Planar graph colorings without short monochromatic cycles

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Abstract

It is well known that every planar graph G is 2-colorable in such a way that no 3-cycle of G is monochromatic. In this paper, we prove that G has a 2-coloring such that no cycle of length 3 or 4 is monochromatic. Equivalently, every cubic bridgeless planar graph G has a 2-factor which intersects every edge-cut of G of size 3 or 4. On the other hand, there are planar graphs with the property that any of their 2-colorings has a monochromatic cycle of length at most 5. In this sense, our result is best possible.

1 Introduction

Let G be a simple graph and $k \geq 3$. Let $\mathcal{C}_{\leq k}(G)$ be the hypergraph on $V(G)$ (the vertex set of G) whose edges are (the vertex sets of) cycles in G of length at most

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k. Similarly, let $\mathcal{C}_{\mathrm{odd}}(G)$ be the hypergraph on $V(G)$ whose edges are the odd cycles of G. For the hypergraph of odd cycles, one has the following result:

Theorem 1.1 *For every planar graph G, the hypergraph $\mathcal{C}_{\mathrm{odd}}(G)$ is 2-colorable.*

Stein [5] gave a straightforward argument to deduce the above claim from the Four Color Theorem (which was still a conjecture at the time). Color vertices of G properly by colors 1, 2, 3, and 4. Recolor the odd-colored vertices by black and the even-colored ones by white. It is easy to see that the new 2-coloring is proper for the hypergraph $\mathcal{C}_{\mathrm{odd}}(G)$.

In particular, the above theorem implies that for a planar graph G, the hypergraph $\mathcal{C}_{\leq 3}(G)$ is 2-colorable. A proof of Theorem 1.1, not based on the Four Color Theorem, is given in [5] for triangulations without separating 3-cycles (see the end of this section for the definition of a separating cycle). It uses the fact that a planar triangulation has a 2-coloring without monochromatic faces if and only if its dual has a 2-factor. This equivalence can be easily generalized to non-facial cycles.

Let us first review a few definitions. The edge set of a graph G is denoted by $E(G)$. Recall that an *edge-cut* in G is a set $A \subset E(G)$ such that $G - A$ is disconnected and A is minimal with this property. Since we shall not be interested in vertex-cuts, we refer to edge-cuts simply as *cuts*. If the size of a cut is k, we also use the term *k-cut*. Similarly, we speak about *k-cycles* and *k-faces*. Vertices of degree k are referred to as *k-vertices*.

The dual of a planar graph G is denoted by G^*. We use the notation e^* to refer to the edge of G^* which corresponds to $e \in E(G)$. If $A \subset E(G)$ is a set of edges, we let $A^* = \{ e^* : e \in A \}$. Also if $w \in V(G^*)$, then w^* denotes the corresponding face of G.

An *even factor* of a graph G is a spanning subgraph in which all degrees are even and non-zero. Thus, for instance, any 2-factor is an even factor. If there is no danger of confusion, we identify factors with their edge sets.

Proposition 1.2 *Let G be a planar graph with faces of size at most k. Then, $\mathcal{C}_{\leq k}(G)$ is 2-colorable if and only if the dual G^* of G has an even factor which intersects every cut of size at most k.*

Proof. Let c be a 2-coloring of $\mathcal{C}_{\leq k}(G)$. Let $\mathcal{C}^*$ be the set of edges e^* of G^* with the property that the endvertices of e are colored differently by c. Every vertex of G^* is incident with a positive even number of edges in $\mathcal{C}^*$. Thus $\mathcal{C}^*$ is an even factor of G^*. Since a cut of G^* corresponds to a cycle in G of the same size (and vice versa), and since c has no monochromatic cycle of length at most k, it follows that $\mathcal{C}^*$ intersects all cuts of size at most k in G^*.

