Mixed Hypercacti

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Abstract

A mixed hypergraph H is a triple $(V, \mathcal{C}, \mathcal{D})$ where V is its vertex set and $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V (called $\mathcal{C}$–edges and $\mathcal{D}$–edges). A vertex coloring of H is proper if each $\mathcal{C}$–edge contains two vertices with the same color and each $\mathcal{D}$–edge contains two vertices with different colors. The feasible set of H is the set of all k's such that there exists a proper coloring using exactly k colors. The feasible set is gap–free if it is an interval of integers.

A graph is a strong/weak cactus if all its cycles are vertex/edge–disjoint. A hypergraph is spanned by a graph (with the same vertex set) if the edges of the hypergraph induce connected subgraphs. A strong/weak hypercactus is spanned by a strong/weak cactus. We prove that the feasible set of any mixed strong hypercactus is gap–free. We find infinitely many mixed weak hypercacti such that the feasible set of any of them contains a gap. For each connected non–planar graph $G \neq K_5$, we find a mixed hypergraph spanned by G whose feasible set contains a gap.

1 Introduction

A strict k–coloring is a proper coloring using precisely k colors. Coloring problems are among the most intensively studied combinatorial problems. The numbers of colors which can be used to color strictly a given combinatorial object often form an interval of integers: this is not the case for all mixed hypergraphs (see below). In this paper, we address the question: Which mixed hypergraphs have the property that the numbers of colors which can be used in their strict colorings form intervals?

A mixed hypergraph G is a triple $(V, \mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of V; the members of V are called vertices, the members of $\mathcal{C}$ are called

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$\mathcal{C}$–edges and the members of $\mathcal{D}$ are called $\mathcal{D}$–edges. A *proper k–coloring* c of G is a mapping $c : V \to \{1, \ldots, k\}$ such that there are two vertices with $\mathcal{D}$ifferent colors in each $\mathcal{D}$–edge and two vertices with a $\mathcal{C}$ommon color in each $\mathcal{C}$–edge. A proper coloring c is a *strict k–coloring* if it uses all k colors (as stated already above). A mixed hypergraph is colorable iff G has a proper coloring. Mixed hypergraphs generalize several previously studied notions of special types of colorings: coloring block designs (see [3, 4]), list–coloring of graphs ([10]) and others.

The *feasible set* $\mathcal{F}(G)$ of a mixed hypergraph G is the set of all k's such that there exists a strict k–coloring of G. The *(lower) chromatic number* $\chi(G)$ of G is the minimum number in $\mathcal{F}(G)$ and the *upper chromatic number* $\bar{\chi}(G)$ of G is the maximum number in $\mathcal{F}(G)$. The feasible set of G is *gap–free* (unbroken) iff $\mathcal{F}(G) = [\chi(G), \bar{\chi}(G)]$; we write $[a, b]$ for the set of integers between a and b (inclusively). If the feasible set of G contains a gap, we say it is broken. An example of a mixed hypergraph with a broken feasible set was first given in [5]. It was further proved in [9] that the feasible sets of mixed hypergraphs with maximum degree two are gap–free and there exists a mixed hypergraph of maximum degree three with a broken feasible set.

A hypergraph $H = (V, \mathcal{E})$ is *spanned* by a graph $G = (V, E)$ if each edge of H induces a connected subgraph of G. We introduce several other definitions and some basic properties of mixed hypergraphs spanned by different classes of graphs in Section 2. We recall several classes of graphs based on the structure of their cycles:

- A connected graph G is a *tree* if it does not contain a cycle.
- A connected graph G is *unicyclic* if it contains exactly one cycle.
- A connected graph G is a *hairy cycle* if it contains exactly one cycle and each vertex not contained in the cycle is adjacent to a vertex of the cycle.
- A connected graph G is a *strong cactus* if each two of its cycles are vertex–disjoint.
- A connected graph G is a *weak cactus* if each two of its cycles are edge–disjoint.

A hypergraph is an *interval hypergraph/hypertree/unicyclic hypergraph/hairy cyclic hypergraph/strong hypercactus/weak hypercactus* if it is spanned by a path/tree/unicyclic graph/hairy cycle/strong cactus/weak cactus. It was proved in [5] that feasible sets of mixed interval hypergraphs are gap–free. It was later proved in [8, 7] that even feasible sets of mixed hypertrees are gap–free. We present another proof of this statement (Corollary 1). We extend these results first to mixed unicyclic hypergraphs (Theorem 1) in Section 3. Then in Section 4, we prove that feasible sets of mixed strong hypercacti are gap-free (Theorem 2). On the other hand, we find in Section 5 infinitely many mixed weak hypercacti with broken feasible sets (Corollary 2). In addition, we present two different sufficient conditions on a graph G in order to span a mixed hypergraph with a broken feasible set in Section 5 (Theorem 3 and Lemma 10).

Let $\mathcal{G}$ be the class of such graphs G that each mixed hypergraph spanned by G has a gap–free feasible set. The problem to characterize the class $\mathcal{G}$ is in [6]. Theorem 2 of Section 4 states that all strong cacti are contained in $\mathcal{G}$. On the other hand, there are infinitely many weak hypercacti not contained in $\mathcal{G}$ as stated in Corollary 2. We prove several positive and negative results about the properties of the class $\mathcal{G}$ in Section 6, e.g. that $\mathcal{G}$ is not closed under contracting edges (Observation 8). Moreover, we prove that the only connected non–planar graph contained in $\mathcal{G}$ is K_5 in Theorem 4, i.e. each connected non–planar graph different from K_5 spans a mixed hypergraph with a broken feasible set. The full characterization of the class $\mathcal{G}$ remains an open problem.

We refer the reader to [1, 2] for basics of graph theory; we recall just some of them in this paragraph. A subgraph induced by a vertex set $W \subseteq V$ of a graph $G = (V, E)$ is its graph $G[W] = (W, \binom{W}{2} \cap E)$. An induced subgraph is proper if $W \neq V$. The graph obtained from G by identifying the vertices u and v is the graph where we replace u and v with a new vertex w and we replace the vertices u and v by w in the edges in which u or v is contained; we remove all the resulting loops and parallel edges. If we identify two vertices which form an edge, we say that we contract that edge.

2 Basic Definitions and Properties

We call a mixed hypergraph H *good* if H satisfies the following two conditions:

- The feasible set of H is gap–free.
- If H is colorable, then its chromatic number is at most three.

We say that a graph G is *good* if each mixed hypergraph spanned by G is good. If a mixed hypergraph or a graph is not good, then it is *bad*. Some easy observations follow:

Observation 1 *A mixed hypergraph H is good if and only if $k \in \mathcal{F}(H)$ implies that $k - 1 \in \mathcal{F}(H)$ for all $k \geq 4$.*

Proof: If H is good, then $k \in \mathcal{F}(H)$ implies $k - 1 \in \mathcal{F}(H)$ for all $k \geq 4$. On the other hand, if $k \in \mathcal{F}(H)$ implies $k - 1 \in \mathcal{F}(H)$, then $[3, \bar{\chi}(H)] \subseteq \mathcal{F}(H)$. If $\chi(H)$ is either two or three, then the feasible set of H is clearly gap–free. If $\chi(H) = 1$, then the mixed hypergraph H consists only of $\mathcal{C}$–edges and thus its feasible set is gap–free.

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Observation 2 *If the graphs $G_1, \ldots, G_l$ are good, then their disjoint union is also good.*

Proof: Let H be any mixed hypergraph spanned by the disjoint union of $G_1,\ldots,G_l$. Let H_i be the restriction of H spanned by G_i. Proper colorings $c_1,\ldots,c_l$ of the mixed hypergraphs $H_1,\ldots,H_l$ using $k_1,\ldots,k_l$ colors can be used to get a proper coloring c using any number of colors between $\max\{k_1,\ldots,k_l\}$ and $k_1+\ldots+k_l$. On the other hand restriction of a proper coloring of H to H_i is also a proper coloring. Thus the feasible set $\mathcal{F}(H)$ of the whole mixed hypergraph is $[\max\{\chi(H_1),\ldots,\chi(H_l)\},\bar{\chi}(H_1)+\ldots+\bar{\chi}(H_l)]$ and the mixed hypergraph H is good.

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Let c be a coloring of the vertices of G. Then the mixed hypergraph H_G^c spanned by G is defined as follows: The $\mathcal{D}$–edges of H_G^c are exactly all the edges uv of G such that $c(u)\neq c(v)$ and the $\mathcal{C}$–edges of H_G^c are exactly all the paths $v_1,\ldots,v_k$ of G such that $c(v_1)=c(v_k)$ and $c(v_i)\neq c(v_j)$ for any $1\le i<j\le k$ such that $(i,j)\neq(1,k)$. We state two easy lemmas about this mixed hypergraph H_G^c derived from a graph G and its vertex coloring c:

Lemma 1 *Let H be any mixed hypergraph spanned by G and let c be any proper coloring of H. Then $\mathcal{F}(H_G^c)\subseteq\mathcal{F}(H)$.*

Proof: It is enough to realize that for each $\mathcal{C}$–/$\mathcal{D}$–edge e of H there is a $\mathcal{C}$–/$\mathcal{D}$–edge e' of H_G^c such that $e'\subseteq e$. This is because each $\mathcal{D}$–edge e of H contains an edge uv of G such that $c(u)\neq c(v)$ (because c is a proper coloring of H); each $\mathcal{C}$–edge e of H contains two vertices colored by c with the same color — let u and v be two nearest ones (measured in the subgraph of G induced by e) colored with the same color; every shortest path between u and v (in the subgraph of G induced by e) satisfies the conditions of the definition H_G^c in order to be a $\mathcal{C}$–edge and thus is included in H_G^c as a $\mathcal{C}$–edge.

