On growth rates of hereditary permutation classes

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Abstract

A class of permutations Π is called hereditary if $\pi \subset \sigma \in \Pi$ implies $\pi \in \Pi$, where the relation $\subset$ is the natural containment of permutations. Let Π_n be the set of all permutations of $1, 2, \ldots, n$ belonging to Π. We investigate the counting functions $n \mapsto |\Pi_n|$ of hereditary classes. Our main result says that if $|\Pi_n| < 2^{n-1}$ for at least one $n \geq 1$, then there is a unique $k \geq 1$ such that $F_{n,k} \leq |\Pi_n| \leq F_{n,k} \cdot n^c$ holds for all $n \geq 1$ with a constant $c > 0$. Here $F_{n,k}$ are the generalized Fibonacci numbers which grow like powers of the largest positive root of $x^k - x^{k-1} - \cdots - 1$. We characterize also the constant and the polynomial growth of hereditary permutation classes and give two more results on these.

1 Introduction

A permutation $\sigma = (b_1, b_2, \ldots, b_n)$ of $[n] = \{1, 2, \ldots, n\}$ *contains* a permutation $\pi = (a_1, a_2, \ldots, a_k)$ of $[k]$, in symbols $\sigma \supset \pi$, if σ has a (not necessarily consecutive) subsequence of length k whose terms induce the same order pattern as π. For example, $(3, 5, 4, 2, 1, 7, 8, 6, 9)$ contains $(2, 1, 3)$, as $(\ldots, 5, \ldots, 1, \ldots, 6, \ldots)$ or as $(\ldots, 4, 2, \ldots, 9)$, but it does not contain $(3, 1, 2)$.

Let $f(n, \pi)$ be the number of the permutations of $[n]$ not containing π. The following conjecture was made by R. P. Stanley and H. S. Wilf (it appeared first in print in Bóna [10, 11, 12]).

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The Stanley–Wilf conjecture. For every permutation π, there is a constant $c > 0$ such that $f(n, \pi) < c^n$ for all $n \geq 1$.

It is known to hold for all π of length at most 4 (Bóna [12]), for all layered π (Bóna [13], see below for the definition of layered permutations), and for *all* π in a weaker form with an almost exponential upper bound (Alon and Friedgut [3]). A permutation π of $[n]$ is called *layered* if $[n]$ can be partitioned into intervals $I_1 < I_2 < \ldots < I_k$ so that every restriction $\pi|I_i$ is decreasing and $\pi(I_1) < \pi(I_2) < \ldots < \pi(I_k)$. (We call π layered also in the case when $\pi(I_1) > \pi(I_2) > \ldots > \pi(I_k)$ and the restrictions $\pi|I_i$ are increasing.) Equivalently, π is layered (in the former sense) if and only if it contains neither $(2, 3, 1)$ nor $(3, 1, 2)$. Other works dealing with the conjecture and/or the containment of permutations are, to name a few, Adin and Roichman [1], Albert et al. [2], Bóna [11], Klazar [18], Stankova-Frenkel and West [29], and West [33].

A class Π of permutations is *hereditary* if, for every π and σ, $\pi \subset \sigma \in \Pi$ implies $\pi \in \Pi$. The symbol Π_n, $n \in \mathbf{N} = \{1, 2, \ldots\}$, denotes the set of all permutations in Π of length n. The *counting function* of Π is the function $n \mapsto |\Pi_n|$ whose value at n is the number of permutations in Π of length n. For example, $n \mapsto 0$ is the counting function of the empty class $\Pi = \emptyset$, while the (hereditary) class of all permutations has the counting function $n \mapsto n!$. The Stanley–Wilf conjecture says, in effect, that except for the latter trivial example, there are no other superexponential counting functions.

A reformulation of the Stanley–Wilf conjecture. Let Π be any hereditary class of permutations different from the class of all permutations. Then $|\Pi_n| < c^n$ for all $n \geq 1$ and a constant $c > 0$.

Indeed, if Π is hereditary and $\pi \notin \Pi$, then $|\Pi_n| \leq f(n, \pi)$ for all $n \geq 1$. On the other hand, for every π the function $n \mapsto f(n, \pi)$ is the counting function of the hereditary class consisting of all permutations not containing π.

If one starts to investigate the realm of hereditary permutation classes from the top, one gets immediately stuck at the question whether every counting function different from the trivial $n \mapsto n!$ has to be at most exponential. In this article we take the other course and start from the bottom, at the empty class $\Pi = \emptyset$. We shall investigate the counting functions of 'small' hereditary permutation classes.

We summarize our results and give a few more definitions. Theorem 2.1 points out two simple set-theoretical facts about the set of all hereditary classes. Theorem 2.2, due to P. Valtr, gives a uniform lower bound on $\liminf_{n \to \infty} f(n, \pi)^{1/n}$. Sections 3 and 4 contain our main results. Theorem 3.4 shows that any counting function grows either at most polynomially or at least as the Fibonacci numbers F_n. Thus $n \mapsto F_n$ is the smallest superpolynomial counting function. Theorem 3.8 classifies the possible exponential growth rates below $n \mapsto 2^{n-1}$: Either $|\Pi_n| \geq 2^{n-1}$ for all $n \geq 1$, or there is a unique $k \geq 1$ such that $|\Pi_n|$ grows, up to a polynomial factor, as the generalized Fibonacci numbers $F_{n,k}$. Theorem 4.2 shows that any counting function is either eventually constant or grows at least as the identity function $n \mapsto n$. Thus $n \mapsto n$ is the smallest unbounded counting function. Theorem 4.4 shows that if the function $n \mapsto |\Pi_n|$ grows polynomially, then it is eventually an integral linear combination of the polynomials $\binom{n-i}{j}$. The concluding part (Section 5) contains some remarks and comments.

Recall that $\mathbf{N}$ denotes $\{1, 2, \ldots\}$, the set of positive integers, and $[n]$ denotes the set $\{1, 2, \ldots, n\}$. More generally, for $a, b \in \mathbf{N}$ and $a \leq b$, the interval $\{a, a+1, \ldots, b\}$ is denoted by $[a, b]$. If π is a permutation of $[n]$, we say that n is its *length* and write $|\pi| = n$. Let $A_1, A_2, B_1, B_2 \subset \mathbf{N}$ be four finite sets of the same cardinality. We call two bijections $f : A_1 \to A_2$ and $g : B_1 \to B_2$ *similar* iff $f(x) = j(g(h(x)))$ holds for every $x \in A_1$, where $h : A_1 \to B_1$ and $j : B_2 \to A_2$ are the unique increasing bijections. In other words, using only the order relation we cannot distinguish the graphs of f and g. Every bijection between two n-element subsets of $\mathbf{N}$ is similar to a unique permutation of $[n]$. For two permutations σ and π, σ contains π iff a subset of σ (regarded as a set of pairs) is similar to π. We take the *restriction* $\pi|X$ of a permutation π of $[n]$ to a subset $X \subset [n]$ to be the unique permutation similar to the usual restriction. For a set of permutations X we define

$$\mathrm{Forb}(X) = \{\pi : \ \pi \text{ contains no } \sigma \in X\}.$$

For any X, this is a hereditary class. Note that for every hereditary class Π there is exactly one set X of permutations pairwise incomparable by $\subset$ (that is, X is an *antichain*) such that $\Pi = \mathrm{Forb}(X)$; the set X consists of the minimal permutations not in Π. Thus the hereditary permutation classes correspond bijectively to antichains of permutations. A function $f : \mathbf{N} \to \mathbf{N}$ *eventually dominates* another function $g : \mathbf{N} \to \mathbf{N}$ iff $f(n) \geq g(n)$ for every $n \geq n_0$.

2 The number of hereditary classes and a lower bound on $f(n, \pi)$

If Π and Π' are hereditary classes of permutations and $\Pi \backslash \Pi'$ is finite then, trivially, $n \mapsto |\Pi'_n|$ eventually dominates $n \mapsto |\Pi_n|$. By the following theorem, there are uncountably many classes such that this trivial comparison does not apply for any two of them.

