Compatible decompositions of eulerian graphs and their transformations *

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Abstract

In this paper some earlier defined local transformations between eulerian trails are generalized to transformations between decompositions of graphs into (possibly more) closed subtrails. For any graph G with a forbidden partition system F, we prove that it is possible to transform any F-compatible decomposition of G into closed subtrails to one another, and at the same time to preserve F-compatibility and not increase the number of subtrails by more than one. From this, several earlier results for eulerian trails easily follow. These results are embedded into the rich spectrum of results of theory of eulerian graphs and their applications. We further apply this statement to digraphs and investigate other possibilities of transforming F-compatible decompositions of digraphs to one another.

1 Introduction

A closed trail in a graph G is a sequence $v_0, e_1, v_1, \ldots, v_{t-1}, e_t, v_t$ of its vertices and edges such that $v_t = v_0$, for every i, e_i is incident with v_{i-1}, v_i, and all edges $e_1, \ldots, e_t$ are distinct. A closed trail which contains all edges of G is called *eulerian trail.* Eulerian trails belong to the very classical combinatorial structures, yet they play a prominent role in many modern branches of graph theory, such as network flows, graph polynomials, coloring problems, nowhere-zero flows, cycle double cover conjecture, to name just a few (see [10] and [14] for many more examples). Eulerian trails provide also some background structure for problems in areas as diverse computational biology [24, 23], knot theory [4, 31] and isotropic systems [6]. Although the basic questions related to eulerian trails (such as existence, number and various

*The research was supported by project LN00A056 of the Ministry of Education of the Czech Republic

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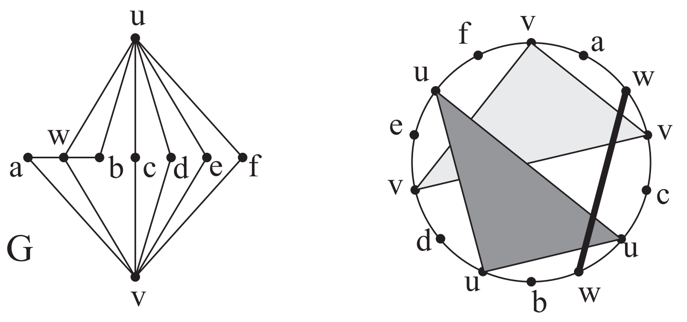

Figure 1: Polygonal diagrams

optimization problems) are well-known and were clarified long time ago, the new applications produce specific problems which are interesting (and sometimes hard). One such area of problems is the study of eulerian trails which satisfy special local conditions such as forbidding special consecutive pairs of edges. This study was initiated by A. Kotzig [16, 1] and continued by G. Sabidussi who related it to the Cycle double cover conjecture and other well-known problems, see [10, 26].

In this paper we address this problem in the full generality and consider the set $\mathcal{S}(G, F)$ of all decompositions of G into closed trails with respect to a given set F of forbidden transitions at every vertex (see Section 2 for definitions). A bit surprisingly, we show on this level of generality that this set forms a single orbit with respect to one simple operation which we call MSJ-transformation (see Theorem 2). For special sets F (such as union of equivalences at each vertex) we can prove more: our operations can be modified to binary operations (splits, joins and segment reversals, see Definition 1) and they preserve not only compatibility with the forbidden equivalences F but also the number of trails in the decomposition (Theorem 1).

Our main result (Theorem 1) implies and gives short proofs of several results obtained earlier (see Sections 4 and 5): Kotzig's theorem on transformations between eulerian trails [1, 16] and theorem of Fleischner et al. concerning transformations between F-compatible eulerian trails [11].

In Section 5 we also discuss corresponding results for directed graphs. As expected, the situation in more complex and results do not hold in full generality. Yet, our techniques yield a simple proof of Pevzner's result concerning transformations of eulerian trails in digraphs [24].

We conclude the paper by remarks on algorithmic complexity of general compatible eulerian trail problem and by relationship of compatible decompositions to the Cycle double cover conjecture [27].

