Complexity of compatible decompositions of eulerian graphs and their transformations *

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Abstract

In this paper some earlier defined local transformations between eulerian trails are generalized to transformations between decompositions of graphs into (possibly more) closed subtrails. For any graph G with a forbidden partition system F, we give an efficient algorithm which transforms any F-compatible decomposition of G into closed subtrails to another one, and at the same time it preserves F-compatibility and does not increase the number of subtrails by more than one. From this, several earlier results for eulerian trails easily follow. These results are embedded into the rich spectrum of results of theory of eulerian graphs and their applications. We further apply this statement to digraphs and discuss the time complexity of enumeration of all F-compatible decompositions (resp. of all F-compatible eulerian trails) in both graphs and digraphs.

1 Introduction

A closed trail in a graph G is a sequence $v_0, e_1, v_1, \ldots, v_{t-1}, e_t, v_t$ of its vertices and edges such that $v_t = v_0$, for every i, e_i is incident with v_{i-1}, v_i, and all edges $e_1, \ldots, e_t$ are distinct. A closed trail which contains all edges of G is called *eulerian trail.* Eulerian trails belong to the very classical combinatorial structures, yet they play a prominent role in many modern branches of graph theory, such as network flows, graph polynomials, coloring problems, nowhere-zero flows, cycle double cover conjecture, to name just a few (see [6] and [9] for many more examples). Eulerian trails provide also some background structure for problems in areas as diverse computational biology [17, 16] and knot theory [2, 22]. Although the basic questions related to eulerian trails (such as existence, number and various optimization problems) are well-known and were clarified long time ago, the new applications produce specific problems which are interesting (and sometimes hard). One such area of problems is the study

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of eulerian trails which satisfy special local conditions such as forbidding special consecutive pairs of edges. This study was initiated by A. Kotzig [11, 1] and continued by G. Sabidussi who related it to the Cycle double cover conjecture and other well-known problems, see [6, 18].

In this paper we address this problem in the full generality and consider the set $\mathcal{S}(G, F)$ of all decompositions of G into closed trails with respect to a given set F of forbidden transitions at every vertex (see Section 2 for definitions). In section 4 we give an efficient algorithm for transforming any two F-compatible decompositions of $\mathcal{S}(G, F)$ (F a partition system in G) to one another with preserving the F-compatibility and not increasing the number of trails in the decomposition by more than one (see Theorem 1). This result also implies and gives short (and algorithmic) proofs of several results obtained earlier: Kotzig's theorem on transformations between eulerian trails [1, 11] (Corollary 2) and theorem of Fleischner et al. concerning transformations between F-compatible eulerian trails [7] (Corollary 1).

Our approach has been partly inspired by chord-diagrams which play an important role in knot theory (see for example [12, 15] for papers related to Vassiliev invariants). Chord diagrams are defined as a circle with additional set of disjoint chords (i.e. a matching). For a given knot, the chords correspond to the crossing points (in the plane projection) and thus to the double occurrence of a vertex in the eulerian trail. The graphs in knot theory are 4-regular planar graphs (or 2-in 2-out planar digraphs).

This can be naturally generalized to *diagrams* of arbitrary eulerian trail $v_0, e_1, \dots, e_m, v_m = v_0$ in graph $G = (V, E)$ as follows: We consider a cycle of length m with vertices $1, \dots m$ and with extra added disjoint complete graphs $\{K_v\}_{v \in V}$. These complete graphs correspond to vertices of G; $V(K_v) = \{v_i \mid v_i = v\}$. The complete graph K_v is specified by the polygon inscribed in the cycle (these polygons may be degenerated). For distinct vertices $u \neq v$ in G, the polygons K_u, K_v have disjoint vertices. The diagrams of eulerian trails could be also called *polygonal diagrams*.