Theorem 2.1 *(1) There exist $2^{\aleph_0}$ (continuum many) distinct hereditary classes of permutations.*

(2) In fact, there exists a set S of $2^{\aleph_0}$ hereditary classes of permutations such that for every $\Pi, \Pi' \in S$, $\Pi \neq \Pi'$, both sets $\Pi \backslash \Pi'$ and $\Pi' \backslash \Pi$ are infinite.

Proof. (1) It is known (see, for example, Spielman and Bóna [28]) that there is an infinite antichain of permutations A. Then

$$\{\mathrm{Forb}(X):\ X \subset A\}$$

is a set of $2^{\aleph_0}$ hereditary classes. Indeed, every $\mathrm{Forb}(X)$ is hereditary and it is easy to see that $X, Y \subset A$, $X \neq Y$ implies $\mathrm{Forb}(X) \neq \mathrm{Forb}(Y)$.

(2) In fact, if $X, Y \subset A$ and $\pi \in X \backslash Y$ then $\pi \in \mathrm{Forb}(Y) \backslash \mathrm{Forb}(X)$. It suffices to show that there is a system of $2^{\aleph_0}$ subsets of A such that the set difference of every two distinct members of the system is infinite. For the notational convenience we identify A with $\mathbf{N}$. Recall that for $X \subset \mathbf{N} = A$, the upper and lower asymptotic densities of X are defined as

$$\overline{d}(X) = \limsup_{n\to\infty} \frac{|X \cap [n]|}{n} \quad \text{and} \quad \underline{d}(X) = \liminf_{n\to\infty} \frac{|X \cap [n]|}{n}.$$

For every real constant c, $0 < c < \frac{1}{2}$, we select a subset $X_c \subset \mathbf{N} = A$ such that $\overline{d}(X_c) = 1 - c$ and $\underline{d}(X_c) = c$. Then

$$S = \{\mathrm{Forb}(X_c):\ 0 < c < \tfrac{1}{2}\}$$

is a set of $2^{\aleph_0}$ hereditary classes with the stated property. Indeed, for every two real constants $c, d \in (0, \frac{1}{2})$, $c \neq d$, the set $X_c \backslash X_d$ is infinite because for $X, Y \subset \mathbf{N}$ with $X \backslash Y$ finite one has $\underline{d}(X) \leq \underline{d}(Y)$ and $\overline{d}(X) \leq \overline{d}(Y)$. □

Of course, there is nothing special about permutations in the previous theorem. It holds for the hereditary classes in any countably infinite poset that has an infinite antichain. Do there exist two hereditary classes of permutations such that their counting functions are incomparable by the eventual dominance? Are there $2^{\aleph_0}$ such hereditary permutation classes?

We take the opportunity to include an unpublished lower bound on the size of a class characterized by a forbidden permutation. The following theorem and its proof are due to Pavel Valtr [32] and are reproduced here with his kind permission.

Theorem 2.2 *Let c be any constant such that $0 < c < \mathrm{e}^{-3} = 0.04978\ldots$ where $\mathrm{e} = 2.71828\ldots$ is Euler number. Then for any permutation π of length k, where $k > k_0 = k_0(c)$, we have*

$$\liminf_{n\to\infty} f(n,\pi)^{1/n} > ck^2.$$

Proof. Let π be a permutation of length k. A random permutation τ of length m contains π with probability

$$\Pr[\tau \supset \pi] \leq \frac{1}{k!}\binom{m}{k} < \frac{m^k}{(k!)^2}.$$

We set $m = \lfloor dk^2 \rfloor$ where $0 < d < \mathrm{e}^{-2}$ is a constant. Then, by the Stirling asymptotics, this probability goes to 0 with $k \to \infty$ and for all sufficiently large k we have

$$f(m, \pi) > \frac{m!}{2}.$$

We can assume that π cannot be split as $[k] = I \cup J$, $I < J$, where both I and J are nonempty and $\pi(I) < \pi(J)$. (Otherwise we replace π with the reversed permutation.) Let $n \in \mathbf{N}$ and $n = mt + u$, where $t \geq 0$ and $0 \leq u < m$ are integers. It follows that none of the $f(m,\pi)^t f(u,\pi)$ permutations

$$(b_1, \ldots, b_u, d_1 + a_1^1, \ldots, d_1 + a_m^1, d_2 + a_1^2, \ldots, d_2 + a_m^2, \ldots, d_t + a_1^t, \ldots, d_t + a_m^t)$$

of length n, where $d_i = u + (i-1)m$ and $(b_1, \ldots, b_u)$ and $(a_1^i, \ldots, a_m^i)$ are permutations not containing π, contains π. Since $m! > (m/\mathrm{e})^m$ for large k,

$$f(n,\pi)^{1/n} \geq f(m,\pi)^{t/n} > \left(\frac{m!}{2}\right)^{t/n} > \left(\frac{m!}{2}\right)^{1/m - 1/n} > \frac{2^{1/n - 1/m}}{(m!)^{1/n}} \cdot \frac{m}{\mathrm{e}}.$$

By the choice of m, for any $\varepsilon > 0$ and $k > k_0 = k_0(\varepsilon)$,

$$\liminf_{n\to\infty} f(n,\pi)^{1/n} > \frac{(1-\varepsilon)d}{\mathrm{e}} \cdot k^2.$$

□

Arratia [4] proved that $\lim_{n\to\infty} f(n,\pi)^{1/n}$ always exists, and therefore in the previous bound we can replace lim inf with lim. For a general permutation π of length k the bound is best possible, up to the constant c, because

$$f(n, (1,2,\ldots,k)) \leq \frac{1}{(k-1)!} \sum_{\substack{0 \leq i_1, \ldots, i_{k-1} \\ i_1 + \cdots + i_{k-1} = n}} \binom{n}{i_1, \ldots, i_{k-1}}^2 \leq \frac{(k-1)^{2n}}{(k-1)!}.$$

The first inequality follows from the fact that by Dilworth's theorem, every permutation with no increasing subsequence of length k can be partitioned in at most $k-1$ decreasing subsequences. The second inequality follows by the multinomial theorem. Thus $\lim_{n\to\infty} f(n, (1,2,\ldots,k))^{1/n} \leq (k-1)^2$. By the exact asymptotics found by Regev [25], $\lim_{n\to\infty} f(n, (1,2,\ldots,k))^{1/n} = (k-1)^2$.

3 Below $n \mapsto 2^{n-1}$ — the Fibonacci growths

In this section we prove Theorem 3.8 which characterizes the exponential growth rates possible for the hereditary permutation classes Π satisfying $|\Pi_n| < 2^{n-1}$ for at least one $n \geq 1$. For the proof of the following classic result see, for example, Lovász [22, Problem 14.25].

Theorem 3.1 (Erdős–Szekeres) *Every sequence of n integers has a monotone subsequence of length $\geq n^{1/2}$.*

A permutation π, $|\pi| = n$, has k *alternations* if there are $2k$ indices $1 \leq i_1 < j_1 < i_2 < j_2 < \ldots < i_k < j_k \leq n$ such that

$$\pi(\{i_1, i_2, \ldots, i_k\}) > \pi(\{j_1, j_2, \ldots, j_k\}).$$

A hereditary permutation class Π *unboundedly alternates* if for every $k \geq 1$ there is a $\pi \in \Pi$ such that π or π^{-1} has k alternations.