Our approach has been partly inspired by chord-diagrams which play an important role in knot theory (see for example [2, 22, 19] for papers related to

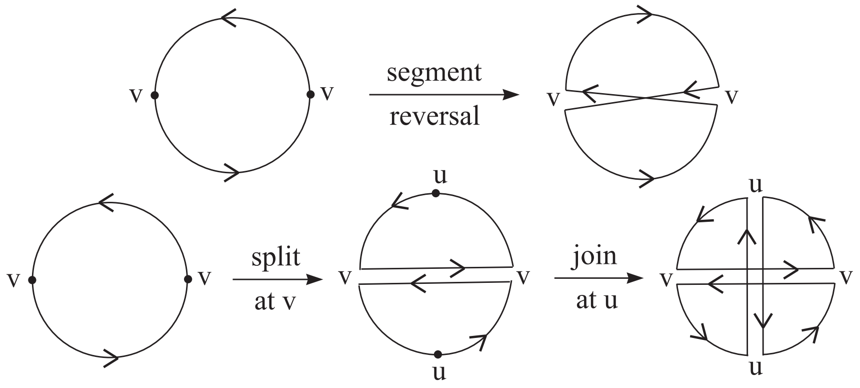

Figure 2: Diagrams of transformations

Vassiliev invariants). Chord diagrams are defined as a circle with additional set of disjoint chords (i.e. a matching). For a given knot, the chords correspond to the crossing points (in the plane projection) and thus to the double occurrence of a vertex in the eulerian trail. The graphs in knot theory are 4-regular planar graphs (or 2-in 2-out planar digraphs).

This can be naturally generalized to *diagrams* of an arbitrary eulerian trail $v_0, e_1, \ldots, e_m, v_m = v_0$ in graph $G = (V, E)$ as follows: We consider a cycle of length m with vertices $1, \ldots m$ and with extra added disjoint complete graphs $\{K_v\}_{v \in V}$. These complete graphs correspond to vertices of G; $V(K_v) = \{v_i \mid v_i = v\}$.

The complete graph K_v is specified by the polygon inscribed in the cycle (these polygons may be degenerated), see Fig. 1. For distinct vertices $u \neq v$ in G, the polygons K_u, K_v have disjoint vertices. The diagrams of eulerian trails could be also called *polygonal diagrams*.

Some questions related to diagrams of eulerian trails are much harder than for chord diagrams (i.e. diagrams of 4-regular planar graphs). One such problem is e.g. the Gauss problem which is already open for graphs with maximum degree at most six (see [25, 12, 13, 8] for the solution for 4-regular graphs). Still the diagram is a useful tool which motivated some of our constructions below. For example, the three main transformations may be schematicly depicted as follows (we skip the detailed description):

2 Basic notions and related work

We use standard graph theory notation, see e.g. [20]. $G = (V, E)$ will always denote an (undirected) graph with the vertex-set V and the edge-set E.

$E(v)$ denotes the set of edges incident with v. Any two-element subset of

$E(v)$ is called a *transition* through v. $\mathcal{T}(v)$ denotes the set of all transitions through v and $\mathcal{T}(G) = \bigcup_{v\in V} \mathcal{T}(v)$ the set of all transitions of G.

If $X(v)$ is a partition of $E(v)$ into two-element classes (i.e. each class of $X(v)$ is a transition through v), then $X(v)$ is called a *transition system through* v and $X = \bigcup_{v\in V} X(v)$ is called a *transition system of* G.

$\mathcal{S}_k(G)$ denotes the set of all decompositions of G into exactly k (edge-disjoint) closed trails. In particular, $\mathcal{S}_1(G)$ denotes the set of all (closed) eulerian trails of G. We set $\mathcal{S}(G) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(G)$. Let us remark that we do not consider cyclic rotations of the sequence of edges in a closed trail T as a trail different from T. Also a closed trail and its complete reversal are treated as the same trail.

Let $S = \{S_1, \ldots, S_k\} \in \mathcal{S}_k(G)$, X_{S_i}, is the set of transitions which correspond to a subsequence of S_i. Then $X_S = \bigcup_{i=1}^{k} X_{S_i}$ is a transition system of G. And conversely, every transition system X of G defines a (unique) $S \in \mathcal{S}(G)$, i.e. there is a unique S such that $X = X_S$. If there is no danger of confusion, we will not distinguish between the transition system and the decomposition into closed trails defined by it.

It follows that it is easy to evaluate the size of $\mathcal{S}(G)$:

$$|\mathcal{S}(G)| = \prod_{v\in V} (\deg_G(v) - 1)!!$$

(as $(n-1)!! = (n-1)(n-3)\ldots$ is the number of all perfect matchings of the complete graph on n vertices, when n is even). This should be compared with the difficulty of determination of $S_1(G)$. It is an open (and well known) problem (see e.g. [21]) whether the computation of $|\mathcal{S}_1(G)|$ is a $\sharp P$-complete problem. It is known that for graphs with degree at most six, there is a fully polynomial randomized approximation scheme (FPRAS) and a fully polynomial almost uniform sampling (FPAUS)(see [29]). Perhaps some of our results will lead to a better understanding of this.

