Remarks on Information Systems

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The purpose of this paper is to throw some new light on D. Scott's information systems ([**8**]) by providing a specific arrangement of proofs of their equivalence with various other concepts which is hoped to make their interconnections more transparent.

Our starting point for this will be the partially ordered set determined by the entailment relation od an information system in the quotient of its collection of consistent finite sets modulo mutual entailment – an approach familiar from other settings which formalize some notion of implication. Augmented by an added zero this turns out to be a *meet semilatice with zero and unit*, and a simple argument shows that any such semilattice arises in this manner. Further, the morphisms between information systems, the approximable mappings of [**8**], can then be described suggestively as homomorphisms between bounded meet semilattices, though not from A to B but rather from A to the lattice $\mathfrak{D}B$ of downsets U of B ($0 \in U$, and $a \in U$ whenever $a \leq b$ and $b \in U$) which we call the *extended* homomorphism from A to B. In all, we obtain a dual equivalence between Scott's category of information systems and the category **ESLat** of bounded meet semilattices and their extended homomorphisms (Section 1).

Following this we establish (Section 2) the equivalence of **ESLat** with the full subcategory of the category **Frm** of frames and their homomorphisms given by the frames $\mathfrak{D}A$ for bounded meet semilattices A which we characterize by simple internal conditions in terms of their supercompact elements. In addition, by the obvious fact that these frames are spatial this provides an equivalence between information systems and the spectra of these $\mathfrak{D}A$, called the *superspectral spaces*, which we subsequently identify as the familiar filter spaces of our semilattices. Next, we establish the dual equivalence of **ESLat** with the category **SDom** of Scott domains, thereby providing simple proofs of the equivalence between domains and (1) information systems and (2) superspectral spaces. In addition (Section 4) we show that **ESLat** is dually equivalent to the category of *conjunctive propositional theories*, introduced here as the conjunctive fragment of propositional geometric theories in order to provide a more purely logical version of information systems and to substantiate formally the intuitive notion that the concept of conjunction is exactly what information systems capture. Further, we discuss the fact (Section 5) that the category **ESLat** is actually the result of a well-known general categorical construction which applies to every so-called monad, namely the associated *Kleisli category* (Mac Lane [**6**], Chapter VI) where the monad relevant here is given by the down-set functor on the category of bounded meet semilattices taking each A to $\mathfrak{D}A$. Given the additional fact that the category **Frm** is the other category canonically constructed from this monad, namely the category of its *Eilenberg-Moore algebras*, this places the information systems into an intimated relationship with frames which may be viewed as an alternative way of saying that they cover exactly the part of propositional geometric logic which is determined by conjunction alone.

We conclude with some observations about number of further topics (Section 6). Thus, we point out how the universal information system described by Scott [**8**] has a very simple and suggestive form in the present setting, give an outline of the counterparts of the results in Section 3 for the more general *algebraic* domains, show that **ESLat** is dually equivalent to Abramsky's category of domain prelocales [**1**], thereby providing an alternative proof of the equivalence of the latter with information systems established in [**1**], and have a brief look at the treatment of finitary propositional theories, concerning binary disjunction together with conjunction, in the same style in which we dealt with the conjunctive case, making a connection with come work by Hulley [**3**].

Regarding background we refer to Mac Lane [**6**] for the few categorical notions involved here and to Johnstone [**4**] and Vickers [**9**] for basic facts concerning frames.

1. Information systems and semilattices

Recall that an *information system* is an entity $S = (E, \mathsf{Con}, \vdash)$ where E is a set, Con a collection of finite subsets $u, v, \ldots$ of E, and $\vdash$ a binary relation between Con and E subject to the following requirements for any $u, v, \ldots$ and elements $x, y, \ldots$ of E.

(IS1) If $u \subseteq v$ and $v \in \mathsf{Con}$ then $u \in \mathsf{Con}$.
(IS2) $\{x\} \in \mathsf{Con}$.
(IS3) If $u \vdash x$ then $u \cup \{x\} \in \mathsf{Con}$.
(IS4) If $u \in \mathsf{Con}$ and $x \in u$ then $u \vdash x$.
(IS5) If $u \vdash x$ for all $x \in v$ and $v \vdash y$ then $u \vdash y$.

Note that this notion is essentially due to Scott [**8**] except for the omission of the distinguished ("least informative") element $\Delta \in E$. We follow Larden-Winskel [**5**] who recognized that the latter is redundant, leading to present simpler but equivalent form.

We use the following notation. For any information system $S = (E, \mathsf{Con}, \vdash)$ we let

$$|S| = E, \quad \mathsf{Con}S = \mathsf{Con}, \quad \vdash_S = \vdash,$$

allowing ommision of the index in the latter whenever suitable.

Regarding *semilattices*, these will always be taken as meet semilattices with zero (=bottom) 0 and unit (=top) e, and their homomorphisms will be assumed to preserve the latter as well as meet. **SLat** will be the corresponding category.

In order to establish our basic connection between these two notions, recall that, for any information system S, the relation $\vdash_S$ is extended to $\mathsf{Con}S$ by setting

$$u \vdash_s v \quad \text{iff} \quad u \vdash_S x \ \text{ for all } \ x \in v,$$

providing a preorder on $\mathsf{Con}S$ (Scott [**8**], 2.2 and 2.3). Hence passing to the quotient $\mathsf{Con}S/\!\dashv\vdash$, $u \dashv\vdash v$ meaning that $u \vdash v \vdash u$, we obtain a partially ordered set, clearly with largest element, given by $\emptyset$: $u \vdash_S \emptyset$ vacuously for all $u \in \mathsf{Con}S$. Moreover, this quotient is a conditional meet semilattice: if $u \vdash_S v$ and $u \vdash_S w$ then $v \cup w \in \mathsf{Con}_S$ by (IS3) and (IS1) so that $u \vdash_S v \cup w$ while $v \cup w \vdash_S v$ and $v \cup w \vdash_S w$ trivially. As a result, it is natural to adjoin a formal bottom $\perp$ to $\mathsf{Con}S/\!\dashv\vdash$ to obtain a *semilattice* $\mathcal{L}S$.

It should be added that the above discussion tacitly assumed that $\mathsf{Con}S$ is non-void while the void case is not to be ignored: the specification $|S| = \emptyset = \mathsf{Con}S$ certainly defines an information system, even if a very trivial one. In this case $\mathsf{Con}S/\!\dashv\vdash$ is not a conditional meet semilattice with unit, being empty, but $\mathcal{L}S$ is then taken as $\{\perp\}$, the trivial (=terminal) semilattice. Incidentally, the condition $|S| = \emptyset$ also allows for $\mathsf{Con}S = \{\emptyset\}$ in which case $\mathcal{L}S$ is the two-element semilattice.

Remark 1. We note that our transition from S to $\mathcal{L}S$ is closely related to the passage, described by Sambin et al. [**7**], from information systems to what they call *information bases*. Indeed, treating the latter in classical rather than in intuitionist logic as done in [**7**] reduces them, up to the relevant notion of isomorphism, exactly to semilattices, with $a > 0$ as the positivity predicate $\mathsf{Pos}(a)$. Further, the manner in which the information base is derived from an information system in [**7**], although formulated in somewhat different terms corresponds exactly to the way we obtain $\mathcal{L}S$ from S.

Our first result is now

Lemma 1. *For any sublattice A there exist information systems S such that $\mathcal{L}S \cong A$.*

Proof: We define S by the following stipulations

$$|S| = A^+ = \{a \in A \mid a > 0\},$$
$$u \in \mathsf{Con}S \ \text{ iff } \ \bigwedge u \neq 0, \quad \text{and}$$
$$u \vdash x \ \text{ iff } \ \bigwedge u \leq x,$$

where $\bigwedge u = a_1 \wedge \cdots \wedge a_n$ if $u = \{a_1, \ldots, a_n\}$.