Some questions related to diagrams of eulerian trails are much harder than for chord diagrams (i.e. diagrams of 4-regular planar graphs). One such problem is e.g. the Gauss problem which is already open for graphs with maximum degree at most six (see [8, 5] for the solution for 4-regular graphs). Still the diagram is a useful tool which motivated some of our constructions below. For example, the three main transformations are schematicly depicted in Fig. 1 (we skip the detailed description).

2 Terminology

We use standard graph theory notation, see e.g. [13]. $G = (V, E)$ will always denote an (undirected) graph with the vertex-set V and the edge-set E.

$E(v)$ denotes the set of edges incident with v. Any two-element subset of $E(v)$ is called a *transition* through v. $\mathcal{T}(v)$ denotes the set of all transitions through v and $\mathcal{T}(G) = \bigcup_{v \in V} \mathcal{T}(v)$ the set of all transitions of G.

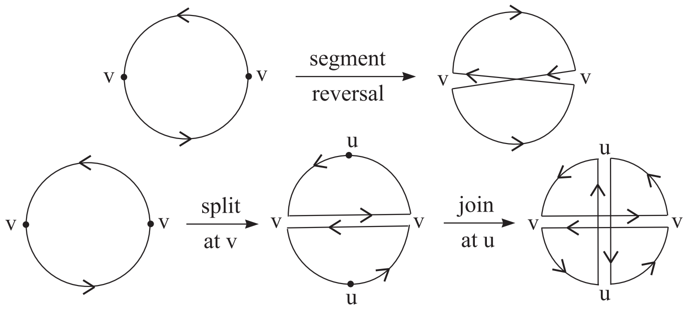

Figure 1: Diagrams of transformations

If $X(v)$ is a partition of $E(v)$ into two-element classes (i.e. each class of $X(v)$ is a transition through v), then $X(v)$ is called a *transition system through* v and $X = \bigcup_{v \in V} X(v)$ is called a *transition system of* G.

$\mathcal{S}_k(G)$ denotes the set of all decompositions of G into exactly k (edge-disjoint) closed trails. In particular, $\mathcal{S}_1(G)$ denotes the set of all (closed) eulerian trails of G. We set $\mathcal{S}(G) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(G)$. Let us remark that two closed trails in an (undirected) graph are considered as equal if one can be obtained from another by a cyclic permutation and possibly by a complete reversal.

Let $S = \{S_1, \ldots, S_k\} \in \mathcal{S}_k(G)$, X_{S_i}, is the set of transitions which correspond to a subsequence of S_i. Then $X_S = \bigcup_{i=1}^{k} X_{S_i}$ is a transition system of G. And conversely, every transition system X of G defines a (unique) $S \in \mathcal{S}(G)$, i.e. there is a unique S such that $X = X_S$. $n(X)$ denotes the number of closed trails in the decomposition of G defined by X (i.e. if $X = X_S$ for $S \in \mathcal{S}_k(G)$, then $n(X) = k$). If there is no danger of confusion, we will not distinguish between the transition system and the decomposition into closed trails defined by it. Let us stress that this correspondence is not between eulerian trails and transitions systems. Of course, any decomposition of the edge set of a connected graph into closed trails may be combined to a single eulerian trail, however at the expense that the transition system is changed. We are interested here in those operations which do not change transition system too much. This is captured by the notion of compatibility. Its origin lies in questions first raised by A. Kotzig for eulerian trails and by G. Sabidussi for cycle decompositions.

Let G be a graph and $F \subseteq \mathcal{T}(G)$ any subset of its transitions. $F = \bigcup_{v \in V} F(v)$, where $F(v) \subseteq \mathcal{T}(v)$. Any transition t of G such that $t \in F$ is said to be *forbidden*, any transition t of G such that $t \notin F$ is said to be *admissible* (or also *compatible*). (G, F) will denote a graph with a given forbidden system F. If for every v, $F(v)$ is a transitive relation on $E(v)$ we say that $F = \bigcup_{v \in V} F(v)$ is a *forbidden partition system of* G. This is consistent with definitions given in [6, 7].