Let us stress that this correspondence is not between eulerian trails and transitions systems. Of course, any decomposition of the edge set of a connected graph into closed trails may be combined to a single eulerian trail, however at the expense that the transition system is changed. We are interested here in those operations which do not change transition system too much. This is captured by the notion of compatibility. Its origin lies in questions first raised by A. Kotzig for eulerian trails and by G. Sabidussi for cycle decompositions [10].

Let G be a graph and $F \subseteq \mathcal{T}(G)$ any subset of its transitions. $F = \bigcup_{v\in V} F(v)$, where $F(v) \subseteq \mathcal{T}(v)$. Any transition t of G such that $t \in F$ is said to be *forbidden*, any transition t of G such that $t \notin F$ is said to be *admissible* (or also *compatible*). (G, F) will denote a graph with a given forbidden system F. If for every v, $F(v)$ is a transitive relation on $E(v)$ we say that $F = \bigcup_{v \in V} F(v)$ is a *forbidden partition system of* G. This is consistent with definitions given in [10, 11].

Another possibility how to view a forbidden system of transition is the following: To every $F(v)$ corresponds a graph $G_{F(v)}$ on the set of neighbors of v with edges corresponding to forbidden transitions through v. More formally, $V(G_{F(v)}) = \{u \in V(G) \mid \{u,v\} \in E(G)\}$, $E(G_{F(v)}) = \{\{u,u'\} \mid \{\{u,v\},\{u',v\}\} \in F(v)\}$. Note that if F is a forbidden partition system in G, then $G_{F(v)}$ is a disjoint union of complete graphs (for every v).

Let (G,F) be given and let $S \in \mathcal{S}_k(G)$, S is said to be *F-compatible* (or simply *compatible*) if $X_S \cap F = \emptyset$. $\mathcal{S}_k(G,F)$ denotes the set of all F-compatible decompositions of G into (exactly) k closed trails. Again, $\mathcal{S}(G,F) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(G,F)$.

These definitions capture several important instances: $\mathcal{S}_1(G)$ is just the set of all eulerian trails in G and if F is a red-blue partition of $E(G)$, then $\mathcal{S}_1(G,F)$ denotes the set of all eulerian trails in G which are alternating in colors [24, 11].

The complexity of evaluation of $|\mathcal{S}(G,F)|$, for a general system F of forbidden transitions, is $\sharp P$-complete, since the problem of counting perfect matchings in a graph is easily reducible to it:

Given a graph $H = (V,E)$, let G be a "friendship" graph with the vertex-set $V(G) = V \times \{+1,-1\} \cup \{0\}$ and the edge-set $E(G) = \{\{0,(v,i)\} \mid i = \pm 1, v \in V\} \cup \{\{(v,+1),(v,-1)\} \mid v \in V\}$. Since all vertices of G except 0 are of degree 2, the only forbidden transitions could be through vertex 0. We define $F = F(0)$ as follows:

$$\{\{(v,+1),0\},\{v',-1),0\}\} \in F(0)\ \forall v,v' \in V(H)$$

$$\{\{(v,+1),0\},\{v',+1),0\}\} \in F(0) \Leftrightarrow \{v,v'\} \notin E(H)$$

$$\{\{(v,-1),0\},\{v',-1),0\}\} \in F(0) \Leftrightarrow \{v,v'\} \notin E(H)$$

It is easy to check that $|\{M \mid M \text{ is a perfect matching in } H\}|^2 = |\mathcal{S}(G,F)|$ (transition system of each element of $|\mathcal{S}(G,F)|$ corresponds to an ordered pair of two perfect matchings in H, one on $(v,+1)$- and one on $(v,-1)$-vertices).

Evaluation of $|\mathcal{S}(G,F)|$ is in fact equivalent to the problem of counting perfect matchings in a graph:

For a given (G,F), we define a graph H such that the number of perfect matchings in H equals to the number of F-compatible decompositions of (G,F). First, for every vertex $v \in V(G)$ we define a graph H_v on the neighbors of v (in G) with edges corresponding to admissible transitions through v in G. More formally,

$$V(H_v) = \{u \in V(G) \mid \{u,v\} \in E(G)\}$$

$$E(H_v) = \{\{u,u'\} \mid \{\{u,v\},\{u',v\}\} \notin F(v)\}$$

Let H be the disjoint union of all H_v, $v \in V(G)$. It is clear that $|\{M \mid M$ is a perfect matching in $H\}| = |\mathcal{S}(G,F)|$.

