Approximate Euler characteristic, dimension, and weak pigeonhole principles

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Abstract

We define the notion of approximate Euler characteristic of definable sets of a first order structure. We show that a structure admits a non-trivial approximate Euler characteristic if it satisfies weak pigeonhole principle WPHP_n^{2n}: two disjoint copies of a non-empty definable set A cannot be definably embedded into A, and principle CC of comparing cardinalities: for any two definable sets A, B either A definably embeds in B or vice versa. Also, a structure admitting a non-trivial approximate Euler characteristic must satisfy WPHP_n^{2n}.

Further we show that a structure admits a non-trivial dimension function on definable sets iff it satisfies weak pigeonhole principle $\mathrm{WPHP}_n^{n^2}$: for no definable set A with more than one element can A^2 definably embed into A.

An abstract Euler characteristic (Ec) on a first order structure assigns to definable sets values in a commutative ring such that basic properties of counting with finite sets are fulfilled (we recall the formal definition from [5] in Section 2). Not all structures admit Ec. For example, if there is a definable bijection between a definable set and the set plus one other point (i.e., the onto-pigeonhole principle ontoPHP fails), then $0 = 1$ in the ring and everything is trivial. In fact, the validity of ontoPHP characterizes structures

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admitting the weak Ec, cf.[5]. The ordinary PHP (a definable set cannot be definably embedded into its proper subset) characterizes structures where the weak Ec is ordered, cf.[8].

Important variants of the pigeonhole principle are the so called weak pigeonhole principles; while PHP principles say in some form that a bigger set cannot embed into a smaller one, WPHP principles assert that a set cannot embed into a much smaller set. The particular definition of the qualification *much* determines various forms of WPHP. We shall consider the two variants WPHP_n^{2n} and $\mathrm{WPHP}_n^{n^2}$ defined earlier that are prominent in proof complexity and in bounded arithmetic.

It is not difficult to see that $\mathrm{WPHP}_n^{n^2}$ principle characterizes structures that admit a non-trivial dimension function (in the sense of Schanuel [11]; we shall recall the definition in Section 3) on their definable sets. The case of WPHP_n^{2n} is more involved and requires an introduction of a new invariant of definable sets which we shall call approximate Euler characteristic (aEc).

The idea of aEc is that similarly as Ec formalizes counting, aEc will formalize an approximate counting: aEc counts the size of definable sets but with a possible error, the error being a negligible percentage. This is an important issue in complexity theory (and in bounded arithmetic) as for some complexity classes the exact counting is hard (e.g., given a polynomial-time set the function - of x - that counts the number of elements in the set below x is generally assumed not to be p-time) while an approximate counting (the function counts the number of elements with some bounded error) is possibly much simpler. We shall show that a structure admits a non-trivial approximate Euler characteristic if it satisfies weak pigeonhole principle WPHP_n^{2n} and principle CC of comparing cardinalities: for any two definable sets A, B either A definably embeds in B or vice versa (this principle, and its two variants, were considered in [8]). Also, a structure admitting a non-trivial approximate Euler characteristic must satisfy WPHP_n^{2n}.

The paper is organized as follows. In Section 1 we give few preliminaries on semirings. Section 2 recalls definitions of (ordered) Ec and facts about them from [5, 8]. The dimension function and $\mathrm{WPHP}_n^{n^2}$ are considered in Section 3. The definition of aEc and the connection to WPHP_n^{2n} are in Section 4. Section 5 offers several examples of structures and their aEc'. More background information can be found in [5, 8, 6].

1 Preliminaries on semirings

A semiring, or a rig in an equivalent terminology, is a structure $(R, 0, 1, +, \cdot)$ having properties like a ring, except that $(R, 0, +)$ may be only a semigroup. We shall consider only commutative rigs. Examples are $\mathbf{N}$, $\mathbf{N}[x]$ or generally ordered rings without the "negatives" (hence the name rig, e.g. in [11]).

