Line transversals to unit disks

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Abstract

We show that if every 3 members of a finite disjoint family of unit disks in the plane have a line transversal, then there is a line transversal to all except at most 12 disks in the family. We derive an analogous result for translates of a general compact convex set, with the constant equal to 58, improving a theorem due to Katchalski and Lewis.

1 Introduction

A prominent role in combinatorial geometry is played by the Helly theorem which states that if every $d+1$ members of a family of compact convex subsets of $\mathbf{R}^d$ intersect, then the whole family is intersecting. This is commonly expressed by saying that the Helly number for intersections of compact convex sets in $\mathbf{R}^d$ is $d+1$. Among the many results and questions inspired by the Helly theorem, there is a result of Tverberg on common transversals which will be our starting point.

A *line transversal* to a collection $\mathcal{F}$ of compact convex subsets of $\mathbf{R}^d$ is a line meeting every member of $\mathcal{F}$. It is natural to ask whether there is an analogue of the Helly theorem involving line transversals instead of common points. (This is a particular case of a question of Vincensini [11].) In general, the answer is in the negative, as shown e.g. by examples in [6]. The situation is different if we restrict ourselves to translates of a compact convex set. Tverberg [10] proved the following conjecture of Grünbaum [5]:

Theorem 1.1 (Tverberg) *If every 5 members in a disjoint family $\mathcal{F}$ of translates of a compact convex set in the plane have a line transversal, then there is a line transversal to $\mathcal{F}$.*

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An account of the history of this result is given e.g. in [2]. For a broader view of the context of Helly-type problems, consult [1], [2] or [4].

In 1980, Katchalski and Lewis [8] showed that much can be said even if we assume only that every 3 members of $\mathcal{F}$ have a line transversal. They proved:

Theorem 1.2 *There is a constant C such that if every 3 members in a finite disjoint family $\mathcal{F}$ of translates of a compact convex set in the plane have a line transversal, then there is a line meeting all but at most C members of $\mathcal{F}$.*

They established the result with $C = 192\pi$ (for some reason also cited as $C = 108$ in the literature) and conjectured that in fact $C = 2$ would do. This is known as the Katchalski–Lewis conjecture.

Focusing on the special case where $\mathcal{F}$ consists of unit disks, we prove the following result.

Theorem 1.3 *If $\mathcal{F}$ is a finite disjoint family of unit disks in the plane such that every three of the disks have a line transversal, then there is a line intersecting all but at most 12 members of $\mathcal{F}$.*

In Section 4, we outline how the method used for disks can be adapted to obtain a proof of Theorem 1.2 with $C = 58$.

2 Preliminaries

A key ingredient of our approach is the following well-known observation. For convenience, we give a short proof.

Lemma 2.1 *Three pairwise disjoint convex sets in $\mathbf{R}^d$ have* no *line transversal if and only if each can be strictly separated from the other two by a hyperplane.*

Proof. Assume that the sets C_1, C_2, C_3 have a line transversal ℓ meeting them in this order. Any hyperplane separating C_2 from C_1 and C_3 has to intersect ℓ twice, which is impossible.

For the converse, assume C_1, C_2, C_3 have no line transversal. Clearly, C_1 is disjoint from the convex hull of $C_2 \cup C_3$, so it can be strictly separated from the latter by a hyperplane. The other two separating hyperplanes can be obtained similarly. $\square$

If $D = \{ D_1, D_2, D_3 \}$ is a set of 3 pairwise disjoint convex sets in $\mathbf{R}^d$ and if S is a triple of hyperplanes such that each D_i is strictly separated from both the D_j's with $j \neq i$ by a hyperplane in S, then we shall call S a (strictly) *separating triple* for D. (See Fig. 1 for a planar illustration.) A related notion is that of a *weakly separating triple*, where we do not require the separating hyperplanes to be strictly separating (so they may touch the sets D_i).