It is immediately obvious that this satisfies the conditions for information systems. Further, $u \dashv\vdash \{\bigwedge u\}$ for any $u \in \mathsf{Con}S$ while $\{a\} \vdash \{b\}$ iff $a \leq b$ for all $a, b \in A^+$. As a result, the partially ordered set $\mathsf{Con}S/\dashv\vdash$ and A^+ are isomorphic, and hence $\mathcal{L}S \cong A$ as desired. $\square$

We note in passing that very different looking information systems S can have the same $\mathcal{L}S$, the point being that an information system is really *a presentation of a semilattice by generators and relations*, the former corresponding to the $\{x\}$ and the latter encoded by $\vdash$, and evidently the same semilattice can have many different such presentations. As a case in point, if $S = (E, \mathsf{Con}, \vdash)$ is such that Con consists of all finite $u \subseteq E$ and $u \vdash x$ iff $x \in u$ then $\mathcal{L}S$ is the *free semilattice on* E but the information systems described for it in the above proof is clearly quite different.

Next recall that an approximable mapping $f : S \to T$ of information systems (Scott [**8**], 5.1), here to be called *map* for short, is a relation $f \subseteq \mathsf{Con}S \times \mathsf{Con}T$ such that

(M1) $\emptyset f \emptyset$.
(M2) If ufv and ufw then $uf(v \cup w)$.
(M3) If $u' \vdash u$, $u \vdash v$, and $v \vdash v'$ then $u'fv'$.

Further, for any such $f : S \to T$ and $g : T \to U$, the composite $gf : S \to U$ is the familiar relational composite $f \circ g$ ([**8**], 5.5) and for each S, $\vdash_S$ is a map $S \to S$ which obviously acts as a unit with respect to composition. As a result, one has a category which we call **InfSyst**.

Given the correspondence $S \mapsto \mathcal{L}S$ from information systems to semilattices, it is natural to wonder how this can be extended meaningfully to the morphisms of this category. Of course there is a perfectly obvious, and somewhat unimaginative, way of doing this by considering the relations $F \subseteq A \times B$ between semilattices specified by conditions strictly parallel to these defining the maps of information systems:

(M'0) $af0$ iff $a = 0$.
(M'1) eFe.
(M'2) If aFb and aFc then $aFb \wedge c$.
(M'3) If $a' \leq a$, aFb, and $b \leq b'$ then $a'Fb'$.

Clearly, any map $f : S \to T$ of information systems determines an $F \subseteq \mathcal{L}S \times \mathcal{L}T$ of this kind such that

$$\bot F \bot, \quad [u]F[v] \text{ iff } ufv$$

where $[\cdot]$ stands for the $\dashv\vdash$-block of $\cdot$. On the other hand, any $F \subseteq \mathcal{L}S \times \mathcal{L}T$ as above determines a map $f : S \to T$ by letting

$$ufv \quad \text{iff} \quad [u]F[v],$$

and the correspondences $f \mapsto F$, $F \mapsto f$ are obviously inverse to each other.

Now, the above relation $F \subseteq A \times B$ has a very suggestive algebraic description, namely as the semilattice homomorphism $\varphi : B \to \mathfrak{D}A$ for the lattice

$$\mathfrak{D}A = \{A \subseteq A \mid 0 \in U,\ a \in U \text{ whenever } a \leq b \text{ and } b \in U\}$$

of *downsets* of A, by the obvious correspondence $F \mapsto \varphi$ and $\varphi \mapsto F$ such that

$$\varphi(b) = \{a \in A \mid aFb\} \quad \text{and} \quad aFb \text{ iff } a \in \gamma(b).$$

It is these $\varphi : B \to \mathfrak{D}A$ that we shall focus on.

For the following, recall some important aspects of the lattices $\mathfrak{D}A$. Firstly, they are complete such that (non-empty) join is union and meet intersection, and consequently they satisfy the distribution law

$$U \wedge \bigvee \mathcal{H} = \bigvee \{U \wedge V \mid V \in \mathcal{H}\}$$

for all $U \in \mathfrak{D}A$ and $\mathcal{H} \subseteq \mathfrak{D}A$. Further, for any semilattice homomorphism $\varphi : A \to L$ from a semilattice into a frame, the map $\varphi : \mathfrak{D}A \to L$ taking $U \in \mathfrak{D}A$ to $\bigvee \gamma[U]$ is a frame homomorphism, by easy calculation, clearly unique such that $\widetilde{\varphi}(\downarrow a) = \varphi(a)$ where $\downarrow a = \{x \in A \mid x \leq a\}$. It follows that the semilattice homomorphism $\varepsilon_A : A \to \mathfrak{D}A$ taking a to $\downarrow a$ is the *universal semilattice homomorphism from A to frames.*

For any semilattices A and B, the semilattice homomorphisms $\sigma : A \to \mathfrak{D}B$ will be called the *extended homomorphisms* from A to B, denoted by $\sigma : A \multimap B$. Now, any $\sigma : A \multimap B$ and $\tau : B \multimap C$ determine the semilattice homomorphism

$$A \xrightarrow{\sigma} \mathfrak{D}B \xrightarrow{\widetilde{\tau}} \mathfrak{D}C,$$

and we may define this as the composite $\tau \cdot \sigma : A \multimap C$ of σ and τ. Note that this operation is associative: for any

$$A \overset{\sigma}{\multimap} B \overset{\tau}{\multimap} C \overset{\rho}{\multimap} D,$$

$\rho \cdot (\tau \cdot \sigma) = \widetilde{\rho}(\tau \cdot \sigma) = \widetilde{\rho}\widetilde{\tau}\sigma = (\widetilde{\widetilde{\rho}\tau})\sigma = (\rho \cdot \tau) \cdot \sigma$, the crucial third step because

$$\widetilde{\rho}\widetilde{\tau}(\downarrow b) = \widetilde{\rho}\tau(b) = (\widetilde{\widetilde{\rho}\tau})(\downarrow b)$$

for all $b \in B$.

Further, the homomorphisms $\varepsilon_A : A \to \mathfrak{D}A$ are the units for this composition:

$$\sigma \cdot \varepsilon_A = \widetilde{\sigma}\varepsilon_A = \sigma, \quad \varepsilon_A \cdot \sigma = \widetilde{\varepsilon}_A\sigma = \sigma,$$

the first by the definition of $\widetilde{\sigma}$ and the second since $\widetilde{\varepsilon}_A = \mathrm{id}_{\mathfrak{D}A}$. As a result we have the category **ESLat** of semilattices and extended semilattice homomorphisms.

Remark 2. Any semilattice homomorphism $h : A \to B$ determines an extended homomorphism $\varepsilon_B h : A \multimap B$, and the correspondence $A \mapsto A$, $h \mapsto \varepsilon_B h$ clearly defines a category embedding **SLat** $\to$ **ESLat** so that the latter may be viewed as an enlargement of the former. Of course, this is *proper*. As a simple example we note that for any frame L generated by its complemented elements (= zero-dimensional) we have $\sigma : L \multimap L$ such that

$$\sigma(a) = \{x \in L \mid x \leq c \leq a \text{ for some complemented } c \in L\},$$

and $\sigma = \varepsilon_L h$ for some semilattice homomorphism $h : L \to L$ iff each $\sigma(a)$ has a largest element iff *each $a \in L$ is complemented*, the latter by zero-dimensionality. Hence any zero-dimensional frame which is not a Boolean algebra provides an example of the desired kind, and the ideal lattice of any infinite Boolean algebra is of this sort.