For us an important semiring is constructed from sets definable in a structure M as follows. Let $\mathrm{Def}(M)$ be the collection of subsets of all M^k, $k \geq 1$, that are definable in M with parameters. Two definable sets A, B are equivalent, $A \sim B$, iff there is a definable bijection between them. The set $\tilde{\mathrm{Def}}(M) := \mathrm{Def}(M)/\sim$, together with zero $0 := \emptyset/\sim$, one $1 := \{a\}/\sim$ (a any element), and operations $+$ - disjoint union, and $\cdot$ - Cartesian product forms a rig. We shall denote the quotient map from $\mathrm{Def}(M)$ to $\tilde{\mathrm{Def}}(M)$ by $[\ldots]$.

A partially ordered ring R is R together with a subset $P \subseteq R$ such that $0, 1 \in P$, $P + P \subseteq P$, $P \cdot P \subseteq P$, and if $0 \neq x \in P$ then $-x \notin P$. P are the non-negative elements of R and the partial ordering is given by $x \leq y \equiv y - x \in P$.

For a an element of a rig and $k \in \mathbf{N}$, $k \cdot a$ denotes the scalar multiple $a + \ldots + a$, k-times.

Definition 1.1 *Let $(S, 0, \oplus, \leq)$ be a partially ordered commutative semigroup and R a linearly ordered rig.*

Partially ordered rig $R[x^S]$ consists of elements that are finite R-linear combinations of expressions x^u, $u \in S$. The addition is coordinate-wise and the multiplication is determined by $x^u \cdot x^v := x^{u \oplus v}$, with $x^0 = 1$.

The partial ordering is induced by $a \cdot x^u < b \cdot x^v$, if $u < v$ in S or $u = v$ and $a < b$ in R.

Note that $R[x^S]$ is a ring if R is a ring. We shall use this construction for R one of $\mathbf{N}$, $\mathbf{Q}$ or $\mathbf{R}$.

2 Euler characteristic

Abstract Euler characteristic on first-order structures has been defined in [5]. It's variant, the weak Ec, is a special case of a general construction by Schanuel [11]. He defined (in our terminology) weak Euler characteristic and dimension on any distributive category. However, I shall follow [5, 8] and consider only the category of definable sets and maps in a structure.

Definition 2.1 ([5, Def.2.1]) *Let M be a first-order structure. Let R be a commutative ring with unity. A function*

$$\chi : Def(M) \longrightarrow R$$

is an Euler characteristic on M over R *iff it satisfies the following conditions:*

1. $\chi(\{a\}) = 1$, *any* $a \in M^k$.
2. $\chi(A \cup B) = \chi(A) + \chi(B)$, *whenever* $A, B, A \cup B \in Def(M)$ *and* A, B *are disjoint.*
3. $\chi(A \times B) = \chi(A) \cdot \chi(B)$, *whenever* $A, B, A \times B \in Def(M)$.
4. $\chi(A) = \chi(B)$, *whenever* $A, B \in Def(M)$ *and there is a definable bijection between* A *and* B.
5. $\chi(A) = c \cdot \chi(B)$, *whenever* $c \in R$, $A, B \in Def(M)$ *and there is a definable map* f *with domain* A *and range* B *such that each its fiber* $f^{(-1)}(b)$, $b \in B$, *has Euler characteristic* $\chi(f^{(-1)}(b)) = c$.

The function χ/R is a weak Euler characteristic iff χ/R satisfies the first four properties. Euler characteristic is sometimes called strong Euler characteristic, in order to distinguish it from the weak one.

One can state the definition of weak Ec in a more algebraic language: weak Ec the quotient map $[\ ] : \mathrm{Def}(M) \to \tilde{\mathrm{Def}}(M)$ composed with a righomomorphism of $\tilde{\mathrm{Def}}(M)$ into a ring. We use the definition explicitly referring to $\mathrm{Def}(M)$ as it allows for the formulation of the fifth, fiber, condition. The condition is important for various applications (e.g. [5]) and has also the nice property that it implies, together with condition 2., all other conditions, cf. [5].