Lemma 3.2 *If a hereditary permutation class Π unboundedly alternates, then $|\Pi_n| \geq 2^{n-1}$ for every $n \geq 1$.*

Proof. We suppose that for any k there is a $\pi \in \Pi$ with k alternations; the case with π^{-1} is analogous. Using the hereditarity of Π and Theorem 3.1, we see that for every $n \geq 1$ there is a $\pi \in \Pi_{2n+1}$ such that the restriction $\pi|\{2i-1 : \ 1 \leq i \leq n+1\}$ is monotone, and $\pi(i) > \pi(j)$ whenever i is odd and j is even. We may assume that the restriction is increasing; the other case when it is decreasing is quite similar. By the hereditarity of Π, for every $X \subset [2, n]$ there is a $\pi_X \in \Pi_n$ such that $\pi_X(i) > \pi_X(1) \Leftrightarrow i \in X$. Distinct X give distinct permutations π_X and thus $|\Pi_n| \geq 2^{n-1}$. $\square$

A word $u = u_1, u_2, \ldots, u_n$ *has no immediate repetitions* if $u_i \neq u_{i+1}$ for each $1 \leq i \leq n-1$. We say that u is *alternating* if $u = ababa\ldots$ for two distinct symbols a and b. For a word u we denote $\ell(u)$ the maximum length of an alternating subsequence of u. Let

$$W_{m,l,n} = \{u \in [m]^* : \ |u| = n \ \& \ \ell(u) \leq l\}$$

be the set of all words over the alphabet $[m]$ of length n which have no alternating subsequence of length $l+1$. Claim (1) of the following lemma is a result of Davenport and Schinzel [15].

Lemma 3.3 *(1) If $u = u_1, u_2, \ldots, u_n$ is a word over $[m]$ which has no immediate repetitions and satisfies $\ell(u) \leq l$, then*

$$n \leq \binom{m}{2}(l-1) + 1.$$

(2) For every $m, l, n \geq 1$ we have

$$|W_{m,l,n}| \leq (m+1)^c \cdot n^c$$

where $c = \binom{m}{2}(l-1) + 1$.

(3) *Suppose that the alphabet $[m]$ is partitioned into r subalphabets $A_1, \ldots, A_r$ and u is a word over $[m]$ such that every subword v_i of u consisting of the occurrences of the letters in A_i satisfies $\ell(v_i) \le l$. Then u can be split into t intervals $u = I_1 I_2 \ldots I_t$ such that every I_i uses at most one letter from every A_j, and*

$$t \le 2\binom{m}{2}(l-1) + 2.$$

Proof. (1) Let $u = u_1, u_2, \ldots, u_n$ be over $[m]$, without immediate repetitions, and let $n \ge \binom{m}{2}(l-1) + 2$. By the pigeon-hole principle, some l of the $n-1$ two-element sets $\{u_i, u_{i+1}\}$ must coincide. It is easy to see that the corresponding positions in u contain an $l+1$-element alternating subsequence.

(2) Every word $u \in W_{m,l,n}$ splits uniquely into intervals $u = I_1 I_2 \ldots I_t$ such that $I_i = a_i a_i \ldots a_i$ consists of repetitions of a single letter a_i and $a_i \ne a_{i+1}$ for $1 \le i \le t-1$. Contracting every I_i into one term, we obtain a word u^* over $[m]$, $|u^*| = t$, with $\ell(u^*) \le l$ and no immediate repetitions. By (1), $t \le \binom{m}{2}(l-1)+1 = c$. Clearly, u^* and the composition $|I_1|, |I_2|, \ldots, |I_t|$ of n determine u uniquely. Thus

$$|W_{m,l,n}| \le (\#u^*) \cdot n^c \le (m+1)^c \cdot n^c.$$

(3) We consider the unique splitting $u = I_1 I_2 \ldots I_t$, where I_1 is the longest initial interval of u using at most one letter from every A_j, I_2 is the longest following interval with the same property, etc. Note that every pair $I_i I_{i+1}$ has a subsequence a, b (where b is the first term of I_{i+1}) such that $a, b \in A_j$ for some j and $a \ne b$. Now arguing similarly as in (1), we see that

$$t \le 2\binom{m}{2}(l-1) + 2.$$

$\square$

The shifted Fibonacci numbers $(F_n)_{n\ge 1} = (1, 2, 3, 5, 8, 13, 21, \ldots)$ are defined by $F_1 = 1, F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$. The explicit formula is

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

By induction, $F_n \le 2^{n-1}$ for every $n \ge 1$.

The next theorem identifies the jump from the polynomial to the exponential growth and shows that $n \mapsto F_n$ is the first superpolynomial growth rate. Although it is fully subsumed in the more general Theorem 3.8, we give a sketch of the proof. We think that it may be interesting and instructive for the reader to compare how the concepts used here develop later in the more complicated proof of Theorem 3.8.

Theorem 3.4 *Let Π be any hereditary class of permutations. Then exactly one of the following possibilities holds.*

(1) There is a constant $c > 0$ such that $|\Pi_n| \le n^c$ for all $n \ge 1$.

(2) $|\Pi_n| \ge F_n$ for all $n \ge 1$.

Proof. (Extended sketch.) We split any permutation π into $\pi = S_1 S_2 \ldots S_m$ where S_1 is the longest initial monotone segment, S_2 is the longest following monotone segment, and so on. We mark the elements in S_i by i and read the marks from bottom to top (that is, from left to right in π^{-1}). This way we obtain a word $u(\pi)$ over the alphabet $[m]$, where $m = m(\pi)$ is the number of the monotone segments S_i. For example,

$$\text{if } \ \pi = (3,5,4,2,1,7,8,6,9) \ \text{ then } \ m(\pi) = 4 \ \text{ and } \ u(\pi) = 2,2,1,2,1,4,3,3,4$$

because $S_1 = 3,5$, $S_2 = 4,2,1$, $S_3 = 7,8$, and $S_4 = 6,9$. Note that π is determined uniquely by $u(\pi)$ and an m-tuple of signs $(\pm,\pm,\ldots,\pm)$ in which $+$ indicates an increasing segment and $-$ a decreasing one. For every pair S_i, S_{i+1} we fix an interval $T_i = T_{\pi,i} = [\min\{a,b,c\}, \max\{a,b,c\}]$ where a,b,c is a non-monotone subsequence of $S_i S_{i+1}$ (such a subsequence certainly exists). In our example, for $i = 3$ we may set $a,b,c = 7,8,6$ and $T_3 = [6,8]$.

For a hereditary permutation class Π and π ranging over Π we distinguish four cases. Case 1a: $m(\pi)$ is bounded and so is $\ell(u(\pi))$. Case 1b: $m(\pi)$ is bounded and $\ell(u(\pi))$ is unbounded. Case 2a: $m(\pi)$ is unbounded and the maximum number of mutually intersecting intervals in the system $S(\pi) = \{T_{\pi,1}, T_{\pi,3}, T_{\pi,5}, \ldots\}$ is unbounded as well. Case 2b: $m(\pi)$ is unbounded and so is the maximum number of mutually disjoint intervals in the system $S(\pi)$.

In case 1a we use (2) of Lemma 3.3 and deduce the polynomial upper bound of claim (1). In case 1b, the class Π unboundedly alternates and, by Lemma 3.2, $|\Pi_n| \ge 2^{n-1} \ge F_n$. In case 2a, the class Π again unboundedly alternates and $|\Pi_n| \ge 2^{n-1} \ge F_n$. In case 2b, it follows by Theorem 3.1 and the definition of $T_{\pi,i}$, that either for every $n \ge 1$ we have $(2,1,4,3,6,5,\ldots,2n,2n-1) \in \Pi$ or for every $n \ge 1$ we have $(2n-1,2n,2n-3,2n-2,\ldots,1,2) \in \Pi$. Using the hereditarity of Π, we conclude that in this case, $|\Pi_n| \ge F_n$. $\square$

To state Theorem 3.8, we need a few more definitions and lemmas. For k an integer and F a power series, $[x^k]F$ denotes the coefficient at x^k in F. We define the family of generalized Fibonacci numbers $F_{n,k} \in \mathbf{N}$, where $k \ge 1$ and n are integers, by

$$F_{n,k} = [x^n]\,\frac{1}{1 - x^k - x^{k-1} - \cdots - x}.$$

In particular, $F_{n,1} = 1$ for every $n \ge 1$ and $F_{n,2} = F_n$. More generally, $F_{n,k} = 0$ for $n < 0$, $F_{0,k} = 1$, and

$$F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k} \ \text{ for } n > 0.$$

Lemma 3.5 *Let $k \geq 1$ be fixed.*

(1) For $n \to \infty$, we have the asymptotics

$$F_{n,k} = c_k \alpha_k^n + O(\beta_k^n), \;\; c_k = \frac{\alpha_k^k(\alpha_k - 1)^2}{\alpha_k^{k+1} - (k+1)\alpha_k + k},$$

where α_k is the largest positive real root of $x^k - x^{k-1} - x^{k-2} - \cdots - 1$ and β_k is a constant such that $0 < \beta_k < \alpha_k$.

