ON NON P-RECURSIVENESS OF NUMBERS OF MATCHINGS (LINEAR CHORD DIAGRAMS) WITH MANY CROSSINGS

MARTIN KLAZAR

Abstract. The number con_n counts matchings X on $\{1, 2, \ldots, 2n\}$, which are partitions into n two-element blocks, such that the crossing graph of X is connected. Similarly, cro_n counts matchings whose crossing graph has no isolated vertex. (If it has no edge, Catalan numbers arise.) We apply generating functions techniques and prove, using a more generally aplicable criterion, that the sequences (con_n) and (cro_n) are not P-recursive. On the other hand, we show that the residues of con_n and cro_n modulo any fixed power of 2 can be determined P-recursively. We consider also the numbers sco_n of symmetric connected matchings. Unfortunately, their generating function satisfies a complicated differential equation which we cannot handle.

Matchings on the vertex set $[2n] = \{1, 2, \ldots, 2n\}$ consist of n mutually disjoint two-element edges. One finds easily that their number mat_n equals $(2n-1)!! = 1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)$. Another classical result tells us that the number ncr_n of *noncrossing matchings* on $[2n]$ (no two edges $\{a, b\}$ and $\{c, d\}$ satisfy $a < c < b < d$) is the nth Catalan number: $\mathrm{ncr}_n = \frac{1}{n+1}\binom{2n}{n}$. How many matchings are there if their crossings are restricted in a more complicated way? In the present article we investigate numbers of such matchings, namely the numbers con_n of *connected matchings* in which each two edges can be connected by a chain of consecutivly crossing edges, the numbers sco_n of *symmetric connected matchings* which, in addition, are mirror symmetric, and the numbers cro_n of *crossing matchings* in which each edge crosses another edge. We concentrate only on P-recursiveness of these numbers. Also, we touch upon some congruence properties. The sequences (mat_n) and (ncr_n) are trivially P-recursive but, as we prove, the sequences (con_n) and (cro_n) are not.

First we remind the definition of P-recursiveness and D-finiteness. Then we introduce $\mathrm{D_A}$-finiteness and review some facts on power series. In Theorem 1 we prove that if a sequence of numbers has an OGF (ordinary generating function) that has zero convergence radius and satisfies a certain differential equation, then the sequence is far from being P-recursive. In Theorem 2 we apply this criterion to the sequences (con_n) and (cro_n). In Theorem 3 a more complicated differential equation is derived for the OGF of the sequence (sco_n). Finally, in Theorem 4 we show that modulo 2^l the sequences (con_n) and (cro_n) coincide with certain P-recursive, in fact algebraic, sequences.

Symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers $\{\ldots, -1, 0, 1, \ldots\}$ and $\{1, 2, \ldots\}$. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. A sequence of complex numbers

Supported by project LN00A056 of the Ministry of Education of the Czech Republic.

$(a_n)_{n\geq 0}$ is called *P-recursive* if there exist polynomials $P_0, P_1, \ldots, P_j \in \mathbb{C}[x]$, $P_0 \neq 0$, such that

$$P_0(n)a_n + P_1(n)a_{n-1} + \cdots + P_j(n)a_{n-j} = 0$$

holds for each integer $n, n \geq j$. Many combinatorial counting sequences are P-recursive, for instance (mat_n), (ncr_n), Schröder numbers, and numbers of labelled k-regular graphs (Gessel [6]). But some are not, for instance Bell numbers and numbers of integer partitions.

We write $\mathbb{C}[[x_1, \ldots, x_k]]$ for the ring of power series with complex coefficients and variables $x_1, \ldots, x_k$. A power series $F \in \mathbb{C}[[x]]$ is *D-finite* if F solves the linear differential equation

$$R_0F^{(m)} + R_1F^{(m-1)} + \cdots + R_{m-1}F' + R_mF + R_{m+1} = 0$$

with polynomial coefficients $R_i \in \mathbb{C}[x]$, $R_0 \neq 0$. A sequence is P-recursive if and only if its OGF is D-finite. For the (easy) proof of this equivalence and further information and references on P-recursiveness and D-finiteness we refer the reader to Stanley [20, chapter 6]. At first these concepts were systematicly investigated in Stanley [19].