If M admits a weak Ec then there exists a universal one with the target ring denoted $K_0(M)$ - the Grothendieck ring of the structure, cf. [5, 8].

The following theorem characterizes when a structure admits a weak Euler characteristic. No such transparent characterization is known for strong Ec, but there is a universal strong Ec (see [8]).

Theorem 2.2 ([5, Thm.3.1]) *M admits a weak Euler characteristic iff it satisfies the onto pigeonhole principle ontoPHP: There is no definable set and a definable bijection between the set and the set plus one other point.*

Definition 2.3 ([8, 4.2]) *A weak Euler characteristic χ/R on M is partially ordered if R is partially ordered and the range of χ is a subset of the non-negative elements of R. Equivalently, if $A \subseteq B$ are definable sets then $\chi(A) \leq \chi(B)$.*

Theorem 2.4 ([8, 4.3]) *M admits an ordered weak Euler characteristic iff it satisfies the ordinary pigeonhole principle PHP: There is no definable set and a definable injection of the set into its proper subset.*

3 Dimension

Dimension (in a special case of a definition by Schanuel [11]) is a rig-homomorphism of $\tilde{\text{Def}}(M)$ into a rig in which $1 + 1 = 1$. Rigs S in which $1 + 1 = 1$ can be transparently given in the "logarithmic" notation as an upper semilattice $(S, -\infty, \leq, 0, \vee, \oplus)$ in which 0 becomes $-\infty$, $+$ becomes the semilattice union $\vee$ with partial ordering $\leq$, 1 becomes 0 and $\cdot$ maps to $\oplus$.

Definition 3.1 ([11],[8, Sec.9]) *An abstract dimension function on M is a function*

$$d : Def(M) \to S$$

where $(S, -\infty, \leq, 0, \vee, \oplus)$ represents a rig in which $1 + 1 = 1$, such that d factors through $[\ ] : Def(M) \to \tilde{Def}(M)$ and $d/\!\sim$ is a rig homomorphism.

We say that dimension d is non-trivial iff $d(A^2) > d(A)$ for all infinite definable sets A.

Note that the non-triviality condition implies that for all $k \geq 1$, $d(M^{k+1}) > d(M^k)$ (otherwise also $d(M^{2k}) \leq d(M^k)$, violating the condition).

Next proposition is a characterization of structures admitting a non-trivial dimension function.

Theorem 3.2 *M admits a non-trivial dimension function iff it satisfies weak pigeonhole principle $WPHP_n^{n^2}$: There is no definable set A with at least two elements and a definable embedding of A^2 into A.*

Proof :

Assume that M admits a non-trivial dimension function. Hence $d(A^2) > d(A)$ for all infinite definable sets A. This inequality prevents a definable embedding of A^2 into A as such an embedding would imply $d(A^2) \leq d(A)$. As $\mathrm{WPHP}_n^{n^2}$ automatically holds for finite sets, M satisfies $\mathrm{WPHP}_n^{n^2}$.

For the opposite direction assume that M satisfies $\mathrm{WPHP}_n^{n^2}$. It is sufficient to show that $d(A^2) > d(A)$ in the universal dimension function $d : \mathrm{Def}(M) \to \mathcal{D}(M)$, defined in [8]. The construction of the universal d is simple. Define equivalence relation $\equiv$ on $\mathrm{Def}(M)$ by: $A \equiv B$ iff for some $k, \ell \in \mathbf{N}$, A definably embeds into k copies of B and B definably embeds into ℓ copies of A.