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Lemma 2 *Let G be any graph. If for each coloring c using $k\ge 4$ colors, the mixed hypergraph H_G^c has a strict $(k-1)$–coloring, then G is a good graph.*

Proof: Let us assume that G is bad. Then there exists a mixed hypergraph H spanned by G such that its feasible set is broken or $\chi(H)>3$; hence there is $k\ge 4$ such that $k\in\mathcal{F}(H)$ and $k-1\notin\mathcal{F}(H)$ due to Observation 1. Let c be a strict k–coloring of H. Then $k-1\in\mathcal{F}(H_G^c)$ follows from the assumption of the lemma, and $k-1\in\mathcal{F}(H)$ follows from Lemma 1; this contradicts the choice of H.

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We introduce two contraction operations in the next two–lemmas; we call a 2–*contraction* the operation of Lemma 3 and we call a 3–*contraction* the operation of Lemma 4. The proofs of both the lemmas are more or less straightforward:

Lemma 3 *Let c be a coloring of a mixed hypergraph H spanned by a graph G and let $\{u, v\}$ be a $\mathcal{C}$–edge of H (note that uv has to be an edge of G). Let G' be a graph obtained from G by identifying the vertices u and v and let H' be the mixed hypergraph obtained from H by identifying the vertices u and v and removing all the $\mathcal{C}$–edges which originally contained both the vertices u and v.*

Then H' is spanned by G' and the proper colorings of H' one–to–one correspond to the proper colorings of H.

Proof: The mixed hypergraph H' is clearly spanned by G'. Let w be the vertex obtained by identifying the vertices u and v. Let c' be a proper coloring of H'. Then the coloring c defined by $c(u) = c(v) = c'(w)$ and $c(x) = c'(x)$ for $x \neq u, v$ is a proper coloring of H: the edges not containing both u and v are colored properly, since they are present in H', and the $\mathcal{C}$–edges containing both u and v are also colored properly because they contain the vertices u and v colored with the same color. On the other hand, let c be a proper coloring of H; it has to assign the same color to the vertices u and v. Then the coloring c' defined by $c'(w) = c(u)$ and $c'(x) = c(x)$ for $x \neq w$ is a proper coloring of H'.

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Lemma 4 *Let c be a coloring of a mixed hypergraph H spanned by a graph G and let $\{u, v\}$ and $\{v, w\}$ be $\mathcal{D}$–edges of H and $\{u, v, w\}$ be a $\mathcal{C}$–edge of H (note that uv and vw have to be edges of G). Let G' be a graph obtained from G by identifying the vertices u and w and let H' be the mixed hypergraph obtained from H by identifying the vertices u and w and removing all the $\mathcal{C}$–edges which originally contained both the vertices u and w.*

Then H' is spanned by G' and the proper colorings of H' one–to–one correspond to the proper colorings of H.

Proof: The mixed hypergraph H' is clearly spanned by G'. Note that the $\mathcal{D}$–edges $\{u, v\}$ and $\{v, w\}$ were replaced with a new $\mathcal{D}$–edge containing the vertex v and the new vertex produced by identifying the vertices u and w. The proof that the colorings of H' and H one–to–one correspond is similar to the proof of Lemma 3 and thus omitted.

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We state two easy lemmas about the effect of 2–/3–contractions on certain classes of previously defined graphs:

Observation 3 *The class of trees and unicyclic graphs is closed under the operations of 2–contractions and 3–contractions. The class of mixed hypertrees and mixed unicyclic hypergraphs is closed under the operations of 2–contractions and 3–contractions.*

Proof: It is enough to prove the first part of the statement of the observation. The class of trees and unicyclic graphs is the class of connected graphs containing at most one cycle. A contraction of an edge (i.e. 2–contraction) cannot increase the number of cycles in any graph and thus the class is closed under this operation. The same is not true for 3–contractions in general, but it is true if we additionally assume that a graph contains at most one cycle. Thus the class is closed also under 3–contractions.

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Observation 4 *The class of strong cacti is closed under the following operations:*

- *2–contraction of an edge contained in a cycle*
- *2–contraction of an edge such that at least one of its end–vertices is not contained in a cycle*
- *3–contraction of two edges uv and vw such that both of them are contained in a cycle (and thus in the same cycle)*
- *3–contraction of two edges uv and vw such that the vertex w is not contained in a cycle*

The class of mixed strong hypercacti is closed under all the above mentioned operations.

Proof: It is enough to prove the first part of the statement of the observation (the one about strong cacti). Let G be a strong cactus and let $C_1, \ldots, C_l$ be its cycles. A contraction of an edge violates the vertex–disjointness of the cycles only if we contract an edge e such that e is joins two different cycles. Thus the class of strong cacti is closed under the 2–contractions described in the statement of the observation. A 3–contraction violates the vertex–disjointness of the cycles only if we 3–contract the edges uv and vw such that u and w are contained in two different cycles of the strong cactus. Thus the class of strong cacti is closed under the 3–contractions described in the statement of the observation.

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We finish this section with a lemma on bridges and mixed hypergraphs H_G^c:

Lemma 5 *Let G be a connected graph, $e = uv$ one of its bridges and c one of its colorings such that $c(u) = c(v)$. Let G_1 and G_2 be the two connected components of $G \setminus e$ and let c_1 and c_2 be the restrictions of c to G_1 and G_2. If $H_{G_1}^{c_1}$ and $H_{G_2}^{c_2}$ are good, then H_G^c is good.*

Proof: Since uv is a bridge, then the only edge of H_G^c containing both u and v is the $\mathcal{C}$–edge $\{u, v\}$. Thus each pair of colorings c'_1 and c'_2 of $H_{G_1}^{c_1}$ and $H_{G_2}^{c_2}$ using k_1 and k_2 colors (respectively) give the colorings c' and c'' of H_G^c using $\max\{k_1, k_2\}$ and $k_1 + k_2 - 1$ colors: In order to construct c', it is enough to assume that c'_1 and c'_2 use the same colors and $c'_1(u) = c'_2(v)$. In order to construct c'', we assume that c'_1 and c'_2 use mutually different colors and then we identify the colors of the vertices u and v. On the other hand, restriction of any proper coloring of H_G^c to $H_{G_i}^{c_i}$ is a proper coloring (for $i = 1, 2$). This implies that $\bar{\chi}(H_G^c) = \bar{\chi}(H_{G_1}^{c_1}) + \bar{\chi}(H_{G_2}^{c_2}) - 1$ and it also implies that if $H_{G_1}^{c_1}$ and $H_{G_2}^{c_2}$ are good, then H_G^c is good.

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3 Feasible Sets of Mixed Unicyclic Hypergraphs

We first state and prove a lemma about articulations in good graphs:

Lemma 6 *Let G be a graph which contains an articulation not contained in any cycle of G. If each proper connected induced subgraph of G is good, then G is good.*

Proof: If G is not connected, then each of its components is good and thus G is good due to Observation 2. Otherwise, let v be the articulation of G not contained in any of the cycles of G and let $G_1, \ldots, G_l$ be the connected components of $G \setminus \{v\}$; note that l is the degree of v in G, since v is not contained in any cycle of G. Let v_i be the neighbor of v contained in G_i for $1 \leq i \leq l$.

Let H be a mixed hypergraph spanned by G which is bad. That means there is $k \geq 4$ such that $k \in \mathcal{F}(H)$ and $k - 1 \notin \mathcal{F}(H)$ due to Observation 1. There cannot be a coloring c of H using k colors which assigns the same color to the vertices v and v_i (one of the neighbors of v): otherwise, since H_G^c is good due to Lemma 5 used for the bridge vv_i, H_G^c can be colored by $k-1$ colors and thus H can also be colored by $k-1$ colors due to Lemma 1. Thus any strict k–coloring of H colors the vertices v and v_i (for $1 \leq i \leq l$) with different colors. Let c be a strict k–coloring of H which colors the most neighbors of v with the same color. We assume w.l.o.g. that $c(v) = 1$ and the most used color on the neighbors of v is the color 2.

We claim that c colors all the neighbors of v with the color 2. Let V_i be the set of vertices of the components G_j for which $c(v_j) = i$ ($3 \leq i$, $1 \leq j \leq l$). If we interchanged the colors 2 and i on the vertices of V_i, we would get either an improper coloring, a strict $(k-1)$–coloring or a strict k–coloring assigning the color 2 to more neighbors of v — the latter cases are impossible due to the choice of H, k and c. Since the obtained coloring is not proper, there has to be a $\mathcal{C}$–edge containing some vertices of V_i and some vertices not of V_i which is not colored properly after the recoloring (all the $\mathcal{D}$–edges of H and all the $\mathcal{C}$–edges either fully contained in $V_i \cup \{v\}$ or disjoint with V_i are still colored properly). Note that the only colors affected by the recoloring are the colors 2 and i. Since the neighbors of v were recolored from the color i to the color 2, the two vertices of this $\mathcal{C}$–edge colored with the same color by the original coloring c were colored by the color i; one of these two vertices is in V_i and one is not in V_i. This means that there exists a vertex $w_i \notin V_i$ such that $c(w_i) = i$.

If we interchange simultaneously the colors 2 and i on all the vertices of V_i for all $i \geq 3$, we get a proper coloring of H: All the $\mathcal{D}$–edges are still colored properly, the $\mathcal{C}$–edges either fully contained in $\{v\} \cup V_3 \cup \ldots \cup V_k$ or completely disjoint with $V_3 \cup \ldots \cup V_k$ are clearly colored properly. Each of the remaining $\mathcal{C}$–edges contains at least two neighbors of v and thus is also colored properly. Due to the existence of w_i for non–empty sets of V_i, the obtained coloring uses still k colors. But this contradicts our choice of c unless c colors all the neighbors of v with the same color.