To prove the other direction, we use an analogue of the argument from [5]. Let $\mathcal{C}^*$ be an even factor in G^* which meets every cut of G^* of size at most k. As a subgraph of G^*, $\mathcal{C}^*$ is spanning and Eulerian. We can properly 2-color the faces of $\mathcal{C}^*$. This coloring induces a (possibly improper) 2-coloring of the faces of G^*, and thus a 2-coloring of the vertices of G. Now, if G has a monochromatic cycle of length $t \leq k$, we obtain a t-cut in G^* which is disjoint from $\mathcal{C}^*$, a contradiction. $\square$

The well-known Petersen theorem asserts that every bridgeless cubic graph has a 2-factor. Schönberger [4] proved the following generalization.

Theorem 1.3 *Let G be a cubic bridgeless multigraph and let e, f be two edges of G. Then, G has a 2-factor which contains both e and f.*

Observe that this is equivalent to the assertion that every edge of a bridgeless cubic graph is contained in a 1-factor (cf. [2], Chapter 4).

As noted in [5], D. Barnette has pointed out that Theorem 1.3 can be used to prove that the hypergraph $\mathcal{C}_{\leq 3}(G)$ is 2-colorable for every planar graph G. A direct inductive proof of this fact was found by Král [3].

In this paper, we prove that for every planar graph G, the hypergraph $\mathcal{C}_{\leq 4}(G)$ is 2-colorable. Equivalently, every bridgeless cubic graph G has a 2-factor which intersects every cut of G of size 3 or 4. However, there exist planar graphs for which $\mathcal{C}_{\leq 5}(G)$ is not 2-colorable. It was noted in [5] that one such graph is the dual of the well-known non-hamiltonian planar cubic graph used by Tutte [6] to disprove Tait's Conjecture. In this sense, our result is best possible.

We conclude this section with a few more definitions. If C is a cycle of a plane graph G, then $\mathrm{Int}(C)$ is the subgraph of G consisting of all vertices and edges which belong to C or are contained inside it (with respect to the fixed embedding of G in the plane). The graph $\mathrm{Out}(C)$ is defined symmetrically. We say that a cycle C is *separating* if both $\mathrm{Int}(C)$ and $\mathrm{Out}(C)$ contain vertices not belonging to C.

2 Colorings and types

Let G be a plane graph. For a given face F of G, an *F-type* is any non-empty subset of $E(F)$ of size 2. A *type vector* τ for G is a mapping which assigns an F-type to each inner 4-face F of G. We denote this F-type by $\tau(F)$.

Let $c : V(G) \to \{1, 2\}$ be a 2-coloring of a planar graph G. For brevity, any cycle of length at most 4 will be called *short*. If no short cycle of G is monochromatic, then c is a *good* coloring. Denote by $D(c)$ the set of edges xy of G with $c(x) \neq c(y)$. A coloring c *crosses* an F-type T if $D(c) \cap E(F) \neq T$ and $D(c) \cap T \neq \emptyset$. We say that c *crosses* a type vector τ if for each inner 4-face F of G, the coloring c crosses $\tau(F)$. Saying that *c crosses τ at F*, where F is a 4-face of G, means simply that c crosses the F-type $\tau(F)$.

A 4-cycle C of a planar graph G is *nice*, if for every 4-cycle C' of G either

(a) C' is a cycle of $\mathrm{Int}(C)$ or $\mathrm{Out}(C)$, or

(b) some edge of C is a diagonal of C'.

In order to prove our main theorem, we will first consider the following special case:

Lemma 2.1 *Let G be a plane graph isomorphic to $K_{2,n}$ $(n \geq 2)$ with outer face O, and let τ be a type vector of G. Then there exists an O-type T such that every good coloring of O which crosses T can be extended to a good coloring of G which crosses τ.*

Proof. Label the vertices of G in such a way that the two partites of G are $\{x, y\}$ and $\{a_1, a_2, \ldots, a_n\}$, and furthermore, $O = xa_1ya_n$ and for each $i = 1, \ldots, n-1$, the 4-cycle xa_iya_{i+1} bounds a face F_i of G. We may assume that for each inner face F_i of G, the type $\tau(F_i)$ contains the edge xa_i (otherwise, replace $\tau(F_i)$ by its complement $E(F_i)\backslash\tau(F_i)$). We may also restrict our attention to colorings of O assigning color 1 to x.