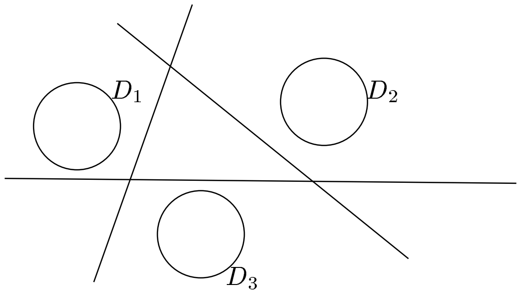

Figure 1: A strictly separating triple.

Let $\mathcal{F}$ be a finite family of pairwise disjoint unit disks in $\mathbf{R}^2$ (that is, disks of unit radius).

We may assume that no three disks in $\mathcal{F}$ have a common tangent which weakly separates one of them. To achieve this, inflate all the disks very slightly, preserving the (non-)existence of a transversal to any subset of $\mathcal{F}$. The inflation can be done so that all common tangents of the above type are eliminated. Finally, rescale the family so that the disks are of unit radius again.

Lemma 2.2 *There is no weakly separating triple for any subset* $\{ D_1, D_2, D_3 \}$ *of* $\mathcal{F}$.

Proof. This follows from our assumption on $\mathcal{F}$: as long as a separating line does not touch all three disks D_i, it can be made strictly separating by a small perturbation. But this would yield a strictly separating triple for the D_i's. $\square$

Now let A and B be two disks from $\mathcal{F}$ whose distance is maximal.

In all that follows, we shall assume (as we may) that A is centered at the origin $O = A_0$, and the center of B is at $B_0 = (b, 0)$, where $b > 2$. Let Z be the parallel strip of width 2, bounded by the two parallel tangents to A and B. The situation is shown in Figure 2.

We shall denote the boundary of a set $C \subset \mathbf{R}^2$ by ∂C and its interior by $\operatorname{int} C$.

Lemma 2.3 *All disks in* $\mathcal{F}$ *intersect the strip* Z.

Proof. Let $D \in \mathcal{F}$ be disjoint from Z. We may take D to lie above the x axis.

Consider two disks A', B' of radius b, concentric with A and B, respectively. Let m be the horizontal line at distance 2 above the x axis, and let H be the open halfplane above m. The center of D must lie in

$$M = A' \cap B' \cap H$$

by the maximality of the distance between A and B, and by the assumption that D does not intersect Z. Let us denote the point in $\partial A' \cap \partial B'$ with positive y-coordinate by X.

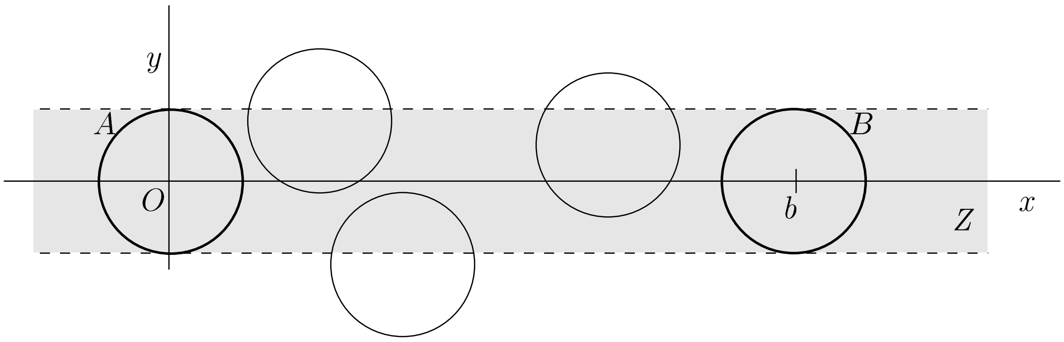

Figure 2: The two disks at maximum distance.