The question whether $|\mathcal{S}(G,F)|$ has FPRAS is clearly related to the FPRAS for counting perfect matchings and thus presently open.

All the above defined notions can be modified also for directed graphs. This is discussed in Section 5.

3 Compatible decompositions into closed trails

Let (G,F) be an eulerian graph with a given forbidden system F. We first answer the question when $\mathcal{S}(G,F) \neq \emptyset$.

Lemma 1 *Let (G,F) be an eulerian graph with an arbitrary forbidden system F. $\mathcal{S}(G,F) \neq \emptyset$ if and only if for every v there is a transition system $X(v)$ through v such that $X(v) \subseteq \mathcal{T}(v) \setminus F(v)$.*

Proof: If there is some $S \in \mathcal{S}(G,F)$ then X_S satisfies $X_S(v) \subseteq \mathcal{T}(v) \setminus F(v)$ for every v. Conversely, if for every v there is some $X(v)$ such that $X(v) \subseteq \mathcal{T}(v) \setminus F(v)$ then $X = \bigcup_{v \in V} X(v)$ is a transition system of G which defines $S \in \mathcal{S}(G,F)$. $\square$

If F is a forbidden partition system of G, then Lemma 1 can be reformulated in the following way:

Lemma 2 *Let (G,F) be an eulerian graph with a forbidden partition system F. $\mathcal{S}(G,F) \neq \emptyset$ if and only if for every v and every equivalence class C of $F(v)$, $|C| \leq 1/2\deg(v)$.*

Our aim now is to define some local transformations between elements of $\mathcal{S}(G,F)$ which preserve F-compatibility. For eulerian trails $\mathcal{S}_1(G)$ this was already done by A. Kotzig (see [1, 16]), for F-compatible eulerian trails $\mathcal{S}_1(G,F)$ by H. Fleischner et al. (see [11]). As we will see, our generalization to decompositions into more closed trails is very natural. Moreover, this approach enables us to prove Theorem 1 from which results of [1, 11, 16] follow as easy corollaries (see Section 4).

Definition 1 *Let $S = \{S_1, \dots, S_k\} \in \mathcal{S}_k(G)$*

1. *Let $S_k = v, x_1, \dots, x_p, v, y_1, \dots y_r, v$. We say that $S' = \{S'_1, \dots, S'_k\} \in \mathcal{S}_k(G)$ arises from S by a v-**segment reversal** if $S_i = S'_i$, for $i = 1, \dots k-1$ and $S'_k = v, x_p, \dots, x_1, v, y_1, \dots y_r, v$. (the two occurrences of v on S_k need not to be consecutive).*

2. *Let $S_k = v, x_1, \dots, x_p, v, y_1, \dots y_r, v$. We say that $S' = \{S'_1, \dots, S'_{k+1}\} \in \mathcal{S}_{k+1}(G)$ arises from S by a* **split** *at v if $S_i = S'_i$, for $i = 1, \dots k-1$, $S'_k = v, x_1, \dots, x_p, v$ and $S'_{k+1} = v, y_1, \dots, y_r, v$.*

3. *Let $S_{k-1} = v, x_1, \dots, x_p, v$ and $S_k = v, y_1, \dots, y_r, v$. We say that $S' = \{S'_1, \dots, S'_{k-1}\} \in \mathcal{S}_{k-1}(G)$ arises from S by a* **join** *at v if $S_i = S'_i$, for $i = 1, \dots k-2$ and $S'_{k-1} = v, x_1, \dots, x_p, v, y_1, \dots y_r, v$.*

Remark Of course, the sequence order of S_i's in S is irrelevant. Thus, to be strict correct, we should define the transformation more carefully: S' arises from S by a segment reversal if there is a permutation π on $\{1, \dots k\}$ and $i_0 \in \{1, \dots k\}$ such that $S_i = S'_{\pi(i)}$ for all $i \neq i_0$, $S_{i_0} = v, x_1, \dots, x_p, v, y_1, \dots y_r, v$ and $S'_{\pi(i_0)} = v, x_p, \dots, x_1, v, y_1, \dots y_r, v$. Definitions of split and join should be also changed in similar way. Nevertheless, we prefer to define the transformations in the simpler way as above. Compare the definitions also with the schematic figures in Section 1.

The next lemma follows immediately from the definitions.