To prove that $d(A^2) > d(A)$ we need to show, for infinite definable A, that A^2 cannot definably embed into k copies of A, any $k \in \mathbf{N}$. Assume f is such a definable embedding. Define another embedding $g : A^4 \to A^2$ by mapping first $(x, y, z, t) \in A^4$ to $(f(x, y), f(z, t)) \in (k \cdot A)^2$, and thinking of $(k \cdot A)^2$ as k^2 copies of A^2 mapping this further by f into k^3 copies of A. Using parameters for k^3 distinct elements of A, $k^3 \cdot A$ embeds into A^2.

q.e.d.

4 Approximate Euler characteristic

In complexity theory the error in approximate counting of $A \subseteq \{0,1\}^n$ is a percentage that gets smaller as n increases. In our ideal situation with infinite ambient space the error will be infinitesimal.

Definition 4.1 *Let R be a partially ordered semiring. Define three relations on R:*

1. $a << b$ *iff for all* $k \in \mathbf{N}$, $k \cdot a < b$.
2. $a \dot{\leq} b$ *iff for any rational* $q > 1$ *there are* $k, \ell \in \mathbf{N}$ *such that* $\frac{\ell}{k} < q$ *and* $k \cdot a < \ell \cdot b$.
3. $a \doteq b$ *iff* $a \dot{\leq} b \wedge b \dot{\leq} a$.

The general idea of axioms of aEc is to replace all original equalities $a = b$ (or inequalities $a \leq b$) in Definitions 2.1 and 2.3 by $a \doteq b$ (or by $a \dot{\leq} b$).

Definition 4.2 *A weak approximate Euler characteristic on M (aEc) is a function*

$$\xi \ : \ Def(M) \to R$$

where R is a partially ordered ring, satisfying the following inequalities for all definable sets A, B:

1. *$\xi(A) = |A|$, for finite A.*
2. *$\xi(A \dot\cup B) \doteq \xi(A) + \xi(B)$.*
3. *$\xi(A \times B) \doteq \xi(A) \cdot \xi(B)$.*
4. *$\xi(A) \dot\leq \xi(B)$, if A is definably embedded into B.*

The aEc is strong if also

5. *If $f : A \mapsto B$ is injective and $c \in R$, then*
   - *$\xi(A) \dot\leq c \cdot \xi(B)$, if $\forall b \in B\ \xi(f^{(-1)}(b)) \dot\leq c$.*
   - *$c \cdot \xi(B) \dot\leq \xi(A)$, if $\forall b \in B\ c \dot\leq \xi(f^{(-1)}(b))$.*

Fiber condition 5. is useful for proving inequalities of the following type. For $C \subseteq A \times B$, if $\xi(C) \leq \frac{1}{2}\xi(A \times B)$ then there is $b \in B$ such that the section $C_b := \{a \in A \mid (a, b) \in C\}$ has aEc $\xi(C_b) \dot\leq \frac{1}{2}\xi(A)$.

Theorem 4.3 *If a structure M admits a non-trivial weak approximate Euler characteristic then it satisfies weak pigeonhole principle $WPHP^{2n}_n$: There is no definable set $A \neq \emptyset$ and a definable embedding of two disjoint copies of A into A.*

On the other hand, if M satisfies $WPHP^{2n}_n$ and principle CC: For any two definable sets A and B, either A definably embeds in B or vice versa, then it admits a non-trivial weak approximate Euler characteristic.

Proof :

Consider the first part. Assume that ξ is a weak aEc. We first observe that $\xi(A) > 0$ for all non-empty A. As $\emptyset \subseteq A$, $\xi(\emptyset) = 0 \dot\leq \xi(A)$ and so $0 \leq \xi(A)$. Further, a singleton embeds into A, i.e. $1 \dot\leq \xi(A)$ and hence $0 \neq \xi(A)$. So $0 < \xi(A)$.