(2) The roots α_k satisfy inequalities $1 = \alpha_1 < \alpha_2 < \alpha_3 < \ldots < 2$, and $\alpha_k \to 2$ as $k \to \infty$.

(3) For all integers m and n,

$$F_{m,k} \cdot F_{n,k} \leq F_{m+n,k}.$$

(4) For every $n \geq 1$ we have

$$F_{n,k} \leq 2^{n-1} \;\; \textit{and} \;\; F_{n,n} = 2^{n-1}.$$

Proof. (1) Since

$$\sum_{n \geq 0} F_{n,k} x^n = \frac{1}{1 - x^k - x^{k-1} - \cdots - x},$$

the asymptotics of $F_{n,k}$ follows by the standard technique of decomposing rational functions into partial fractions (see, for example, Stanley [30, p. 202]). We need to prove only that α_k is a simple root of the reciprocal polynomial $p_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - 1$ and that on the complex circle $|z| = \alpha_k$, the polynomial p_k has no other root besides α_k. The constant β_k can then be set to ε+the second largest modulus of a root of p_k. The form of the coefficient c_k follows by a simple manipulation from the identity $c_k = \alpha_k^{k-1}/p_k'(\alpha_k)$ provided by the partial fractions decomposition.

Clearly, $1 \leq \alpha_k < 2$. Since $xp_k' - kp_k = x^{k-1} + 2x^{k-2} + \cdots + k$, we have $p_k'(\alpha_k) > k(k+1)/4$ and α_k is a simple root of p_k. Since $p_k = (x^{k+1} - 2x^k + 1)/(x-1)$, $p_k(x) = 0$ is equivalent to $x^k = 1/(2-x)$. It is clear that no z, $|z| = \alpha_k$, $z \neq \alpha_k$, satisfies this equation.

(2) This is immediate from the identity $\alpha_k^k = 1/(2 - \alpha_k)$ used in (1).

(3) and (4): These are easy to verify inductively by the recurrence for $F_{n,k}$. We only prove (3). We proceed by induction on $m + n$. For $m < 0$ or $n < 0$ the inequality is true. It also holds for $m = n = 0$. Let $m \geq 0$ and $n \geq 1$. Then

$$F_{m,k} F_{n,k} = F_{m,k} \sum_{i=n-k}^{n-1} F_{i,k} \leq \sum_{i=m+n-k}^{m+n-1} F_{i,k} = F_{m+n,k}.$$

$\square$

We list approximate values of the first few roots α_k:

k	2	3	4	5	6	10
α_k	1.61803	1.83928	1.92756	1.96594	1.98358	1.99901

Let A be a finite alphabet equipped with a weight function $w : A \to \mathbf{N}$. The weight $w(u)$ of a word $u = u_1, u_2, \ldots, u_m \in A^*$ is the sum $w(u_1) + w(u_2) + \cdots + w(u_m)$. We set

$$p(w, n) = \#\{u \in A^* : \ w(u) = n\}.$$

Lemma 3.6 *Let $k \geq 1$ be fixed.*

(1) If $A = \{a_1, a_2, \ldots, a_k\}$ and $w(a_i) = i$ for $i = 1, \ldots, k$, then $p(w, n) = F_{n,k}$ for every $n \geq 1$.

(2) If $A = \{a_1, a_2, \ldots, a_{k+1}\}$ and $w(a_i) = i$ for $i = 1, \ldots, k$ and $w(a_{k+1}) = k$, then $p(w, n) \geq 2^{n-1}$ for every $n \geq 1$.

Proof. In the general situation we have the identity

$$\sum_{n=0}^{\infty} p(w, n) x^n = \frac{1}{1 - \sum_{a \in A} x^{w(a)}}.$$

Now (1) is clear since then $\sum_{a \in A} x^{w(a)} = x^k + x^{k-1} + \cdots + x$.

In (2), we have

$$\sum_{n=0}^{\infty} p(w, n) x^n = \frac{1}{1 - (2x^k + x^{k-1} + \cdots + x)}$$

and the inequality $p(w, n) \geq 2^{n-1}$ follows by induction from the recurrence

$$p(w, n) = p(w, n-1) + \cdots + p(w, n-k+1) + 2p(w, n-k) \ (n > 0)$$

starting from $p(w, n) = 0$ for $n < 0$ and $p(w, 0) = 1$. □

In (2), one might be interested in a more precise bound. Since $1 - (2x^k + x^{k-1} + \cdots + x) = (1 - 2x)(x^{k-1} + x^{k-2} + \cdots + 1)$, the decomposition into partial fractions gives

$$p(w, n) = [x^n] \left(\frac{\alpha}{1 - 2x} + \frac{\beta_1}{1 - x/\zeta_1} + \cdots + \frac{\beta_{k-1}}{1 - x/\zeta_{k-1}} \right)$$

where $\alpha, \beta_1, \ldots \beta_{k-1} \in \mathbf{C}$ are suitable constants and ζ_i are the k-th roots of unity distinct from 1. Thus $\alpha = 1/\sum_{i=0}^{k-1} (\frac{1}{2})^i$ and, for $n \to \infty$, we obtain the asymptotics

$$p(w, n) = \left(\frac{1}{2} + \frac{1}{2^{k+1} - 2} \right) \cdot 2^n + O(1).$$

An *upward reduction* of a permutation π, $|\pi| = n$, is a partition $[n] = [1,r] \cup [r+1,n]$, where $1 \leq r < n$, such that $\pi([1,r]) < \pi([r+1,n])$. If π has no upward reduction, we say that π is *upward irreducible*. The set Irr^+ consists of all upward irreducible permutations and $\mathrm{Irr}^+_n = \{\pi \in \mathrm{Irr}^+ : \ |\pi| = n\}$. Every permutation π of $[n]$ has a unique decomposition $\pi|I_1, \ldots, \pi|I_m$, called the *upward decomposition* of π, in which $I_1 < I_2 < \ldots < I_m$ are intervals partitioning $[n]$ such that $\pi(I_1) < \pi(I_2) < \ldots < \pi(I_m)$ and every restriction $\pi|I_i$ is upward irreducible. (This decomposition can be obtained by iterating the upward reductions). We call the permutations $\pi|I_i$ the *upward blocks* of π. Notions symmetric to these are obtained in the obvious way, replacing the appropriate signs $<$ by the opposite signs $>$. Thus we get the definitions of *downward reductions*, *downward irreducibility*, *downward decompositions*, *downward blocks*, and the sets Irr^- and Irr^-_n.

We prove that one can delete an entry from any upward irreducible permutation in such a way that the result is upward irreducible. Needless to say, the same holds for downward irreducible permutations.

Lemma 3.7 *For every $\pi \in \mathrm{Irr}^+_n$, $n > 1$, there is some $i \in [n]$ such that $\pi|([n]\backslash\{i\})$ is in Irr^+_{n-1}.*

Proof. For a permutation π of $[n]$ and $i \in [n]$ we say that i is a *record* of π if $\pi(j) < \pi(i)$ for every $j < i$. Let $1 = r_1 < r_2 < \ldots < r_m \leq n$ be the records of π. It is easy to see that π is upward irreducible if and only if for every $i = 1, 2, \ldots, m-1$ there is a j, $r_{i+1} < j \leq n$, with $\pi(j) < \pi(r_i)$. Suppose that π, $|\pi| = n \geq 2$, is upward irreducible and consider the set $A = \{i \in [n] : \ r_m < i \leq n \ \& \ \pi(i) > \pi(r_{m-1})\}$; if $m = 1$, we set $A = [2, n]$. If $A \neq \emptyset$, the deletion of any $i \in A$ leaves an upward irreducible permutation. If $A = \emptyset$, we delete $i = r_m$. $\square$

We remark that it is easy to find examples of permutations of arbitrary length such that the statement of Lemma 3.7 is satisfied with only two indices i.