We now turn to the maps of information systems. For any such $f : S \to T$,

$$\mathcal{L}f([v]) = \{[u] \in \mathcal{L}S \mid ufv\} \cup \{\bot\}, \quad \mathcal{L}f(\bot) = \{\bot\}$$

defines an extended homomorphism $\mathcal{L}f : \mathcal{L}T \multimap \mathcal{L}S$. Clearly, these sets are downsets of $\mathcal{L}S$ and $\mathcal{L}f$ preserves zero, unit, and partial order. Further,

$$\mathcal{L}f([v]) \cap \mathcal{L}f([w]) = \{[u] \in \mathcal{L}S \mid ufv \text{ and } ufw\} \cup \{\bot\} \subseteq \mathcal{L}f([v] \wedge [w])$$

since ufv and ufw implies $uf(v \cup w)$ and $[v \cup w] = [v] \wedge [w]$, showing that $\mathcal{L}f([v] \wedge [w]) = \mathcal{L}f([v]) \cap \mathcal{L}f([w])$, and hence $\mathcal{L}f$ preserves finite meets.

Next we have

Proposition 1. $\mathcal{L} : \mathbf{InfSyst} \to \mathbf{ESLat}$ *is a dual equivalence.*

Proof: In view of Lemma 1 it remains to show that (1) $\mathcal{L}$ is a contravariant functor which is (2) full and (3) faithful.

Re (1). For the identity maps $\vdash_S : S \to S$,

$$\mathcal{L}(\vdash_S)([u]) = \{[v] \in \mathcal{L}S \mid v \vdash_S u\} \cup \{\bot\} = \downarrow [u]$$

so that $\mathcal{L}(\vdash_S) = \varepsilon_{\mathcal{L}S}$, the identity morphism $\mathcal{L}S \to \mathcal{L}S$. Further, if $f : S \to T$ and $g : T \to U$ then

$$\begin{aligned}(\mathcal{L}f) \cdot (\mathcal{L}g)([w]) &= \widetilde{\mathcal{L}f}(\{[v] \in \mathcal{L}T \mid vgw\} \cup \{\bot\} = \bigcup \{\mathcal{L}f(v) \mid vgw\} \cup \{\bot\} \\ &= \bigcup \{[u] \in \mathcal{L}S \mid ufv \text{ and } vgw\} \cup \{\bot\} = \mathcal{L}(gf)([w])\end{aligned}$$

for any $[w] \in \mathcal{L}U$, showing that $\mathcal{L}(gf) = (\mathcal{L}f) \cdot (\mathcal{L}g)$.

Re (2). Given any $\varphi : \mathcal{L}T \multimap \mathcal{L}S$, $f \subseteq \mathsf{Con}S \times \mathsf{Con}T$ such that ufw iff $[u] \in \varphi([v])$ clearly is a map $f : S \to T$ and $\mathcal{L}f = \varphi$ since

$$\mathcal{L}f([v] = \{[u] \in \mathcal{L}S \mid [u] \in \varphi([v])\} \cup \{\bot\} = \varphi([v])$$

for each $[v] \in \mathcal{L}T$.

Re (3). If $\mathcal{L}f = \mathcal{L}g$ for some $f, g : S \to T$ then

$$\{[u] \in \mathcal{L}S \mid ufv\} = \{[u] \in \mathcal{L}S \mid ugv\}$$

for each $v \in \mathsf{Con}T$, and since $[u] = [w]$ and wgv implies ugv it follows that ufv implies ugv and conversely, showing $f = g$. $\square$

Remark 3. The adjoint of $\mathcal{L}$ is easily obtained by augmenting the construction of Lemma 1. For each semilattice A, let $\mathcal{S}A$ be the information system described there, and for any $\sigma : A \multimap B$ in **ESLat** define $f_\sigma : \mathcal{S}B \to \mathcal{S}A$ by $uf_\sigma v$ iff $\bigwedge u \in \sigma(\bigwedge v)$. Simple calculations then show that $A \mapsto \mathcal{S}A$, $\sigma \mapsto f_\sigma$ is the desired functor. In particular, the adjunction unit $A \multimap \mathcal{L}\mathcal{S}A$ is provided by the isomorphism of Lemma 1, taking $a \neq 0$ to $\downarrow[\{a\}]$, while the corresponding $S \to \mathcal{S}\mathcal{L}S$ is $f \subseteq \mathsf{Con}S \times \mathsf{Con}\mathcal{S}\mathcal{L}S$ such that $uf\{[v_1], \ldots, [v_n]\}$ iff $u \vdash v_1 \cup \cdots \cup v_n$.

2. Frames

In this section we relate the category **ESLat**, and consequently by Proposition 1 also the category **InfSyst**, to a certain category of frames and then derive some consequences from this.

First some terminology. An element a in a frame L is called *supercompact* if $a \leq \bigvee S$ for any non-void $S \subseteq L$ implies that $a \leq t$ for some $t \in S$, and a frame will be called *supercoherent* if

(1) every element is a join of supercompact elements, and
(2) any meet of finitely many supercompact elements is supercompact.

Of course, the second condition is equivalent to saying that the unit and $a \wedge b$ for any supercompact a and b are supercompact. In the following, **SupCoh** will be the full subcategory of **Frm** given by the suprcoherent frames.

Remark 4. Note that the exclusion of $S = \emptyset$ in the above terminology is at variance with terminology elsewhere (Banaschewski-Niefield [**2**]). It has the effect of making 0 supercompact which otherwise it is not since $0 = \bigvee \emptyset$. In particular this affects the notion of supercoherence; for instance, the four-element Boolean algebra with new added top is supercoherent in the present sense but not in that of [**2**]. In any event in our context it is more natural to impose the condition $S \neq \emptyset$.

Remark 5. Supercompactness and supercoherence should be compared with the more familiar notions of compactness and coherence, which result by taking the above S *updirected* rather than arbitrary non-void. Of course coherence is weaker than supercoherence: any finite Boolean algebra B is trivially coherent but B is supercoherent iff it has at most two elements.

Of relevance is the following characterization which is a variant of a result by Banaschewski - Niefield [**2**].

Lemma 2. *A frame is supercoherent iff it is isomorphic to $\mathfrak{D}A$ for some semilattice A.*

Proof: ($\Leftarrow$) The supercompact $U \in \mathfrak{D}A$ are evidently the $\downarrow a$, any $U \in \mathfrak{D}A$ is trivially the union of all $\downarrow a \subseteq U$, and $\downarrow a \cap \downarrow b = \downarrow(a \wedge b)$.

($\Rightarrow$) Let A be the set of all supercompact elements of L with partial order induced from L. This is a subsemilattice of L by supercoherence, and the map $\mathfrak{D}A \to L$ by taking joins is an isomorphism: trivially order preserving, it is onto by supercoherence and order reflecting since $a \leq \bigvee U$ implies $a \in U$, for any $a \in A$ and $U \in \mathfrak{D}A$. $\square$

Now recall from earlier considerations that any $\sigma : A \multimap B$ determines a unique frame homomorphism $\widetilde{\sigma} : \mathfrak{D}A \to \mathfrak{D}B$ such that $\widetilde{\sigma}\varepsilon_A = \sigma$ and that $\widetilde{(\tau \cdot \sigma)} = \widetilde{(\widetilde{\tau}\sigma)} = \widetilde{\tau}\widetilde{\sigma}$ for the composite $\tau \cdot \sigma : A \multimap C$ of any $\sigma : A \multimap B$ and $\tau : B \multimap C$. It follows that the correspondence $A \mapsto \mathfrak{D}A$, $\sigma \mapsto \widetilde{\sigma}$ is a functor which we call $\mathfrak{D}$. Further, we have

LEMMA 3. $\mathfrak{D} : \mathbf{ESLat} \to \mathbf{SupCoh}$ *is an equivalence.*

PROOF: By Lemma 2, $\mathfrak{D}$ goes into the category indicated and is relplete, that is, every object of the second category is isomorphic to an image object by $\mathfrak{D}$. On the other hand, the correspondences

$$\begin{aligned} \sigma : A \multimap B \text{ in } \mathbf{ESLat} \quad &\mapsto \quad \widetilde{\sigma} : \mathfrak{D}A \to \mathfrak{D}B \text{ in } \mathbf{Frm} \\ h : \mathfrak{D}A \to \mathfrak{D}B \text{ in } \mathbf{Frm} \quad &\mapsto \quad h\varepsilon_a : A \multimap B \text{ in } \mathbf{ESLat} \end{aligned}$$

are obviously inverse to each other, showing that the functor $\mathfrak{D}$ is faithful. $\square$

In all, this has proved

PROPOSITION 2. $\mathfrak{D}\mathcal{L} : \mathbf{InfSyst} \to \mathbf{SupCoh}$ *is a dual equivalence.*

Remark 6. As an aside concerning the relation between extended and ordinary semilattice homomorphisms we note the following corollary of Lemma 3.