Lemma 3 *Let $S, S' \in \mathcal{S}(G)$. S' can be obtained from S by a segment reversal if and only if S can be obtained from S' by a segment reversal. S' can be obtained from S by a split if and only if S can be obtained from S' by a join.*

Lemma 4 *Let X, Y be two transition systems of G such that $X \setminus Y = \{t_1, t_2\}$ for some $t_1, t_2 \in X$. $X = X_S$ for $S \in \mathcal{S}_k(G)$. Then one of the following condition holds:*

1. *$Y = X_{S'}$ for some $S' \in \mathcal{S}_k(G)$ and S' can be obtained from S by a segment reversal.*

2. *$Y = X_{S'}$ for some $S' \in \mathcal{S}_{k+1}(G)$ and S' can be obtained from S by a split.*

3. *$Y = X_{S'}$ for some $S' \in \mathcal{S}_{k-1}(G)$ and S' can be obtained from S by a join.*

Proof: Clearly, t_1, t_2 are transitions through the same vertex, $t_1, t_2 \in X(v)$. We first consider the case when $t_1 = \{e, e'\}, t_2 = \{f, f'\}$ occur in the same closed trail of S, without loss of generality in order $\dots, e, e', \dots, f, f', \dots$. Then either $Y = X_S \setminus \{t_1, t_2\} \cup \{\{e, f\}, \{e', f'\}\}$ and S' arises from S by a segment reversal, or $Y = X_S \setminus \{t_1, t_2\} \cup \{\{e, f'\}, \{e', f\}\}$ and S' arises from S by a split.

If $t_1 = \{e, e'\}, t_2 = \{f, f'\}$ occur in different elements of S, $\{e, e'\} \in X_{S_i}$ and $\{f, f'\} \in X_{S_j}$, $i \neq j$. Then again either $Y = X_S \setminus \{t_1, t_2\} \cup \{\{e, f\}, \{e', f'\}\}$, or $Y = X_S \setminus \{t_1, t_2\} \cup \{\{e, f'\}, \{e', f\}\}$. In both cases S' arises from S by a join. $\square$

It follows that we can give an equivalent definition of our operations:

Let $S \in \mathcal{S}_k(G)$ and $S' \in \mathcal{S}_\ell(G)$ such that $|X_S \setminus X_{S'}| = 2$, that is X_S and $X_{S'}$ differ in only two transitions.

1. *If $k = \ell$, then S' arises from S be a* **segment reversal**.
2. *If $\ell = k+1$, then S' arises from S be a* **split**.
3. *If $\ell = k-1$, then S' arises from S be a* **join**.

Lemma 5 *Let $S = \{S_1, \ldots, S_k\} \in \mathcal{S}_k(G,F)$ where F is a forbidden partition system of G. Let there be two transitions through a vertex v such that $t_1 = \{e, e'\} \in X_{S_1}$ and $t_2 = \{f, f'\} \in X_{S_2}$. Suppose further that transition $\{e, f\} \notin F$. Then there is $S' \in \mathcal{S}_{k-1}(G,F)$ such that S' can be obtained from S by a join at v and $X_S \setminus X_{S'} = \{t_1, t_2\}$.*

Proof: Essentially, two different joins are possible:

$$X_{S'} = X_S \setminus \{\{e,e'\},\{f,f'\}\} \cup \{\{e,f\},\{e',f'\}\} \text{ or}$$

$$X_{S'} = X_S \setminus \{\{e,e'\},\{f,f'\}\} \cup \{\{e,f'\},\{e',f\}\}.$$

Thus, in order to prove the lemma, we have to show that either $\{e,f\},\{e',f'\}$ are both F-compatible or $\{e,f'\},\{f,e'\}$ are both F-compatible. If $\{e',f'\} \notin F$, then we are done since $\{e,f\} \notin F$ by assumptions. So we may suppose that $\{e',f'\} \in F$. If $\{e,f'\} \in F$ then also $\{e,e'\} \in F$ (since F is a partition system), a contradiction. If $\{e',f\} \in F$ then also $\{f,f'\} \in F$, a contradiction. $\square$

Lemma 6 *Let F be a forbidden partition system in a graph G and let $S = \{S_1, \ldots, S_k\} \in \mathcal{S}_k(G,F)$, $S' = \{S'_1, \ldots, S'_\ell\} \in \mathcal{S}_\ell(G,F)$, $k > \ell$. Then there exists $S^* \in \mathcal{S}_{k-1}(G,F)$ such that S^* arises from S by a join and $|X_{S^*} \cap X_{S'}| \geq |X_S \cap X_{S'}|$.*