If π is a permutation, $|\pi| = n$, and $I_1 < I_2 < \ldots < I_m$ is a partition of $[n]$ into m nonempty intervals, we associate with π (as in the sketched proof of Theorem 3.4) the word $u(\pi) = u_1, u_2, \ldots, u_n$ over the alphabet $[m]$ by setting $u_i = j$ if $\pi^{-1}(i) \in I_j$. Note that π is uniquely determined by $u(\pi)$ and the m restrictions $\pi|I_i$.

Also, we associate with π the word $v^+(\pi)$ over the alphabet Irr^+ describing the upward decomposition of π. By $h^+(\pi) \in \mathbf{N}$ we denote the maximum size of an upward block of π appearing in the upward decomposition of π. Thus if $h^+(\pi) = k$ then $v^+(\pi) \in (\mathrm{Irr}^+_1 \cup \ldots \cup \mathrm{Irr}^+_k)^*$. In the analogous way we define $v^-(\pi)$ and $h^-(\pi)$.

Theorem 3.8 *Let Π be any hereditary class of permutations. Then either Π is finite, or exactly one of the following possibilities holds.*

(1) *There is a unique* $k \geq 1$ *and a constant* $c > 0$ *such that* $F_{n,k} \leq |\Pi_n| \leq F_{n,k} \cdot n^c$ *for all* $n \geq 1$.

(2) $|\Pi_n| \geq 2^{n-1}$ *for all* $n \geq 1$.

Proof. The *k-decomposition*, where $k \geq 2$ is an integer, of a permutation π, $|\pi| = n$, is the unique partition of $[n]$ into the intervals $U_1 < U_2 < \ldots < U_m$ such that U_1 is the longest initial interval of $[n]$ with $h^+(\pi|U_1) < k$ or $h^-(\pi|U_1) < k$, U_2 is the longest following interval with the same property, etc. We call the intervals U_i the *k-segments* of π. The number m of k-segments of π is denoted by $s_k(\pi)$.

Let Π be an infinite hereditary permutation class. Let $s_k(\Pi) = \max\{s_k(\pi) : \pi \in \Pi\}$. We set $s_1(\Pi) = \infty$. For every fixed $k \geq 1$ we prove the following claims.

Claim A. If $s_k(\Pi) = \infty$ then $|\Pi_n| \geq F_{n,k}$ for every $n \geq 1$.

Claim B. If $s_k(\Pi) < \infty$ then either $|\Pi_n| \geq 2^{n-1}$ for every $n \geq 1$ or $|\Pi_n| \leq F_{n,k-1} \cdot c_1 n^{c_2}$ for every $n \geq 1$ and some constants $c_1, c_2 > 0$.

This will prove the theorem. To see this, note that either $s_k(\Pi) = \infty$ for every $k \geq 1$ or there is a $k \geq 1$ such that $s_k(\Pi) = \infty$ but $s_{k+1}(\Pi) < \infty$. In the former case, claim A implies that $|\Pi_n| \geq F_{n,n} = 2^{n-1}$ for every $n \geq 1$ (by (4) of Lemma 3.5). In the latter case, we apply claim A with k and claim B with $k+1$ and conclude that either again $|\Pi_n| \geq 2^{n-1}$ for every $n \geq 1$ or that $F_{n,k} \leq |\Pi_n| \leq F_{n,k} \cdot n^c$ for every $n \geq 1$ (c_1 was absorbed in the enlarged c_2).

Proof of Claim A. For a $\pi \in \Pi$ with the k-segments $U_1 < U_2 < \ldots < U_{s_k(\pi)}$, we set $T_{\pi,i} = [\min(\pi(U_iU_{i+1})), \max(\pi(U_iU_{i+1}))]$. Note that, by the definition of k-segments and Lemma 3.7, every restriction $\pi|U_iU_{i+1}$ contains a member of Irr_k^+ and a member of Irr_k^-. We consider the system of intervals

$$S(\pi) = \{T_{\pi,1}, T_{\pi,3}, T_{\pi,5}, \ldots, T_{\pi,r}\}, \quad \text{where } r = 2\lceil (s_k(\pi) - 1)/2 \rceil - 1.$$

By the Ramsey theorem, either for every $m \geq 1$ there is a $\pi \in \Pi$ such that $S(\pi)$ contains m mutually intersecting intervals or for every $m \geq 1$ the same holds with mutually disjoint intervals. In the former case, it is easy to see that π must have at least $m/2$ alternations, since all members of a system of mutually intersecting intervals must have a point in common. By Lemma 3.2 and (4) of Lemma 3.5, $|\Pi_n| \geq 2^{n-1} \geq F_{n,k}$ for every $n \geq 1$.

In the latter case, for every $m \geq 1$ there is a $\pi \in \Pi$ for which $[|\pi|]$ can be partitioned into m intervals $I_1 < I_2 < \ldots < I_m$ such that every restriction $\pi|I_i$ contains a member of Irr_k^+ and a member of Irr_k^-, and for every $i \neq j$ we have $\pi(I_i) > \pi(I_j)$ or $\pi(I_i) < \pi(I_j)$. By Theorem 3.1, we may assume that $\pi(I_1) < \pi(I_2) < \ldots < \pi(I_m)$ or $\pi(I_1) > \pi(I_2) > \ldots > \pi(I_m)$. Let $\pi(I_1) < \pi(I_2) < \ldots < \pi(I_m)$; the other case is similar. Since m may be arbitrarily large and $|\mathrm{Irr}_k^+| \leq k!$, we may use the pigeon-hole principle and assume that there is one fixed $\sigma \in \mathrm{Irr}_k^+$ that is contained, for every $m \geq 1$, in every $\pi|I_i$, $1 \leq i \leq m$. By Lemma 3.7, there is a set of permutations $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ such that $\sigma_i \in \mathrm{Irr}_i^+$, $\sigma_i \subset \sigma_{i+1}$, and $\sigma_k = \sigma$. Since Π is hereditary, for every word u over the alphabet Σ there is a $\tau \in \Pi$ (contained in π) such that $v^+(\tau) = u$. Clearly, different words u determine different permutations τ. Using (1) of Lemma 3.6 (where the weight function is $w(\sigma_i) = |\sigma_i| = i$), we conclude that $|\Pi_n| \geq F_{n,k}$ for every $n \geq 1$. This finishes the proof of Claim A.

Proof of Claim B. We have $k \geq 2$ and there is a constant K such that $s_k(\pi) \leq K$ for every $\pi \in \Pi$. If $\pi \in \Pi_n$ and $U_1 < U_2 < \ldots < U_{s_k(\pi)}$ is the partition of $[n]$ into the k-segments of π, we consider the word $u(\pi)$ over $[K]$ as defined above the theorem.

For $1 \leq i \leq s_k(\pi)$ and $1 \leq j \leq k-1$ we define $v_{i,j}^+(\pi)$ as the subword of $v^+(\pi|U_i)$ consisting of the occurrences of the letters from the subalphabet Irr_j^+. The word $v_{i,j}^-(\pi)$ is defined in the obvious symmetric manner. Recall that $\ell(u)$ is the length of the longest alternating subsequence of u. Let $\ell(u(\Pi)) = \max\{\ell(u(\pi)) : \pi \in \Pi\}$ and, for $1 \leq i \leq K$ and $1 \leq j \leq k-1$, $\ell(v_{i,j}^+(\Pi)) = \max\{\ell(v_{i,j}^+(\pi)) : \pi \in \Pi\}$. The quantity $\ell(v_{i,j}^-(\Pi))$ is defined analogously. (For $i > s_k(\pi)$ we set $\ell(v_{i,j}^+(\pi)) = \ell(v_{i,j}^-(\pi)) = 0$.) We distinguish two complementary cases.