Note first that the coloring c of $V(O)$ given by $c(x) = c(y) = 1$, $c(a_1) = c(a_n) = 2$ can always be extended to a good coloring $\tilde{c}$ of G which crosses τ. Indeed, one can set $\tilde{c}(a_i) = 2$ for all $i \in \{2, \ldots, n-1\}$. The coloring $\tilde{c}$ is good, and since for each i, $D(\tilde{c}) \supset E(F_i)$, it follows that $\tilde{c}$ crosses τ. Also note that no matter which O-type T will be chosen, the coloring $\tilde{c}$ crosses T. At this point, we distinguish the following two cases.

Case 1. *For some k, $\tau(F_k) = \{xa_k, ya_k\}$.*

In this case, we set $T = \{xa_1, ya_1\}$. Let c be a coloring of $V(O)$ which crosses T. Necessarily $c(x) \neq c(y)$. It follows that no extension of c has a monochromatic cycle. We extend c to a coloring $\tilde{c}$ by the following rule. First, color $a_2, \ldots, a_k$, one by one in the given order, in such a way that $\tilde{c}$ crosses τ at F_i for all $i < k$. This is always possible since of the two choices for $\tilde{c}(a_{i+1})$, at most one fails to produce a coloring which crosses τ at F_i.

Similarly, color $a_{n-1}, \ldots, a_{k+1}$ (in this order), making sure that $\tilde{c}$ crosses τ at F_i for all $i > k$. It remains to check the face F_k. The fact that $c(x) \neq c(y)$ implies that exactly one of the edges xa_k, ya_k is in $D(\tilde{c})$. Since $\tau(F_k) = \{xa_k, ya_k\}$, $\tilde{c}$ necessarily crosses the F_k-type $\tau(F_k)$. Hence, $\tilde{c}$ crosses τ as desired.

Case 2. *For all i, $\tau(F_i)$ is different from $\{xa_i, ya_i\}$.*

Assume first that we wish to extend the coloring d of O given by $d(x) = d(y) = d(a_1) = 1$ and $d(a_n) = 2$. Setting $\tilde{d}(a_i) = 2$ for $i = 2, \ldots, n-1$, we clearly obtain a good coloring. Moreover, for $i \geq 2$, $E(F_i) \subset D(\tilde{d})$; thus to verify that $\tilde{d}$ crosses τ, it is sufficient to show that it crosses τ at F_1. This is immediate from the fact that $\tau(F_1) \neq \{xa_1, ya_1\}$.

By symmetry, we conclude that any coloring of O which assigns the same color to x and y, and distinct colors to a_1 and a_n, extends to a good coloring which crosses τ.

To find other colorings with this property, let c_1 be the coloring of $\{x, y, a_1\}$ given by $c_1(x) = c_1(a_1) = 1$, $c_1(y) = 2$. There is a unique extension $\tilde{c}_1$ of c_1 to G which crosses τ. To see this, extend c_1 to $a_2, \ldots, a_n$ in sequence and note that at each step, setting $\tilde{c}_1(a_{i+1}) = \tilde{c}_1(a_i)$ if $\tau(F_i) = \{xa_i, ya_{i+1}\}$, and $\tilde{c}_1(a_{i+1}) \neq \tilde{c}_1(a_i)$ otherwise, is the only choice for which $\tilde{c}_1$ crosses τ at F_i. The resulting coloring is the unique extension of c_1.