Another possibility how to view a forbidden system of transition is the following: To every $F(v)$, there is a corresponding graph $G_{F(v)}$ on the set of neighbors of v with edges corresponding to forbidden transitions through

v. More formally, $V(G_{F(v)}) = \{u \in V(G) \mid \{u,v\} \in E(G)\}$, $E(G_{F(v)}) = \{\{u,u'\} \mid \{\{u,v\},\{u',v\}\} \in F(v)\}$. Note that if F is a forbidden partition system in G, then $G_{F(v)}$ is a disjoint union of complete graphs (for every v).

Let (G,F) be given and let $S \in \mathcal{S}_k(G)$, S is said to be F-*compatible* (or simply *compatible*) if $X_S \cap F = \emptyset$. $\mathcal{S}_k(G,F)$ denotes the set of all F-compatible decompositions of G into (exactly) k closed trails. Again, $\mathcal{S}(G,F) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(G,F)$.

These definitions capture several important instances. $\mathcal{S}_1(G)$ is just the set of all eulerian trails in G and if F is a red-blue partition of $E(G)$, then $\mathcal{S}_1(G,F)$ denotes the set of all eulerian trails in G which are alternating in colors [17, 7].

All the above defined notions can be easily modified for directed graphs. In this paper, $D = (V,A)$ always denotes a directed graph, $A^+(v)$ (resp. $A^-(v)$) denotes the set of all arcs outgoing from (resp. incoming to) v. A *transition* through v in D is any two-element subset of $A(v)$ with one element in $A^+(v)$ and the other one in $A^-(v)$. We skip precise definitions of other notions, they are similar to the undirected case.

Our aim now is to define some local transformations between elements of $\mathcal{S}(G,F)$ which preserve F-compatibility. As we will see, our generalization to decompositions into more closed trails is very natural. For simplicity we define the transformation between transition systems, but of course, they uniquely determine also transformations between elements of $\mathcal{S}(G)$.

Definition 1 *Let X, X' be two transition systems of a graph G such that $|X \setminus X'| = 2$, that is X and X' differ in only two transitions.*

1. *If $n(X') = n(X)$, then we say that X' arises from X be a* **segment reversal**.
2. *If $n(X') = n(X) + 1$, then we say that X' arises from X be a* **split**.
3. *If $n(X') = n(X) - 1$, then we say that X' arises from X be a* **join**.

Note that whenever X and X' differ in only two transitions, then these are transitions through the same vertex (i.e. there is a vertex $v \in V(G)$ such that $X(u) = X'(u)$ for every $u \neq v$) and furthermore $n(X') \in \{n(X), n(X) \pm 1\}$. Hence, $|X \setminus X'| = 2$, if and only if X' arises from X by a segment reversal, by a split or by a join.

The above defined transformations have their natural meanings, *segment reversal* corresponds to reversing a segment (i.e. a part of one trail between two occurrences of the same vertex), *split* corresponds to dividing one trail of S into two closed trails and *join* corresponds to joining two elements of S into one closed trail.

Later we will need the following technical definition. Let X be a transition system in a graph G such that $\{g,g'\},\{h,h'\} \in X(v)$ and $\{g,h\} \notin F(v)$. We define a new transition system $Y = join(X,g,h)$ by:

$$Y = \begin{cases} X \setminus \{\{g,g'\},\{h,h'\}\} \cup \{\{g,h\},\{g',h'\}\} & \text{if } \{g',h'\} \notin F(v) \\ X \setminus \{\{g,g'\},\{h,h'\}\} \cup \{\{g,h'\},\{g',h\}\} & \text{otherwise} \end{cases}$$

Note that if F is a partition system and $X \cap F = \emptyset$ then also $Y \cap F = \emptyset$.