To prove WPHP^{2n}_n, assume for the sake of contradiction that two copies of a non-empty A embed into A. Then $\xi(A \dot\cup A) \dot\leq \xi(A)$. We have $2\xi(A) \doteq \xi(A \dot\cup A)$ and so $2\xi(A) \dot\leq \xi(A)$. This means that there are $k, \ell \in \mathbf{N}$ with $\frac{\ell}{k} > 1$ arbitrarily close to 1, such that $2k\xi(A) \leq \ell\xi(A)$. Taking such k, ℓ with $\frac{\ell}{k} < 2$ gives a contradiction with $0 < \xi(A)$.

The proof of the second part of the theorem is divided into few claims. Recall the universal dimension function d on $\mathrm{Def}(M)$ from the proof of Theorem 3.2. As d factors through the quotient map $[\ldots] : \mathrm{Def}(M) \to$

$\tilde{\mathrm{Def}}(M)$, we shall simplify the notation and consider function $dim := d/\sim$: $\tilde{\mathrm{Def}}(M) \to \mathcal{D}(M)$, given by taking quotient by:

$$\exists k, \ell \geq 1; a \leq k \cdot b \wedge b \leq \ell \cdot a$$

where $\leq$ is the partial ordering in $\tilde{\mathrm{Def}}(M)$ induced by the embedability in $\mathrm{Def}(M)$. By CC the ordering is, in fact, linear.

Fix $d \in \mathcal{D}(M) \setminus \{-\infty\}$; i.e. d is the dimension of a non-empty set. Let $E_d \subseteq \tilde{\mathrm{Def}}(M)$ be the set of all $a \in \tilde{\mathrm{Def}}(M)$ of dimension d, together with 0. Clearly, $(E_d, 0, +)$ is a sub-semigroup of the additive semigroup of $\tilde{\mathrm{Def}}(M)$.

Claim 1: $(E_d, 0, +)$ homomorphically maps onto a sub-semigroup of the additive semigroup of the non-negative reals $(\mathbf{R}^{\geq 0}, 0, +)$, such that only 0 maps to 0.

To prove the claim fix an element $e \in E_d$. For any non-zero $a \in E_d$ define the set of rational numbers

$$C_e(a) := \{\frac{\ell}{k} \mid k \cdot a \leq \ell \cdot e, \text{ both } k, l \geq 1\}$$

As the dimensions of a and e are the same, there is some ℓ such that $a \leq \ell \cdot e$ and so $C_e(a)$ is non-empty. Define the real number $c_e(a) \geq 0$ to be the infimum of $C_e(a)$.

First note that actually $c_e(a) > 0$: We have $e \leq m \cdot a$ for some $m \geq 1$, and so $k \cdot a \leq \ell \cdot e$ implies $k \cdot a \leq (\ell m) \cdot a$. If it were that $c_e(a) = 0$ we could find $k, \ell \geq 1$ such that $\frac{\ell}{k} < \frac{1}{2m}$, i.e. $2\ell m < k$ and the inequality $k \cdot a \leq (\ell m) \cdot a$ would show that WPHP^{2n}_n fails in M for a set A for which $[A] = (\ell m) \cdot a$.

Define $c_e(0) := 0$. The map

$$c_e : (E_d, 0, +) \to (\mathbf{R}^{\geq 0}, 0, +)$$

is a homomorphism: If $k \cdot a \leq \ell \cdot e$ and $k' \cdot a' \leq \ell' \cdot e$, then $(kk') \cdot (a + a') \leq (k'\ell + \ell' k) \cdot e$. So $C_e(a) + C_e(a') \subseteq C_e(a + a')$. But similarly one shows that $C_e(a) + C_e(a')$ is downward cofinal in $C_e(a + a')$. Assume $k \cdot (a + a') \leq \ell \cdot a$. We need to demonstrate that $u \cdot a \leq v \cdot e$, $u' \cdot a' \leq v' \cdot e$ for some $u, v, u', v' \geq 1$ such that $\frac{v}{u} + \frac{v'}{u'}$ is arbitrarily close to $\frac{\ell}{k}$. To simplify the notation let us write $a \leq c_{a'}(a) \cdot a'$ (meaning that we can find $m, n \geq 1$ with $\frac{n}{m}$ arbitrarily close to $c_{a'}(a)$ such that $m \cdot a \leq n \cdot a'$) and similarly $a' \leq c_a(a') \cdot a$. Then the assumption $k \cdot (a + a') \leq \ell \cdot e$ implies $k(1 + c_{a'}(a)) \cdot a' \leq \ell \cdot e$ (in the proper formulation with m, n this is $k(m + n) \cdot a' \leq \ell m \cdot e$ obtained from