The ring $\mathbb{C}[[x_1, \ldots, x_k]]$ is contained in the ring of *Laurent series* $\mathbb{C}((x_1, \ldots, x_k))$ whose elements are formal sums

$$F = \sum_{i_1,\ldots,i_k \in \mathbb{Z}} a_{i_1,\ldots,i_k} x_1^{i_1} \ldots x_k^{i_k}$$

with $a_{i_1,\ldots,i_k} \in \mathbb{C}$ and such that, for some $p \in \mathbb{N}$ (depending on F), $(x_1 \ldots x_k)^p F \in \mathbb{C}[[x_1, \ldots, x_k]]$. For $k = 1$ the ring $\mathbb{C}((x_1, \ldots, x_k))$ is a field (but not for $k > 1$). We use notation

$$\begin{aligned} [x_1^{i_1} \ldots x_k^{i_k}]F &= a_{i_1,\ldots,i_k} \text{ and} \\ \mathrm{ord}_{x_j}(F) &= \min\{i_j \in \mathbb{Z} : \ a_{i_1,\ldots,i_k} \neq 0 \text{ for some } i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k \in \mathbb{Z}\}. \end{aligned}$$

By the definition, for every j and nonzero $F \in \mathbb{C}((x_1, \ldots, x_k))$ we have $\mathrm{ord}_{x_j}(F) > -\infty$. We set $\mathrm{ord}_{x_j}(0) = \infty$.

A power series $F \in \mathbb{C}[[x_1, \ldots, x_k]]$ is *analytic* if there is a real constant $\delta > 0$ such that the series $F(\alpha_1, \ldots, \alpha_k)$ absolutely converges for all $\alpha_i \in \mathbb{C}$, $|\alpha_i| < \delta$. Analytic power series form a ring denoted $\mathbb{C}\{x_1, \ldots, x_k\}$. This ring is closed also on division (if defined) and derivatives. Generalized coefficients of $F \in \mathbb{C}\{x_1, \ldots, x_k\}$ (obtained by distinguishing a subset of variables) are again analytic. For more details see Ruiz [17] or Fischer [4]. A Laurent series $F \in \mathbb{C}((x_1, \ldots, x_k))$ is analytic if, for some $p \in \mathbb{N}$, $(x_1 \ldots x_k)^p F \in \mathbb{C}\{x_1, \ldots, x_k\}$. Analytic Laurent series form again a ring closed on division, derivatives, and generalized coefficients. We say that $F \in \mathbb{C}[[x]]$ is *D_A-finite* if F solves the linear differential equation

$$R_0F^{(m)} + R_1F^{(m-1)} + \cdots + R_{m-1}F' + R_mF + R_{m+1} = 0$$

with analytic coefficients $R_i \in \mathbb{C}\{x\}$, $R_0 \neq 0$.

Fact. If $G \in \mathbb{C}[[x,y]]$, $G(0,0) = 0$, is nonzero and analytic, then every solution $F \in \mathbb{C}[[x]]$, $F(0) = 0$, of the equation

$$G(x, F) = 0$$

is analytic.

With the condition $\frac{\partial G}{\partial y}(0,0) \neq 0$ this is the implicit function theorem for analytic functions (power series) and F is unique; see, for example, Hille [7] or Markushevich [12]. Without it one proceeds as follows. By the Weierstrass preparation theorem, see [17, p. 14] or [4, p. 107], $G(x, F) = 0$ is equivalent to a polynomial equation $P(x, F) = 0$ where $P \in \mathbb{C}\{x\}[y]$. By the standard results of algebraic geometry on local parametrizations (Puiseux series), all solutions of $P(x, F) = 0$ are again analytic. A very readable account on these matters is given in Fischer [4, chapter 7]. The more complicated situation of analytic systems of equations in several variables and M. Artin's approximation theorem (which implies the Fact too) are discussed in Ruiz [17, pp. 94 and 106]. It is clear that in the Fact we can extend $\mathbb{C}[[x,y]]$ to $\mathbb{C}((x,y))$ (conditions $G(0,0) = 0$ and $F(0) = 0$ must be then modified accordingly).

Theorem 1. *Let $F \in \mathbb{C}[[x]]$ satisfy the differential equation*

$$F' = G(x, F) \tag{1}$$

where $G \in \mathbb{C}((x,y))$. Suppose that (i) F is not analytic, (ii) G is analytic, and (iii) $\mathrm{ord}_y(G) < 0$. Then F is not $\mathrm{D_A}$-finite, the less D-finite.