3 Formulation of main problems and results

We are interested in the possibility of transforming two F-compatible transition systems into one another by means of the transformations defined above such that the F-compatibility is preserved throughout this process and the number of closed trails of any decomposition is not increased by more than one. More formally, we define:

REACHABILITY PROBLEM

INPUT: (G, F) a graph with a forbidden system
X, X' two F-compatible transition systems of G.

QUESTION: Does there exist a sequence $X = X_0, X_1, \ldots, X_n = X'$ such that for all $0 = 1, \ldots, n$:

1. X_i is F-compatible transition system of G,
2. $n(X_i) \leq \max\{n(X), n(X')\} + 1$,
3. $|X_i \setminus X_{i+1}| = 2$,
4. if $n(X_i) = \max\{n(X), n(X')\} + 1$, then $n(X_{i+1}) = n(X_i) - 1$.

Theorem 1 *Let F be a forbidden partition system in G. Then for any transition system X, X' compatible with F, there is a polynomial algorithm which finds the transforming sequence $X = X_0, X_1, \ldots, X_n = X'$ satisfying conditions 1.-4. above.*

In particular, Theorem 1 implies that when F is a partition system, the answer to REACHABILITY PROBLEM is always "yes". A polynomial algorithm for the REACHABILITY PROBLEM is presented below.

In Section 6 we discuss algorithmic complexity of the following decision and evaluation problems:

GRAPH COMPATIBLE DECOMPOSITION PROBLEM (GCD)
INPUT: (G, F) a graph with a forbidden system
QUESTION: Is $\mathcal{S}(G, F) \neq \emptyset$?

$\sharp$GRAPH COMPATIBLE DECOMPOSITION PROBLEM ($\sharp$GCD)
INPUT: (G, F) a graph with a forbidden system
QUESTION: Determine $|\mathcal{S}(G, F)|$.

Similarly, we can define GRAPH COMPATIBLE TRAIL PROBLEM (GCT) and $\sharp$GRAPH COMPATIBLE TRAIL PROBLEM ($\sharp$GCT) concerning the existence and number of F-compatible eulerian trails. For the lack of space we skip definitions of analogous problems for digraphs: DIGRAPH COMPATIBLE DECOMPOSITION PROBLEM (DCD), $\sharp$DIGRAPH COMPATIBLE DECOMPOSITION PROBLEM ($\sharp$DCD), DIGRAPH COMPATIBLE TRAIL PROBLEM (DCT), and $\sharp$DIGRAPH COMPATIBLE TRAIL PROBLEM ($\sharp$DCT).

Algorithm for REACHABILITY PROBLEM for partition systems

INPUT: (G, F) a graph with a forbidden partition system
$X \neq X'$ transition systems in G such that

$$X \cap F = \emptyset, \quad X' \cap F = \emptyset \text{ and } n(X), n(X') \leq \ell$$

OUTPUT: A sequence $X = X_0, X_1, \ldots, X_k = X'_k, \ldots, X'_1, X'_0 = X'$.

Step 1:
Set $i := 0$, $X_i := X$ and $X'_i := X'$.

Step 2:
Choose a vertex v such that $X_i(v) \neq X'_i(v)$ and edges $a, b, c, d, e \in E(v)$ (possibly $d = e$) such that $\{a, b\}, \{c, d\} \in X_i \setminus X'_i$ and $\{a, c\}, \{b, e\} \in X'_i \setminus X_i$. Set
$Y_1 := X_i \setminus \{\{a, b\}, \{c, d\}\} \cup \{\{a, c\}, \{b, d\}\}$,
$Y_2 := X_i \setminus \{\{a, b\}, \{c, d\}\} \cup \{\{a, d\}, \{b, c\}\}$,
$Y_3 := X'_i \setminus \{\{a, c\}, \{b, e\}\} \cup \{\{a, b\}, \{c, e\}\}$,
$Y_4 := X'_i \setminus \{\{a, c\}, \{b, e\}\} \cup \{\{a, e\}, \{b, c\}\}$.