Thus the coloring c colors the vertex v with the color 1 and all its neighbors with the color 2. Let G'_i be the subgraph of G induced by the vertices of G_i and the vertex v (for $1 \leq i \leq l$). Let c_i be the coloring c restricted to G'_i (for $1 \leq i \leq l$); note that $H^{c_i}_{G'_i}$ contains exactly the edges of the mixed hypergraph H^c_G fully contained in G'_i. We define a proper coloring c'_i of $H^{c_i}_{G'_i}$ as follows: If c_i does not use the color k, then we set $c'_i = c_i$. If c_i uses the color k, but it omits a color k' for $3 \leq k' \leq k$ (it cannot omit the color 1 and the color 2 since $c_i(v) = 1$ and $c_i(v_i) = 2$), then we interchange the colors k' and k. If c_i is a strict k–coloring, then we set c'_i to any strict $(k-1)$–coloring of $H^{c_i}_{G'_i}$ which exists since G'_i is good due to the assumption of the lemma; we assume w.l.o.g. in this case that $c'_i(v) = 1$ and $c'_i(v_i) = 2$. Note that the coloring c'_i is a proper coloring of $H^{c_i}_{G'_i}$. Let c' be the coloring defined by $c'(x) = c'_i(x)$ for $x \in G_i, 1 \leq i \leq l$ and $c'(v) = 1$. The coloring c' is a strict $(k-1)$–coloring: It clearly uses $k-1$ colors and the only colored improperly edges might be the $\mathcal{C}$–edges contained in several graphs G'_i's but these $\mathcal{C}$–edges are colored properly since each of them contains at least two neighbors of v, both of them colored by the color 2. Hence $k - 1 \in \mathcal{F}(H)$ due to Lemma 1. This contradicts the choice of H.

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The immediate corollary of Lemma 6 is the following:

Corollary 1 *Each tree is good, in particular the feasible set of any mixed hypertree is gap–free.*

Proof: Let T be a tree with the smallest number of vertices which is bad. Due to the choice of T, T cannot contain an articulation due to Lemma 6. This implies that T is either a single vertex or a single edge, but both these trees are clearly good.

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Before we prove the main theorem of this section, we state a lemma which we use in the beginning of the proof of the main theorem:

Lemma 7 *Let H be a mixed hairy cyclic hypergraph spanned by a hairy cycle G such that each edge of G is a $\mathcal{D}$–edge of H and each $\mathcal{C}$–edge of H has size at least three. Let c be a strict k–coloring of H $(4 \leq k)$ which uses at most $k-1$ colors to color the vertices of the cycle of G. Then $k - 1 \in \mathcal{F}(H)$.*

Proof: Let W be the set of vertices contained in the cycle of G. Let $l < k$ be the number of colors used by c for coloring the vertices in W. Let us assume w.l.o.g. that the colors used on W are $1, \ldots, l$. We call the nearest vertex of W to the vertex v the *root* of v (if $v \in W$, then v is the root of itself). Let W_i be the set of vertices v colored with the color i by c; note that $W \subseteq W_1 \cup \ldots \cup W_l$. We define the coloring $\tilde{c}$ of H using $k-1$ colors:

- $\tilde{c}(v) := i$ if $v \in W_i$ and $i < k$
- $\tilde{c}(v) := 1$ if $v \in W_k$ and the root of v is not colored by 1
- $\tilde{c}(v) := 2$ if $v \in W_k$ and the root of v is colored by 1

We prove that $\tilde{c}$ is a $(k-1)$–strict coloring of H.

The coloring $\tilde{c}$ clearly uses exactly $k-1$ colors. Due to the definition of $\tilde{c}$ different colors have been assigned to the vertices of each edge of G and thus each $\mathcal{D}$–edge of H contains two vertices colored with different colors by $\tilde{c}$. Let e be a $\mathcal{C}$–edge and let u and v be two vertices of e colored by c with the same color. If $c(u) = c(v) < k$, then e is colored properly by $\tilde{c}$, since $\tilde{c}(u) = \tilde{c}(v) = c(u) = c(v)$. If $c(u) = c(v) = k$ (and thus u and v are not contained in the cycle of G), we distinguish two cases:

- The roots of u and v are different. Let $\tilde{u}$ be the root of u and $\tilde{v}$ be the root v. Since G spans H, it holds that $\{u, v, \tilde{u}, \tilde{v}\} \subseteq e$. On the other hand either $\tilde{c}(u)$ or $\tilde{c}(\tilde{u})$ is equal to 1 and either $\tilde{c}(v)$ or $\tilde{c}(\tilde{v})$ is equal to 1 due to the definition of $\tilde{c}$. Thus e contains two vertices colored by $\tilde{c}$ with the same color.
- The roots of u and v are the same vertex. Then the coloring $\tilde{c}$ assigns u and v the same color.

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We are ready to prove the main theorem of this section:

Theorem 1 *Each unicyclic graph is good. In particular the feasible set of any mixed unicyclic hypergraph is gap–free.*

Proof: Let G be a unicyclic graph with the smallest number of vertices which is bad. Then G cannot contain an articulation not contained in a cycle due to Lemma 6. Thus G is a hairy cycle. We prove that the mixed hypergraph H_G^c has a strict $(k-1)$–coloring for each coloring c using $k \geq 4$ colors (this is enough due to Lemma 2). If H contains a $\mathcal{C}$–edge of size two, then we can perform a 2–contraction (the class of unicyclic graphs and trees is closed under 2–contractions of edges due to Observation 3) and we get either a mixed unicyclic hypergraph H' spanned by a smaller unicyclic graph or a mixed hypertree; thus H' is good and H_G^c is also good due to Lemma 3. Recall that mixed hypertrees are good due to Corollary 1. Thus the mixed hypergraph H_G^c contains no $\mathcal{C}$–edge of size two and all the edges of G are the $\mathcal{D}$–edges of H_G^c due to the definition of H_G^c.

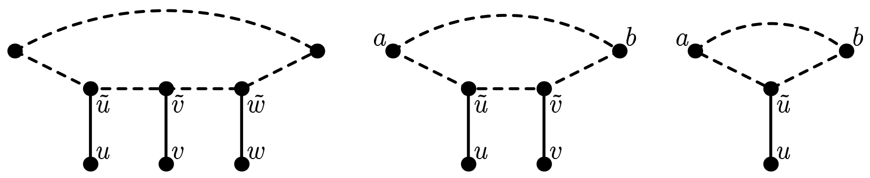

Figure 1: Notation used in the proof of Theorem 1 in case that there exists three/just two/no $\mathcal{C}$–neighbors.

If H_G^c contains a $\mathcal{C}$–edge of size three, then we can perform a 3–contraction (the class of unicyclic graphs and trees is closed under 3–contractions due to Observation 3 and all the edges of G are $\mathcal{D}$–edges of H_G^c as stated above) and we get either a mixed unicyclic hypergraph H' spanned by a smaller unicyclic graph or a mixed hypertree; thus H' is good and H_G^c is also good due to Lemma 4. If c uses only at most $k-1$ colors to color the vertices on the cycle, then H_G^c can be colored by $k-1$ colors due to Lemma 7. We conclude this paragraph with stating that the $\mathcal{D}$–edges of H_G^c are precisely the edges of G, that each $\mathcal{C}$–edge of H_G^c has size at least four and that each of the colors is used to color a vertex of the cycle of G.

We adopt the terminology of root vertices from the proof of Lemma 7. Note that if the roots of the vertices u and v are the same, then $c(u) \neq c(v)$ (otherwise u and v would have been 3–contracted). We call vertices u and v *c–neighbors*, if neither u nor v is in the cycle of G and at least one of the two paths between u and v is a $\mathcal{C}$–edge of H_G^c. Note that if u and v are c–neighbors and v and w are c–neighbors, then u and w are c–neighbors, too. We distinguish four cases in the proof:

- **There exists three or more mutual disjoint $\mathcal{C}$–neighbors in H_G^c. Let u, v and w be such $\mathcal{C}$–neighbors that the distance between u and v and v and w is minimal in G.**
  Let $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ be the roots of u, v and w (see Figure 1). The $\mathcal{C}$–edge corresponding to the path between u and w contains $\tilde{v}$ (i.e. v is in the middle between u and w) and each $\mathcal{C}$–edge containing v contains also either $\tilde{u}$ or $\tilde{w}$ due to the choice of the vertices u, v and w as the nearest ones. Let H' be the unicyclic mixed hypergraph obtained from H_G^c by deleting the vertex v and all the $\mathcal{C}$–edges and the $\mathcal{D}$–edge that contain it. Let $\tilde{c}$ be the coloring of H' using $k-1$ colors (its existence follows from the choice of G). The path between u and w (containing $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$) is a $\mathcal{C}$–edge of H' and it has to contain vertices x and y such that $\tilde{c}(x) = \tilde{c}(y)$. Let e_u be the $\mathcal{C}$–edge corresponding to the path between v and u and let e_w be the $\mathcal{C}$–edge corresponding to the path between v and w. We extend the coloring $\tilde{c}$ to v as follows (the $\mathcal{D}$–edge consisting of v and $\tilde{v}$ is always colored properly due to the way in which we set $\tilde{c}(v)$; thus we take care below only of $\mathcal{C}$–edges):
  - $\tilde{c}(v) := \tilde{c}(u)$ if both x and y are contained in the path between $\tilde{v}$ and

w (inclusively) and $\tilde{c}(\tilde{v}) \neq \tilde{c}(u)$
Let e be any $\mathcal{C}$–edge containing v and let e correspond to a path between v and z; e contains either $\tilde{u}$ or $\tilde{w}$. If $e = e_u$ (thus $z = u$), e is colored properly, since it contains u and v; if $e = e_w$ (thus $z = w$), e is colored properly, since it contains x and y. Assume further that z is neither u nor w. If e contains $\tilde{u}$, let $\tilde{e}$ be the $\mathcal{C}$–edge corresponding to the path between u and z; note that $\tilde{e} \setminus e = \{u\}$. Since $\tilde{e}$ is colored properly and the color of v is the same as the color of u, e is also colored properly. If e contains $\tilde{w}$, let $\tilde{e}$ be the $\mathcal{C}$–edge corresponding to the path between w and z. If w is neither x nor y, then $\tilde{e}$ is colored properly since it contains x and y. Otherwise, assume w.l.o.g. that w is y. Note that $\tilde{e} \setminus e = \{w\}$. Since $\tilde{e}$ is colored properly and the color of $x \in e$ is the same as the color of $w = y$, e is also colored properly.