Proof: Since $k > \ell$ there is a vertex v end edges e, f incident with v such that $\{e,f\} \in X_{S'}$ and $e \in E(S_i)$, $f \in E(S_j)$, $i \neq j$. Hence there are edges e', f' incident with v such that $\{e,e'\} \in X_{S_i}$ and $\{f,f'\} \in X_{S_j}$. We apply Lemma 5 and obtain some $S^* \in \mathcal{S}_{k-1}(G,F)$ such that $X_S \setminus X_{S^*} = \{\{e,e'\},\{f,f'\}\}$. Since $\{e,e'\},\{f,f'\} \notin X_{S'}$, it is clear that $|X_{S^*} \cap X_{S'}| \geq |X_S \cap X_{S'}|$. $\square$

Now we are ready to state and prove our main result.

Theorem 1 *Let F be a forbidden partition system in a graph G and let $S \in \mathcal{S}_k(G,F)$ and $S' \in \mathcal{S}_\ell(G,F)$, $k \le \ell$. Then there is a sequence $S_0 = S, S_1, \dots, S_n = S'$ such that*

1. *for every $i = 0, \dots, n$, $S_i \in \mathcal{S}_j(G,F)$ and $j \le \ell+1$,*
2. *for every $i = 1, \dots, n$, $|X_{S_i} \setminus X_{S_{i-1}}| = 2$ (i.e. S_i arises from S_{i-1} by a segment reversal, by a split or by a join),*
3. *if for some i, $S_i \in \mathcal{S}_{\ell+1}(G,F)$ then $S_{i+1} \in \mathcal{S}_\ell(G,F)$*

Proof: Suppose for contradiction, there are some $S, S' \in \mathcal{S}(G)$ for which there is no sequence with the prescribed properties. We choose S, S' such that $|X_S \cap X_{S'}|$ is maximal possible.

Since $S \ne S'$ there must be a vertex v such that $X_S(v) \ne X_{S'}(v)$. It means that there are edges a, b, c, d, e (possibly $d = e$) incident with v such that $\{a,b\}, \{c,d\} \in X_S \setminus X_{S'}$ and $\{a,c\}, \{b,e\} \in X_{S'} \setminus X_S$.

We follow a similar strategy as in [11]. We consider four new transition systems:
$X_1 := X_S \setminus \{\{a,b\},\{c,d\}\} \cup \{\{a,c\},\{b,d\}\}$,
$X_2 := X_S \setminus \{\{a,b\},\{c,d\}\} \cup \{\{a,d\},\{b,c\}\}$,
$X_3 := X_{S'} \setminus \{\{a,c\},\{b,e\}\} \cup \{\{a,b\},\{c,e\}\}$,
$X_4 := X_{S'} \setminus \{\{a,c\},\{b,e\}\} \cup \{\{a,e\},\{b,c\}\}$.

At least one of X_1 and X_2 is F-compatible: Suppose X_1 is not, then $\{b,d\} \in F$. If X_2 is not F-compatible then $\{a,d\}$ or $\{b,c\}$ must be F-forbidden. If $\{a,d\} \in F$ then also $\{a,b\} \in F$ (since F is a partition system), a contradiction. If $\{b,c\} \in F$ then also $\{c,d\} \in F$, a contradiction. Similarly, also at least one of X_3 and X_4 is F-compatible.

In the following we consider several cases distinguished by F-compatibility of X_i's. In each case we find a new pair of transition systems (both in $\bigcup_{j\le\ell} \mathcal{S}_j(G,F)$) such that their intersection is strictly greater than $X_S \cap X_{S'}$ and the former can be obtained from X_S by a sequence with the prescribed properties while the latter can be obtained from $X_{S'}$ in such a way. This will give us a contradiction with the choice of S and S'. Note also that by Lemma 4, X_1, X_2 (resp. X_3, X_4) arose from S (resp. S') by a segment reversal, by a split or by a join and thus $X_1, X_2 \in \mathcal{S}_j(G)$, $j \in \{k-1, k, k+1\}$ and $X_3, X_4 \in \mathcal{S}_j(G)$, $j \in \{\ell-1, \ell, \ell+1\}$.

Case 1: X_1 is F-compatible. If $X_1 \in \mathcal{S}_j(G,F)$, $j \le \ell$, then the pair X_1, S' contradicts the choice of S, S'. Indeed, $|X_1 \cap X_{S'}| > |X_S \cap X_{S'}|$ since $\{a,c\} \in X_1 \cap X_{S'}$.