Step 3:

3.1. If $Y_1 \cap F = \emptyset$, then $X_{i+1} := Y_1$, $X'_{i+1} := X'_i$ and go to Step 4.

3.2. If $Y_3 \cap F = \emptyset$, then $X_{i+1} := X_i$, $X'_{i+1} := Y_3$ and go to Step 4.

3.3. If $\min\{n(Y_2), n(Y_4)\} \leq \ell$, then $X_{i+1} := Y_2$, $X'_{i+1} := Y_4$ and go to Step 4.

3.4. If $n(Y_2) = n(Y_4) = \ell + 1$, then

if a and e appear in different trails of X, then $X'_{i+2} := X'_{i+1} := Y_4$, $X_{i+1} := Y_2$, $X_{i+2} := join(Y_2, a, e)$ $i := i + 1$ and go to Step 4,

otherwise $X_{i+1} := Y_2$, $X_{i+2} := join(Y_2, b, e)$, $X'_{i+2} := X'_{i+1} := X'_i$, $i := i + 1$ and go to Step 4.

Step 4:
If $n(X_{i+1}) = \ell + 1$, then choose edges g, h such that $\{g, h\} \in X'_{i+1}$ and g, h appear in different trails of X_{i+1}. Set $X_{i+2} := join(X_{i+1}, g, h)$, $X'_{i+2} := X'_{i+1}$, $i := i + 1$.

If $n(X'_{i+1}) = \ell + 1$, then choose edges g, h such that $\{g, h\} \in X_{i+1}$ and g, h appear in different trails of X'_{i+1}. Set $X'_{i+2} := join(X'_{i+1}, g, h)$, $X_{i+2} := X_{i+1}$, $i := i + 1$.

Step 5:
If $X_{i+1} = X'_{i+1}$ then STOP, otherwise $i := i + 1$ and go to Step 2.

4 Proof of Theorem 1

We prove that the transforming sequence constructed by the above algorithm satisfies the required properties 1. − 4. Because of the lack of space, let us only stress some most important parts of the proof: Note that at least one of Y_1 and Y_2 (and similarly) also at least one of Y_3 and Y_4) is F-compatible. All X_i's and X_i''s are F-compatible, but it might happen that $n(X_{i+1}) = \ell + 1$ or $n(X'_{i+1}) = \ell + 1$. Still, after Step 3, the current pair X_{i+1} and X'_{i+1} satisfies $\min\{n(X_{i+1}), n(X'_{i+1})\} \leq \ell$ and $|X_{i+1} \cap X'_{i+1}|$ is increased. After Step 4, the current pair X_{i+1} and X'_{i+1} satisfies $n(X_{i+1}), n(X'_{i+1}) \leq \ell$ and $|X_{i+1} \cap X'_{i+1}|$ is again increased or remains the same. Thus, $X = X_0, X_1, \ldots X_k = X'_k \ldots, X'_0 = X'$ (after possibly deleting repetitive consecutive members of the sequence) satisfies conditions 1. − 4. required by Theorem 1.

As to the time complexity of the algorithm, the number of repetitions of Steps 2 to 4 is at most $|X \setminus X'| \leq m$, where m is the number of edges of G. Furthermore (by brute force counting) it is quite clear that each Step 1 to 5 can be done in time $O(m)$. Thus, the time complexity of our algorithm is $O(m^2)$ in the worst case.

5 Applications to eulerian trails

As F-compatible eulerian trails are special cases of F-compatible decompositions, Theorem 1 can be applied also to eulerian trails. In particular, as its consequence for $k = \ell = 1$, we obtain results of Fleischner et al. [7] and Kotzig [11]. Our corollaries are even stronger than the original results which proved only existence of the transforming sequence, we give a polynomial algorithm which constructs it.