In particular, this implies that another good coloring $\tilde{c}_2$ of G which crosses τ can be obtained from $\tilde{c}_1$ by changing its value on all the vertices a_i. In fact, $\tilde{c}_2$ is the only other coloring of G which crosses τ and agrees with $\tilde{c}_1$ on x and y. It is easy to check that $D(\tilde{c}_1)$ and $D(\tilde{c}_2)$ intersect $E(O)$ in non-empty disjoint subsets. Consequently, $\tilde{c}_1$ crosses an O-type if and only if $\tilde{c}_2$ does.

If $\tilde{c}_1(a_n) = 1$, then set $T = \{xa_1, ya_n\}$; otherwise, set $T = \{xa_1, xa_n\}$. Observe that in both cases, the above discussion implies that T meets the requirement of the theorem.

□

We are now ready to prove our main result.

Theorem 2.2 *Let G be a (simple) plane graph with each face of size 3 or 4. Let O be the outer face of G and τ be a type vector of G. Then,*

(a) If O is a triangle, then every good coloring of O can be extended to a good coloring of G which crosses τ.

(b) If O is a quadrangle, then there exists an O-type T such that every (good) coloring of O which crosses T can be extended to a good coloring of G which crosses τ.

Proof. By contradiction. Let G, τ be a counterexample with $|V(G)| + |E(G)|$ minimum.

Claim 1. *G has no separating 3-cycles.*

Suppose that the claim is false, i.e. G has a separating 3-cycle C. Since every inner 4-face of $\mathrm{Int}(C)$ or $\mathrm{Out}(C)$ is a face of G, it follows that τ induces type vectors τ_{int} and τ_{out} in $\mathrm{Int}(C)$ and $\mathrm{Out}(C)$, respectively.

Assume first that O is of length 3. Let c be an arbitrary good coloring of O. Then, by the minimality, extend c to a good coloring of $\mathrm{Out}(C)$ which crosses τ_{out}. Note that C is not monochromatic. Now, again by the minimality, we can extend c

(or rather, its restriction to C) to a good coloring of $\mathrm{Int}(C)$ which crosses τ_{int}. We claim that the resulting coloring of G is good. Certainly, no 4-cycle contained in $\mathrm{Int}(C)$ or $\mathrm{Out}(C)$ is monochromatic. All the other 4-cycles have a diagonal which is an edge of C. Consider the two triangles formed by this diagonal and the edges of the 4-cycle. Since one of the triangles is contained in $\mathrm{Int}(C)$ (and the other one is contained in $\mathrm{Out}(C)$), it follows that the 4-cycle cannot be monochromatic. So c is a good coloring of G which crosses τ.

Suppose now that O is of length 4. By the minimality, there exists an O-type T_{out} such that every good coloring c of O which crosses T_{out} can be extended to a good coloring of $\mathrm{Out}(C)$ which crosses τ_{out}. Afterwards, extend the coloring c of C to a good coloring of $\mathrm{Int}(C)$ which crosses τ_{int}. Since, by the above, c is a good coloring of G which crosses τ, just set $T = T_{\mathrm{out}}$ to establish this case.

Claim 2. *G has no nice separating 4-cycles.*

Assume that the claim is false and C is a nice separating 4-cycle. The fact that C is nice allows us to use induction, since every 4-cycle C' of G is contained either in $\mathrm{Int}(C)$ or in $\mathrm{Out}(C)$, or else some edge of C is a diagonal of C', in which case three vertices of C' form a triangle in $\mathrm{Int}(C)$. Denote by τ_{int} the type vector induced by τ on $\mathrm{Int}(C)$. By the minimality, there exists a C-type T_{int} which satisfies requirements of part (b) of this theorem for the graph $\mathrm{Int}(C)$ and the type vector τ_{int}. Denote by τ_{out} the type vector induced by τ on $\mathrm{Out}(C)$ with the addition that $\tau_{\mathrm{out}}(C) = T_{\mathrm{int}}$.