- we set $\tilde{c}(v)$ to an arbitrary color different from $\tilde{c}(\tilde{v})$ if both x and y are contained in the path between $\tilde{v}$ and w (inclusively) and $\tilde{c}(\tilde{v}) = \tilde{c}(u)$
  The $\mathcal{D}$–edge consisting of v and $\tilde{v}$ is clearly colored properly. Let e be any $\mathcal{C}$–edge containing v and let e correspond to a path between v and z; e contains either $\tilde{u}$ or $\tilde{w}$. If $e = e_u$ (thus $z = u$), e is colored properly since it contains u and $\tilde{v}$; if $e = e_w$ (thus $z = w$), e is colored properly since it contains x and y. Assume further that z is neither u nor w. If e contains $\tilde{u}$, let $\tilde{e}$ be the $\mathcal{C}$–edge corresponding to the path between u and z; note that $\tilde{e} \setminus e = \{u\}$. Since $\tilde{e}$ is colored properly and the color of $\tilde{v}$ is the same as the color of u, e is also colored properly. If e contains $\tilde{w}$, let $\tilde{e}$ be the $\mathcal{C}$–edge corresponding to the path between w and z. If w is neither x nor y, then $\tilde{e}$ is colored properly since it contains x and y. Otherwise, assume w.l.o.g. that w is y. Note that $\tilde{e} \setminus e = \{w\}$. Since $\tilde{e}$ is colored properly and the color of $x \in e$ is the same as the color of $w = y$, e is also colored properly.
- $\tilde{c}(v) := \tilde{c}(w)$ if both x and y are contained in the path between $\tilde{v}$ and u (inclusively) and $\tilde{c}(\tilde{v}) \neq \tilde{c}(w)$
  This case is symmetric to the first case.
- we set $\tilde{c}(v)$ to an arbitrary color different from $\tilde{c}(\tilde{v})$ if both x and y are contained in the path between $\tilde{v}$ and u (inclusively) and $\tilde{c}(\tilde{v}) = \tilde{c}(w)$
  This case is symmetric to the second case.
- $\tilde{c}(v) := \tilde{c}(x) = \tilde{c}(y)$ if $\tilde{v}$ is contained in the path between x and y (exclusively) and $\tilde{c}(\tilde{v}) \neq \tilde{c}(x) = \tilde{c}(y)$
  Let us assume w.l.o.g. that x is contained in the path between u and $\tilde{v}$ and y in the path between w and $\tilde{v}$. Let e be any $\mathcal{C}$–edge containing v and let e correspond to a path between v and z; e contains either $\tilde{u}$ or $\tilde{w}$. If $e = e_u$ (thus $z = u$), e is colored properly since it contains v and x; if $e = e_w$ (thus $z = w$), e is colored properly since it contains v and y. Let us assume further that z is neither u nor w. If e contains $\tilde{u}$, then if $u \neq x$, e is properly colored since it contains v and x, and

otherwise $\tilde{e} \setminus e = \{x\}$. But the vertex $v \in e$ is colored with the same color as the vertex x and thus e is colored properly. We proceed similarly in case that e contains $\tilde{w}$.

- we set $\tilde{c}(v)$ to an arbitrary color different from $\tilde{c}(\tilde{v})$ if $\tilde{v}$ is contained in the path between x and y (exclusively) and $\tilde{c}(\tilde{v}) = \tilde{c}(x) = \tilde{c}(y)$
  Let us assume w.l.o.g. that x is contained in the path between u and $\tilde{v}$ and y in the path between w and $\tilde{v}$. Let e be any $\mathcal{C}$–edge containing v and let e correspond to a path between v and z; e contains either $\tilde{u}$ or $\tilde{w}$. If $e = e_u$ (thus $z = u$), e is colored properly since it contains $\tilde{v}$ and x; if $e = e_w$ (thus $z = w$), e is colored properly since it contains $\tilde{v}$ and y. Let us assume further that z is neither u nor w. If e contains $\tilde{u}$, then if $u \neq x$, e is properly colored since it contains $\tilde{v}$ and x, and otherwise $\tilde{e} \setminus e = \{x\}$. But the vertex $\tilde{v} \in e$ is colored with the same color as the vertex x and thus e is colored properly. We proceed similarly in case that e contains $\tilde{w}$.

- **There exist c–neighbors u and v but there are no three c–neighbors in H^c_G.**
  There are exactly three $\mathcal{C}$–edges containing u (otherwise there would exist another c–neighbor of u). The same holds for v. Let a and b be the end vertices of the paths corresponding to $\mathcal{C}$–edges different from the $\mathcal{C}$–edge corresponding to the path between u and v; note that it could possibly hold that $a = b$ (see Figure 1). Let a be nearer to u and let b be nearer to v. Let $\tilde{u}$ be the root of u and $\tilde{v}$ the root of v. Let H' be the unicyclic mixed hypergraph obtained from H^c_G by deleting the vertex v and the three $\mathcal{C}$–edges and the $\mathcal{D}$–edge which contain it. Let $\tilde{c}$ be a coloring of H' using $k-1$ colors (its existence follows from the choice of G). The path between u and b (containing $\tilde{u}$ and $\tilde{v}$) is a $\mathcal{C}$–edge of H' and it has to contain vertices x and y such that $\tilde{c}(x) = \tilde{c}(y)$. There are three $\mathcal{C}$–edges containing v; let e_u be that corresponding to the path between v and u, let e_a be that corresponding to the path between v and a and let e_b be that corresponding to the path between v and b. Let $\tilde{e}$ be the $\mathcal{C}$–edge corresponding to the path between u and a. There is only one $\mathcal{D}$–edge containing v; let e_D be this $\mathcal{D}$–edge consisting of v and $\tilde{v}$. We extend the coloring $\tilde{c}$ to v as follows:

  - $\tilde{c}(v) := \tilde{c}(u)$ if both x and y are contained in the path between $\tilde{v}$ and b (inclusively) and $\tilde{c}(\tilde{v}) \neq \tilde{c}(u)$
    The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_b is colored properly since it contains both x and y, the $\mathcal{C}$–edge e_u is colored properly since it contains u and v and the $\mathcal{C}$–edge e_a is colored properly since $\tilde{e} \setminus e_a = \{u\}$ and the colors of u and v are the same.

  - we set $\tilde{c}(v)$ to an arbitrary color different from $\tilde{c}(\tilde{v})$ if both x and y are contained in the path between $\tilde{v}$ and b (inclusively) and $\tilde{c}(\tilde{v}) = \tilde{c}(u)$
    The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_b is colored properly since it contains both x and y, the $\mathcal{C}$–edge e_u is colored properly

since it contains u and $\tilde{v}$ and the $\mathcal{C}$–edge e_a is colored properly since $\tilde{e} \setminus e_a = \{u\}$ and the colors of u and $\tilde{v}$ are the same.

- $\tilde{c}(v) := \tilde{c}(b)$ if both x and y are contained in the path between $\tilde{v}$ and u (inclusively) and $\tilde{c}(\tilde{v}) \neq \tilde{c}(b)$
  The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_u is colored properly since it contains both x and y, the $\mathcal{C}$–edge e_a is colored properly since it either contains both x and y or (in case that u is x or y) since $\tilde{e}$ is colored properly, $\tilde{e} \setminus e_a = \{u\}$ and e_a contains one of the vertices x and y which is colored with the same color as u. The $\mathcal{C}$–edge e_b is colored properly since it contains v and b.
- we set $\tilde{c}(v)$ to an arbitrary color different from $\tilde{c}(\tilde{v})$ if both x and y are contained in the path between $\tilde{v}$ and u (inclusively) and $\tilde{c}(\tilde{v}) = \tilde{c}(b)$
  The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_u is colored properly since it contains both x and y, the $\mathcal{C}$–edge e_a is colored properly since it either contains both x and y or (in case that u is either x or y), since $\tilde{e}$ is colored properly, $\tilde{e} \setminus e_a = \{u\}$ and e_a contains one of the vertices x and y which is colored with the same color as u. The $\mathcal{C}$–edge e_b is colored properly since it contains $\tilde{v}$ and b.
- $\tilde{c}(v) := \tilde{c}(x) = \tilde{c}(y)$ if $\tilde{v}$ is contained in the path between x and y (exclusively) and $\tilde{c}(\tilde{v}) \neq \tilde{c}(x) = \tilde{c}(y)$
  The $\mathcal{D}$–edge e_D is colored properly. The $\mathcal{C}$–edge e_u is colored properly since it contains v and either x or y. The $\mathcal{C}$–edge e_b is colored properly since it contains v and either x or y. If u is neither x nor y, then the $\mathcal{C}$–edge e_a is colored properly since it contains v and either x and y; otherwise (let us assume w.l.o.g. u is x) it is colored properly since $\tilde{e} \setminus e_a = \{u\} = \{x\}$ and the colors of u and v are the same.
- we set $\tilde{c}(v)$ to an arbitrary color different from $\tilde{c}(\tilde{v})$ if $\tilde{v}$ is contained in the path between x and y (exclusively) and $\tilde{c}(\tilde{v}) = \tilde{c}(x) = \tilde{c}(y)$
  The $\mathcal{D}$–edge e_D is colored properly. The $\mathcal{C}$–edge e_u is colored properly since it contains $\tilde{v}$ and either x or y. The $\mathcal{C}$–edge e_b is colored properly since it contains $\tilde{v}$ and either x or y. If u is neither x nor y, then the $\mathcal{C}$–edge e_a is colored properly since it contains $\tilde{v}$ and either x or y; otherwise (let us assume w.l.o.g. u is x) it is colored properly since $\tilde{e} \setminus e_a = \{u\} = \{x\}$ and the colors of u and $\tilde{v}$ are the same.