Proof. Differentiating repeatedly (1) and substituting $G(x, F)$ for F', we express the derivatives of F as

$$F^{(k)} = G_k(x, F) \ \text{ where } \ G_{k+1}(x,y) = \frac{\partial G_k}{\partial x} + G \cdot \frac{\partial G_k}{\partial y} \quad (\text{and } G_1 = G).$$

By condition (iii), $p = \mathrm{ord}_y(G) < 0$. Using condition (ii) and induction on k, we see that each $G_k \in \mathbb{C}((x,y))$ is analytic and

$$\mathrm{ord}_y(G_k) = k(p-1) + 1 < 0.$$

We suppose for contradiction that F is $\mathrm{D_A}$-finite and satisfies an equation

$$R_0 F^{(m)} + R_1 F^{(m-1)} + \cdots + R_m F + R_{m+1} = 0$$

with $R_i \in \mathbb{C}\{x\}$, $R_0 \neq 0$. We may assume that $m > 0$ because $m = 0$ implies $F = -R_1/R_0 \in \mathbb{C}\{x\}$ and this contradicts condition (i). Replacing each $F^{(k)}$ by $G_k(x, F)$, we obtain equation

$$H(x, F) = 0 \ \text{ where } \ H(x,y) = R_0(x)G_m(x,y) + \cdots + R_m(x)y + R_{m+1}(x).$$

The analyticity of the G_ks and R_is implies the analyticity of H. It is clear that $H \neq 0$ because $\mathrm{ord}_y(H) = \mathrm{ord}_y(R_0 G_m) = m(p-1)+1 < 0$. Thus, by the above Fact, $F \in \mathbb{C}\{x\}$, in contradiction with condition (i). $\square$

We have introduced the class of $\mathrm{D_A}$-finite power series primarily to put our arguments in a natural setting — why to restrict to polynomials when the key lies in the analytic×nonanalytic dichotomy — but one could investigate the closure properties of the class as well. Clearly, $\mathrm{D_A}$-finite power series are closed to addition and multiplication, the proof being the same as for D-finite power series, see [20, chapter 6]. D-finite power series are closed also to the Hadamard product that is defined as the coefficientwise multiplication. Are $\mathrm{D_A}$-finite power series closed to the Hadamard product?

To the "non-Catalan" aspect of matching enumeration (many crossings, nonanalytic OGFs) are relevant articles (listed in chronological order) by Touchard [26, 27, 28], Kleitman [10], Hsieh [8], Riordan [16], Stein [21], Stein and Everett [22], Nijenhuis and Wilf [13], Read [15], Ismail, Stanton and Viennot [9], Penaud [14], Li and Sun [11], Cori and Marcus [3], Flajolet and Noy [5], Stoimenow [25], and Sawada [18]. We attempted to make this list of references complete or nearly complete.

Matchings are also called *complete pairings* or *linear(ized) chord diagrams*. In about 1993, their close relatives *(circular) chord diagrams* started to play, together with chord diagram algebras, an important role in studying knot invariants, especially Vassiliev knot invariants. By now more than 30 articles applying chord diagrams in knot theory have been published, of which we mention only Bar-Natan [1] and Stoimenow [23, 24]. This lower bound could be probably improved because Bar-Natan's bibliography on Vassiliev invariants, see [2], contains more than 400 items. Chord diagrams are orbits of the $2n$-element cyclic group acting on matchings by cyclically reordering vertices in $[2n]$. They appear in knot theory as follows. Suppose K is a singular knot, which is a smooth embedding of the circle S^1 in $\mathbb{R}^3$ with n transversal selfintersections (double points). Fix an orientation of S^1 and associate with K a chord diagram X: mark on S^1 the $2n$ preimages of the selfintersections of K and connect by chord each two of them that correspond to one selfintersection. In fact, in these applications only those chord diagrams are important in which each chord crosses another chord (cf. cro_n). We refer to [1] for further information.

Recall that con_n and cro_n are numbers of connected and crossing matchings with n edges. We can rephrase their definition in terms of *crossing graphs*. For a matching X, this is a graph $G(X) = (X, E)$ with the vertex set X and the edge set $E = \{\{A, B\} : \ A, B \in X \text{ cross }\}$. Connected (crossing) matchings are matchings X for which $G(X)$ is connected (has no isolated vertex).