We argue similarly as in Claim 1. Assume first that O is a triangle. Given a good coloring c of O, extend it first to $\mathrm{Out}(G)$. Since $\tau_{\mathrm{out}}(C) = T_{\mathrm{int}}$, it follows that c crosses T_{int}. By the minimality, we can extend the coloring induced by c on C to a good coloring of $\mathrm{Int}(C)$. We obtain a good coloring of G which crosses τ.

Suppose now that O is a quadrangle. By the minimality, there exists an O-type T_{out} such that every good coloring c of O which crosses T_{out} can be extended to a good coloring of $\mathrm{Out}(C)$. As above, we extend c to $\mathrm{Int}(C)$ and obtain the required coloring of G. Finally, we set $T = T_{\mathrm{out}}$.

Claim 3. *G has no separating 4-cycles.*

Assume $C = x_1x_2x_3x_4$ is a separating 4-cycle. By Claim 2, it follows that C is not nice. In other words, there exists a 4-cycle C' of G which is neither contained in $\mathrm{Int}(C)$ nor in $\mathrm{Out}(C)$ and no edge of C is a diagonal of C'. It is easy to observe that there are essentially two possibilities as shown in Fig. 1.

In case (a), an edge of C' is a diagonal of C, say x_1x_3. We may assume that $C' = zyx_1x_3$ and z, y are both in $\mathrm{Int}(C) - C$ or $\mathrm{Out}(C) - C$. As C is a separating 4-cycle, we infer that one of the 3-cycles $x_1x_3x_2$, $x_1x_3x_4$ is separating, which is a contradiction by Claim 1.

In case (b), C' has two vertices such that one is in $\mathrm{Int}(C) - C$ and the other one is in $\mathrm{Out}(C) - C$. Without loss of generality we may assume that $C' = zx_1yx_3$.

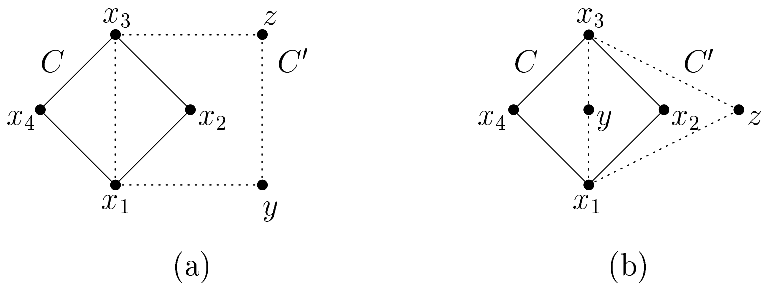

Figure 1: The two possibilities for C and C' in Claim 3.

Consider a complete bipartite (plane) subgraph B of G such that one of its partites is $B_1 = \{x_1, x_3\}$ and the size of the other partite B_2 is as large as possible. Enumerate the vertices of B_2 as $a_1, a_2, \dots, a_n$ in such a way that for each $i \in \{1, \dots, n-1\}$, the 4-cycle $F_i = x_3 a_i x_1 a_{i+1}$ is an inner face of B. Since $x_2, x_4, y, z \in B_2$, we have $n \geq 4$. Note that x_1x_3 is not an edge of G, for otherwise we would obtain a separating 3-cycle.

We shall show that no face F_i of B is a separating cycle in G. Assume the opposite. Claim 2 implies that F_i is not a nice 4-cycle. By the definition, there is a 4-cycle H which is contained neither in $\mathrm{Int}(F_i)$ nor in $\mathrm{Out}(F_i)$. Once again, we have two possibilities as in Fig. 1. In case (a), since $x_1x_3 \notin E(G)$, it follows that a_i and a_{i+1} are adjacent. But then we obtain a separating 3-cycle in G. In case (b), either $x_1, x_3 \in V(H)$, or $a_i, a_{i+1} \in V(H)$. The former possibility is ruled out by the maximality of B. The latter one would contradict the planarity of G since H separates x_1 from x_3, while they are known to have common neighbors other than those in $V(H)$. We have shown that F_i cannot be separating in G. It follows that for each i, either $a_i a_{i+1} \in E(G)$ or F_i is a 4-face of G. Note also that B is necessarily a spanning subgraph of G, i.e. $V(B) = V(G)$.