- **There exists a vertex u not contained in the cycle of G, but there are no two c–neighbors.**
  There are exactly two $\mathcal{C}$–edges containing u (otherwise there would exist a c–neighbor of u). Let a and b be the other end vertices of the paths corresponding to these two $\mathcal{C}$–edges; note that it could possibly hold that $a = b$ (see Figure 1). Let $\tilde{u}$ be the root of u. Let H' be the unicyclic mixed hypergraph obtained from H_G^c by deleting the vertex u and the two $\mathcal{C}$–edges and the $\mathcal{D}$–edge which contain it. Let $\tilde{c}$ be the coloring of H' using $k-1$ colors (its existence follows from the choice of G). We extend $\tilde{c}$ to u now. If $\tilde{c}(a) = \tilde{c}(b)$, then we set $\tilde{c}(u)$ to $\tilde{c}(a) = \tilde{c}(b)$ in

case that $\tilde{c}(\tilde{u}) \neq \tilde{c}(a)$ and we set $\tilde{c}(u)$ to an arbitrary color different from $\tilde{c}(\tilde{u})$ in case that $\tilde{c}(\tilde{u}) = \tilde{c}(a) = \tilde{c}(b)$; hence H_G^c is properly colored by $k-1$ colors. Otherwise $\tilde{c}(a) \neq \tilde{c}(b)$ and thus $a \neq b$. The path between a and b (containing $\tilde{u}$) is a $\mathcal{C}$–edge of H' and it has to contain two vertices x and y such that $\tilde{c}(x) = \tilde{c}(y)$. There are two $\mathcal{C}$–edges containing v; let e_a be that corresponding to the path between u and a and let e_b be that corresponding to the path between u and b. Let $\tilde{e}$ be the $\mathcal{C}$–edge corresponding to the path between a and b. There is only one $\mathcal{D}$–edge containing v; let e_D be this $\mathcal{D}$–edge consisting of u and $\tilde{u}$. We extend the coloring $\tilde{c}$ to the vertex u as follows:

- $\tilde{c}(u) := \tilde{c}(a)$ if both x and y are contained in the path between $\tilde{u}$ and b (inclusively) and $\tilde{c}(\tilde{u}) \neq \tilde{c}(a)$
  The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_a is colored properly since it contains u and a and the $\mathcal{C}$–edge e_b is colored properly since it contains x and y.
- we set $\tilde{c}(u)$ to an arbitrary color different from $\tilde{c}(\tilde{u})$ if both x and y are contained in the path between $\tilde{u}$ and b (inclusively) and $\tilde{c}(\tilde{u}) = \tilde{c}(a)$
  The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_a is colored properly since it contains $\tilde{u}$ and a and the $\mathcal{C}$–edge e_b is properly colored since it contains x and y.
- $\tilde{c}(u) := \tilde{c}(b)$ if both x and y are contained in the path between $\tilde{u}$ and a (inclusively) and $\tilde{c}(\tilde{u}) \neq \tilde{c}(b)$
  This case is symmetric to the first case.
- we set $\tilde{c}(u)$ to an arbitrary color different from $\tilde{c}(\tilde{u})$ if both x and y are contained in the path between $\tilde{u}$ and a (inclusively) and $\tilde{c}(\tilde{u}) = \tilde{c}(b)$
  This case is symmetric to the second case.
- $\tilde{c}(u) := \tilde{c}(x) = \tilde{c}(y)$ if $\tilde{u}$ is contained in the path between x and y (exclusively) and $\tilde{c}(\tilde{u}) \neq \tilde{c}(x) = \tilde{c}(y)$
  The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_a is colored properly since it contains u and either x or y and the $\mathcal{C}$–edge e_b is properly colored since it contains u and either x or y.
- we set $\tilde{c}(u)$ to an arbitrary color different from $\tilde{c}(\tilde{u})$ if $\tilde{u}$ is contained in the path between x and y (exclusively) and $\tilde{c}(\tilde{u}) = \tilde{c}(x) = \tilde{c}(y)$
  The $\mathcal{D}$–edge e_D is colored properly, the $\mathcal{C}$–edge e_a is colored properly since it contains $\tilde{u}$ and either x or y and the $\mathcal{C}$–edge e_b is colored properly since it contains $\tilde{u}$ and either x or y.

- **All the vertices of H_G^c are the vertices of the cycle of G, i.e. G is a cycle.**
  We first deal with the case that there is a color i used by c for coloring only one vertex (say v) of H_G^c; we assume w.l.o.g. that $i = k$. We recolor the vertex v with one of the colors 1, 2 or 3 which is different from the colors of its two neighbors (if there are more possible choices, we choose

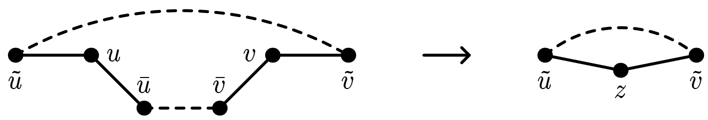

Figure 2: Notation used in the proof of Theorem 1 in case that the graph is a cycle and it contains two vertices u and v colored with the same color.

arbitrarily). The just obtained coloring is proper, since there are no $\mathcal{C}$–edges of H_G^c ending in v and the two $\mathcal{D}$–edges of H_G^c containing v are colored properly. It remains to deal with the case that each color is used at least twice.

Let u and v be the nearest vertices of H_G^c (measured in G) colored by c with the same color. Note that the distance between u and v measured as the number of the edges between them is at least 3 (since otherwise H_G^c would contain a $\mathcal{C}$–edge of size less than 4) and is at most k, since c uses only k colors. Let H' be a mixed unicyclic hypergraph obtained from H_G^c by identification of u and v (let z be the vertex obtained by this identification) and removing the (shorter) path between u and v and removing all the $\mathcal{C}$–edges and $\mathcal{D}$–edges containing at least one vertex of the removed path including the $\mathcal{C}$–edge corresponding to the removed path between u and v. The coloring c is proper for H' and it uses exactly k colors (we decrease the number of vertices colored by each color by at most one; otherwise u and v would not be the nearest vertices colored with the same color). Let c' be a coloring of H' using $k-1$ colors (its existence follows from the choice of G). Let $\tilde{u}$ and $\tilde{v}$ be the neighbors (in G) of u and v not contained in the removed path between u and v and $\bar{u}$ and $\bar{v}$ be their other neighbors in G (see Figure 2). We set $\tilde{c}$ to c' on the vertices of H' different from z, we set $\tilde{c}(u) = \tilde{c}(v) := c'(z)$ and we extend $\tilde{c}$ to the vertices of the removed path between u and v. In order to do this, we distinguish two cases according to the distance between the vertices u and v:

- The distance between u and v is three (it cannot be less as explained above). Note that $\bar{u}$ and $\bar{v}$ are neighbors in this case. We set $\tilde{c}(\bar{u})$ to $\tilde{c}(\tilde{u})$. If $\tilde{c}(\tilde{u}) \neq \tilde{c}(\tilde{v})$, we set $\tilde{c}(\bar{v})$ to $\tilde{c}(\tilde{v})$; otherwise, if $\tilde{c}(w) \neq \tilde{c}(v)$ we set $\tilde{c}(\bar{v})$ to $\tilde{c}(w)$ where w is the neighbor of $\tilde{v}$ different from v; if $\tilde{c}(w) = \tilde{c}(v)$, we set $\tilde{c}(\bar{v})$ to an arbitrary color different from $\tilde{c}(\bar{u})$ and $\tilde{c}(v)$.

  Any $\mathcal{C}$–edge containing a vertex $\bar{u}$ or $\bar{v}$ contains $\tilde{u}, u, \bar{u}$ or $u, \bar{u}, \bar{v}, v$ or $\bar{u}, \bar{v}, v, \tilde{v}$ or $\bar{v}, v, \tilde{v}, w$ since its size has to be at least four. Thus all such $\mathcal{C}$–edges are colored properly.

- The distance between u and v measured as the number of the edges on the path is at least four. We set $\tilde{c}(\bar{u})$ to $\tilde{c}(\tilde{u})$ and we set $\tilde{c}(\bar{v})$ to $\tilde{c}(\tilde{v})$. The remaining vertices of the removed path between u and

v will be colored with the following three colors $\tilde{c}(u) = \tilde{c}(v)$, $\tilde{c}(\tilde{u})$ and $\tilde{c}(\tilde{v})$; in case that $\tilde{c}(\tilde{u}) = \tilde{c}(\tilde{v})$ we can use another arbitrary color (recall $3 \leq k-1$).
All the $\mathcal{D}$–edges are colored properly. Any $\mathcal{C}$–edge entirely contained in the path between u and v (inclusively), is colored properly, since its size is at least four and there are only three colors used to color the vertices of the path between u and v. If the $\mathcal{C}$–edge contains a vertex of the path between $\bar{u}$ and $\bar{v}$ (inclusively) and is not contained entirely in the path between u and v (inclusively), then it has to contain both $\tilde{u}$ and $\bar{u}$ or both $\tilde{v}$ and $\bar{v}$ and thus it is colored properly.

■

4 Feasible Sets of Mixed Strong Hypercacti

We first prove that mixed hairy cyclic hypergraphs have strict 3–colorings of special type:

Lemma 8 *Let H be a mixed hairy cyclic hypergraph spanned by a hairy cycle G, let v_0 be a vertex of H contained in the cycle of G and let H satisfy the following conditions:*

- *Each edge of G is a $\mathcal{D}$–edge of H.*
- *H contains no $\mathcal{C}$–edges of size two.*
- *Each $\mathcal{C}$–edge of H of size three contains the vertex v_0 as its middle vertex.*
- *The cycle of G is not a triangle.*

Then there exists a proper coloring c of H using at most three colors such that all the neighbors of v_0 are colored with the same color.

Proof: Color the vertex v_0 with the color 1 and all its neighbors with the color 2. Extend this coloring arbitrary to a proper 3–coloring c of the graph G. We claim that the coloring c is also a proper coloring of the mixed hypergraph H. Each $\mathcal{D}$–edge of H is colored properly due to the way of construction of the coloring c and each $\mathcal{C}$–edge of H of size at least four is colored properly due to the pigeon–hole principle. The $\mathcal{C}$–edges of H of size three contain two neighbors of v_0 and thus they are colored properly.

■

Theorem 2 *Strong cacti are good graphs. In particular, any mixed strong hypercactus has a gap–free feasible set.*

Proof: Let G be a strong cactus with the smallest number of vertices which is not good. We prove that if any strong cactus with a smaller number of vertices is good, then G is good. It is enough to prove due to Observation 1 and Lemma 2 that for each coloring c of the vertices of G using exactly k colors $(4 \leq k)$ there is a strict $(k-1)$–coloring of H_G^c. Note that it is enough in order to prove this to show that H_G^c is good.