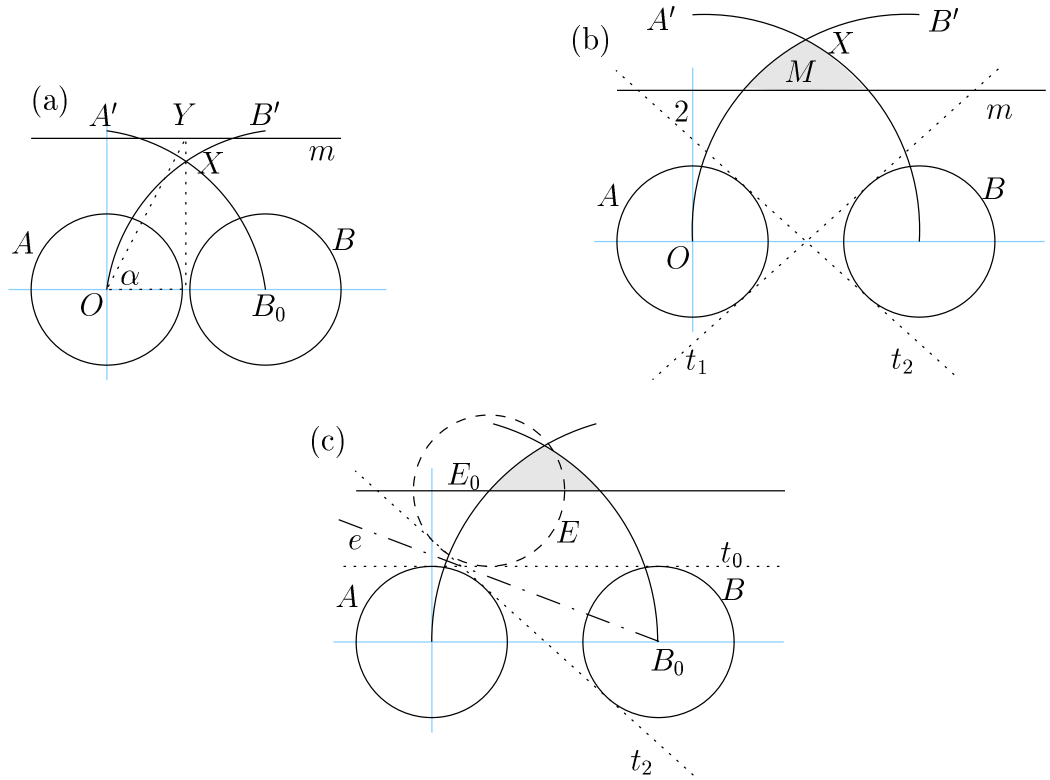

Figure 3: The region M is (a) empty for $b \leq 4/\sqrt{3}$, (b) non-empty for $b > 4/\sqrt{3}$. (c) The bottom left corner E_0 of M is at distance 1 from t_2.

Assume first that $b \le 4/\sqrt{3}$ (Fig. 3a). Let $Y = (b/2, 2)$ and $\alpha = \angle XAA'$. Considering the dotted triangle in Fig. 3a, we see that

$$\tan\alpha = \frac{2}{b/2} \ge \sqrt{3}$$

and so $\alpha \ge \pi/3$. Since the angle at O in the triangle XOB_0 is $\pi/3$, it follows that X lies at m or below it. But then $A' \cap B'$ is disjoint from H, a contradiction.

Thus we may assume $b > 4/\sqrt{3}$, in which case the (upper) crossing of $\partial A'$ and $\partial B'$ lies in H (see Fig. 3b). Consider the two non-horizontal common tangents t_1, t_2 to A and B. We claim that the distance between M and, say, t_2 is 1.

It is easy to see that the intersection E_0 of m with $\partial B'$ (more precisely, the one above the horizontal axis) is closer to t_2 than any point of M is. We wish to show that E_0 is at distance 1 from t_2. Let t_0 be the horizontal common tangent of A and B passing through $(1, 0)$. Let e be the line passing through B_0 and $t_0 \cap t_2$. Imagine a unit disk E centered at E_0. (See Fig. 3c.) The symmetry about e takes A to E and t_2 to t_0. Since t_0 is tangent to A, t_2 must be tangent to E. In particular, the distance of t_2 from the center E_0 of E is 1.