$k \cdot (a + a') \leq \ell \cdot e$ by first multiplying both sides by m and then replacing $m \cdot a$ by $n \cdot a'$) and $k(1 + c_a(a')) \cdot a \leq \ell \cdot e$. So it is enough to show that

$$\frac{\ell}{k(1 + c_{a'}(a))} + \frac{\ell}{k(1 + c_a(a'))} = \frac{\ell}{k}$$

This follows from $c_a(a') \cdot c_{a'}(a) = 1$: By CC either $m \cdot a \leq n \cdot a'$ or $n \cdot a' \leq m \cdot a$, i.e. either $\frac{n}{m} \in C_{a'}(a)$ or $\frac{m}{n} \in C_a(a')$. This proves the claim.

Claim 2: $\mathcal{D}(M)$ is linearly ordered. Further, if $dim(A) < dim(B)$, then k copies of A embed in B, for any $k \geq 1$.

$\mathcal{D}(M)$ is clearly linearly ordered. Assume that k copies of A do not embed in B. Then, by CC, B embeds into k copies of A and so $dim(B) \leq dim(A)$ which contradicts the assumption that $dim(A) < dim(B)$.

The construction of Claim 1 and Claim 2 yield

Claim 3: For $a, b \in E_d$ define $a \equiv b$ iff there are definable sets A and B with $[A] = a$ and $[B] = b$ and $d(A \triangle B) < d$ ($A \triangle B$ is the symmetric difference). Then $a \equiv b$ iff $c_a(b) = 1$.

Denote by $[A]^*$ the $\equiv$-class of $[A]$. By Claims 1 and 3, $[A]^*$ maps additive semigroup $(E_d, 0, +)$ into $(\mathbf{R}^{\geq 0}, 0, +)$. So we want to define the aEc as $[A]^*$ taking the value in the direct sum of $(\mathbf{R}^{\geq 0}, 0, +)$, graded by Def$(M)$, i.e. in $\mathbf{R}[x^{\mathcal{D}(M)}]$. A homomorphism of $(E_d, +)$ into $(\mathbf{R}^{>0}, +)$ is a positive multiple of $c_a(x)$, any $a \in E_d$ by Claim 1, and hence we define the aEc in the following way.

Let $\{e_d\}_d$ be a fixed set of points, one from each E_d for each dimension $d \neq -\infty$. Let function $\mathcal{E} : d \in \mathcal{D}(M) \setminus \{-\infty\} \to \epsilon_d \in \mathbf{R}^{>0}$ be arbitrary. Define map

$$\xi_{\mathcal{E}} : \mathrm{Def}(M) \to \mathbf{R}[x^{\mathcal{D}(M)}]$$

by

$$\xi_{\mathcal{E}}(A) := \epsilon_d c_{e_d}([A]) \cdot x^{dim([A])}$$

Intuitively, one would like to have e_d's the "unit d-dimensional cubes" in which case all ϵ_d's could be equal to 1. However, in a general situation there is no such object and so we pick e_d's arbitrarily and use ϵ_d's for rectification of the values, so that they obey conditions 1. and 3. of Definition 4.2.

Claim 4: Assume e_0 is the $[\ldots]$-class of a finite set with $n \geq 1$ elements and $\epsilon_0 = n$. Then for any finite A: $\xi_{\mathcal{E}}(A) = |A|$.