Let us assume (for a while) that G contains an edge e such that the coloring c colors the end–vertices of e with the same color. Either e is contained in a cycle or e is a bridge. If e is contained in a cycle of G, then the mixed strong hypercactus H' obtained from H_G^c by 2–contraction of e (cf. Observation 4) is good due to the choice of G and thus H_G^c is good due to Lemma 3. If e is a bridge, then H_G^c is good due to Lemma 5 and the choice of G. Thus we may assume that the end–vertices of any edge of G are colored by different colors and the $\mathcal{D}$–edges of H_G^c are precisely the edges of G.

If H_G^c contains a $\mathcal{C}$–edge C of size three such that either C is fully contained in a cycle of G or C has an end–vertex not contained in a cycle of G, then we can perform 3–contraction on C; the obtained mixed hypergraph H' is a mixed strong hypercactus due to Observation 4 and H' is good due to the choice of G. Thus H_G^c is good due to Lemma 4. If G contains a vertex not contained in a cycle, then this vertex has to be an articulation of G and H_G^c is good due to Lemma 6 and the choice of G. If G contains at most one cycle, then H_G^c is good due to Theorem 1. We conclude these paragraphs with stating several properties of G and H_G^c:

- G contains at least two cycles. All the cycles of G are vertex–disjoint.
- Each vertex of G either has degree one or is contained in a cycle.
- The $\mathcal{D}$–edges of H_G^c are precisely the edges of G.
- The $\mathcal{C}$–edges of H_G^c of size three contain only vertices contained in cycles of G and each of them meets at least two different cycles of G. The remaining $\mathcal{C}$–edges of H_G^c have size at least four.

We focus on proving the existence of a strict $(k-1)$–coloring of H_G^c in the rest of this proof. Let $e = uv$ be a bridge of G such that one of the components of $G \setminus e$ contains exactly one cycle. Let G_1 and G_2 be these two components and let G_1 be the component containing exactly one cycle. Let us assume w.l.o.g. that G_1 contains the vertex u. Let G_1' and G_2' be the components G_1 and G_2 together with the edge e. Let k_1 be the number of colors used by c on the vertices of G_1' and let k_2 be the number of colors used by c on the vertices of G_2'. If $k_1 < k$ and $k_2 < k$ (k is the number of colors used by c), then there exists a color i_1 not used in G_1' and a color i_2 not used in G_2'. Note that neither the color i_1 nor the color i_2 is used to color the vertices of the edge e. Thus if we recolor with the color i_1 all the vertices of G_2' colored by c with the color i_2, then we get a strict $(k-1)$–coloring of H_G^c: Each $\mathcal{D}$–edge is clearly colored properly and each $\mathcal{C}$–edge is also colored properly (because we joined two color classes). Thus it holds $k_1 = k$ or $k_2 = k$. We may assume w.l.o.g. that $k_2 = k$;

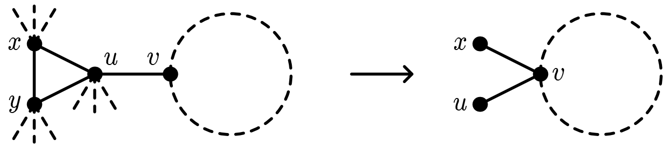

Figure 3: Notation used in the proof of Theorem 2 in case that G_1 contains a triangle.

we choose otherwise instead of the bridge e a bridge e' separating one of the cycles of G_2 from the remaining cycles of G (note that if G contains exactly two cycles, then both these bridges are the same); the bridge e' separates a cycle from the rest of G which contains G_1 and thus c uses at least $k_1 = k$ colors to color the vertices of the rest of G. Thus we may assume w.l.o.g. that $k_2 = k$.

If the cycle of G_1 is not a triangle, we proceed as follows: Let H_i be the mixed hypergraph with the vertex set equal to the vertices of G'_i which contains exactly the edges of H^c_G fully contained in G'_i. Let c_1 be a proper 3–coloring of H_1 such that all the neighbors of u are colored with the same color (existence of such coloring is assured by Lemma 8) and let c_2 be a strict $(k-1)$–coloring of H_2 (which exists, since G was a minimal bad strong hypercactus). Construct the coloring c based on c_1 and c_2 as follows: Identify the colors $c_1(u)$ and $c_2(u)$, the colors $c_1(v)$ and $c_2(v)$ and the possible third color used by c_1 with a color used by c_2 different from colors $c_2(u)$ and $c_2(v)$. Let c' be the obtained coloring of H^c_G. We claim that c' is a strict $(k-1)$–coloring. The coloring c' clearly uses exactly $k-1$ colors. Any $\mathcal{D}$–edge is clearly colored properly and any $\mathcal{C}$–edge fully contained in either G'_1 or G'_2 is also colored properly. Each $\mathcal{C}$–edge which is only partially contained in G'_2 has to contain the vertex u together with at least two of its neighbors and thus is also colored properly.

We deal with the remaining case that G_1 contains a triangle. Let the vertices x and y form together with the vertex u the triangle of G_1 (see Figure 3). Note that each vertex of G_1 except of u, x and y is a neighbor of exactly one of the vertices of u, x and y. Let us consider the coloring c' of H^c_G obtained from the coloring c by recoloring all the leaves adjacent to x or y with the color $c(u)$. The coloring c' is a strict k–coloring of H^c_G. It clearly uses exactly k colors (the vertices of G_2 were not affected by the recoloring). All the $\mathcal{D}$–edges are clearly colored properly and also all the $\mathcal{C}$–edges are colored properly, since any $\mathcal{C}$–edge which contains a leaf adjacent to x or y has size at least four and thus it contains that leaf together with the vertex u or together with another leaf adjacent to x or y. If G really contains a leaf adjacent to x or y, then this leaf together with its neighbor and the vertex u can be 3–contracted in $H^{c'}_G$ (note that they form a $\mathcal{C}$–edge in $H^{c'}_G$ and we get a smaller mixed strong hypercactus. Thus $H^{c'}_G$ is good and it has a strict $(k-1)$–coloring. Since c' is a proper coloring of H^c_G, it holds that $\mathcal{F}(H^{c'}_G) \subseteq \mathcal{F}(H^c_G)$ due to Lemma 1 and $k-1 \in \mathcal{F}(H^c_G)$. Thus we assume further w.l.o.g. that there are no leaves adjacent to x or y in G, i.e. the degrees of x and y are two.

Let us consider the coloring c'' of H_G^c obtained from the coloring c by recoloring all the leaves adjacent to u with the color $c(v)$. The coloring c'' is a strict k–coloring of H_G^c. It clearly uses exactly k colors (the vertices of G_2 were not affected by the recoloring). All the $\mathcal{D}$–edges are clearly colored properly and also all the $\mathcal{C}$–edges are colored properly, since any $\mathcal{C}$–edge which contains a leaf adjacent to u has to contain the vertex v (otherwise, it has to be a $\mathcal{C}$–edge of size three containing together with that leaf the vertex u and one of the vertices x and y — but such a $\mathcal{C}$–edge is not contained in H_G^c as stated above). If G really contains a leaf adjacent to u, then this leaf together with u and v can be 3–contracted in $H_G^{c''}$ (note that they form a $\mathcal{C}$–edge in $H_G^{c''}$ and we get a smaller mixed strong hypercactus. Thus $H_G^{c''}$ is good and it has a strict $(k-1)$–coloring. Since c'' is a proper coloring of H_G^c, it holds that $\mathcal{F}(H_G^{c''}) \subseteq \mathcal{F}(H_G^c)$ due to Lemma 1 and $k-1 \in \mathcal{F}(H_G^c)$. Thus we assume further w.l.o.g. that there are no leaves adjacent to u. Since uv is a bridge separating the triangle uxy from the remaining cycles, the vertex u has degree three and its only neighbors are the vertices x, y and v.

We deal with the remaining case that G_1 is equal to the triangle xyu in this paragraph. The colors $c(x)$, $c(y)$ and $c(u)$ are mutually different since xyu form a triangle in G. We may assume w.l.o.g. that $c(y) = c(v)$ (otherwise either we interchange x and y or we recolor $c(y)$ with the color $c(v)$ and we get a strict k–coloring c''' and if we find a strict $(k-1)$–coloring of $H_G^{c'''}$, we find also a strict $(k-1)$–coloring of H_G^c due to Lemma 1). Consider the graph G^* obtained from G by removing the vertex y from G and replacing the edge xu with an edge xv. Note that G^* is a strong cactus with a smaller number of vertices than G. Let H^* be the mixed strong hypercactus obtained from H_G^c by adding the $\mathcal{D}$–edge $\{x, v\}$, removing the $\mathcal{D}$–edge $\{x, u\}$, removing all the edges containing the vertex y and removing the vertex u from the $\mathcal{C}$–edges containing the vertex x (note that each $\mathcal{C}$–edge of H_G^c containing x contains u). Note that H^* is spanned by G^*. Let c^* be the coloring c restricted to the vertices of H^*; c^* is clearly a strict k–coloring of H^*. Since G^* is good due to the choice of G, H^* is good and thus it has a strict $(k-1)$–coloring c_0^* due to Observation 1. Define the coloring c_0 of H_G^c as follows: The coloring c_0 is the same as c_0^* on all the vertices with possible exception of x. The coloring c_0 assigns the vertex y the color $c_0^*(v)$ and thus $c_0(v) = c_0(y)$. If $c_0^*(x) \neq c_0^*(u)$, then $c_0(x) = c_0^*(x)$; otherwise we set $c_0(x)$ to any color used by c_0^* different from $c_0(u)$ and $c_0(y)$. We claim that c_0 is a strict $(k-1)$–coloring of H_G^c. It clearly uses exactly $k-1$ colors. It remains to prove that it is proper: Any $\mathcal{D}$–edge of H_G^c is clearly colored properly. Any $\mathcal{C}$–edge not containing x or y is affected neither by turning H_G^c to H^* nor by changing c_0^* to c_0 and thus it is colored properly. Any $\mathcal{C}$–edge of H_G^c containing y contains also v and thus it is colored properly. Any $\mathcal{C}$–edge of H_G^c containing x (and thus containing u) was present in H^* as the same $\mathcal{C}$–edge with the removed vertex u — if $c_0^*(u) = c_0^*(x)$, then it is clearly colored properly (since it contains in H_G^c the vertex u which it does not contain in H^*); otherwise $c_0(x) = c_0^*(x)$ and it is also colored properly. We conclude that c_0 is a strict $(k-1)$–coloring of H_G^c.