If some 4-cycle in G is edge-disjoint from B, then all of its vertices must belong to B_2. The planarity of G implies easily that there are no other vertices in B_2, so that $n = 4$ and G must be the octahedron. Thus, O is triangular and G has no 4-faces. Fig. 2 exhibits a good coloring of the octahedron; by symmetry, any good coloring of the outer face extends to the whole graph. Henceforth, we assume that every 4-cycle of G intersects $E(B)$.

Suppose first that O is a triangle, say $O = x_3 a_1 a_n$. Then observe that the 4-cycle $C = x_3 a_1 x_1 a_n$ is nice (but not separating). We argue similarly as in Claim 2. By the minimality, there exists a C-type T_{int} such that any good coloring of C which crosses T_{int} can be extended to a good coloring of $\mathrm{Int}(C)$ which crosses τ_{int} (in this case, $\tau_{\mathrm{int}} = \tau$). Finally, observe that for each good coloring c of O, one can choose a color $c(x_1)$ so that $D(c) \cap E(C)$ crosses T_{int}.

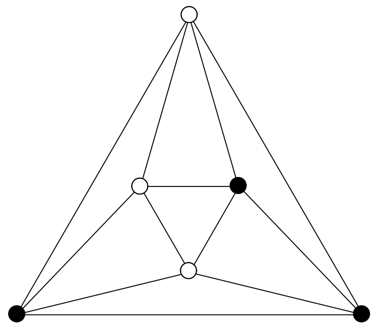

Figure 2: A good coloring of the octahedron.

Assume now that O is a 4-face. Thus, $O = x_1a_1x_3a_n$. If some a_ia_{i+1} is an edge of G then let $G' = G - a_ia_{i+1}$ and let τ' be the type vector for G' obtained from τ by setting $\tau(F_i) = \{x_1a_i, x_1a_{i+1}\}$. By the minimality, there exists some O-type T' for the pair G', τ'. Since a good coloring of G which crosses τ is a good coloring of G' which crosses τ', just set $T = T'$ to complete this case.

The remaining possibility is that no a_ia_{i+1} is an edge of G. In this case, $G = B$, and Claim 3 follows by Lemma 2.1.

By Claims 1 and 3, we may assume that no short cycle of G is separating. It follows that any 2-coloring of G without monochromatic faces is a good coloring of G. We use the following notation. Recall that for $A \subset E(G)$, A^* is defined as the set of the corresponding edges in the dual. If F is an inner 4-face of G, we abbreviate $(\tau(F))^*$ as $\tau^*(F)$. The vertex of G^* corresponding to the outer face of G is denoted by O^*.

We define the graph G^*_τ as follows. Each 4-vertex $w \neq O^*$ of G^* is split into two adjacent vertices w_1, w_2 of degree 3, such that w_1 is adjacent to the two edges in $\tau^*(F)$ (where $F = w^*$ is the face of G corresponding to w), and w_2 is adjacent to the remaining two edges in $E(w)$. The process is illustrated in Fig. 3.

The resulting graph G^*_τ has at most one vertex of degree 4, namely the vertex O^*. Note that G^*_τ need not be planar. We claim that it is bridgeless. To begin with, G is 2-connected, since it is simple with each face of size 3 or 4. It is well known that the dual of any 2-connected plane graph is 2-connected (or consists of 2 vertices and at least one edge). Furthermore, it is easily checked that the splitting of any vertex of degree 4, as above, does not introduce any cut-vertex. Since G^*_τ arises by a series of such splittings, it must be 2-connected and hence bridgeless.