Case B1. *One of the $2K(k-1)+1$ quantities $\ell(u(\Pi))$, $\ell(v_{i,j}^+(\Pi))$, and $\ell(v_{i,j}^-(\Pi))$ equals ∞.*

We prove that then always

$$|\Pi_n| \geq 2^{n-1} \text{ for all } n \geq 1.$$

For unbounded $\ell(u(\pi))$ we can find a $\pi \in \Pi$ with as many alternations in π^{-1} as we wish and thus $|\Pi_n| \geq 2^{n-1}$ for every $n \geq 1$ by Lemma 3.2. For unbounded $\ell(v_{i,j}^+(\pi))$ (the argument for $v_{i,j}^-(\pi)$ is the same) there is a $j \in [k-1]$ (in fact, necessarily $j \in [3, k-1]$) and two distinct permutations $\tau, \sigma \in \mathrm{Irr}_j^+$ such that for every alternating word v over $\{\sigma, \tau\}$ there is a $\pi \in \Pi$ with $v^+(\pi) = v$. Using Lemma 3.7 again, we can take a set of permutations $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_j\}$ such that $\sigma_i \in \mathrm{Irr}_i^+$, $\sigma_i \subset \sigma_{i+1}$, and $\sigma_j = \sigma$. By the hereditarity of Π, for every word v over the alphabet $\Sigma \cup \{\tau\}$ there is a $\pi \in \Pi$ with $v^+(\pi) = v$. By (2) of Lemma 3.6, $|\Pi_n| \geq 2^{n-1}$ for every $n \geq 1$. This finishes the proof of case B1.

Case B2. *There is a constant $L > 0$ such that $\ell(u(\Pi)) \leq L$ and $\ell(v_{i,j}^+(\Pi)) \leq L$, $\ell(v_{i,j}^-(\Pi)) \leq L$ for every $1 \leq i \leq K, 1 \leq j \leq k-1$.*

We prove the upper bound

$$|\Pi_n| \leq F_{n,k-1} \cdot c_1 n^{c_2} \text{ for all } n \geq 1.$$

Every $\pi \in \Pi_n$ is uniquely determined by the word $u(\pi) \in [K]^*$ together with the $s_k(\pi) \le K$ restrictions $\pi|U_i$. For $s_k(\pi) < i \le K$ we set $U_i = \emptyset$. Let $R(m)$ be the number of possible restrictions $\pi|U_i$ with $|U_i| = m$. If we prove that $R(m) \le F_{m,k-1} \cdot c_3 m^{c_4}$ for all $m \ge 1$ and constants $c_3, c_4 > 0$ (depending only on K and L), we are done since (3) of Lemma 3.5 and (2) of Lemma 3.3 imply that

$$\begin{aligned}
|\Pi_n| &\le \sum_{u \in W_{K,L,n}} R(|U_1|)R(|U_2|)\dots R(|U_K|) \quad (\text{note that } |U_1| + \cdots + |U_K| = n) \\
&\le \sum_{u \in W_{K,L,n}} F_{n,k-1} \cdot c_3^K n^{c_4 K} \\
&\le F_{n,k-1} \cdot c_3^K n^{c_4 K} \cdot (K+1)^{c_5} n^{c_5} \quad \left(\text{where } \ c_5 = \binom{K}{2}(L-1)+1\right) \\
&\le F_{n,k-1} \cdot c_1 n^{c_2}.
\end{aligned}$$

It remains to show that $R(m) \le F_{m,k-1} \cdot c_3 m^{c_4}$. Let σ be a generic restriction $\pi|U_i$ with $|U_i| = m$ and π ranging over Π_n. We have that $h^+(\sigma) < k$ or $h^-(\sigma) < k$; we may assume the former. For $1 \le j \le k-1$ we write $v_j^+(\sigma)$ for $v_{i,j}^+(\pi)$. By the hypothesis of case B2, $\ell(v_j^+(\sigma)) \le L$ for every $1 \le j \le k-1$. Since $v^+(\sigma) \in (\mathrm{Irr}_1^+ \cup \ldots \cup \mathrm{Irr}_{k-1}^+)^*$, we apply (3) of Lemma 3.3 and conclude that there is a partition into intervals

$$v^+(\sigma) = J_1 J_2 \dots J_M \ \text{ with } \ M \le 2\binom{1! + \cdots + (k-1)!}{2}(L-1) + 2 = N$$

such that every J_i uses at most one letter from any subalphabet Irr_j^+. Let $Q(m)$ be the number of $\tau \in \Pi_m$ such that $h^+(\tau) < k$ and $v^+(\tau)$ uses at most one letter from every Irr_j^+. Then, by (1) of Lemma 3.6,

$$Q(m) \le 1! \cdot 2! \cdot \ldots \cdot (k-1)! \cdot F_{m,k-1} = c_6 \cdot F_{m,k-1}$$

because $v^+(\tau) \in \Sigma^*$ for a transversal Σ of the $k-1$ sets $\mathrm{Irr}_1^+, \ldots, \mathrm{Irr}_{k-1}^+$, and there are at most c_6 such transversals.

So, by the bound on $Q(m)$, (3) of Lemma 3.5, and the bound $M \le N$, the number of σs satisfies

$$\begin{aligned}
R(m) &\le \sum_{m_1 + \cdots + m_N = m} Q(m_1)Q(m_2)\dots Q(m_N) \quad (\text{summing over } m_i \ge 0) \\
&\le \sum_{m_1 + \cdots + m_N = m} F_{m,k-1} \cdot c_6^N \le F_{m,k-1} \cdot c_6^N (m+1)^N \\
&\le F_{m,k-1} \cdot c_3 m^{c_4}.
\end{aligned}$$

This finishes the proof of case B2, of claim B, and of the whole theorem. □

The growth rate $n \mapsto F_{n,k}$ is attained by the hereditary class of permutations π whose upward blocks are decreasing sequences of length at most k.

Let $a_n = |\mathrm{Irr}_n^+| = |\mathrm{Irr}_n^-|$. It follows from the upward decompositions that the numbers a_n of upward irreducible permutations satisfy the recurrence $a_n = n! - \sum_{k=1}^{n-1} a_k(n-k)!$ where $n \geq 2$ and $a_1 = 1$. From this we calculate that

$$(|\mathrm{Irr}_n^+|)_{n\geq 1} = (1, 1, 3, 13, 71, 461, 3447, 29093, 273343, 2829325, \ldots).$$

This is sequence A003319 of Sloane [27]. It appears three times in Comtet's textbook [14, pp. 84, 262, and 295]. Since

$$1 \geq \frac{a_n}{n!} = 1 - \sum_{k=1}^{n-1} \frac{a_k}{k!} \cdot \frac{1}{\binom{n}{k}} \geq 1 - \left(\frac{2}{n} + \frac{2(n-3)}{n(n-1)}\right) = 1 - \frac{4n-8}{n(n-1)}$$

for every $n \geq 3$, we conclude that $a_n = |\mathrm{Irr}_n^+| = |\mathrm{Irr}_n^-| = n!(1 - O(1/n))$. See [14, p. 295] for a more precise asymptotic expansion. Thus almost every permutation is upward irreducible (downward irreducible). For $|\mathrm{Irr}_n^+ \cap \mathrm{Irr}_n^-|$ we have the same asymptotics, because $|\mathrm{Irr}_n^+ \cap \mathrm{Irr}_n^-| = 2a_n - n!$. It follows that almost every permutation is both upward irreducible and downward irreducible.

4 Constant and polynomial growths

We look in more detail on the slow growths and begin with the constant growth. Let π be a permutation of $[n]$. For $r \in \mathbf{N}$, we say that π has the *r-intrusion property* if there are subsets $X, Y \subset [n]$ and an element $x \in [n]$ such that $X < x < Y$, $|X|, |Y| \geq r$, and $\pi|(X \cup Y)$ is monotone but $\pi|(X \cup Y \cup \{x\})$ is not. We say that π has the *r-union property* if there are subset $X, Y \subset [n]$ such that $X < Y$, $|X|, |Y| \geq r$, and both restrictions $\pi|X$ and $\pi|Y$ are monotone but $\pi|(X \cup Y)$ is not.