Corollary 1 *Let F be a forbidden partition system in a graph G and let $T, T' \in \mathcal{S}_1(G, F)$. Then there is a polynomial algorithm which constructs a sequence $X_0 = X_T, X_1, \ldots, X_n = X_{T'}$ such that for every $i = 0, \ldots, n$:*

1. *X_i defines F-compatible decomposition of G into at most 2 trails*
2. *$|X_i \setminus X_{i+1}| = 2$,*
3. *if $n(X_i) = 2$ then $n(X_{i+1}) = 1$.*

Corollary 2 *Let $T, T' \in \mathcal{S}_1(G)$. Then there is a polynomial algorithm which constructs a sequence $T_0 = T, T_1, \ldots, T_n = T'$ such that for every $i = 1, \ldots, n$, T_i arises from T_{i-1} by a segment reversal.*

Corollary 1 is an immediate consequence of Theorem 1 for $n(X) = n(X') = 1$. For the proof of Corollary 2 one has to prove a little more, namely that a split and a join (performed consecutively) can be replaced by a finite sequence of segment reversals (which is easy to see).

Theorem 1 can be also applied to digraphs (note that we cannot use segment reversals):

Definition 2 *Let X, X' be two transition systems of a digraph D such that $|X \setminus X'| = 2$.*

1. *If $n(X') = n(X) + 1$, then we say that X' arises from X be a* **split**.
2. *If $n(X') = n(X) - 1$, then we say that X' arises from X be a* **join**.

Again, whenever X and X' differ in only two transitions, then these are transitions through the same vertex and furthermore $n(X') = n(X) \pm 1$. Hence, for digraphs, $|X \setminus X'| = 2$, if and only if X' arises from X by a split or by a join.

Let D be a digraph and G its underlying graph. The orientation of D defines a natural forbidden transition system F in G ($F(v)$ consists of two equivalence classes corresponding to $A^+(v)$ and $A^-(v)$). Clearly there is a 1-1 correspondence between elements of $\mathcal{S}_k(G, F)$ and $\mathcal{S}_k(D)$. Thus, if we apply Theorem 1 and observe that segment reversals can be never used we obtain the following corollary. The statement for $n(X) = n(X') = 1$ was also observed in [17, 7].

Corollary 3 *For any transition system X, X' in a digraph D there is a polynomial algorithm which constructs a sequence $X = X_0, X_1, \ldots, X_n = X'$ such that for every i:*

1. *X_i transition system of D,*
2. *$n(X_i) \leq \max\{n(X), n(X')\} + 1$,*
3. *$|X_i \setminus X_{i+1}| = 2$,*
4. *if $n(X_i) = \max\{n(X), n(X')\} + 1$, then $n(X_{i+1}) = n(X_i) - 1$.*

6 Complexity remarks

In this section we consider the complexity of enumeration problems $\sharp$GCD (resp. $\sharp$DCD, $\sharp$GCT, $\sharp$DCT) for eulerian graphs and digraphs. First let us consider the seemingly simplest case when there are no forbidden transitions.

As there is a 1-1 correspondence between elements of $\mathcal{S}(G)$ and transition systems in G (and the same holds also for digraphs) and $|X(v)|$ equals to the number of all perfect matching of a complete graph on $\deg_G(v)$ vertices (which is $(\deg_G(v) - 1)!!$ if $\deg_G(v)$ is even) we have

$$|\mathcal{S}(G)| = \prod_{v \in V(G)} (\deg_G(v) - 1)!!.$$

Similarly, by counting the number of perfect matchings in a complete bipartite graph, one gets

$$|\mathcal{S}(D)| = \prod_{v \in V(D)} (\deg_D^+(v))!.$$

This should be compared with the difficulty of determination of $|S_1(G)|$. It is an open (and well known) problem whether this problem is a $\sharp P$-complete or not (see [14]). It is known that for graphs with maximum degree at most six, there is a fully polynomial randomized approximation scheme (FPRAS) and a fully polynomial almost uniform sampling (FPAUS) [20]. Perhaps some of our results will lead to a better understanding of this. The number $|\mathcal{S}_1(D)|$ (of eulerian trails in a digraph D) is of course counted by the classical BEST-formula [4, 19].