If $X_1 \in \mathcal{S}_{\ell+1}(G,F)$, then by Lemma 6 applied on X_1 and $X_{S'}$ there exists $S^* \in \mathcal{S}_\ell(G,F)$ such that S^* arises from X_1 by a join and $|X_{S^*} \cap X_{S'}| \ge |X_1 \cap X_{S'}| > |X_S \cap X_{S'}|$. Thus the pair S^*, S' gives a contradiction with the choice of S, S'.

Case 2: X_3 is F-compatible. This case can be handled analogously as Case 1.

Case 3: None of X_1, X_3 is F-compatible. Hence, $e \neq d$, $\{b,d\} \in F$, $\{c,e\} \in F$.

Subcase 3.1: $X_2 \in \mathcal{S}_i(G,F)$, $X_4 \in \mathcal{S}_j(G,F)$, both $i,j \leq \ell$.

Then $|X_2 \cap X_4| > |X_S \cap X_{S'}|$ (because $\{b,c\} \in X_2 \cap X_4$). Thus the pair X_2, X_4 gives a contradiction with the choice of S, S'.

Subcase 3.2: $X_2 \in \mathcal{S}_{\ell+1}(G,F)$, $X_4 \in \mathcal{S}_j(G,F)$, $j \leq \ell$ (or conversely).

Then by Lemma 6 applied on X_2 and X_4 there exists $S^* \in \mathcal{S}_\ell(G,F)$ such that S^* arises from X_2 by a join and $|X_{S^*} \cap X_4| \geq |X_2 \cap X_4| > |X_S \cap X_{S'}|$. Thus the pair S^*, X_4 gives a contradiction with the choice of S, S'.

Subcase 3.3: $X_2, X_4 \in \mathcal{S}_{\ell+1}(G,F)$.

Let $\bar{S} = \{\bar{S}_1, \ldots, \bar{S}_{\ell+1}\}$ be the decomposition defined by X_2 (i.e. $X_2 = X_{\bar{S}}$). And similarly let $\hat{S} = \{\hat{S}_1, \ldots \hat{S}_{\ell+1}\}$ be the decomposition defined by X_4 (i.e. $X_4 = X_{\hat{S}}$).

We may suppose that $\{a,d\} \in X_{\bar{S}_1}$ and $\{b,c\} \in X_{\bar{S}_2}$ (because X_2 arose from S by a split). Further there must be some $i \in \{1, \ldots \ell+1\}$ and $f \in E(v)$ such that $\{e,f\} \in X_{\bar{S}_i}$.

If $i \neq 1$, then we can perform a join at v (joining $\bar{S}_1$ in $\{a,d\}$ with $\bar{S}_i$ in $\{e,f\}$) obtaining $S^* \in \mathcal{S}_\ell(G,F)$. Since $\{a,e\} \in X_4$ (and thus $\{a,e\} \notin F$) this is possible due to Lemma 5. Then $|X_{S^*} \cap X_4| \geq |X_2 \cap X_4| > |X_S \cap X_{S'}|$. As $S^* \in \mathcal{S}_\ell(G,F)$ and $\hat{S} \in \mathcal{S}_{\ell+1}(G,F)$ (where $X_4 = X_{\hat{S}}$), there are edges g,h such that the transitions $\{g,h\} \in X_{S^*}$ (and thus $\{g,h\} \notin F$) while $g \in E(\hat{S}_p)$, $h \in E(\hat{S}_r)$. According to Lemma 5 we can join $\hat{S}_p$ and $\hat{S}_r$ obtaining $S^\sharp \in \mathcal{S}_\ell(G,F)$ with the property that $|X_{S^*} \cap X_{S^\sharp}| \geq |X_{S^*} \cap X_4| > |X_S \cap X_{S'}|$. Thus the pair $S^*, S^\sharp$ gives a contradiction with the choice of S, S'.