Lemma 4.1 *Let Π be any hereditary class of permutations.*

(1) If for every $r \geq 1$ there is a $\pi \in \Pi$ such that π or π^{-1} has the r-intrusion property, then $|\Pi_n| \geq n$ for all $n \geq 1$.

(2) If for every $r \geq 1$ there is a $\pi \in \Pi$ with the r-union property, then $|\Pi_n| \geq n$ for all $n \geq 1$.

(3) Suppose $\pi \neq \tau$ are two permutations of $[n]$ and $I \subset [n]$ is such that all three sets $I, \pi(I)$, and $\tau(I)$ are intervals in $[n]$ and both restrictions $\pi|I$ and $\tau|I$ are monotone. Then for every subset $J \subset I$ such that $|I| - |J| \geq 2$ we have $\pi|([n]\backslash J) \neq \tau|([n]\backslash J)$.

Proof. (1) We may assume that for every $r \geq 1$ there is a $\pi \in \Pi_{2r+1}$ such that $\pi|([2r+1]\backslash\{r+1\})$ is increasing but $\pi(r) > \pi(r+1)$; the other possible cases are very similar. Thus for every n and m, $1 \leq m \leq n-1$, there is a $\pi_m \in \Pi_n$

such that $\pi|[m]$ is increasing but $\pi(m) > \pi(m+1)$. The permutations π_m are mutually distinct and together with $(1, 2, \ldots, n) \in \Pi_n$ they show that $|\Pi_n| \geq n$.

(2) If π, $|\pi| = 2n$, is such that $\pi|[n]$ and $\pi|[n+1, 2n]$ are monotone but π is not, then $\pi(n-1), \pi(n), \pi(n+1)$ or $\pi(n), \pi(n+1), \pi(n+2)$ is non-monotone. From this it easily follows, as in (1), that there are n distinct permutations σ, $|\sigma| = n$, such that $\sigma \subset \pi$. Thus $|\Pi_n| \geq n$ for all $n \geq 1$.

(3) The restrictions of π and τ on $[n]\backslash J$ must be different because at least 2 terms remained from the monotone sequences $\pi|I$ and $\tau|I$ and thus π and τ can be completely reconstructed from the restrictions. $\Box$

Theorem 4.2 *Let Π be any hereditary class of permutations. Then exactly one of the following possibilities holds.*

(1) $|\Pi_n|$ is constant for $n \geq n_0$.

(2) $|\Pi_n| \geq n$ for all $n \geq 1$.

Proof. We may assume that Π is a hereditary permutation class such that, for some $r \geq 1$, for every $\pi \in \Pi$ neither π nor π^{-1} has the r-intrusion property and π does not have the r-union property. If r does not exist, we are done because (1) and (2) of Lemma 4.1 then imply that (2) holds. Now let $\pi \in \Pi_n$ be arbitrary and let $n \geq 9r^2$. Consider the longest monotone subsequence of π, determined by $X \subset [n]$. Thus $\pi|X$ is monotone, $|X|$ is maximum, and, by Theorem 3.1, $|X| \geq 3r$. We partition X into the sets $X_1 < Y < X_2$ where $|X_1| = |X_2| = r$. So $|Y| \geq r$. Since neither π nor π^{-1} has the r-intrusion property and $|X|$ is maximum, both Y and $\pi(Y)$ must form an interval in $[n]$. Let $A = [\min Y - 1]\backslash X_1$ and $B = [\max Y + 1, n]\backslash X_2$. Using the assumption that π does not have the r-union property and invoking Theorem 3.1, we see that $|A|, |B| \leq r^2$. We conclude that for every $\pi \in \Pi_n$, $n \geq 9r^2$, there is a subset $Y \subset [n]$ such that $\pi|Y$ is monotone, both Y and $\pi(Y)$ form an interval in $[n]$, and $n - |Y| \leq 2(r^2 + r) = R$. If R is enlarged to $R = 9r^2$, the conclusion holds for every $n \geq 1$.

For all $n \geq 1$, $|\Pi_n| \leq 2(R+1)^2 R!$ because we have two possibilities for $\pi|Y$, at most $R+1$ ways to place Y, the same number of ways to place $\pi(Y)$, and at most $R!$ possibilities for $\pi|([n]\backslash Y)$. Let

$$k = \limsup_{n \to \infty} |\Pi_n|$$

and $n_0 \in \mathbf{N}$ be such that $n_0 \geq 2R + 2$ and $|\Pi_n| \leq k$ for $n \geq n_0$. We show that in fact, $|\Pi_n| = k$ for $n \geq n_0$. Let $n \geq n_0$ be arbitrary, $m \geq n$ be such that $|\Pi_m| = k$, and let $\Pi_m = \{\pi_1, \pi_2, \ldots, \pi_k\}$. These k permutations satisfy the hypothesis of (3) of Lemma 4.1 with $I = [R+1, m-R]$. For $J = [n-R+1, m-R]$ we have $J \subset I$, $|I| - |J| \geq 2$, and $\sigma_i = \pi_i|([m]\backslash J) \in \Pi_n$ for $1 \leq i \leq k$. By (3) of Lemma 4.1, all σ_i are distinct. Hence $|\Pi_n| \geq k$ and $|\Pi_n| = k$. $\Box$

All possible constant growths are attained. The growth $n \mapsto 0$ is attained by $\Pi = \emptyset$ and $n \mapsto k$, $n \geq k$, is attained by $\{\pi \subset (1, 2, \ldots, n, n+k, n+k-1, \ldots, n+1) : n \in \mathbf{N}\}$. Similarly, $n \mapsto n$ is attained by $\{\pi \subset (1, 2, \ldots, n, 2n, 2n-1, \ldots, n+1) : n \in \mathbf{N}\}$.

We proceed to the polynomial growth and partially characterize polynomially growing counting functions of hereditary permutation classes. For this, we need to look first at the partial order $(\mathbf{N}^m, \leq)$, where $m \in \mathbf{N}$ and $a = (a_1, \ldots, a_m) \leq (b_1, \ldots, b_m) = b$ means that $a_i \leq b_i$ for every $i = 1, \ldots, m$. We say that a subset $S \subset \mathbf{N}^m$ is *hereditary* if $a \leq b \in S$ always implies $a \in S$. For $a \in \mathbf{N}^m$ we define $\|a\| = a_1 + a_2 + \cdots + a_m$. For a (hereditary) subset $S \subset \mathbf{N}^m$ and $n \in \mathbf{N}$ we set

$$S_n = \{a \in S : \ \|a\| = n\}.$$

The elements of S_n can be represented by partitions of $[n]$ into m nonempty intervals, and therefore the next lemma is a particular case of Theorem 3.1 in [17]. However, the direct proof is not too difficult; and for the sake of completeness, we give it here.

Lemma 4.3 *For every hereditary set $S \subset \mathbf{N}^m$ there is a number $M \in \mathbf{N}$ and $(M+1)^2$ integers $a_{i,j}$, $0 \leq i, j \leq M$, so that for every $n \geq n_0$ we have*

$$|S_n| = \sum_{i,j=0}^{M} a_{i,j} \binom{n-i}{j}.$$

Proof. We say that $S \subset \mathbf{N}^m$ is *canonical* if there is an m-tuple $b \in (\mathbf{N} \cup \{\infty\})^m$ so that

$$S = \{a \in \mathbf{N}^m : \ a_i < b_i \text{ for every } i = 1, \ldots, m\},$$

where $a_i < \infty$ means that the i-th coordinate of a is unrestricted. Canonical sets are hereditary and for a canonical S determined by b we have

$$|S_n| = [x^n] \left(\frac{x}{1-x}\right)^{\#(b_i=\infty)} \prod_{b_i<\infty} \sum_{j=1}^{b_i-1} x^j \ \text{ for } n \geq 1,$$

where the empty sum equals 0. If $b_i = \infty$ for no i, then $|S_n| = 0$ for $n \geq n_0 = \|b\| - m + 1$, and the formula is true with all $a_{i,j}$ zero. Otherwise, the formula follows by the binomial theorem. Thus the lemma holds in the case when S is a canonical set, even with nonnegative $a_{i,j}$. It is clear that every intersection of canonical sets is again a canonical set. Therefore, by the inclusion-exclusion principle, the lemma holds more generally for every finite union of canonical sets (now we may get negative $a_{i,j}$).