Theorem 2. *The power series*

$$\begin{aligned} E &= \sum_{n\ge1} \mathrm{con}_n x^n = x + x^2 + 4x^3 + 27x^4 + \cdots \quad \textit{and} \\ F &= \sum_{n\ge0} \mathrm{cro}_n x^n = 1 + x^2 + 4x^3 + 31x^4 + \cdots \end{aligned}$$

satisfy the differential equations

$$E' = \frac{E^2+E-x}{2xE} \quad \text{and} \tag{2}$$

$$F' = \frac{-x^2F^3+F-1}{2x^3F^2+2x^2F}. \tag{3}$$

By Theorem 1, E and F are not D_A-finite. In particular, the sequences (con_n) and (cro_n) are not P-recursive.

Proof. By the *spaces* in a matching X on $[2n]$ we mean the $2n-1$ *inner spaces* between the consecutive elements of $[2n]$ and the two spaces before and after $[2n]$. Alltogether X has $2n+1$ spaces. Let X be a connected matching on $[2n]$ with the first edge $\{1,a\} \in X$. Deleting it from X, the graph $G(X)$ falls apart into k components to which correspond nonempty matchings $X_1,\ldots,X_k$. X_i has $n_i > 0$ edges and $X_1,\ldots,X_k$ are *nested*, which means that X_{i+1} lies in one of the $2n_i-1$ inner spaces of X_i. Also, a lies in one of the $2n_k-1$ inner spaces of X_k. There are no other restrictions on the X_is and a. In terms of E,

$$E = x\sum_{k\ge 0}(2xE'-E)^k.$$

Summing the geometric series and solving the result for E', we obtain equation (2).

Let X be a generic crossing matching on $[2n]$. We shall construct X by considering the edges which cross *only* the first edge $\{1,a\} \in X$. Suppose there are k such edges: $\{b_i,c_i\} \in X$, $1 < b_1 < b_2 < \cdots < b_k < a < c_k < \cdots < c_2 < c_1 \le 2n$. The remaining edges of X lie either inside or outside each $\{b_i,c_i\}$ and the $n_i \ge 0$ edges sandwiched between $\{b_{i-1},c_{i-1}\}$ and $\{b_i,c_i\}$ form, for $i=1,2,\ldots,k+1$, a (possibly empty) crossing matching X_i. We distinguish the cases $k>0$ and $k=0$.

Let $k>0$. X arises by choosing a crossing matching X_1 with $n_1 \ge 0$ edges, inserting $\{b_1,c_1\}$ in one of the $2n_1+1$ spaces of X_1, inserting a crossing matching X_2 inside $\{b_1,c_1\}$, inserting $\{b_2,c_2\}$ in one of the $2n_2+1$ spaces of X_2 (inside $\{b_1,c_1\}$), inserting a crossing matching X_3 inside $\{b_2,c_2\}$, and so on. In the end X_{k+1} is inserted inside $\{b_k,c_k\}$, 1 is put in the beginning, and a is inserted in one of the $2n_{k+1}+1$ spaces of X_{k+1} (and inside $\{b_k,c_k\}$, of course).

If $k=0$, just $\{1,a\}$ is added to X_1. However, now the difference is that a cannot be inserted in each of the $2n_1+1$ spaces of X_1. Since $\{1,a\}$ must cross an edge, allowed are only spaces not separating X_1 into two crossing matchings.

In terms of F,

$$F = 1+\sum_{k\ge 0}(x(2xF'+F))^{k+1} - xF^2.$$

Term xF^2 subtracts the number of spaces forbidden for a if $k=0$. Summing the geometric series and solving the result for F', we obtain equation (3).

The conditions of Theorem 1 are easy to check. The right hand sides G of (2) and (3) are $(2xy)^{-1}(-x+y+y^2)$ and $(2x^2y)^{-1}(-1+y-x^2y^3)(1+xy)^{-1}$, respectively. Thus both G belong to $\mathbb{C}(x,y)\cap\mathbb{C}((x,y))$ (are rational Laurent series) and for both $\mathrm{ord}_y(G)=-1$. Conditions (ii) and (iii) are satisfied. Clearly, $\mathrm{con}_n, \mathrm{cro}_n \ge (n-1)!$

for every $n \in \mathbb{N}$; just intersect one edge by the others. Condition (i) is satisfied too and Theorem 1 applies. $\square$

Writing (2) as $E = E \cdot (2xE' - E) + x$, we obtain the recurrence

$$\begin{aligned}\mathrm{con}_n &= \sum_{k=1}^{n-1} (2k-1) \cdot \mathrm{con}_k \cdot \mathrm{con}_{n-k} \\ &= (n-1) \sum_{k=1}^{n-1} \mathrm{con}_k \cdot \mathrm{con}_{n-k} \quad (n \geq 2 \;\text{ and }\; \mathrm{con}_1 = 1)\end{aligned}$$

found in [21]. It was derived first by Stein and Riordan and then a simple bijective proof was provided in [13].