For any isomorphism $\sigma : A \multimap B$ there exists an isomorphism $h : A \to B$ such that $\sigma = \varepsilon_B h$.

Indeed, $\widetilde{\sigma} : \mathfrak{D}A \to \mathfrak{D}B$ is an isomorphism, hence $\sigma(a) = \widetilde{\sigma}(\downarrow a)$ is supercompact and consequently has a largest element. It follows that $\sigma(a) = \downarrow h(a)$ for some map $h : A \to B$, and one easily sees that this is a semilattice isomorphism.

The following considerations provide an alternative to Proposition 2. Note first that the frames $\mathfrak{D}A$ are spatial since

$$\{U \smallsetminus \{0\} \mid U \in \mathfrak{D}A\}$$

is clearly a topology on $|A| \smallsetminus \{0\}$ isomorphic to $\mathfrak{D}A$. Consequently, the usual spectrum functor Σ from $\mathbf{Frm}$ to the category $\mathbf{Top}$ of topological spaces and continuous maps induces a dual equivalence between $\mathbf{SupCoh}$ and the full subcategory of $\mathbf{Top}$ given by the spectra $\Sigma\mathfrak{D}A$, that is, the sober spaces X for which the frame $\Omega(X)$ of open sets is supercoherent, sober meaning each frame homomorphism $\xi : \Omega(X) \to 2$ is given by a unique $x \in X$. We call these the *superspectral spaces*, in analogy with the familiar notion of spectral spaces, and let $\mathbf{SupSpec}$ be the corresponding full subcategory of $\mathbf{Top}$.

With this we immediately have

Proposition 3. $\Sigma\mathfrak{D}\mathcal{L}$: **InfSyst** $\to$ **SupSpec** *is an equivalence*

There is a suggestive direct description of the composite functor $\Sigma\mathfrak{D}$ on the category **ESLat** as follows. The points of $\Sigma\mathfrak{D}A$, that is, the frame homomorphisms $\xi : \mathfrak{D}A \to 2$ correspond exactly to the semilattice homomorphisms $A \to 2$ by the universality property of $\varepsilon_A : A \to \mathfrak{D}A$. On the other hand, the latter are just the characteristic functions of the filters (always understood as *proper*) in A, that is, the $F \subseteq A$ such that

$0 \notin F$, $e \in F$
$a \in F$ implies $b \in F$ whenever $a \leq b$
$a \in F$ and $b \in F$ implies $a \wedge b \in F$,

and if ΦA is the familiar filter space of A, with basic open sets $\Phi_a = \{F \in \Phi A \mid a \in F\}$ we have a homeomorphism $\gamma_A : \Sigma\mathfrak{D}A \to \Phi A$ given by

$$\gamma_A(\xi) = \{a \in A \mid \xi({\downarrow}a) = 1\}.$$

It follows immediately that the correspondence $A \mapsto \Phi A$ is functorial on **ESLat** such that

$$\sigma : A \to B \quad \mapsto \quad \gamma_A(\Sigma\widetilde{\sigma})\gamma_B^{-1} : \Phi B \to \Phi A.$$

Actually, the latter is easily described directly: For any $F \in \phi B$ let $\xi = \gamma_B^{-1}(F)$ so that $\xi(U) = 1$ iff $U \cap F \neq \emptyset$; then

$$\begin{aligned}\gamma_A(\Sigma\widetilde{\sigma})\gamma_B^{-1}(F) &= \gamma_A(\Sigma\widetilde{\sigma})(\xi) = \gamma_A(\xi\widetilde{\sigma}) = \{a \in A \mid \xi\widetilde{\sigma}({\downarrow}a) = 1\} \\ &= \{a \in A \mid \xi\sigma(a) = 1\} = \{a \in A \mid \sigma(a) \cup F \neq \emptyset\}\end{aligned}$$

showing that $\Phi\sigma(F) = \{a \in A \mid \sigma(a) \cup F \neq \emptyset\}$ which may be viewed as some kind of inverse image of F relative to σ.

Of course all this shows that the γ_A provide a natural isomorphism $\Sigma\mathfrak{D} \cong \Phi$ and hence the following variant of Proposition 3:

Corollary. $\Phi\mathcal{L}$: **InfSyst** $\to$ **SupSpec** *is an equivalence.*

2. Scott domains

Recall that a *Scott domain* (to be called *domain* for short) is a partially ordered set in which

(D1) every subset bounded above has a join,
(D2) every updirected subset has a join, and
(D3) every element is a join of compact elements.

As is well known, there are a variety of equivalent conditions but these seem to be the most convenient in our context. As morphisms between domains one takes the maps $f : D \to E$ of underlying sets which preserve all updirected joins, and **SDom** will then be the resulting category.

As a familiar fact, we note that the set ΨA of (proper) filters in a semilattice A, partially ordered by inclusion, is a domain: indeed, we have

(D1) since $\bigcap \mathcal{H}$ is a filter for any non-void $\mathcal{H} \subseteq \psi(A)$,
(D2) by the fact that updirected join is union, and
(D3) because the compact $F \in \Psi A$ are exactly the principal filters $\uparrow a = \{c \in A \mid c \geq a\}$.

Furthermore we have

Lemma 3. *For any domain D there exists a semilattice A such that $\Psi A \cong D$*

Proof: Let KD be ths set of all compact elements of D, with partial order induced from D. Since any finite join of compact elements is compact this is a conditional join semilattice with zero so that adding a top $\top$ produces a bounded join semilattice $(KD)^\top$ and we let A be its dual $((KD)^\top)^{\mathrm{opp}}$. We claim that $\Psi A \cong D$. Noting that the proper filters are exactly the proper ideals of $(KD)^\top$ we have the map

$$f : \Psi A \to D, \quad F \mapsto \bigvee F$$

which is an order isomorphism because it preserves order trivially, reflects order by the properties of compact elements, and is onto by the properties of domains. □

Now recall from our discussion introducing the filter space functor Φ that any $\sigma : A \multimap B$ determines a map taking filters in B to filters in A given by

$$(*) \qquad F \mapsto \widetilde{F} = \{a \in A \mid \sigma(a) \cap F \neq \emptyset\}.$$

Trivially, this preserves updirected union and hence the correspondence $A \mapsto \Psi A$ is contravariantly functorial on **ESLat**, with $\Psi\sigma$ given by $(*)$.

More precisely, we have

Proposition 4. Ψ : **ESLat** $\to$ **SDom** *is a dual equivalence.*

Proof: By Lemma 3 it is enough to show that Ψ is (1) faithful and (2) full, but (1) is immediate by the connection between Φ and Ψ so that it remains to consider (2).