Let $S \subset \mathbf{N}^m$ be any hereditary set. It suffices to show that S is a finite union of canonical sets. S is determined by the set B of minimal elements in $(\mathbf{N}^m \backslash S, \leq)$. The set B is an antichain. It is known, and not too difficult to prove, that every antichain in $(\mathbf{N}^m, \leq)$ is finite (this is the combinatorial core of Hilbert's basis theorem), see (for example) Nash-Williams [23, Lemma 1] for a more general result. So $B = \{b^1, b^2, \ldots, b^r\}$ and

$$S = \{a \in \mathbf{N}^m : \ a \not\geq b^i \text{ for every } i = 1, \ldots, r\}.$$

Thus

$$a \in S \iff \bigwedge_{i=1}^{r} \bigvee_{j=1}^{m} (a_j < b_j^i).$$

The right hand side of the equivalence is equivalent to

$$\bigvee_{j_1,\ldots,j_r} \bigwedge_{i=1}^{r} (a_{j_i} < b_{j_i}^i)$$

where in the disjunction the j_i's range over all r-tuples from $[m]$. Since every conjuction $\bigwedge_{i=1}^{r}(a_{j_i} < b_{j_i}^i)$ defines a canonical set, S is indeed a union of m^r canonical sets and the lemma follows. $\square$

A *canonical form* of a word u is the form $u = u_1^{a_1}, u_2^{a_2}, \ldots, u_m^{a_m}$ where $a_i \in \mathbf{N}$, $u_i \neq u_{i+1}$ for $i = 1, 2, \ldots, m-1$, and $u_i^{a_i}$ abbreviates a_i repetitions of the letter u_i. Let $u(\pi) = u_1, u_2, \ldots, u_n$ be the word over $[m]$ determined by the 2-decomposition $U_1 < U_2 < \ldots < U_m$ of a permutation π of length n; $u_i = j \Leftrightarrow \pi^{-1}(i) \in U_j$ and every $\pi|U_i$ is monotone (see the proofs of Theorems 3.4 and 3.8). Recall that π is uniquely determined by $u(\pi)$ and the m-tuple $s(\pi) \in \{+,-\}^m$ whose i-th component records whether $\pi|U_i$ is increasing or decreasing. Every $i < m$ appears in $u(\pi)$ at least twice and $i = m$ may appear only once. Let $u(\pi) = w_1^{a_1}, w_2^{a_2}, \ldots, w_r^{a_r}$ be the canonical form of $u(\pi)$. The *reduced* form of $u(\pi)$ is the word

$$u(\pi)^* = w = w_1^{b_1}, w_2^{b_2}, \ldots, w_r^{b_r}$$

where $b_i = 1$ if $w_i = m$ or if $w_j = w_i$ for some $j \neq i$, else $b_i = 2$. We define $E(w) = \{i \in [r] : \ b_i = 2\}$ and $e(\pi) \in \mathbf{N}^r$ by $e_i = a_i$ if $i \notin E(w)$ and $e_i = a_i - 1$ if $i \in E(w)$.

Let Π be any hereditary permutation class. We split Π into (nonhereditary) subclasses by the equivalence $\sim$: $\pi \sim \sigma$ iff $u(\pi)^* = u(\sigma)^* = w = w_1^{b_1}, w_2^{b_2}, \ldots, w_r^{b_r}$ and $s(\pi) = s(\sigma)$. In one equivalence subclass X, the permutations are fully determined by the r-tuples

$$e(X) = \{e(\pi) : \ \pi \in X\} \subset \mathbf{N}^r.$$

Note that $e(X)$ is hereditary (in the sense of the previous lemma) and that, for $\pi \in X$, $|\pi| = \|e(\pi)\| + |E(w)|$. By Lemma 4.3, there are $(M+1)^2$ integers $a_{i,j}$ such that

$$|X_n| = \#\{\pi \in X : \ |\pi| = n\} = \sum_{i,j=0}^{M} a_{i,j} \binom{n-i}{j}$$

for all $n \geq n_0$.

Theorem 4.4 *If Π is a hereditary class of permutations such that $|\Pi_n| \le n^c$ for all $n \ge 1$ and a constant $c > 0$, then there is a number $M \in \mathbf{N}$ and $(M+1)^2$ integers $a_{i,j}$, $0 \le i, j \le M$, such that for all $n \ge n_0$ we have*

$$|\Pi_n| = \sum_{i,j=0}^{M} a_{i,j} \binom{n-i}{j}.$$

Proof. The proof of Theorem 3.8 (case B2, $k = 2$) shows that if $|\Pi_n| \le n^c$ for all $n \ge 1$, then for all $\pi \in \Pi$ we have $m = s_2(\pi) \le K$ and $\ell(u(\pi)) \le L$ for some constants $K, L > 0$. Thus by (1) of Lemma 3.3, the length of the reduced form of $u(\pi)$ is bounded by some $d > 0$, and we have at most $(K+1)^d$ possible reduced forms $u(\pi)^*, \pi \in \Pi$. Hence the equivalence $\sim$ on Π has at most $2^K(K+1)^d$ subclasses. We select n_0 large enough so that for every of the subclasses X the above argument applies and for $n \ge n_0$ the number $|X_n|$ has the form $\sum_{i,j=0}^{M} a_{i,j}\binom{n-i}{j}$, with integral $a_{i,j}$ (M and $a_{i,j}$ depend on X). Then for $n \ge n_0$ also the finite sum

$$|\Pi_n| = \sum_X |X_n|$$

has the form $\sum_{i,j=0}^{M} a_{i,j}\binom{n-i}{j}$. □

It is an interesting question to fully characterize those polynomials that can be realized (for $n \ge n_0$) as $n \mapsto |\Pi_n|$. For example, it follows from Theorem 4.2 that no polynomial $\binom{n-i}{1} = n - i$, $i \in \mathbf{N}$, can be realized as a counting function. Note that a part of Theorem 4.2, the fact that every bounded counting function must be eventually constant, is an immediate corollary of the last theorem.

5 Concluding remarks

The fact that the containment order of permutations admits an infinite antichain has been known for a long time. The earliest references are Laver [21, p. 9], Pratt [24], and Tarjan [31]. Kruskal [20, p. 304] mentions Laver's (counter)example four years earlier and Laver himself "[uses] a construction of Jenkyns and Nash-Williams" [16]. Thus the idea seems to go back to the late 1960s. The recent reference is Spielman and Bóna [28]. See Atkinson, Murphy and Ruškuc [6] for further results on permutation antichains.

Hereditary classes of permutations and their counting functions have been investigated before by Atkinson [5] who, together with West [33], gives the counting function $n \mapsto |\Pi_n|$ for every hereditary Π of the form $\Pi = \mathrm{Forb}(\{\alpha, \beta\})$ where $|\alpha| = 3, |\beta| = 4$, and $\alpha \not\subset \beta$. Our approach is much inspired by the works of Scheinerman and Zito [26] and Balogh, Bollobás and Weinreich [7, 8, 9] on the hereditary classes and monotone classes of graphs. (For graphs, hereditarity means that the class is closed with respect to induced subgraphs, while monotonicity means that it is closed with respect to all subgraphs.) As far as we know, graphs are the only combinatorial structures for which the counting functions of hereditary classes have been systematically investigated from a 'global' viewpoint. One global result (although cast in the 'local' Forb(X) language) on hereditary classes of set partitions is in Klazar [17, Theorem 3.1]. The counting functions of the hereditary classes of set partitions are further investigated in [19].

The question posed after Theorem 2.1, whether there are $2^{\aleph_0}$ hereditary permutation classes with counting functions mutually incomparable by the eventual dominance, has a positive answer for graph hereditary classes, see [8, Theorem 10]. Note also that if we restrict our attention to polynomially growing permutation classes, then by Theorem 4.4, the eventual dominance is a linear order.

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