We have just finished the proof that if H_G^c has a strict k–coloring c ($k \geq 4$), then it has a strict $(k-1)$–coloring. This implies that G is proper due to Observation 1. Thus there cannot exist a strong cactus which is bad.

■

5 Mixed Hypergraphs with Broken Feasible Sets

We first state an easy lemma about subgraphs spanning mixed hypergraphs (note that the condition on G to be connected cannot be removed from the statement of the lemma):

Lemma 9 *Let G be a connected graph, G' be a subgraph of G and H' a mixed hypergraph spanned by G'. Then there exists a mixed hypergraph H spanned by G such that the proper colorings of H one–to–one correspond to the proper colorings of H'. In particular, $\mathcal{F}(H) = \mathcal{F}(H')$.*

Proof: We prove the following claim which immediately implies the lemma (we can apply this claim several times to subgraphs of G each time adding one vertex to it): If G' is a subgraph of a (possibly disconnected) graph G such that $G \setminus G' = \{v\}$ and v has degree at least one in G and if a mixed hypergraph H' is spanned by G', then there exists a mixed hypergraph H spanned by G such that the proper colorings of H one–to–one correspond to the proper colorings of H'.

Let w be a neighbor of v in G and let H be the mixed hypergraph obtained from H' by adding the vertex w and the $\mathcal{C}$–edge $\{v, w\}$. H is clearly spanned by G and since it may be 2–contracted to H', its proper colorings one–to–one correspond to the proper colorings of H' due to Lemma 3.

■

We state a simple sufficient condition on a graph in order to span a mixed hypergraph with a broken feasible set:

Theorem 3 *Let G be a connected graph containing an edge uv and three cycles C_1, C_2 and C_3 such that $v \in C_i$ and $u \notin C_i$ for $i = 1, 2, 3$ and $\{v\} = C_i \cap C_j$ for all $i \neq j$. Then there exists a mixed hypergraph H spanned by G whose feasible set is broken (see Figure 4).*

Proof: Let l_i be the length of the cycle C_i and let $C_i = v, v_1^i, \ldots, v_{l_i-1}^i$. Let us consider the mixed hypergraph H with the following edges:

- The set $\{v_a^i, v_{a+1}^i\}$ is a $\mathcal{C}$–edge for all $2 \leq a \leq l_i - 2$ and $i = 1, 2, 3$.
- The triples $\{u, v, v_1^i\}$ and $\{u, v, v_{l_1-1}^i\}$ are both $\mathcal{C}$–edges and $\mathcal{D}$–edges for $i = 1, 2, 3$.
- The triples $\{v, v_1^1, v_{l_3-1}^3\}$, $\{v, v_1^2, v_{l_1-1}^1\}$ and $\{v, v_1^3, v_{l_2-1}^2\}$ are $\mathcal{C}$–edges.

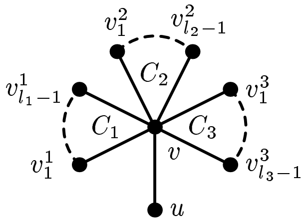

Figure 4: Notation of Theorem 3.

- The pairs $\{v_1^1, v_2^1\}$, $\{v_1^2, v_2^2\}$ and $\{v_1^3, v_2^3\}$ are $\mathcal{D}$–edges.

We prove that $\mathcal{F}(H) = \{2, 4\}$. Note that H is spanned by $\{u, v\} \cup C_1 \cup C_2 \cup C_3$ (a subgraph of G).

Let us consider a proper coloring c of H. Note that in any proper coloring of H, it holds that $c(v_2^i) = c(v_3^i) = \cdots = c(v_{l_i-1}^i)$ for all $i = 1, 2, 3$. Let $c_1^i = c(v_1^i)$ and $c_2^i = c(v_2^i)$. The only colors used to color the vertices of H are $c(u), c(v), c_1^i, c_2^i$ for $i = 1, 2, 3$. We distinguish two cases:

- $c(u) \neq c(v)$
  Then $c_j^i \in \{c(u), c(v)\}$ for all $i = 1, 2, 3$ and $j = 1, 2$ due to the presence of $\mathcal{C}$–edges $\{u, v, v_1^1\}$, $\{u, v, v_{l_1-1}^1\}$, $\{u, v, v_1^2\}$, $\{u, v, v_{l_2-1}^2\}$, $\{u, v, v_1^3\}$ and $\{u, v, v_{l_3-1}^3\}$. Thus c can use at most two colors. On the other hand, setting c_1^i to the color $c(u)$ and c_2^i to the color $c(v)$ yields a strict 2–coloring of H.

- $c(u) = c(v)$
  Then $c_j^i \neq c(u) = c(v)$ for $i = 1, 2, 3$ and $j = 1, 2$ due to the presence of $\mathcal{D}$–edges $\{u, v, v_1^1\}$, $\{u, v, v_{l_1-1}^1\}$, $\{u, v, v_1^2\}$, $\{u, v, v_{l_2-1}^2\}$, $\{u, v, v_1^3\}$ and $\{u, v, v_{l_3-1}^3\}$. Due to the presence of $\mathcal{C}$–edges $\{v, v_1^1, v_{l_3-1}^3\}$, $\{v, v_1^2, v_{l_1-1}^1\}$ and $\{v, v_1^3, v_{l_2-1}^2\}$, it has to be $c_1^1 = c_2^3$, $c_1^2 = c_2^1$ and $c_1^3 = c_2^2$. Finally due to the presence of $\mathcal{D}$–edges $\{v_1^1, v_2^1\}$, $\{v_1^2, v_2^2\}$ and $\{v_1^3, v_2^3\}$, the colors c_1^1, c_1^2 and c_1^3 are mutually different. Thus c has to be a strict 4–coloring. On the other hand, setting the colors as described above yields a strict 4–coloring.

Since H is spanned by $\{u, v\} \cup C_1 \cup C_2 \cup C_3$ and G is connected and contains $\{u, v\} \cup C_1 \cup C_2 \cup C_3$ as a subgraph, we can extend H to H' spanned by G such that $\mathcal{F}(H) = \mathcal{F}(H')$ due to Lemma 9.

■

The immediate corollary of Theorem 3 is following:

Corollary 2 *There is an infinite set Σ of weak cacti with the property: For each member G of Σ, there exists a mixed hypergraph H spanned by G whose feasible set contains a gap.*

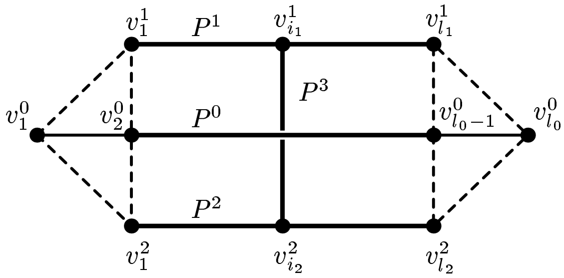

Figure 5: Notation of Lemma 10.

We state another sufficient condition on a graph in order to span a mixed hypergraph with a broken feasible set:

Lemma 10 *Let G be a connected graph satisfying the following conditions: G contains vertex disjoint paths $P^0 = v_1^0, \ldots, v_{l_0}^0$, $P^1 = v_1^1, \ldots, v_{l_1}^1$ and $P^2 = v_1^2, \ldots, v_{l_2}^2$ (l_1 and l_2 may be 1) which satisfy the following conditions (see Figure 5):*

- *The number l_0 is even and $l_0 \geq 4$.*
- *There exist $1 \leq i_1 \leq l_1$ and $1 \leq i_2 \leq l_2$ such that the vertices $v_{i_1}^1$ and $v_{i_2}^2$ are joined by a path whose all internal vertices are different from the vertices contained in the paths P^0, P^1 and P^2. (This path may consist of a single edge.)*
- *The vertex v_1^k is adjacent to v_1^0 or v_2^0 for $k = 1, 2$.*
- *The vertex $v_{l_k}^k$ is adjacent $v_{l_0-1}^0$ or $v_{l_0}^0$ for $k = 1, 2$.*

Then there exists a mixed hypergraph H spanned by G such that the feasible set of H contains a gap.

Proof: Let $P^3 = v_{i_1}^1 = v_1^3, v_2^3, \ldots, v_{l_3-1}^3, v_{l_3}^3 = v_{i_2}^2$ be the path between the vertices $v_{i_1}^1$ and $v_{i_2}^2$ from the statement of the lemma. The subgraph G' induced by the vertices of P^0, P^1, P^2 and P^3 is a connected subgraph of G. It is enough to find a mixed hypergraph H' with a broken feasible set which is spanned by G' due to Lemma 9.

Consider a mixed hypergraph H' spanned by G' with the following edges:

$$\mathcal{C}\text{–edges } \{v_i^0, v_{i+1}^0, v_{i+2}^0\} \text{ for } 1 \leq i \leq l_0 - 2 \tag{1}$$
$$\mathcal{D}\text{–edges } \{v_i^0, v_{i+1}^0\} \text{ for } 2 \leq i \leq l_0 - 2 \tag{2}$$
$$\mathcal{C}\text{–edges } \{v_i^k, v_{i+1}^k\} \text{ for } k = 1, 2 \text{ and } 1 \leq i \leq l_k - 1 \tag{3}$$
$$\mathcal{C}\text{–edges } \{v_i^3, v_{i+1}^3\} \text{ for } 2 \leq i \leq l_3 - 1 \tag{4}$$
$$\mathcal{C}\text{–edges and } \mathcal{D}\text{–edges } \{v_1^0, v_2^0, v_1^k\} \text{ and } \{v_{l_0-1}^0, v_{l_0}^0, v_{l_k}^k\} \text{ for } k = 1, 2 \tag{5}$$
$$\text{a } \mathcal{D}\text{–edge } \{v_1^3, v_2^3\} \tag{6}$$

The mixed hypergraph H' is clearly spanned by the graph G'. We prove that $\mathcal{F}(H') = \{2, 4\}$.