Given any $f : \Psi B \to \Psi A$, put

$$\sigma(a) = \{b \in B \mid a \in f(\uparrow b) \text{ or } b = 0\}.$$

We first claim this defines an extended homomorphism $A \multimap B$. To begin with, the set indicated is a downset: if $0 < c \leq b$ for $b \in \sigma(a)$ then $\uparrow b \subseteq \uparrow c$ so that $f(\uparrow b) \subseteq f(\uparrow c)$ and hence $a \in f(\uparrow b)$ implies $a \in f(\uparrow c)$. Further, $\sigma(0) = \{0\}$ since there is no $b \in B$

such that $0 \subset f(\uparrow b)$, and $\sigma(e) = B$ since $e \in f(\uparrow b)$ for each $b > 0$ in B. Next, if $a \leq c$ then $a \in f(\uparrow b)$ implies $c \in f(\uparrow b)$ for any $b > 0$ in B, showing that $\sigma(a) \subseteq \sigma(c)$. Finally, if $b \in \sigma(a) \cap \sigma(c)$ and $b > 0$ then $a \in f(\uparrow b)$ and $c \in f(\uparrow b)$, hence $a \wedge c \in f(\uparrow b)$ and therefore $b \in \sigma(a \wedge c)$, proving that $\sigma(a) \cap \sigma(c) = \sigma(a \wedge c)$ since σ preserves order.

Now, for any filter $F \subseteq B$,

$$\begin{aligned}\Psi\sigma(F) &= \{a \in A \mid \sigma(a) \cap F \neq \emptyset\} = \{a \in A \mid a \in f(\uparrow b) \text{ for some } b \in F\} \\ &= \bigcup\{f(\uparrow b) \mid b \in F\} = f(F)\end{aligned}$$

since $F = \bigcup\{\uparrow b \mid b \in F\}$ and F preserves this. Hence $\Psi\sigma = f$ as desired. $\square$

Remark 7. The adjoint Γ of Ψ is easily obtained by augmenting the construction in the proof of Lemma 3. Thus, for any domain D, $\Gamma D = ((KD)^\top)^{\mathrm{opp}}$ as described there, and for any domain map $f : D \to E$, $\Gamma f : \Gamma E \to \Gamma D$ takes each $a \in KE$ to

$$\{b \in KD \mid a \leq f(b)\} \cup \{\top\}.$$

Further, the adjunction isomorphisms are given by

$$A \multimap \Gamma\Psi A, \quad a \mapsto \uparrow a \ (a > 0) \ \text{ and } \ 0 \mapsto \top$$

and

$$D \to \Psi\Gamma D, \quad x \mapsto (KD) \cap \downarrow x.$$

Remark 8. The equivalence between domains and *information bases* established by Sambin et al. [**7**] is essentially the same as Proposition 4. Indeed, classically, the category of information bases and translations introduced there (Definition 3.1) is the dual od our **ESLat**, and the functor Pt (definition 2.3, Lemma 3.3) is essentially the present Ψ.

As immediate further consequences of Proposition 4 we have

Corollary. (1) $\Psi\mathcal{L}$: **InfSyst** $\to$ **SDom** *is an equivalence.*
(2) $\Phi\Gamma$: **SDom** $\to$ **SupSpec** *is an equivalence.*

An analysis of the second equivalence above shows that it takes a domain D to the space $\mathbb{S}D$ determined by the *Scott topology* on D, consisting of the $U \subseteq D$ such that

$$\begin{aligned}&a \in U \text{ implies } b \in U \text{ whenever } a \leq b, \\ &\bigvee S \in U \text{ for updirected } S \text{ implies } S \cap U \neq \emptyset.\end{aligned}$$

Moreover, it is easy enough to show directly that the correspondence $D \mapsto \mathbb{S}D$ defines a full and faithfull functor, leaving the maps unchanged, by the familiar fact that a set map between domains is a domain map iff it is Scott continuous. Consequently, the equivalence in question can also be proved by showing that the spaces $\mathbb{S}D$ are exactly the supercoherent spaces which, again, can certainly be done directly; the point here is to place this equivalence in the context of Proposition 4.

In conclusion, we note that the equivalence between information systems and domains is essentially due to Scott [**8**] who described the transition from the former to the latter, that is, the functor $\Psi\mathcal{L}$, directly, mentioning or at least implying its fullness and repleteness. Our splitting of this into the composite $\Psi\mathcal{L}$ is believed to offer a certain amount of added insight.

Actually, the way [**8**] derives domains from information systems is formally somewhat different from ours. The precise connection between the two approaches is provided by the fact that, given a generating set E of a semilattice A, any filter $F \subseteq A$ is already determined by $F \cap E$, and the sets $X \subseteq E$ of this type are evidently characterized by the conditions

if $u \subseteq X$ then $\bigwedge u \neq 0$, and
if $u \subseteq X$, $v \subseteq E$, and $\bigwedge u \leq \bigwedge v$ then $v \subseteq X$,

for any finite $u, v \subseteq E$. Note that, in terms of information systems S this allows the filters of $\mathcal{L}S$ to be described as the analogous subsets of the underlying set of S – which is precisely what is done in [**8**] (Definition3.1).

4. Propositional logic

The following describes an alternative to the category of informaton systems which is formulated in more specifically logical terms. To put our approach in this perspective we first recall a familiar more general notion.

A *propositional geometric theory* is given by the following ingredients:

• its *propositions*, generated from a given set B of basic propositions by binary conjunction $\wedge$ and disjunction $\bigvee$ of arbitrary sets of propositions, $\perp$ (false) and $\top$ (true).

• its relation of *entailment*, a binary relation $\vdash$ between proposition such that, for any propositions a, b, c and any set S of propositions, the following hold:

$a \vdash a$; and if $a \vdash b$ and $b \vdash c$ then $a \vdash c$;
$a \wedge b \vdash a$, $a \wedge b \vdash b$; if $c \vdash a$ and $c \vdash b$ then $c \vdash a \wedge b$;
$a \vdash \top$;
$t \vdash \bigvee S$ for each $t \in S$; if $t \vdash a$ for each $t \in S$ then $\bigvee S \vdash a$;
$\bigvee \emptyset = \perp$;
$a \wedge \bigvee S \vdash \bigvee \{a \wedge t \mid t \in S\}$

• its *axioms*, a set of specified entailments.

The Lindenbaum algebra associated with such a theory, that is, its propositions modulo mutual entailment $\vdash\dashv$ (which do form a *set*) with partial order induced by

$\vdash$, is a frame, namely the free frame generated by B modulo the relations expressing the axioms. Propositional geometric theories go back to C.J. Mulvey who began to consider them in the early 1970s as the propositional counterpart of geometric theories in general which had evolved in the setting of topos theory. One of their particular uses is as a device to provide suggestive definitions for various kinds of frames.

Now, for present purposes, it is natural to consider the conjunctive fragment of this notion, obtained by removing anything involving disjunction. Thus, we are led to the following definition:

A *conjunctive propositional theory* is given by

- its *propositions*, generated from a given set B of basic propositions by binary conjunction, together with $\perp$ (false) and $\top$ (true).
- its *entailment* $\vdash$ subject to the conditions

  $a \vdash a$; and if $a \vdash b$ and $b \vdash c$ then $a \vdash c$;
  $a \wedge b \vdash a$, $a \wedge b \vdash b$; if $c \vdash a$ and $c \vdash b$ then $c \vdash a \wedge b$;
  $\perp \vdash a$, $a \vdash \top$;

- its *axioms*, a set of specified entailments.

The Lindenbaum algebra $\mathbb{L}T$ of a theory T of this kind is evidently a semilattice which may be described as the semilattice generated by B subject to the relations expressed by the axioms, with $\perp$ and $\top$ determining the zero and unit, respectively.

Note that any semilattice A is isomorphic to some $\mathbb{L}T$. Given A, let T have as its basic propositions elements x_a on one-one corespondence with the $a \in A$, and impose the axioms

$$x_0 \vdash \perp, \quad \top \vdash x_a, \quad x_a \wedge x_b \vdash x_c \text{ iff } a \wedge b \leq c.$$

Clearly, $x_a \mapsto a$ then defines a semilattice homomorphism $\mathbb{L}T \to A$ which is easily seen to be an isomorphism by proving first that $x_a \vdash x_b$ iff $a \leq b$ and then, inductively, that $x_{a_1} \wedge \cdots \wedge x_{a_n} \dashv\vdash x_{a_1 \wedge \cdots \wedge a_n}$.