The value of $\xi_{\mathcal{E}}(A)$ is n times the infimum of all $\frac{\ell}{k}$ such that k copies of A embed into an ℓn-element set. The infimum is clearly $\frac{|A|}{n}$.

The next claim follows from Claim 3 and the fact that the map $[\ldots]^*$ commutes with the Cartesian product. (This is, by the claim, just another way of saying that ratios between sets are preserved by the Cartesian product: $c_{[A]}([B]) \cdot c_{[A']}([B']) = c_{[A \times A']}([B \times B'])$, whenever both sides are defined.)

Claim 5: For any ϵ there is $\mathcal{E}$ such that $\epsilon_0 = \epsilon$ and such that $\xi_{\mathcal{E}}$ satisfies the multiplicative condition 3. of Definition 4.2.

We are ready now to prove the second part of the theorem. Take $\mathcal{E}$ provided by Claim 5, with ϵ_0 chosen so as to satisfy the hypothesis of Claim 4. Hence the map $\xi_{\mathcal{E}}$ satisfies conditions 1. and 3. of Definition 4.2, by Claims 4 and 5.

Condition 4. is satisfied for any $\mathcal{E}$. Assume A embeds into B. Then either $dim(A) < dim(B)$ or $dim(A) = dim(B)$. In the former case, by Claim 2, even $dim(A) << dim(B)$. In the latter case $c_{[B]}([A]) \leq 1$, so $c_e([A]) \leq c_e([B])$ for $e = e_{dim(A)}$, and so $\xi_{\mathcal{E}}(A) \leq \xi_{\mathcal{E}}(B)$.

It remains to verify condition 2.; let A, B be two disjoint definable sets and assume first that $dim(A) < dim(B)$. Then $\xi_{\mathcal{E}}(A \cup B) = \xi_{\mathcal{E}}(B)$ and $\xi_{\mathcal{E}}(A) << \xi_{\mathcal{E}}(B)$, so $\xi_{\mathcal{E}}(A \cup B) \doteq \xi_{\mathcal{E}}(A) + \xi_{\mathcal{E}}(B)$ (as $x^{dim(A)} << x^{dim(B)}$).

Next assume $dim(A) = dim(B)$. Then even $\xi_{\mathcal{E}}(A \cup B) = \xi_{\mathcal{E}}(A) + \xi_{\mathcal{E}}(B)$ by Claim 1.

q.e.d.

It would be interesting to avoid using principle CC in the theorem (it was used in Claims 1 and 2). Also, when are the values of ξ in $\mathbf{Q}[x^{\mathcal{D}(M)}]$ rather than in $\mathbf{R}[x^{\mathcal{D}(M)}]$?

5 Examples

1. Example: reals R.

The real closed field $\mathbf{R}$ satisfies the ontoPHP (as bijections need to preserve Ec and Ec of a set and the set plus one point differ; $\mathbf{R}$ admits Ec - see e.g. [3]) but clearly not PHP or WPHP_n^{2n}. Hence $\mathbf{R}$ admits neither ordered Ec nor aEc. On the other hand $\mathbf{R}$ admits a dimension function (see e.g. [3]) and satisfies $\mathrm{WPHP}_n^{n^2}$ (by dimension reasons). $\mathbf{R}$ also satisfies CC, cf. [8].

2. Example: complex numbers C.

Complex numbers satisfy PHP (by a theorem of Ax [1]) and hence the other three principles too. The universal Ec is thus ordered and so it is also aEc. The Grothendieck ring $K_0(\mathbf{C})$ is huge; for example, it contains the ring of complex polynomials in continuum many unknowns (cf. [8]). Principle CC fails for $\mathbf{C}$.

3. Example: structures with global ranks.

Let $d : \mathrm{Def}(M) \to S$ be a dimension function with $(S, -\infty, 0, \vee, \leq, \oplus)$ linearly ordered. Define

$$deg_d \ : \ \mathrm{Def}(M) \to \mathbf{N} \cup \{\infty\}$$

as the maximum $n \in \mathbf{N}$ s.t. there are n disjoint subsets of A each of the same d-dimension as A, and $deg_d(A) := \infty$, if no such n exists.