We have shown that all points of M are at distance more than 1 from t_2 and, by symmetry, from t_1. In other words, t_1 and t_2 are disjoint from the disk D. Clearly, $\{ t_0, t_1, t_2 \}$ is then a separating triple for $\{ A, B, D \}$, a contradiction. □

3 Proof of the theorem

We now prove Theorem 1.3. We introduce some terminology first.

If p is an oriented line, then H_p^+ and H_p^- denote the positive and negative closed halfplanes of p, respectively.

Let $J = [-\pi/2, \pi/2]$. For $\alpha \in J$, we define an oriented line $p(\alpha)$ as follows. Take an oriented line p such that the angle between the x axis and p (in this order) is α, and the whole of $\mathcal{F}$ is contained in H_p^-. Shift p to the first position where H_p^+ contains some disk from $\mathcal{F}$. The resulting line is $p(\alpha)$. Note that $H_{p(\alpha)}^+$ may contain more than one disk.

Let $\ell(\alpha)$ be the leftmost vertical tangent to a disk contained in $H_{p(\alpha)}^+$. Similarly, $r(\alpha)$ is defined as the rightmost such tangent. They are both oriented upwards. Refer to Fig. 4 for an illustration.

We shall use the term *leftmost (rightmost) disk touching* $p(\alpha)$ to denote the disk contained in $H_{p(\alpha)}^+$ which is first (last) in the ordering induced by the orientation of $p(\alpha)$. Together, these disks will be referred to as the *extremal disks touching* $p(\alpha)$.

Define the *left region* $L(\alpha)$ and the *right region* $R(\alpha)$ of α by

$$L(\alpha) = H_{p(\alpha)}^- \cap H_{\ell(\alpha)}^- \quad \text{and} \quad R(\alpha) = H_{p(\alpha)}^- \cap H_{r(\alpha)}^+.$$

(See Fig. 4.) Note that $L(-\pi/2)$ and $R(\pi/2)$ are both empty.

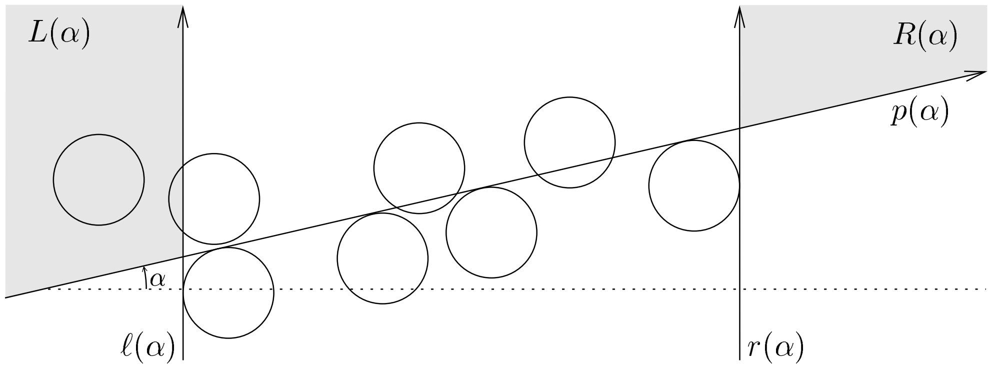

Figure 4: The left and right regions of α.

Let L be the subset of J consisting of points α such that the interior of $L(\alpha)$ contains a member of $\mathcal{F}$. Define $R \subset J$ similarly, with $R(\alpha)$ in place of $L(\alpha)$.

Proposition 3.1 *L and R are non-empty disjoint open subsets of J.*

Proof. To see that L and R are disjoint, consider a point α in their intersection. There is a disk D_1 in $\operatorname{int} L(\alpha)$ and a disk D_2 in $\operatorname{int} R(\alpha)$. Take any disk C in $H^+_{p_\alpha}$. Observe that $p(\alpha)$, $\ell(\alpha)$ and $r(\alpha)$ form a weakly separating triple for D_1, D_2 and C (see Fig. 5). Lemma 2.2 provides the contradiction.