Next we introduce maps $f : T \to U$ between conjunctive propositional theories as relations $f \subseteq \mathsf{P}T \times \mathsf{P}U$, P for the set of propoitions, such that

$af\perp$ iff $a \vdash \perp$; $\top f \top$;
if afb and afc then $af(b \wedge c)$;
if $a' \vdash a$, afb and $b \vdash b'$ then $a'fb'$.

Just as in the case of information systems, the entailment of a theory T is a map $T \to T$, suggesting that the maps $T \to U$ should be viewed as externally provided entailments linking T with U. Further, it is clear that, for any $f : T \to U$ and $g : U \to V$, the relational composite $f \circ g$ is a map $T \to V$, and we define this to be the composite gf. Finally, for each T, its entailment is the identity map $T \to T$. In all, then, we have the category **CPTh** of conjunctive propositional theories.

Now, for any map $f : T \to U$ in **CPTh** and $b \in \mathsf{P}U$ put

$$\mathbb{L}f([b]) = \{[a] \in \mathbb{L}T \mid afb\},$$

again, $[\cdot]$ indicating the $\dashv\vdash$-block of $\cdot$. This obviously determines a morphism $\mathbb{L}f : \mathbb{L}U \multimap \mathbb{L}T$, and we have the following analogue of Proposition 1.

PROPOSITION 5. $\mathbb{L} : \mathbf{CPTh} \to \mathbf{ESLat}$ *is a dual equivalence.*

PROOF: Given that any semilattice is isomorphic to some $\mathbb{L}T$, as noted already, it only has to be shown that $\mathbb{L}$ is a contravariant functor which is full and faithful, but the details of this correspond exactly to the three steps in the proof of Proposition 1 and may therefore be omitted. $\square$

As an immediate consequence we further have:

COROLLARY. **CPTh** *is equivalent to* **InfSyst**

The correspondence from propositional theories T to information systems S resulting here is easily seen to have the following form:

$|S| = \{a \in B \mid a \not\vdash \bot\}$
$\mathsf{Con}S$ consists of all finite $u = \{a_1, \dots, a_n\} \subseteq |S|$ such that $a_1 \wedge \cdots \wedge a_n \not\vdash \bot$
$\{a_1, \dots, a_n\} \vdash_S a$ iff $a_1 \wedge \cdots \wedge a_n \vdash a$ in T.

5. Categorical aspects

The purpose here is to place the category **ESLat** into a larger context, exhibiting it as a particular instance of a general construction which occurs in a wide range of different situations. For this, recall the following (Mac Lane [**6**], Chapter VI).

A *monad* in a category $\mathbb{K}$ is an entity $\mathbb{T} = (T, \mu, \varepsilon)$ where $T : \mathbb{K} \to \mathbb{K}$ is an endofunctor of $\mathbb{K}$ and $\mu : T^2 \to T$ and $\varepsilon : \mathrm{Id}_{\mathbb{K}} \to T$ are natural transformations such that

$$\mu_A T\mu_A = \mu_A \mu_{TA} \quad \text{and} \quad \mu_A \varepsilon_{TA} = \mathrm{id}_{TA} = \mu_A T\varepsilon_A$$

for each $A \in \mathbb{K}$. Any adjoint situation $(L : \mathbb{K} \to \mathbb{M}) \dashv (R : \mathbb{M} \to \mathbb{K})$, with units $\varepsilon : \mathrm{Id}_{\mathbb{K}} \to RL$, $\lambda : LR \to \mathrm{Id}_{\mathbb{M}}$, determines a monad $(T = RL, \mu = R\lambda L, \varepsilon)$. Conversely, for any monad $\mathbb{T}$ there exist adjoint situations which determine it in this way, and among these there is a largest one, $\mathbb{K} \rightleftarrows \mathbb{K}^{\mathbb{T}}$, and a smallest one, $\mathbb{K} \rightleftarrows \mathbb{K}_{\mathbb{T}}$ (see [**6**], Chapter VI). The categories $\mathbb{K}^{\mathbb{T}}$ and $\mathbb{K}_{\mathbb{T}}$ are constructed as follows.

(1) The category $\mathbb{K}^{\mathbb{T}}$ of *Eilenberg-Moore algebras* of $\mathbb{T}$ with

- *objects* (A, α) where $A \in \mathbb{K}$ and $\alpha : TA \to A$ in $\mathbb{K}$ such that $\alpha T\alpha = \alpha\mu_A$ and $\alpha\varepsilon_A = \mathrm{id}_A$, and
- *morphisms* $h : (A, \alpha) \to (B, \beta)$ where $h : A \to B$ such that $h\alpha = \beta Th$.

Here, the usual perception is that $\mathbb{T}$ is a kind of monoid, with multiplication μ and unit ε, in the category of endofunctors of $\mathbb{K}$, and the above $\alpha : TA \to A$ are actions of $\mathbb{T}$ on A, the morphisms involved then being morphisms in $\mathbb{K}$ compatible with the given actions.

Trivially, the correspondence $(A,\alpha) \mapsto A$, $h \mapsto h$ provides a functor $U : \mathbb{K}^{\mathbb{T}} \to \mathbb{K}$, the underlying object functor. Moreover, U has a left adjoint $F : \mathbb{K} \to \mathbb{K}^{\mathbb{T}}$ such that

$$FA = (TA, \mu_A) \quad \text{and} \quad Fh = Th \text{ for } h : A \to B,$$

where FA is viewed as the *free* Eilenberg-Moore algebra on A.

Regarding the scope of this notion we see that all equational classes of algebras fit into this scheme but various other categories do as well, a celebrated non-algebraic instance being the category of compact Hausdorff spaces and continuous maps.

(2) The *Kleisli category* $\mathbb{K}_{\mathbb{T}}$ has

- the *same objects* as $\mathbb{K}$, but
- its *morphisms* $\sigma : A \multimap B$ are the morphisms $\sigma : A \to TB$ in $\mathbb{K}$, where the composite of $\sigma : A \multimap B$ and $\tau : B \multimap C$ is defined via the commuting square

$$\begin{array}{ccc} TB & \xrightarrow{T\tau} & TTC \\ {\scriptstyle \varepsilon_B}\uparrow & & \downarrow{\scriptstyle \mu_C} \\ B & \xrightarrow[\tau]{} & TC \end{array}$$

as

$$\tau \cdot \sigma = \overline{\tau}\sigma, \ \bar{t} = \mu_C T\tau.$$

Further, the unit $A \multimap A$ is $\varepsilon_A : A \to TA$.

It is an important fact that $\mathbb{K}_{\mathbb{T}}$ is equivalent to the subcategory of $\mathbb{K}^{\mathbb{T}}$ given by the free algebras. This equivalence results from the functor

$$\begin{array}{lll} A \mapsto & FA = (TA, \mu_A) \\ A \overset{\sigma}{\multimap} B \ \mapsto & TA \xrightarrow{\overline{\sigma} = \mu_B T\sigma} TB. \end{array}$$

Now, the relevance of this for the present situation is the following. On the category **SLat** we have the endofunctor $\mathfrak{D}$ such that

$$A \mapsto \mathfrak{D}A, \qquad h : A \to B \ \mapsto \ \mathfrak{D}h : \mathfrak{D}A \to \mathfrak{D}B$$

where

$$\mathfrak{D}h(U) = \bigcup\{{\downarrow}h(a) \mid a \in U\},$$

the downset of B generated by the image $h[U]$. Further,there is the universal semilattice homomorphism $\varepsilon_A : A \to \mathfrak{D}A$ of A to frames for each semilattice A, clearly natural in A.