A good example of such d is Morley rank RM; values of deg_{RM} are in $\mathbf{N}$, for example, for strongly minimal structures. As a simple specific example we take the structure $M := (\mathbf{N}, suc)$. As noted in [8, Sec.8], M satisfies WPHPs while obviously not ontoPHP. Definable subsets $A \subseteq M^n$ are disjoint unions of sets A_i of the form

$$U_1 \times \ldots \times U_n$$

where each $U_i \subseteq \mathbf{N}$ is either finite or co-finite. Define $d(A)$ to be the maximum number of co-finite in some A_i. deg_d has values in $\mathbf{N}$.

Define function $\xi \ : \ \mathrm{Def}(M) \to \mathbf{N}[x^S]$ by

$$\xi(A) := deg_d(A) \cdot x^{d(A)}$$

Then ξ is an aEc.

To see this note that for A finite, $d(A) = 0$ and $deg_d(A) = |A|$. So $\xi(A) = |A| \cdot x^0 = |A|$.

For the additivity conditions let $d(A) = u \leq v = d(B)$. If $u < v$, $deg_d(A \dot\cup B) = deg_d(B)$, so

$$\xi(A \dot\cup B) = deg_d(A \dot\cup B) x^v = \xi(B) \leq \xi(A) + \xi(B) \ .$$

If $u = v$, $deg_d(A \dot\cup B) = deg_d(A) + deg_d(B)$, and

$$\xi(A \dot\cup B) = (deg_d(A) + deg_d(B)) x^v = \xi(A) + \xi(B) \ .$$

For the opposite inequality $\xi(A) + \xi(B) \dot\leq \xi(A \dot\cup B)$ write

$$\xi(A) + \xi(B) = deg_d(A) x^u \ + \ deg_d(B) x^v$$

If $u < v$, $deg_d(A)x^u << x^v$, so $\xi(A) + \xi(B) \dot{\leq} \xi(B) \leq \xi(A \dot{\cup} B)$. If $u = v$, $\xi(A) + \xi(B) = \xi(A \dot{\cup} B)$.

I leave it to the reader to check the other two properties required from aEc.

4. Example: pseudo-finite fields.

Let F be an ultraproduct of finite fields $\mathbf{F}_q$'s. Chatzidakis, van den Dries and Macintyre [2] assign to all definable A in F a pair (μ, d), with μ a positive rational and d a natural number such that for some X in the ultrafilter and all $q \in X$ it holds:

$$|A(\mathbf{F}_q) - \mu \cdot q^d| = O(q^{d-\frac{1}{2}})$$

This also yields an aEc

$$A \longrightarrow \mu \cdot x^d \in \mathbf{Q}[x]$$

(changing values for finite A to $|A|$).

5. Example: bounded arithmetic.

This is another example of a structure satisfying WPHP^{2n}_n but not ontoPHP. Let $(M, 0, 1, \oplus, \odot)$ be a countable non-standard model of true arithmetic with ternary relations $\oplus$ and $\odot$ for graphs of additions and multiplication instead of the functions themselves. Let $n \in M$ be a non-standard number and M_n the substructure with the universe $[0, \ldots, n]$.

One can find a bijection f (not definable in M_n) between $[0, \ldots, n]$ and $[0, \ldots, n-1]$ (and so enforcing a failure of ontoPHP) such that the expanded structure still satisfies WPHP^{2n}_n (and also induction for all formulas, possibly containing symbol f); this follows from [9, 7, 10], see also [8].

It is an interesting open problem in bounded arithmetic whether we can similarly violate by f one of the weak pigeonhole principles, but in a way so that induction for all formulas is maintained. This has close relations to propositional proof complexity too.

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