For general forbidden systems of transitions we have:

Theorem 2 *Problem $\sharp$GCD (for a general forbidden system) is $\sharp P$-complete.*

Proof: In fact we prove even more. We prove that $\sharp$GCD is equivalent to the problem of counting of all perfect matchings in a graph (which is known to be $\sharp P$-complete [21]).

Given a graph $H = (V, E)$, we define G by

$$V(G) = V \times \{+1, -1\} \cup \{0\},$$
$$E(G) = \{\{0, (v, i)\} \mid i = \pm 1, v \in V\} \cup \{\{(v, +1), (v, -1)\} \mid v \in V\}.$$

Since all vertices of G except 0 are of degree 2, the only forbidden transitions could be through vertex 0. We define $F = F(0)$ as follows:

$$\{\{(v, +1), 0\}, \{(v', -1), 0\}\} \in F(0) \ \forall v, v' \in V(H)$$
$$\{\{(v, +1), 0\}, \{(v', +1), 0\}\} \in F(0) \Leftrightarrow \{v, v'\} \notin E(H)$$
$$\{\{(v, -1), 0\}, \{(v', -1), 0\}\} \in F(0) \Leftrightarrow \{v, v'\} \notin E(H)$$

It is easy to check that $|\{M \mid M \text{ is a perfect matching in } H\}|^2 = |\mathcal{S}(G, F)|$ (transition system of each element of $|\mathcal{S}(G, F)|$ corresponds to an ordered pair of two perfect matchings in H, one on $(v, +1)$- and one on $(v, -1)$-vertices).

Conversely, for a given (G, F), we define a graph H such that the number of perfect matchings in H equals to the number of F-compatible decompositions of (G, F): First, for every vertex $v \in V(G)$ we define a graph H_v on the neighbors of v (in G) with edges corresponding to admissible transitions through v in G. More formally,

$$V(H_v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$$
$$E(H_v) = \{\{u, u'\} \mid \{\{u, v\}, \{u', v\}\} \notin F(v)\}$$

We put H to be the disjoint union of H_v, $v \in V(G)$. It is clear that $|\{M \mid M$ is a perfect matching in $H\}| = |\mathcal{S}(G, F)|$. □

Similarly, one can also prove that $\sharp$DCD is a $\sharp P$-complete problem. The question whether problems $\sharp$GCD and $\sharp$DCD have FPRAS is clearly related to the FPRAS for counting perfect matchings and thus presently open. We have seen that the exact evaluation problems as $\sharp$GCD and $\sharp$DCD are hard while the decision problems GCD and DCD are easy (since the question whether a graph has a perfect matching can be answered in polynomial time). For the decision problems GCT and DCT, the situation is harder. As we will see below, to decide whether a graph G has an F-compatible eulerian trail is an NP-complete problem (and the same holds also for digraphs). Moreover, from the reduction will follow that the corresponding evaluation problems $\sharp$GCT and $\sharp$DCT are $\sharp P$-complete. These seem to be the first NP-completeness and $\sharp P$-completeness results related to the compatible eulerian trails.

However, if we restrict the input to (G, F), F is a forbidden partition system, then testing whether $\mathcal{S}_1(G, F) \neq \emptyset$ can be done in polynomial time (for undirected graphs). This follows immediately from the fact that $\mathcal{S}_1(G, F) \neq \emptyset \Leftrightarrow \mathcal{S}(G, F) \neq \emptyset$, if F is a partition system [11]. Moreover, assuming $\mathcal{S}(G, F) \neq \emptyset$, one can find an F-compatible eulerian trail by a polynomial algorithm (see [3]).