But there is more, namely the frame homomorphism $\mu_A : \mathfrak{D}\mathfrak{D}A \to \mathfrak{D}A$ given by union: $\mu_A(\mathcal{H}) = \bigcup \mathcal{H}$, also expressable as $\widetilde{\varepsilon}_{\mathfrak{D}A}$ in earlier notation. Quite clearly, this is natural in A as well, and we have

LEMMA 4. *$\mathbb{D} = (\mathfrak{D}, \mu, \varepsilon)$ is a monad in* **SLat**.

PROOF: For any $\Lambda \in \mathfrak{D}^2 A$

$$\mu_a \mathfrak{D}\mu_A(\Lambda) = \bigcup(\bigcup\{\downarrow(\bigcup \mathcal{H}) \mid \mathcal{H} \in \Lambda\})$$

while

$$\mu_A \mu_{\mathfrak{D}A}(\Lambda) = \bigcup(\bigcup \Lambda) = \bigcup(\bigcup\{\mathcal{H} \mid \mathcal{H} \in \Lambda\}).$$

and these are equal since both consist of the $a \in A$ such that

$$a \in U \ \text{ for some } \ U \in \mathcal{H} \ \text{ for some } \ \mathcal{H} \in \Lambda.$$

Similarly, $\mu_A \varepsilon_{\mathfrak{D}A}(U) = \bigcup \downarrow U = U$ and $\mu_A \mathfrak{D}\varepsilon_A(U) = \bigcup\{\varepsilon_A(a) \mid a \in U\} = U$. $\square$

We call the above monad the *downset monad*.

Now, by its very definition, **ESLat** has the same objects and morphisms as the Kleisli category of this monad and one easily sees that the compositions are the same: for $\sigma : A \multimap B$ and $\tau : B \multimap C$ in **ESLat** the composite $\tau \cdot \sigma : A \multimap C$ was defined as $\widetilde{\tau}\sigma$ where $\widetilde{(U)} = \bigcup\{\tau(b) \mid b \in U\}$, but the latter is exactly $\mu_C \mathfrak{D}\tau(U)$, as checking for $U = \downarrow b$ shows. Hence

ESLat *is the Kleisli category of the downset monad.*

Note that, while this result is purely formal, our equivalence of **ESLat** with **SupCoh** is of rather a different nature. The following provides a link betwen the latter and the present context.

The category of Eilenberg-Moore algebras of the downset monad is **Frm**.

One would think this well-known but since we have not been able to locate any reference for it we present the following

PROOF: First, any frame L determines an algebra (L, λ) where $\lambda : \mathfrak{D}L \to L$ is given by taking joins: for any $\mathcal{H} \in \mathfrak{D}\mathfrak{D}L$,

$$\begin{aligned} \lambda(\mathfrak{D}\lambda(\mathcal{H})) &= \lambda(\bigcup\{\downarrow\lambda(U) \mid U \in \mathcal{U}\}) = \bigvee(\bigcup\{\downarrow(\bigvee U) \mid U \in \mathcal{H}\} \\ &= \bigvee\{\bigvee U \mid U \in \mathcal{H}\} = \bigvee(\bigcup \mathcal{H}) = \lambda\mu_L(\mathcal{H}) \end{aligned}$$

by the familiar properties of joins, and for any $a \in L$, $\lambda\varepsilon_L(a) = \bigvee \downarrow a = a$.

Also, for any frame homomorphism $h : L \to M$, if λ and μ are the respective join maps then

$$(*) \qquad h\lambda = \lambda' \mathfrak{D} h$$

since

$$h(\bigvee U) = \bigvee h[U] = \bigvee(\bigcup\{\downarrow h(a) \mid a \in U\}).$$

As a result, we have a functor $\mathbf{Frm} \to \mathbf{SLat}^{\mathbb{D}}$, obviously faithful, taking each frame L to (L, λ) and leaving the maps unchanged. Furthermore, this is full: any semilattice homomorphis $h : L \to M$ which satisfies $(*)$ is clearly a frame homomorphism.

It remains to see that every Eilenberg-Moore algebra (A, α) here is a frame with its join map. First, A is complete: for any $U \in \mathfrak{D}A$, if $a \in U$ and b is any upper bound of U then $\downarrow a \subseteq U \subseteq \downarrow b$ and hence $a \leq \alpha(U) \leq b$, showing U has join $\alpha(U)$, and this makes A complete. Further A is a frame as a consequence of the frame law in $\mathfrak{D}A$:

$$a \wedge \bigvee S = \alpha(\downarrow a \cup \bigcup\{\downarrow t \mid t \in S\} = \alpha(\bigcup\{\downarrow(a \wedge t) \mid t \in S\}) = \bigvee\{a \wedge t \mid t \in S\}. \quad \square$$

6. Concluding remarks

It stands to reason that the view of information systems provided here should have useful consequences for the various constructions that are important in this context. As a particular instance we note that, on account of Propositions 2 and 4, the *universal* information system corresponds to the frame $\mathfrak{D}F$ for the Boolean algebra F on ω generators. F is isomorphic to the algebra of open-closed subsets of the Cantor space 2^ω and hence ΨF is isomorphic to the *domain of proper open subsets* of 2^ω – which identifies $\mathfrak{D}F$ as corresponding to U in a remark in [**8**]. The crucial property of U then amounts to saying that any countably generated supercoherent frame is a *retract* of $\mathfrak{D}F$, and this is easily seen as follows. Obviously, any semilattice A is a subsemilattice of a Boolean algebra B, countable if A is countable, and $\mathfrak{D}A$ is then a retract of $\mathfrak{D}B$ since the embedding $\mathfrak{D}A \to \mathfrak{D}B$ induced by the identical embedding $A \to B$ has $U \mapsto U \cap A$ as left inverse. On the other hand, by the basic fact that every countable Boolean algebra is a retract of F, $\mathfrak{D}B$ is a retract of $\mathfrak{D}F$, making $\mathfrak{D}A$ a retract of $\mathfrak{D}F$. It seems that a systematic treatment of the constructions discussed in [**8**] on the present basis should be a worthwhile undertaking.

2. The equivalence of **ESLat** with **SupCoh** and its dual equivalence with **SDom** are easily seen to be special cases of considerably more general results. To describe these we use the following terminology.

(1) A frame will be called *superalgebraic* if every element is a join of supercompact elements and (which may not always be required but is needed here) its unit is supercompact.

(2) An *algebraic* domain is understood to be a partially ordered set with zero in which every updirected set has a join and the set of compact elements below any element a is updirected and has join a.

(3) For any bounded partially ordered set P and any frame, a map $\varphi : P \to L$ is called *flat* if it preserves zero, unit and partial order, and for any $a, b \in P$

$$\varphi(a) \wedge \varphi(b) = \bigvee\{\varphi(c) \mid c \leq a \text{ and } c \leq b\}.$$

Then, for the frame $\mathfrak{D}P$ of downsets of P, defined as for semilattices, the map $\varepsilon_P : P \to \mathfrak{D}P$ taking a to $\downarrow a$ is trivially flat, and any flat $\varphi : P \to L$ determines a frame homomorphism $\widetilde{\varphi} : \mathfrak{D}P \to L$ such that $\widetilde{\varphi}(U) = \bigvee \varphi[U]$, unique such that $\widetilde{\varphi}\varepsilon_P = \varphi$. Thus $\varepsilon_P : P \to \mathfrak{D}P$ is the universal flat map to frames.

Corresponding to these notions we then have

(1) **SupAlg** – the full subcategory of **Frm** given by the superalgebraic frames,

(2) **ADom** – the category of algebraic domains and maps preserving updirected joins, and

(3) **EPoS** –the category of bounded partially ordered sets and flat maps $\sigma : P \to \mathfrak{D}Q$ with composition and units given in exactly the same way as in the case of **ESLat**.

As an aside, we note that the flat maps $\sigma : Q \to \mathfrak{D}P$ can also be represented by the binary relations $f \subseteq P \times Q$ for which

$af0$ iff $a = 0$, efe,
if afb and afc then afd for some $d \leq b, c$,
if $a' \leq a$, afb, and $b \leq b'$ then $a'fb'$.