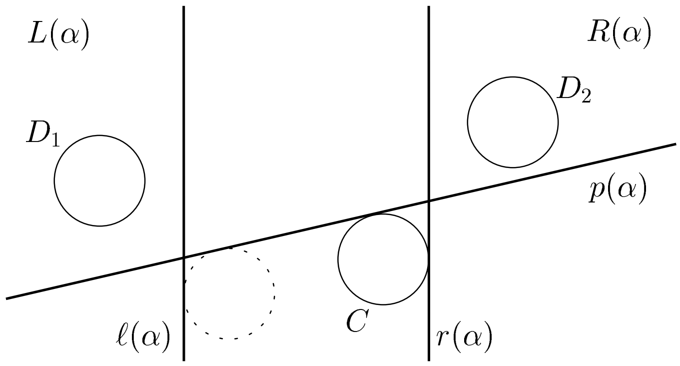

Figure 5: A weakly separating triple (in bold) showing that $L \cap R = \emptyset$.

Note that L is non-empty since $A \subset L(\pi/2)$ and so $\pi/2 \in L$. Similarly, $-\pi/2 \in R$.

Let us prove that L (say) is open in J. We need to show that for any point $\alpha \in L$, L contains a neighbourhood of α.

Assume that $\alpha \in L$, with a disk D contained in $L(\alpha)$. Let C be the rightmost disk touching $p(\alpha)$. For a small $\varepsilon > 0$ and any $\beta \in (\alpha, \alpha + \varepsilon)$, $H^+_{p(\beta)}$ contains C

as well, so $\ell(\beta) = \ell(\alpha)$, and it is not hard to see that $L(\beta) \supset L(\alpha)$ and hence $\beta \in L$.

Finally, take $\beta \in (\alpha - \varepsilon, \alpha)$. If ε is small enough, $p(\beta)$ touches the leftmost disk touching α, and so again, $\ell(\beta) = \ell(\alpha)$. Furthermore, ε may be taken small enough so that $p(\beta)$ does not intersect D. Therefore, D is in the interior of $L(\beta)$, implying $\beta \in L$ as required.

The statement that R is open can be proved in a symmetric way. $\square$

Proposition 3.1 implies that the connected space J cannot be covered by L and R. In other words, there is a point $\alpha_0 \in J$ such that neither $\operatorname{int} L(\alpha_0)$ nor $\operatorname{int} R(\alpha_0)$ contain a member of $\mathcal{F}$. From now on, let us denote $p(\alpha_0)$ simply as p.

Let C_1, C_2 be the leftmost and rightmost disks touching p, respectively. Define the vertical closed strip S_i $(i = 1, 2)$ to be the union of all vertical lines intersecting C_i. Clearly S_i is of width 2.

Proposition 3.2 *Every disk in $\mathcal{F}$ which is not met by p must intersect $S_1 \cup S_2$.*

Proof. Let $D \in \mathcal{F}$ avoid both p and $S_1 \cup S_2$. By the definition of $p(\alpha)$, D must lie in H_p^-. If H_p^+ contains a single disk (i.e. $S_1 = S_2$), we are done, since $S_1, L(\alpha_0)$ and $R(\alpha_0)$ together cover all of H_p^-, so there is no choice for D but to intersect S_1.

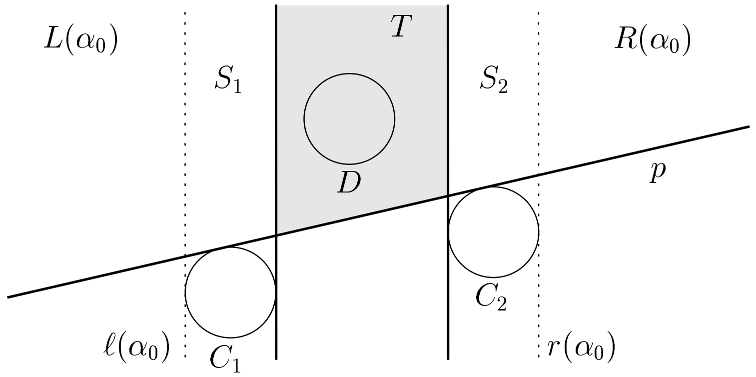

Figure 6: A weakly separating triple (in bold) showing that no disk can be contained in T.