Theorem 3 *Problem* DCT *(i.e. testing $\mathcal{S}_1(D, F) \neq \emptyset$ for general forbidden system F in a digraph D) is NP-complete.*

Proof: Let D be a given digraph, we construct a digraph D'' and a forbidden system F in D'' such that D has a hamiltonian cycle if an only if (D'', F) has an F-compatible eulerian trail. This finishes the proof as testing hamiltonicity is an NP-complete problem, see [10].

First of all we define a digraph D' by $V(D') = \{v^1, v^2 \mid v \in V(D)\}$ and $A(D') = A^1(D') \cup A^2(D')$, where $A^1(D') = \{(v^1, v^2) \mid v \in V(D)\}$ and $A^2(D') = \{(u^2, v^1) \mid (u, v) \in A(D)\}$. Further, let D'' arises from D' by contracting all arcs in $A^2(D')$.

Note that $|A(D'')| = |V(D)|$ (arcs in $A^1(D')$ correspond to vertices of D) and if $(u, v) \in A(D)$ then $u^2 = v^1$ in D''. Hence if D is hamiltonian, then D'' is eulerian. But not conversely. For the other implication we need to define a forbidden system F in D''. We set $F = \{\{(u^1, u^2), (v^1, v^2)\} \mid u^2 = v^1 \text{ in } D'' \text{ and } (u, v) \notin A(D)\}$. It follows from the choice of F that (D'', F) has an F-compatible eulerian trail if and only if D has a hamiltonian cycle. It may happen (in fact it mostly happens) that D'' is a directed multigraph. If we insist on D'' being a simple digraph, we subdivide each arc of D'' with a vertex of degree two. $\square$

Corollary 4 *Problem* GCT *(for general forbidden system F) is NP-complete.*

Proof: We will show that DCT can be reduced to our problem. This together with the previous Theorem 3 will finish the proof.

Suppose that (D, F) (a digraph with an arbitrary forbidden system F) is given. We will construct an undirected graph G and a forbidden system F' in G such that $\mathcal{S}_1(D, F) \neq \emptyset \Leftrightarrow \mathcal{S}_1(G, F') \neq \emptyset$.

We set G to be the corresponding underlying graph of D and let $E^+(v)$ (resp. $E^-(v)$) denote the edges of G corresponding to $A^+(v)$ (resp. $A^-(v)$). If $e \in E(G)$, then the corresponding arc in D is denoted by $\overrightarrow{e}$. We set:

$$F_1(v) = \{\{e, f\} \mid \{e, f\} \subseteq E^-(v) \text{ or } \{e, f\} \subseteq E^+(v)\},$$
$$F_2(v) = \{\{e, f\} \mid \{\overrightarrow{e}, \overrightarrow{f}\} \in F(v)\},$$
$$F' = F_1 \cup F_2.$$

It is easy to see that there is a 1-1 correspondence between F-compatible eulerian trails in D and F'-compatible eulerian trails in G. Indeed, if T is an

F-compatible eulerian trail in D, then it is also an F'-compatible eulerian trail in G (since $F_2 \subseteq F'$). Conversely, if T is an F'-compatible eulerian trail in G, then the choice of F_1 guarantees that it is also an eulerian trail in D and the choice of F_2 that it is F-compatible. Hence, $\mathcal{S}_1(D, F) \neq \emptyset \Leftrightarrow \mathcal{S}_1(G, F') \neq \emptyset$. $\square$

It follows immediately from the proof of Theorem 3, that there is a 1-1 correspondence between hamiltonian circuits in the digraph D and F-compatible eulerian trails in D''. Thus the evaluation problem $\sharp$DCT (and by the same argumentation also $\sharp$GCT) can be reduced to counting the number of hamiltonian circuits in a digraph (which is obviously equivalent to counting the number of hamiltonian circuits in an undirected graph, and thus a $\sharp P$-complete problem).

Corollary 5 $\sharp$DCT *and* $\sharp$GCT *(for general forbidden systems F) are $\sharp P$-complete problems.*

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