We give an alternative derivation for (2). Every matching X with $n > 0$ edges decomposes uniquely into a connected matching Y (the component of $G(X)$ containing the vertex corresponding to the first edge of X) with $m > 0$ edges and $2m$ matchings Z_i, $i = 1 \ldots 2m$, which have l_i edges each and are inserted in the $2m + 1$ spaces of Y except the first one. Each Z_i may be empty, no Z_i lies before Y, and $m + l_1 + \cdots + l_{2m} = n$. Thus, if E is the above OGF of (con_n) and $M = \sum_{n \geq 0} \mathrm{mat}_n x^n = 1 + x + 3x^2 + 15x^3 + \cdots$,

$$E(xM^2) = M - 1. \tag{4}$$

Formula $\mathrm{mat}_n = (2n-1)!!$ implies

$$M' = \frac{(1-x)M - 1}{2x^2}. \tag{5}$$

Here $\mathrm{ord}_y(G)$ is nonnegative, of course.

We write E_0 for $E(xM^2)$ and E_1 for $E'(xM^2)$. Thus (4) reads as $M = E_0 + 1$. Differentiating (4), solving the result for M', and replacing M by $E_0 + 1$, we obtain the equation $M' = E_1(E_0 + 1)^2/(1 - 2xE_1(E_0 + 1))$. Substituting in (5) for M' the latter expression and for M the expression $E_0 + 1$, we obtain a polynomial equation $P(x, E_0, E_1) = 0$. Rewriting it as $Q(x(E_0+1)^2, E_0, E_1) = 0$ and substituting for x the power series inverse of $xM^2 = x(E_0 + 1)^2$, we get a polynomial equation $Q(x, E(x), E(x)') = 0$. It turns out to be (2).

This method may look complicated but its advantage is that it applies without much change to the natural generalization of the problem to r-matchings. These are partitions, for a fixed $r \geq 2$, of $\{1, 2, \ldots, rn\}$ into r-element sets. Generalization of (non)crossing for $r > 2$ is easy (viz. noncrossing partitions). The formula for the number mat_n^r of all r-matchings and the analogue of (5) are straightforward. So is the analogue of (4). A differential equation for the OGF E_r of numbers con_n^r of connected r-matchings then can be derived by the method we have just indicated for $r = 2$. The equation involves derivatives of E_r to the order $r - 1$. The more combinatorial argument from the first part of the proof of Theorem 2 is still possible but for $r > 2$ becomes quite cumbersome.

From (4) also a relation between the numbers mat_n and con_n can be derived that, though standard, seems not to appear in the literature. Namely, writing M_0 for xM^2,

(4) can be restated as $x(1+E(M_0))^2 = M_0$ and Lagrange inversion formula gives $[x^n]M_0(x) = n^{-1}[x^{n-1}](1+E(x))^{2n}$. In terms of the coefficients,

$$(n+1)\sum_{i=0}^{n}(2i-1)!!\cdot(2n-2i-1)!! = \sum \mathrm{con}_{a_1}\cdot\mathrm{con}_{a_2}\cdot\ldots\cdot\mathrm{con}_{a_{2n+2}}.$$

Here $(-1)!! = \mathrm{con}_0 = 1$ and the latter summation goes over all $2n+2$-tuples such that $a_i \geq 0$ and $a_1 + a_2 + \cdots + a_{2n+2} = n$.