Let c be a proper coloring of H'. The presence of $\mathcal{C}$–edges (3) forces that $c(v_1^1) = \ldots = c(v_{l_1)}^1$ and $c(v_1^2) = \ldots = c(v_{l_2}^2)$; let c^1 be the common color of the vertices of P^1 and let c^2 be the common color of the vertices of P^2. Note that all the vertices $v_2^3, v_3^3, \ldots, v_{l_3}^3 = v_{i_2}^2$ share also the color c^2 due to the $\mathcal{C}$–edges (4). The $\mathcal{D}$–edges (2) together with $\mathcal{C}$–edges (1) force that $c(v_2^0) = c(v_4^0) = c(v_6^0) = \ldots = c(v_{l_0-2}^0)$ and $c(v_3^0) = c(v_5^0) = c(v_7^0) = \ldots = c(v_{l_0-1}^0)$. Let c^B be the common color of the vertices $v_2^0, v_4^0, \ldots, v_{l_0-2}^0$ and let c^C be the common color of the vertices $v_3^0, v_5^0, \ldots, v_{l_0-1}^0$. Let c^A be the color of the vertex v_1^0 and let c^D be the color of the vertex $v_{l_0}^0$. The $\mathcal{C}$–edges $\{v_1^0, v_2^0, v_3^0\}$ and $\{v_{l_0-2}^0, v_{l_0-1}^0, v_{l_0}^0\}$ force that $c^A \in \{c^B, c^C\}$ and $c^D \in \{c^B, c^C\}$ (these $\mathcal{C}$–edges are among $\mathcal{C}$–edges (1)); note that $c^B \neq c^C$. We further distinguish four cases:

- $c^A = c^B \neq c^C = c^D$
  The $\mathcal{D}$–edges (5) force $c^1 \notin \{c^B, c^C\}$ and $c^2 \notin \{c^B, c^C\}$; moreover, the $\mathcal{D}$–edge (6) forces that $c^1 \neq c^2$. Thus the coloring c uses four different colors $c^A = c^B$, $c^C = c^D$, c^1 and c^2. Note that the just described coloring is really a proper coloring of H'.

- $c^B \neq c^C = c^A = c^D$ (or $c^A = c^D = c^B \neq c^C$)
  The $\mathcal{C}$–edges (5) force $c^1 \in \{c^B, c^C\}$ and $c^2 \in \{c^B, c^C\}$. Then the $\mathcal{D}$–edges (5) force $c^1 = c^2 = c^B$ ($c^1 = c^2 = c^C$) but this is impossible due to the $\mathcal{D}$–edge (6).

- $c^D = c^B \neq c^C = c^A$
  The $\mathcal{C}$–edges (5) force $c^1 \in \{c^B, c^C\}$ and $c^2 \in \{c^B, c^C\}$. Setting either $c^1 = c^B$ and $c^2 = c^C$ or $c^1 = c^C$ and $c^2 = c^B$ yields a strict 2–coloring of H' and these are also the only possibilities what the proper coloring c can be.

The just finished discussion proves $\mathcal{F}(H') = \{2, 4\}$.

■

6 Mixed Hypergraphs Spanned by Graphs

In this section, we state some positive and negative results about the class $\mathcal{G}$ of graphs G such that each mixed hypergraph spanned by G has a gap–free feasible set.

Since $\mathcal{G}$ contains all good graphs, the immediate corollary of Theorem 2 is the following:

Observation 5 *The class $\mathcal{G}$ contains all strong cacti.*

Strong cacti are good graphs but the class $\mathcal{G}$ does not consist only of good graphs: The complete graph on four vertices, K_4, is clearly not a good graph but since each mixed hypergraph with a broken feasible set has at least six vertices (cf. [5]), it is contained in the class $\mathcal{G}$.

On the other hand, Corollary 2 states that some simple families of graphs are not contained in $\mathcal{G}$:

Observation 6 *The class $\mathcal{G}$ does not contain all weak cacti; in particular it does not contain all graphs with treewidth* 2.

In some sense, the class $\mathcal{G}$ contains only very few graphs:

Theorem 4 *K_5 is the only connected non–planar graph contained in $\mathcal{G}$.*

Proof: The graph K_5 is contained in $\mathcal{G}$, since it has less than 6 vertices and each mixed hypergraph with a broken feasible set has at least six vertices as stated above (cf. [5]). Let G be a connected non–planar graph different from K_5. Then G contains a subdivision of K_5 or $K_{3,3}$ due to the famous Kuratowski Theorem ([11], see also [1, 2]). We distinguish four cases:

- G contains K_5 as a subgraph.
  Let v_1, v_2, v_3, v_4 and v_5 be the vertices of K_5. Since G is connected and different from K_5, it contains another vertex v_0 adjacent to the copy of K_5. We assume w.l.o.g. that v_0 is adjacent to the vertex v_1. It is enough to apply Lemma 10 to the paths $P^0 = v_0, v_1, v_2, v_3$, $P^1 = v_4$ and $P^2 = v_5$. The path between the vertices of P^1 and P^2 is v_4v_5.

- G contains a proper subdivision of K_5 as a subgraph.
  Let v_1, v_2, v_3, v_4 and v_5 be the vertices of K_5, let $v_i \leftrightarrow v_j$ be the path between v_i and v_j corresponding to the subdivided edge between these two vertices and let $[v_i] \leftrightarrow v_j$ be the path $v_i \leftrightarrow v_j$ without the vertex v_i. We assume w.l.o.g. that the edge v_1v_2 is subdivided (at least one of the edges of K_5 is subdivided). It is enough to apply Lemma 10 to the paths $P^1 = [v_1] \leftrightarrow v_4 \leftrightarrow [v_2]$, $P^2 = [v_1] \leftrightarrow v_5 \leftrightarrow [v_2]$ and to the path P^0 which is either $v_1 \leftrightarrow v_2$ or $v_1 \leftrightarrow v_2, w$ where w is the vertex of $v_2 \leftrightarrow v_3$ adjacent to v_2 (one of these two paths has an even number l_0 of vertices, $4 \leq l_0$). The path between the vertices of P^1 and P^2 is $v_4 \leftrightarrow v_5$.

- G contains $K_{3,3}$ as a subgraph.
  Let u_1, u_2, u_3, v_1, v_2 and v_3 be the vertices of $K_{3,3}$. It is enough to apply Lemma 10 to the paths $P^0 = u_1, v_1, u_2, v_2$, $P^1 = u_3$ and $P^2 = v_3$. The path between the vertices of P^1 and P^2 is u_3v_3.

- G contains a proper subdivision of $K_{3,3}$ as a subgraph.
  Let u_1, u_2, u_3, v_1, v_2 and v_3 be the vertices of $K_{3,3}$, let $u_i \leftrightarrow v_j$ be the path between u_i and v_j corresponding to the subdivided edge between these two vertices and let $[u_i] \leftrightarrow v_j$ be the path $u_i \leftrightarrow v_j$ without the vertex u_i. Let w be the vertex of the path $u_2 \leftrightarrow v_1$ adjacent to v_1. We assume w.l.o.g. that the edge u_1v_1 is subdivided (at least one of the edges of $K_{3,3}$ is subdivided). It is enough to apply Lemma 10 to one of the two following triples of paths:

- $P^0 = u_1 \leftrightarrow v_1$, $P^1 = [u_1] \leftrightarrow v_2 \leftrightarrow u_2 \leftrightarrow [v_1]$ and $P^2 = [u_1] \leftrightarrow v_3 \leftrightarrow u_3 \leftrightarrow [v_1]$
- $P^0 = u_1 \leftrightarrow v_1, w$, $P^1 = [u_1] \leftrightarrow v_2 \leftrightarrow u_2 \leftrightarrow [w]$ (in case that $u_2 = w$ the path P^1 is $[u_1] \leftrightarrow v_2 \leftrightarrow [u_2]$) and $P^2 = [u_1] \leftrightarrow v_3 \leftrightarrow u_3 \leftrightarrow [v_1]$

The path between the vertices of P^1 and P^2 is $u_3 \leftrightarrow v_2$. Note that the number of the vertices of P^0 is even and at least four in one of the two above cases.

■

It might be interesting to study under which operations $\mathcal{G}$ is closed and under which it is not closed (the proof of Observation 7 is trivial):

Observation 7 *The class $\mathcal{G}$ is closed under the operation of disjoint union.*

Observation 8 *The class $\mathcal{G}$ is not closed under contractions of edges and thus it is not closed under taking minors.*

Proof: It is enough to realize that any weak cactus can be obtained from a strong cactus by contracting suitable edges.

■

Observation 8 could be strengthen to the following statement: The class $\mathcal{G}$ is not even closed under contraction of edges such that both its end–vertices have degree two. Let K_4^i be the graph obtained from K_4 by subdividing one of its edges i times. The graphs K_4^2, K_4^4, K_4^6, etc., span mixed hypergraphs with broken feasible sets due to Lemma 10. On the other hand, we have a computer–based proof that $K_4^3 \in \mathcal{G}$.

7 Conclusion

We proved that the feasible set of any mixed strong hypercactus is gap–free. We found infinitely many mixed weak hypercacti whose feasible sets are broken. We investigated basic properties of the class $\mathcal{G}$ of graphs G such that each mixed hypergraph spanned by G has a gap–free feasible set. Our results can be restated: All strong cacti are contained in $\mathcal{G}$, but not all weak cacti are contained in $\mathcal{G}$. On the other hand we prove that the only connected non–planar graph contained in $\mathcal{G}$ is K_5. The problem of complete characterization of the (connected) graphs of $\mathcal{G}$ is still open.

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