We consider separately the cases of O being a triangle or a quadrangle, respectively. Assume first that O is triangular. In this case, G^*_τ is cubic. Let c be a good coloring of O. Clearly, $(D(c))^*$ consists of two edges of G^* incident with O^*. By Theorem 1.3, the corresponding pair of edges of G^*_τ is contained in a 2-factor $\mathcal{Q}$ of

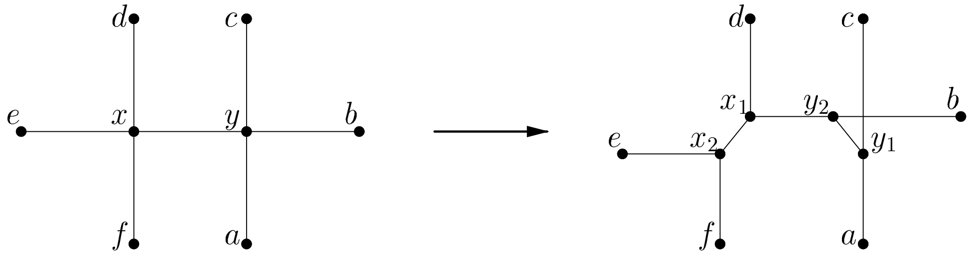

Figure 3: The splitting of 4-vertices x,y, where $\tau^*(x^*) = \{xd, xy\}$ and $\tau^*(y^*) = \{ya, yc\}$.

G^*_τ.

Performing the obvious identification of $E(G^*)$ with the corresponding subset of $E(G^*_\tau)$, consider the set $\mathcal{Q}' = \mathcal{Q} \cap E(G^*)$. This is an even factor of G^*. As in the proof of Proposition 1.2, $\mathcal{Q}'$ induces a 2-coloring $c_\mathcal{Q}$ of G. It is necessarily a good coloring as it has no monochromatic faces ($\mathcal{Q}$ is a 2-factor), while G has no separating short cycles. Furthermore, we claim that $c_\mathcal{Q}$ crosses τ. Consider an inner 4-face F of G. By the construction, $\mathcal{Q}' \cap E(F^*)$ must be different from both $\tau^*(F)$ and $E(F^*) \setminus \tau^*(F)$, for otherwise $\mathcal{Q}$ would not cover one of the two vertices into which F^* was split. Hence $c_\mathcal{Q}$ crosses τ as claimed. This concludes the proof of the first subcase.

If O is a quadrangle, the situation is a little more complicated. We start by defining possible splittings of the single remaining 4-vertex O^* of G^*_τ. Let S be a subset of $E(O^*)$ of size 2. The graph $G^*_{\tau,S}$ is obtained by replacing O^* by two new adjacent 3-vertices O^*_S, O^*_{-S}, making O^*_S incident with the two edges in S, and making O^*_{-S} incident with the remaining two edges in $E(O^*)$. Any $G^*_{\tau,S}$ is a bridgeless cubic graph.

If there is some $S \subset E(O^*)$ of size 2 such that one cannot find any even factor $\mathcal{Q}$ of G^*_τ with the property that $\mathcal{Q} \cap E(O^*) = S$, then set $T = S^*$ (where T is as in the theorem). Otherwise, choose T to be an arbitrary O-type.

Assume that we are given a coloring c of O which crosses T. If $D(c) = E(O)$, then the required even factor $\mathcal{Q}$ of G^*_τ is obtained by extending any pair $S \subset E(O^*)$ to a 2-factor of $G^*_{\tau,S}$ (by Theorem 1.3), and contracting the edge $O^*_S O^*_{-S}$. Since $\mathcal{Q}$ obviously contains all of $E(O^*)$, the associated coloring $c_\mathcal{Q}$ extends c. By the argument of the preceding case, it is good and crosses τ.

It remains to discuss the possibility that $D(c)$ is of size 2. If every pair $S \subset E(O^*)$ can be obtained as the intersection of an even factor of G^*_τ with $E(O^*)$ (that is, if T was chosen arbitrarily), then we simply choose such an even factor $\mathcal{Q}$ for $S = D^*(c)$ and we are done.