Restating equation (3) as $F\cdot(1+xF)\cdot(1-xF-2x^2F') = 1$ leads, after rearrangements and shifts of the indices, to the recurrence

$$\mathrm{cro}_n = \sum (2m+1)\cdot\mathrm{cro}_k\cdot\mathrm{cro}_l\cdot\mathrm{cro}_m \quad (n\geq 1 \text{ and } \mathrm{cro}_{-1} = \mathrm{cro}_0 = 1)$$

where we sum over $-1 \leq k,l,m \leq n-1$ & $k \neq -1$ & $k+l+m = n-2$. This is a more explicit relation than the formulas in Stoimenow [25, p. 217] where cro_n is denoted $\bar{\psi}_n$. The first few values of these numbers are:

$$\begin{aligned}(\mathrm{con}_n)_{n\geq 2} &= (1,4,27,248,2830,38232,593859,10401712,\ldots)\\ (\mathrm{cro}_n)_{n\geq 2} &= (1,4,31,288,3272,43580,666143,11491696,\ldots).\end{aligned}$$

The matchings X on $[2n]$ such that the graph $G(X)$ is connected and X does not change when the linear order of $[2n]$ is reversed were investigated first in [21]. Stein calls them *irreducible symmetric diagrams* and denotes their number, sco_n in our notation, by σ_{2n}.

Theorem 3. *The power series*

$$F = \sum_{n\geq 1}\mathrm{sco}_n x^n = x + x^2 + 2x^3 + 7x^4 + \cdots$$

is related to the power series E of Theorem 2 by

$$F^2 - (1+x)F + E(x^2)\cdot(2xF' - F + 1) + x = 0 \tag{6}$$

and satisfies the second order differential equation

$$F''A_0 + (F')^3A_1 + (F')^2A_2 + F'A_3 + A_4 = 0 \tag{7}$$

where $A_i \in \mathbb{C}[x,F]$, namely $A_0 = -2x^2F^4 + (4x^2+4x^3)F^3 - (2x^2+8x^3+2x^4)F^2 + (4x^3+4x^4)F - 2x^4$, $A_1 = 8x^5$, $A_2 = 4x^2F^3 - (2x^2+6x^3)F^2 - (2x^2-4x^3+10x^4)F + 2x^3 + 10x^4$, $A_3 = -5xF^4 + (7x+7x^2)F^3 + (x-11x^2+4x^3)F^2 + (-3x+x^2-8x^3)F - x + 3x^2 + 4x^3$, and $A_4 = F^5 - (2+x)F^4 + (2x-x^2)F^3 + (2+3x^2)F^2 - (1+2x+3x^2)F + x + x^2$.

Proof. We say that a matching X is a *chain* if the connected matchings corresponding to the components of $G(X)$ are nested as in the proof of Theorem 2. If X is in addition symmetric, we say that X is a *symmetric chain*. Now let X be a generic symmetric connected matching on $[2n]$ with the first edge $\alpha = \{1,a\}$ and the last edge $\beta = \{2n-a+1, 2n\}$. Note that for $n > 1$ always $\alpha \neq \beta$. We delete α and β from X and see what happens. After a while we see (since $X\backslash\{\alpha,\beta\}$ must be symmetric) that the connected matchings corresponding to the components of $G(X)\backslash\{\alpha,\beta\}$ have the following structure.

- On the top there is a (possibly empty) symmetric chain consisting of $k \geq 0$ symmetric connected matchings $X_1, X_2, \ldots, X_k$ where X_k is the innermost one.
- Then there are two (possibly empty) chains Y and Z such that Y precedes Z, Y and Z lie in a centrally symmetric pair of (not necessarily distinct) inner spaces of X_k, and Y and Z are reflections of one another.
- The endvertex of α lies in an inner space of Y and the first vertex of β lies in an inner space of Z, or vice versa, and these spaces move one on the other upon reflection.
- If α and β do not cross then $k \geq 1$ (for X to be connected) else $k \geq 0$.

This is a complete description of the structure of X, that is to say, arranging α, β, X_i, Y, and Z in any way meeting the above conditions, we get a symmetric connected matching. After another while we see that the conditions translate into the GFs F and E as

$$F = 2E(x^2) \cdot xF' \cdot \sum_{k \geq 1} F^{k-1} + E(x^2) + x.$$

Factor 2 accounts for crossing and noncrossing of α and β, the first $E(x^2)$ counts $Y \cup Z \cup \{\alpha, \beta\}$, the term xF' counts the positions of Y and Z in X_k, F^{k-1} and xF' count the nonempty top symmetric chains, the second $E(x^2)$ accounts for the case $k = 0$, and the last x accounts for $n = 1$. Summing the geometric series and rearranging the result, we obtain equation (6).

