i \notin T$. The edges E are all pairs of the vertices of the form $\{(i, \phi), (i', \phi')\}$ where $i \neq i'$ and $\phi(T) \neq \phi'(T)$ whenever triple T contains both i and i'. We shall prove that the graph H has all the properties claimed by Theorem 1.2.

The graph H has clearly no triangle as if $(i, \phi), (i', \phi'), (i'', \phi'')$ are 3 vertices of H then considering the triple $T = \{i, i', i''\}$ we see that at least two of the values $\phi(T), \phi'(T), \phi''(T)$ coincide and thus the corresponding vertices do not form an edge of H. Next we prove that H is a bound for the class $\mathcal{C}_3$ of all triangle free graphs in $\mathcal{C}$. Towards this end let $G \in \mathcal{C}_3$ and let $c : V(G) \to \{1, \dots q\}$ be a proper coloring of G guaranteed by Theorem 1.3. Given a triple $T \subset \{1, \dots q\}$ the subgraph G_T of G induced by the set $c^{-1}(T)$ is bipartite and thus there exists a homomorphism (coloring) $\phi_T : G_T \to K_2$. Define the mapping $f : V(G) \to V(H)$ as follows: $f(v) = (c(v), \phi)$ where $\phi(T)(v) = \phi_T(v)$ providing $v \in T$ and $\phi(T)(v) = 1$ otherwise. It is easy to check that this is a homomorphisms $G \to H$. Finally suppose that all graphs from the class $\mathcal{C}$ are k-colorable. Then the graph $H \times K_k$ is k-colorable triangle free bound for $\mathcal{C}_3$. This completes the proof of Theorem 1.2. □

¿From Theorem 1.2, one can deduce the following:

Corollary 4. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any minor closed class of graph $\mathcal{C}$ with maximum chromatic number k, any triangle free graph $G \in \mathcal{C}$ may be properly colored in $f(k)$ colors, in such a way that any subgraph H of G gets a number of colors at least equal to the minimum number of vertices of a triangle free graph with chromatic number $\chi(H)$.*

Proof. According to Theorem 1.2, there exists a triangle free graph U with order $f(k)$, such that $G < U$, for any $G \in \mathcal{C}$. Color the vertices of G according to their image by a homomorphism from G to U. Then, any subgraph H of G is mapped into a subgraph of U_H of U. The graph H has $|V(U_H)|$ colors and chromatic number $\chi(H) \leq \chi(U_H)$. □

Corollary 5. *There exists a constant c, such that, for any triangle free graph G, there exists a coloration of G with $f(\max_{H \prec G} \chi(H))$ colors, such that any subgraph H of G with chromatic number k gets at least $ck^2 \log k$ colors.*

Proof. According to Erdös and Hajnal (1985)[5], a triangle free graph with chromatic number k has order at least $ck^2 \log k$. □

6. Remarks

1. The notion of folding is an interesting notion as it is sandwiched between locally injective homomorphisms (studied e.g. in [15] and [7] from complexity point of view) and homomorphisms of bounded degree graphs. Because of this we formulated Proposition 1 in a greater generality (than necessary for our proof of Theorem 1.2. More results are going to appear in our forthcoming paper [16].

2. Our Conjecture 1 may be seen as a finitary approximation to Hadwiger conjecture, see e.g. [10]. In our language Hadwiger conjecture may be expressed as follows:

Conjecture 2. *(Hadwiger)*

Any minor closed class $\mathcal{C}$ with bounded chromatic number has greatest element which is a complete graph.

In the other words, if a minor closed class $\mathcal{C}$ is bounded (by a finite graph, for example by a large complete graph) then it has the greatest element which is

a complete graph. In this context one may see our Conjecture 1 as an approximation to Hadwiger conjecture: instead of asking for the greatest element of class $\mathcal{C}$ we ask for a bound with local properties similar to those in $\mathcal{C}$ (such as not containing a given complete graph). On the other hand the following naturally arises as a weaker form of Hadwiger conjecture:

Conjecture 3. *Any minor closed class $\mathcal{C}$ with bounded chromatic number has greatest element.*

3. The following is a consequence of Theorem 1.1

Theorem 6.1. *(Characterization of bounded minor closed classes)*
Let $\mathcal{C}$ be a a minor closed class of graphs. The following statements are equivalent
i. the acyclic chromatic number $\chi_a(G)$ is bounded for $G \in \mathcal{C}$;
ii. the oriented chromatic number $\vec{\chi}(G)$ is bounded for $G \in \mathcal{C}$;
iii. the star chromatic number $\chi_{st}(G)$ is bounded for $G \in \mathcal{C}$;
iv. colored mixed graphs in $\mathcal{C}$ may be colored by a fixed number of colors (in the sense of [1, 18]*);*
v. the chromatic number $\chi(G)$ is bounded for all $G \in \mathcal{C}$;
vi. the clique number $\omega(G)$ is bounded for all $G \in \mathcal{C}$;
vii. the edge density of all graphs G is bounded for all $G \in \mathcal{C}$.
vii. $\mathcal{C}$ is proper *minor closed class of graphs (i.e. $\mathcal{C}$ is not the class of all graphs).*

We shall comment on this more extensively in [16].