It remains to discuss the case where H_p^+ contains more than one disk. The set

$$T = \operatorname{int} H_p^- - \Big(L(\alpha_0) \cup S_1 \cup S_2 \cup R(\alpha_0)\Big)$$

is then nonempty. However, assuming that the disk D is contained in T, it is easy to find a separating triple for C_1, C_2 and D: just take the 2 vertical lines bounding T, together with p as in Fig. 6. This concludes the proof. $\square$

Before proving the main result, we need a technical lemma based on the following classical theorem of L. Fejés Tóth [3] (see also [9]).

Theorem 3.3 (L. Fejés Tóth) *Let H be a convex hexagon, let C be a convex compact set with non-empty interior, and let P_6 denote a hexagon of minimum area circumscribed about C. If n pairwise disjoint translates of C are packed in H, then*

$$n \leq \frac{A(H)}{A(P_6)}$$

where $A(\cdot)$ denotes the area.

Lemma 3.4 *Let D be a unit disk centered at the origin, and let Q be an axis-parallel square with sides of length 6, whose two opposite corners lie at $(-3, -1)$ and $(3, 5)$. For any oriented line s with $D \subset H_s^+$, the region*

$$Q_s = Q \cap H_s^-$$

can hold at most 6 pairwise disjoint unit disks.

Proof. The situation is pictured in Fig. 7.

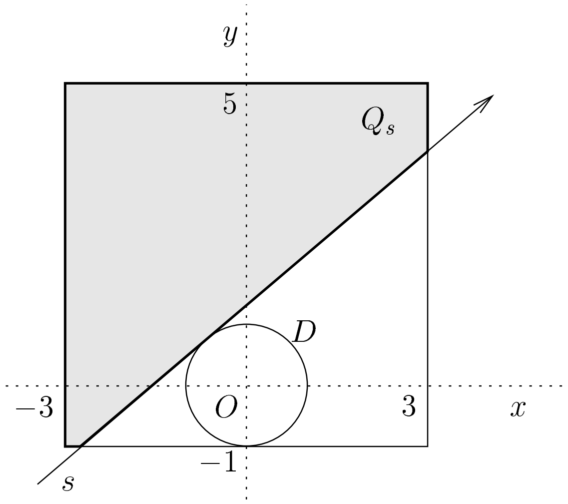

Figure 7: The region Q_s.

It is sufficient to consider the lines s which are tangent to D.

Note first that every (non-empty) Q_s is a convex polygon with less than 6 sides. It is easy to see that Theorem 3.3 applies in this case just as well. It is known that the hexagon P_6 of minimum area circumscribed about a disk is the regular hexagon; if the disk has unit radius, then the area of P_6 is $2\sqrt{3}$. The inequality of Theorem 3.3 thus becomes

$$n \leq \frac{A(Q_s)}{2\sqrt{3}}$$

where n is the number of unit disks packed inside Q_s.

Furthermore, the area $A(Q_s)$ is maximized when s is a horizontal tangent to D, oriented to the right, in which case $A(Q_s) = 24$.

We get that for any s, the number of disks packed inside Q_s is

$$n \leq \left\lfloor \frac{A(Q_s)}{2\sqrt{3}} \right\rfloor \leq \left\lfloor \frac{24}{2\sqrt{3}} \right\rfloor = 6.$$

□

Proof of Theorem 1.3 will be complete when we prove the following proposition.

Proposition 3.5 *The line p intersects all but at most 12 members of $\mathcal{F}$.*

Proof. Take any disk $D \in \mathcal{F}$. By Lemma 2.3, D must be contained in the closed horizontal strip Z' of width 6, whose center line is the x axis. On the other hand, if D is not intersected by p, then by Proposition 3.2, D must be contained in one of the closed vertical strips S'_1, S'_2 of width 6 whose center lines coincide with those of S_1 and S_2, respectively.