Equation (7) arises by solving (6) for $E(x^2)$, differentiating the result, and substituting in (2) the expressions for $E(x^2)$ and $E'(x^2)$ obtained. $\square$

Equating the coefficients at x^n in (6) to zero leads to the formula

$$\begin{aligned} \mathrm{sco}_n &= \sum_{i=1}^{n-2} \mathrm{sco}_i \cdot \mathrm{sco}_{n-i} + \sum_{i=1}^{\lfloor n/2 \rfloor} (2n - 4i - 1) \cdot \mathrm{con}_i \cdot \mathrm{sco}_{n-2i} \\ & \quad \text{where } n \geq 2,\ \mathrm{sco}_0 = -1, \text{and } \mathrm{sco}_1 = 1. \end{aligned}$$

This recurrence is simpler and more transparent than the one in the end of [21]. We have

$$(\mathrm{sco}_n)_{n \geq 2} = (1, 2, 7, 22, 96, 380, 1853, 8510, 44940, \ldots),$$

in agreement with the table in [21].

We say that a sequence (a_n) of integers is *P-recursive modulo* $m \in \mathbb{N}$ if there is a P-recursive sequence of integers (b_n) such that, for each n, $a_n \equiv b_n \bmod m$.

Theorem 4. *The sequences* (con_n) *and* (cro_n) *are P-recursive modulo* 2^k *for each* $k \in \mathbb{N}$.

Proof. We rewrite equation (2) for $E = \sum_{n \geq 1} \mathrm{con}_n x^n$ as $E^2 + E - x = 2xEE'$. Differentiating it, replacing E' on the left by $(E^2 + E - x)/(2xE)$, and multiplying all by $2xE$, we obtain equation $2E^3 + 3E^2 + (1 - 4x)E - x = P_2$ where $P_2 \in 4\mathbb{Z}[x, E, E', E'']$. Continuing this way, we derive for each $k \in \mathbb{N}$ an equation

$$Q_k(x, E) = P_k(x, E, E', \ldots, E^{(k)})$$

where Q_k and P_k are integral polynomials and P_k vanishes identically modulo 2^k. It is easy to see that the polynomials $[E^1]Q_k \in \mathbb{Z}[x]$ and $[E^0]Q_k \in \mathbb{Z}[x]$ have constant terms 1 and 0, respectively. Thus the equation

$$Q_k(x, A_k) = 0$$

has a unique power series solution $A_k \in \mathbb{Z}[[x]]$ with $A_k(0) = 0$. It follows by the induction on n that $\mathrm{con}_n = [x^n]E \equiv [x^n]A_k \bmod 2^k$ for each n. A_k is algebraic over $\mathbb{Z}(x)$ and thus D-finite ([20, Theorem 6.4.6]). Hence the coefficients of A_k form a P-recursive sequence and (con_n) is P-recursive modulo 2^k.

The proof for the sequence (cro_n) is similar and we omit it. □

Lagrange inversion formula yields some explicit congruences. We give one example. Reducing (3) modulo 2 we obtain equation $x^2F^3 - F + 1 = 0$. With $G = xF$ it becomes $G = x(1 - G^2)^{-1}$ and Lagrange inversion shows that cro_{2n+1} is always even and

$$\mathrm{cro}_{2n} \equiv \frac{1}{2n+1}\binom{3n}{n} \equiv \binom{3n}{n} \bmod 2.$$

The well known result saying that the highest power of a prime p dividing $n!$ has the exponent $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$ gives a more explicit criterion: cro_{2n} is even if and only if there is an $r \in \mathbb{N}$ such that the residue of n on division by 2^r is greater than $2^{r+1}/3$. For example, cro_{304} is even (set $r = 5$) but cro_{296} is odd.

Conclusion. As we have indicated, the problem of enumeration of matchings with restricted crossings can be generalized to set partitions with r-element blocks (or to set partitions without any restriction on block sizes). However, then differential equations of more complicated types than (1) arise, such as (7). Probably, in these generalizations less amateur approach is necessary, with some techniques from the differential algebra. Or maybe from the geometry, as suggested by an anonymous referee. We conjecture that neither the sequence (sco_n) is P-recursive. Another research direction is to investigate the behaviour of con_n and cro_n with respect to other moduli. We conjecture that these numbers are P-recursive modulo m only if $m = 2^k$.

References

[1] Bar-Natan, D., "On the Vassiliev knot invariants." *Topology* **34**(1995), 423–472.