4. The following is an interesting consequence of Theorem 1.2 presented as a problem in [13]: For a graph $G = (V, E)$ and a positive integer t define the graph $G^{(t)} = (V, E^{(t)})$ by $\{x, y\} \in E^{(t)}$ iff $d_G(x, y) = t$ ($d_G(x, y)$ is the distance of x and y in G). Note that for an even t the graph $G^{(t)}$ may contain an arbitrarily large complete graph even for a simple graph G (consider subdivision of a star). However for $t = 3$ we have the following, perhaps surprising, general result:

Corollary 6. *For every minor closed class $\mathcal{C}$ the following two statements are equivalent:*
i. The chromatic number of triangle free graphs from $\mathcal{C}$ is bounded;
ii. The chromatic number of graphs $\chi(G^{(3)}), G \in \mathcal{C}, G$ triangle free is bounded.

Proof. Any homomorphism $f : G \to H$ for a triangle free graph H may be viewed as a coloring of $G^{(3)}$ by $|V(G)|$ colors. Thus $i.$ implies $ii.$ by Theorem 1.2. In the reverse direction suppose contrary: let $\chi(G^{(3)} \leq k$ for any triangle free $G \in \mathcal{C}$ and assume that there exists a triangle free graph G in $\mathcal{C}$ with $\chi(G) \geq 3k$. Let $V(G) = V_1 \cup \ldots \cup V_k$ be a coloring of the graph $G^{(3)}$. It follows that there exists $i, 1 \leq i \leq k$ such that the subgraph G_i of G induced by the set V_i is not bipartite. As G_i is triangle free and as V_i is color class of a coloring of $G^{(3)}$ G_i does not contain an induced copy of the path of length 3. However it is well known that then G_i is a perfect graph which is a contradiction. □

(Note that the implication $ii. \Rightarrow i.$ holds for any class of graphs. The reverse implication does not hold generally. This proposition may be formulated in terms of multiplication of incidence matrices.) Clearly many further questions may be asked. We decided to stop here (and continue in [16]).

5. It is a bit surprising that in order to prove the main Theorem 1.2 for planar graphs one can proceed in purely combinatorial way for all minor closed classes. Presently, this is an exception and we do not know how to extend the geometrical techniques of [13] to minor closed classes. Particularly, the following is presently open:

Conjecture 4. *For any $k \geq 3$ there exists a function f_k such that every graph G has a proper coloring with at most $f_k(\max\{\chi(H); H \prec G\})$ colors such that any k-color critical subgraph G of G is either isomorphic to K_k or gets at least $k+1$ colors.*

This is true for $k = 3$ (by Theorem 1.2) and for $k = 4$ for the class of planar graphs ([13]). It is proved in [13] that the the validity of Conjecture 4 implies the existence of a K_k-free bound. For the completeness we give here a simple proof of this particular case.

Lemma 4. *Let $\mathcal{C}$ be a minor closed class of graphs, such that $\chi(G), G \in \mathcal{C}$ is bounded by q. Let $3 \leq k \leq q$ be an integer. Assume there exists a function f such that any graph $G \in \mathcal{C}$ has a coloring with $f(q)$ colors such that any k color-critical subgraph of G is either isomorphic to K_k or gets at least $k+1$ colors.*

Then, the class $\mathcal{C}_k$ of all K_k free graphs in $\mathcal{C}$ is bounded by a q-colorable K_k free graph.

Proof. We construct graph $U = (V, E)$ as follows: The vertices V are all pairs of the form (i, ϕ) where $1 \leq i \leq f(q)$ and ϕ is a function which assigns to every k-uple $X = \{i_1, \ldots, i_k\}; 1 \leq i_1 < \cdots < i_k \leq f(q)$ value $\phi(X) \in \{1, \ldots, k-1\}$. The edges E are all pairs of the vertices of the form $\{(i, \phi), (i', \phi')\}$ where $i \neq i'$ and $\phi(X) \neq \phi'(X)$ whenever k-uple X contains both i and i'. We shall prove that the graph U has all the properties claimed by Lemma 4.

The graph U has clearly no subgraph isomorphic to K_k as if $(i_1, \phi_1), \ldots, (i_k, \phi_k)$ are k vertices of U, then considering the k-uple $X = \{i_1, \ldots, i_k\}$ we see that at least two of the values $\phi_1(X), \ldots, \phi_k(X)$ coincide and thus the corresponding vertices do not form an edge of U. Next we prove that U is a bound for the class $\mathcal{C}$. Towards this end let $G \in \mathcal{C}_k$ and let $c : V(G) \to \{1, \ldots f(q)\}$ be a proper coloring of G guaranteed by the assumptions of the lemma. Given a k-tuple $X \subset \{1, \ldots f(q)\}$ the subgraph G_X of G induced by the set $c^{-1}(X)$ may not have chromatic number k. Otherwise, G_X would have a k color-critical subgraph isomorphic to K_k. Hence, G_X is $(k-1)$-colorable and thus there exists a homomorphism (coloring) $\phi_X : G_X \to K_{k-1}$. Define the mapping $g : V(G) \to V(U)$ as follows: $g(v) = (c(v), \phi)$ where $\phi(X)(v) = \phi_X(v)$ providing $v \in X$ and $\phi(X)(v) = 1$ otherwise. It is easy to check that this is a homomorphisms $G \to U$. Then the graph $U \times K_q$ is q-colorable K_k free bound for $\mathcal{C}_k$. This completes the proof of Theorem 4. $\square$

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Department of Applied Mathematics, and, Institute of Theoretical Computer Science (ITI), Charles University, Malostranské nám.25, 11800 Praha 1, Czech Republic

E-mail address: nesetril@kam.ms.mff.cuni.cz

Centre d'Analyse et de Mathématiques Sociales, CNRS, UMR 8557, 54 Bd Raspail, 75006 Paris, France

E-mail address: pom@ehess.fr