Thus we may assume that there is no even factor of G^*_τ whose intersection with

$E(O^*)$ is T^*. Since c crosses T, the symmetric difference $S = T^* \oplus D^*(c)$ is of size 2. Use Theorem 1.3 to find a 2-factor $\mathcal{Q}'$ of $G^*_{\tau,S}$ containing both $O^*_S O^*_{-S}$ and the unique edge e in $T^* \cap D^*(c)$. Let $\mathcal{Q}$ be the even factor of G^*_τ obtained by contracting the edge $O^*_S O^*_{-S}$. Since $O^*_S O^*_{-S} \in \mathcal{Q}'$, the intersection $I = \mathcal{Q} \cap E(O^*)$ has size 2. The remaining element of I cannot be the edge which is missing in both T^* and $D^*(c)$, for O^*_{-S} would have degree 3 in $\mathcal{Q}'$. Further, if I contained the edge in $T^* \setminus \{e\}$, we would get a contradiction with the way we chose T. We conclude that $I = D^*(c)$. But this implies that the coloring $c_\mathcal{Q}$ associated to $\mathcal{Q}$ extends c. The above arguments show that $c_\mathcal{Q}$ has all the required properties. The proof of the theorem is complete. $\square$

From the last theorem, we immediately obtain the following result:

Theorem 2.3 *Any planar graph G has a 2-coloring in which no cycle of length at most 4 is monochromatic.*

3 Remarks

Theorem 2.3 shows that the hypergraph $\mathcal{C}_{\leq 4}(G)$ is 2-colorable for every planar graph G. Combining it with Proposition 1.2, we obtain the following result. (Note that the result cannot be extended to 2-cuts, as shown by the cubic bridgeless graph in Fig. 4 which has no 2-factor intersecting every 2-cut.)

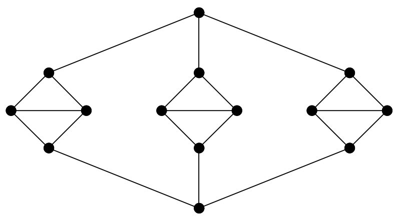

Figure 4: A bridgeless cubic graph with no 2-factor meeting every 2-cut.

Theorem 3.1 *Any cubic bridgeless planar graph G has a 2-factor which intersects every edge-cut of size 3 or 4.*

Proof. The dual G^* of G is a triangulation but it may not be a simple graph. Let H^* be the graph obtained by removing multiple edges of G^*. Note that H^* is a simple plane graph with each face of size 3 or 4. By Theorem 2.3, H^* has a 2-coloring c without monochromatic cycles of length 3 or 4. Observe that c has the same property as a coloring of G. As in the proof of Proposition 1.2, one can show that the properly colored edges of G^* induce a 2-factor in G which meets every edge-cut of size 3 or 4. $\square$

Our last remark concerns Theorem 1.1. Let G be a planar graph. Each of the color classes of a 2-coloring of $\mathcal{C}_{\mathrm{odd}}(G)$ induces a bipartite subgraph in G, by which we easily construct a proper 4-coloring of G. Thus, Theorem 1.1 is equivalent to the Four Color Theorem. Going further, to get rid of planarity, one can define that an integer k-flow f of a directed graph G is *no-odd-cut-zero* (or shortly *NOCZ*) if there is no odd edge-cut such that f is zero on all of its edges. Using the concept of even spanning subgraphs introduced by Archdeacon [1], one can easily show that a graph admits a nowhere zero $2k$-flow if and only it admits NOCZ k-flow. Thus, the 4-Flow Conjecture is equivalent to the claim that if a bridgeless graph does not contain the Petersen graph as a minor, then it admits a NOCZ 2-flow. This is a generalization of the equivalence between the Four Color Theorem and Theorem 1.1.

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