[2] http://www.ma.huji.ac.il/ drorbn/

[3] Cori, R., and Marcus, M., "Counting non-isomorphic chord diagrams." *Theoret. Comput. Sci.* **204**(1998), 55–73.

[4] G. Fischer, *Plane Algebraic Curves.* AMS, Providence, RI, 2001.

[5] Flajolet, P., and Noy, M., "Analytic combinatorics of chord diagrams." In: Krob, D., Mikhalev, A. A., and Mikhalev, A. V. (editors), *Formal Power Series and Algebraic Combinatorics. 12th International Conference, FPSAC'00, Moscow, Russia, June 2000, Proceedings.* Springer, Berlin, 2000, pp. 191–201.

[6] Gessel, I. M., "Symmetric functions and P-recursiveness." *J. Comb. Theory, Ser. A* **53**(1990), 257–285.

[7] Hille, E., *Analytic Function Theory, Vol. I.* Ginn & Company, Boston etc., 1959.

[8] Hsieh, W. N., "Proportions of irreducible diagrams." *Stud. Appl. Math.* **52**(1973), 277–283.

[9] Ismail, M. E. H., Stanton, D., and Viennot, G., “The combinatorics of q-Hermite polynomials and the Askey–Wilson integral.” *Eur. J. Comb.* **8**(1987), 379–392.

[10] Kleitman, D. J., “Proportions of irreducible diagrams.” *Stud. Appl. Math.* **49**(1970), 297–299.

[11] Li, B., and Sun, H., “Exact number of chord diagrams and estimation of the number of spine diagrams of order n.” *Chinese Sci. Bull.* **42**(1997), 705–720.

[12] Markushevich, A. I., *Theory of Functions of a Complex Variable, Vol. II.* Prentice Hall, Englewood Cliffs, NJ, 1965.

[13] Nijenhuis, A., and Wilf, H. S., “The enumeration of connected graphs and linked diagrams.” *J. Comb. Theory, Ser. A* **27**(1979), 356–359.

[14] Penaud, J.-G., “Une preuve bijective d’une formule de Touchard–Riordan.” *Discrete Math.* **139**(1995), 347–360.

[15] Read, R. C., “The chord intersection problem.” *Ann. New York Acad. Sci.* **319**(1979), 444–454.

[16] Riordan, J., “The distribution of crossings of chords joining pairs of $2n$ points on a circle.” *Math. Comput.* **29**(1975), 215–222.

[17] Ruiz, J. M., *The Basic Theory of Power Series.* Friedrich Vieweg & Sohn, Braunschweig/Wiesbaden, 1993.

[18] Sawada, J., “A fast algorithm for generating non-isomorphic chord diagrams.” submitted.

[19] Stanley, R. P., “Differentiably finite power series.” *Eur. J. Comb.* **1**(1980), 175–188.

[20] Stanley, R. P., *Enumerative Combinatorics, Vol. II.* Cambridge University Press, Cambridge, UK, 1999.

[21] Stein, P. R., “On a class of linked diagrams, I. Enumeration.” *J. Comb. Theory, Ser. A* **24**(1978), 357–366.

[22] Stein, P. R., and Everett, J. A., “On a class of linked diagrams, II. Asymptotics.” *Discrete Math.* **21**(1978), 309–318.

[23] Stoimenow, A., *On Enumeration of Chord Diagrams and Asymptotics of Vassiliev Invariants.* Ph.D. thesis, Freie Universität Berlin, Department of Mathematics and Computer Science, Berlin, 1998.

[24] Stoimenow, A., “Enumeration of chord diagrams and upper bound for Vassiliev invariants.” *J. Knot Theory Ramifications* **7**(1998), 93–114.

[25] Stoimenow, A., “On the number of chord diagrams.” *Discrete Math.* **218**(2000), 209–233.

[26] Touchard, J., “Sur un problème de configurations.” *C. R. Acad. Sci. Paris* **230**(1950), 1997–1998.

[27] Touchard, J., “Contribution à l’étude du problème des timbres-poste.” *Can. J. Math.* **2**(1950), 385–398.

[28] Touchard, J., “Sur un problème de configurations et sur les fractions continues.” *Can. J. Math.* **4**(1952), 2–25.

E-mail address: klazar@kam.ms.mff.cuni.cz

Department of Applied Mathematics (KAM) and Institute for Theoretical Computer Science (ITI), Charles University, Malostranské náměstí 25, 118 00 Praha, Czech Republic