This would suffice to prove that D is contained in one of two squares $T_i = Z' \cap S'_i$ $(i = 1, 2)$ of area 36 each. We can improve this by observing that D must actually be contained in the intersection of these squares with H_p^-. In each square T_i, the situation is precisely as in Lemma 3.4. The lemma implies that each $T_i \cap H_p^-$ can contain no more than 6 disks. The total number of disks not intersected by p is thus at most 12 as claimed. □

4 A generalization to translates

The method used above for disks can be adapted to work for translates of a compact convex set, thus providing an improvement on the constant in the Katchalski–Lewis theorem:

Theorem 4.1 *If every 3 members of a finite disjoint family $\mathcal{F}$ of translates of a compact convex set C in $\mathbf{R}^2$ have a line transversal, then there is a line meeting all but* 58 *members of $\mathcal{F}$.*

There is an unsatisfactory gap between the value of the constant here and in Theorem 1.3. The proof of the more general statement involves some rather crude estimates which we have not been able to avoid so far. As it stands, Theorem 4.1 is inferior to a result of A. Holmsen [7], who succeeded in proving the same statement with 58 replaced by 26 using different methods. For this reason, we only sketch the necessary modifications to the proof of Theorem 1.3 and skip the precise details, hoping that the interested reader will be able to reconstruct the argument.

Let $\mathcal{F}$ be a finite family of translates of a compact convex set C. We may assume that C has an interior point; in the opposite case it is a straight line segment and we can approximate it by a very thin rectangle.

We begin with the following easy observation:

Lemma 4.2 *Any compact convex set C can be inscribed in a rectangle whose area is at most twice that of C.*

Proof. (sketch) Take a segment XY which is contained in C and whose length is equal to the diameter of C. The rectangle bounded by 4 lines touching C, two of them parallel to XY and the other two perpendicular, has the desired property. $\square$

Fix the containing rectangle $R(C)$ as in Lemma 4.2 for each member of C, in such a way that the rectangles are congruent. We may in fact assume that the rectangles are squares, performing the obvious affine transformation of the plane if necessary. The transformation preserves the relevant properties of $\mathcal{F}$ and keeps the ratio between the area of $R(C)$ and that of C unchanged.

Inscribe each $R(C)$ in a disk $D(C)$ and let $\mathcal{F}' = \{\, D(C) \mid C \in \mathcal{F} \,\}$. Since a separating triple for three disks $D(C_i)$ $(1 \leq i \leq 3)$ is necessarily a separating triple for the sets C_i, we may use Lemma 2.3 for $\mathcal{F}'$. We obtain a parallel strip Z intersected by any $D(C)$. The width of Z is at most the length d of the diagonal in the square $R(C)$.

We proceed as in Section 3 (working with the family $\mathcal{F}$) and obtain the line $p = p(\alpha_0)$ and two strips S_1, S_2 as in Proposition 3.2. These strips are intersected by any member of $\mathcal{F}$ which is not met by p. The width of the strips is at most the diameter of C, which is bounded from above by d. It follows that every member of $\mathcal{F}$ not intersected by p is contained in one of two squares T_1, T_2 of side $3d$. The area of, say, T_1 is $9d^2 \leq 36A(C)$ (where $A(C)$ is the area of C). Taking into account the fact that a part of T_1 is cut off by p, we can use an analogue of Lemma 3.4 to reduce the area by a factor of $1/6$. We find that in each T_i, at most 29 copies of C can be packed. Theorem 4.1 follows.

5 Open problems

An obvious open problem is the original Katchalski–Lewis conjecture (mentioned in the Introduction), even for disjoint unit disks.

Another question is whether an analogue of Theorem 1.2 might hold in the 3-space, say for disjoint unit balls:

Question 5.1 Is there a constant C such that for any finite disjoint family of unit balls in $\mathbf{R}^3$ in which every 3 members have a line transversal, there is a line intersecting all except at most C of the balls?

This question is even more interesting in view of the fact that it is not known whether there exists a finite Helly number for line transversals to disjoint unit balls in $\mathbf{R}^3$ (that is, whether an analogue of Theorem 1.1 holds in this case, with a suitable constant).

Acknowledgments

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