Clearly, for any such f, $\sigma(b) = \{a \in P \mid afb\}$ defines a flat map $\sigma : Q \to \mathfrak{D}P$ while any such σ determines an F of this kind by the stipulation that afb iff $a \in \sigma(b)$.

Now, obvious simple modifications of our earlier arguments show that

EPoS *is equivalent to* **SupAlg** by the obvious functor taking P to $\mathfrak{D}P$, and

EPoS *is dually equivalent to* **ADom** by the functor taking P to the set ΨP of proper filters (down directed upsets) of P, partially ordered by inclusion.

Finally, the original results mentioned earlier immediately follow from these, simply by restriction of the appropriate subcategories: **ESLat** is the full subcategory of **EPoS** given by the semilattices, a superalgebraic frame is supercoherent iff $a \wedge b$ is supercompact for any supercompact a and b, and an algebraic domain is a Scott domain iff any two compact elements bounded above have a join.

We are indebted to J.J.C. Vermeulen for drawing our attention to the notions considered here.

3. Abramsky ([**1**], 3.2) establishes an equivalence between Scott domains and certain lattices as follows. The *lattices* in question are the bounded distributive lattices L in which the $\vee$-prime elements form a semilattice $\mathsf{Pr}(L)$ generating L, and the *maps* $f : L \to M$ between such lattices are the relations $f \subseteq L \times M$ such that, for all u, v, w in L or M

$uf0$ iff $u = 0$, efe,
if ufv and ufw then $uf(v \wedge w)$,
if ufw and vfw then $(u \vee v)fw$,
if $u' \leq u$, ufv, and $v \leq v'$ then $u'fv'$,
for any $a \in \mathsf{Pr}(L)$, if $af(u \vee v)$ then afu or afv.

Note that $\vee$-prime here means that $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$.

The category **DPL** defined by this, with composition given by the relational composite (in reverse order) and the partial orders as units, is then shown to be equivalent to **SDom**. In the present setting this result is easily recaptured by showing that

DPL *is dually equivalent to* **ESLat**.

PROOF: Given any $f : L \to M$ in **DPL**, let $A = \mathsf{Pr}(L)$, $B = \mathsf{Pr}(M)$, $a \in A$, b and b_i in B, $u \in L$, $v \in M$ and

$$\varphi(b) = \{a \in A \mid afb\}.$$

It is immediate that $\varphi : B \to A$ and the correspondence $f \mapsto \varphi$ is one-one since ufv iff $v = b_1 \vee \cdots \vee b_n$ and for each $a \leq u$ we have afb_i for some i. Further, it is easily checked that $L \mapsto A$, $f \mapsto \varphi$ defines a functor $\mathbf{DPL} \to \mathbf{ESLat}$. Moreover, this is full besides being faithful: any $\psi : B \multimap A$ determines the map $f : L \to M$ such that

$$ufv \text{ iff } v = b_1 \vee \cdots \vee b_n \text{ and for each } a \leq u,\ a \in \psi(b_i) \text{ for some } i,$$

and evidently ψ is then the map $B \multimap A$ derived from f as above. Finally, any semilattice A is isomorphic to $\mathsf{Pr}(L)$ for some $L \in \mathbf{DPL}$: consider the sublattice of $\mathfrak{D}A$ generated by $\{\downarrow a \mid a \in A\} \cong A$. $\square$

Given that Abramsky [**1**] describes the equivalence of **DPL** with **SDom** by the functor $\mathbf{DPL} \to \mathbf{SDom}$ taking each L to the domain of its *prime filters* it might be added that the above duality can be viewed as an expression of the simple fact that, for any $L \in \mathbf{DPL}$, *the domain of prime filters of* L is isomorphic to *the domain of filters of* $\mathsf{Pr}(L)$.

In a similar vein, the difference between Proposition 4 and Abramsky's equivalence of **DPL** with **SDom** amounts to focussing on the *semilattice of supercompact* Scott open sets of a domain instead of its *lattice of compact* Scott open sets. This evidently produces considerable simplification, apart from the fact that it is more faithful to Scott's original treatment of domains via information systems in [**8**].

4. The following augments the logical discussion of Section 3. There is a natural intermediate concept between conjunctive propositional theories and full scale propositional geometric theories, namely the *finitary* fragments of the latter which deals with binary conjunction $\wedge$ and binary disjunction $\vee$ (rather than the full disjunction $\bigvee$). Obviously, this is determined by adding the following conditions for entailment to those in the conjunctive case:

$a \vdash a \vee b$, $b \vdash a \vee b$
if $a \vdash c$ and $b \vdash c$ then $a \vee b \vdash c$
$a \wedge (b \vee c) \vdash (a \wedge b) \vee (a \wedge c)$,

apart from the fact that propositions are now generated from the basic ones by $\wedge$ and $\vee$.

Further, one has the obvious notion of map between such theories, again expressing the intuitive notion of externally given entailment; explicitly, what has to be added to the conditions in the conjunctive case is the following

if afc and bfc then $(a \vee b)fc$, and
if $af(b \vee c)$ then $a \vdash a' \vee a''$ for some a' and a'' such that $a'fb$ and $a''fc$.

As a result we have the category **FPTh** of *finitary propositional theories*, again with composition of maps given by relational composition (in reverse order) and the entailment relations as identity maps.

Here, the Lindenbaum algebras are distributive lattices, always understood as bounded, and the analogue of Proposition 5 is then the dual equivalence between **FPTh** and the category **EDLat** of distributive lattices and their *extended* homomorphisms, those taken as lattice homomorphisms $A \to \mathfrak{J}B$ where the latter is the *ideal lattice* of B. Again, the composition of these maps is based on the fact that the embedding $A \to \mathfrak{J}A$ taking a to $\downarrow a$ is the universal lattice homomorphism to frames.

Further, as counterpart of Lemma 3, we have the equivalence between **EDLat** and the full subcategory of **Frm** determined by the *coherent frames* which in turn is dually equivalent to the category of *spectral spaces* and their continuous maps (provided we assume the Boolean Prime Ideal Theorem), making **FPTh** equivalent to this category.

Finally, the considerations of Section 5 apply here as well – as opposed to the fact that they do *not* apply to the situation discussed in 2 above. Obvious simple modifications of arguments for semilattices and the downset functor show: the endofunctor of the category of distributive lattices given by $A \mapsto \mathfrak{J}A$ determines a monad such that **EDLat** is its Kleisli category and **Frm** the category of its Eilenberg-Moore algebras.

We note that the category **FPTh** is also considered in Hulley [**3**] (where its objects are called *pre-locales* but the matters considered there are quite different from the topics considered here.

Completing the picture it might be added that the obvious category of propositional geometric theories, where maps satisfy the $\bigvee$-counterparts of the above conditions for $\vee$ is dually equivalent to the entire category **Frm**: the relations $f : L \times M$ between frames which result from the theory maps by passage to the Lindenbaum algebras correspond to the frame homomorphisms $\varphi : M \to L$ by the formulae

$$\varphi(c) = \bigvee \{a \in L \mid afc\}, \quad afc \text{ iff } a \leq \varphi(c).$$

Acknowledgements. The authors are grateful fo financial assistence by the following:

the Category Theory and Topology Research Group at the University of Cape Town for supporting a visit by B. Banaschewski from September 2000 to January 2001 during which part of this work was done,

the Natural Sciences and Engineering Research Council of Canada for ongoing support of B. Banaschewski in the form of a research grant,

the Grant Agency of the Czech Republic for Grant 201/99/0310 to A. Pultr, and

Institut of Theoretical Computer Science (ITI) at Charles University in Prague.

Further thanks go to J.J.C. Vermeulen for helpful discussions on various topics related to this work and to H. Hulley for providing some useful background material.

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