If $i = 1$, that is $\{a,d\}, \{e,f\} \in X_{\bar{S}_1}$, then we join $\bar{S}_1$ in $\{e,f\}$ with $\bar{S}_2$ in $\{b,c\}$ obtaining $S^* \in \mathcal{S}_\ell(G,F)$. Since $\{b,e\} \in X_{S'}$ (and thus $\{b,e\} \notin F$) this is possible due to Lemma 5. Hence $X_{S^*} = X_2 \setminus \{\{e,f\},\{b,c\}\} \cup \{\{b,e\},\{c,f\}\}$ (because $X_{S^*} = X_2 \setminus \{\{e,f\},\{b,c\}\} \cup \{\{b,f\},\{c,e\}\}$ is not possible since $\{c,e\} \in F$ in Case 3). Since $\{b,e\} \in X_{S^*} \cap X_{S'}$, we have that $|X_{S^*} \cap X_{S'}| > |X_S \cap X_{S'}|$. Thus the pair S^*, S' gives a contradiction with the choice of S, S'.
□

Example: The following demonstrates that Theorem 1 cannot be strengthened considering arbitrary F (i.e. not necessarily a partition system). Take $G = K_{6,2}$ with the vertex set $V = \{a_1, \ldots a_6, b_1, b_2\}$. Let

$$T_1 = a_1, b_1, a_3, b_2, a_2, b_1, a_5, b_2, a_4, b_1, a_6, b_2, a_1,$$

$$T_2 = a_1, b_1, a_3, b_2, a_4, b_1, a_6, b_2, a_2, b_1, a_5, b_2, a_1.$$

We set $F = \mathcal{T}(G) \setminus \{X_{T_1} \cup X_{T_2}\}$, i.e. F includes all possible transition in G which are neither in T_1 nor in T_2. Then clearly $T_1, T_2 \in \mathcal{S}_1(G,F)$. It is not hard to check there is no transition system X_S such that $S \in \mathcal{S}(G,F)$ and $|X_{T_1} \setminus X_S| = 2$. This is because $X_S(b_1)$ is uniquely determined (also all $X_S(a_i)$) and by changing any two transitions through b_2 we obtain some F-forbidden transitions.

In order to obtain a statement analogous to Theorem 1 for arbitrary F we have to introduce a more complex transformation and put only less restrictive requirements on our transforming sequence.

Definition 2 *Let $S = \{S_1, \ldots, S_k\} \in \mathcal{S}_k(G)$ and $S' = \{S'_1, \ldots, S'_\ell\} \in \mathcal{S}_\ell(G)$, $\ell \geq k$. We say that S' arises from S by a* **multi-split** *at vertex v if there is a set $\tilde{V}$ of some occurrences of v such that the following holds for every $i = 1, \ldots k$:*

1. *If S_i does not contain any element of $\tilde{V}$, then $S_i \in S'$.*
2. *If S_i contains exactly c elements of $\tilde{V}$, say $v^1, v^2, \ldots, v^c$ (in this order) then every segment of S_i from v^j to $v^{j+1(mod c)}$ (which itself is a closed subtrail in G) is an element of S'. (i.e. we split S_i into c closed trails which correspond to the segments of S_i between elements of $\tilde{V}$).*

If S' arises from S by a multi-split at vertex v we also say that S arises from S' by a **multi-join** *at v.*

If S' arises from S by a multi-split at v and S'' arises from S' by a multi-join at the same vertex v, then we say that S'' arises from S by an **MSJ-transformation** *at v.*

Lemma 7 *Let $S, S' \in \mathcal{S}(G)$. S' arises from S by an MSJ-transformation at vertex v if and only if $X_S(u) = X_{S'}(u)$ whenever $u \neq v$.*

Proof: MSJ-transformation at v change of only transitions through v. Thus, one implication is trivial.

Conversely, let $S, S' \in \mathcal{S}(G)$ such that $X_S(u) = X_{S'}(u)$ for every $u \neq v$. We set $\tilde{V}$ to be the set of all occurrences of v on S. We perform a multi-split on S at v (given by $\tilde{V}$) obtaining $S_1 \in \mathcal{S}(G)$. Similarly we perform a multi-split on S' at v (given by $\tilde{V}$) obtaining $S_2 \in \mathcal{S}(G)$. Since $X_S(u) = X_{S'}(u)$ for every $u \neq v$, $S_1 = S_2$. Hence, S' arises from S by performing first a multi-split and then a multi-join at v, or in other words S' arises from S by an MSJ-transformation. □

Theorem 2 *Let $S, S' \in \mathcal{S}(G,F)$, F an arbitrary forbidden system of G. Then there is a sequence $S_0 = S, S_1, \ldots, S_n = S'$ such that*

1. *for every $i = 0, \ldots, n$, $S_i \in \mathcal{S}(G,F)$*
2. *for every $i = 1, \ldots, n$, S_i arises from S_{i-1} by an MSJ-transformation.*

Proof: The statement follows immediately from Lemma 7 applied consecutively on every vertex of G. Let $V(G) = \{v_1, \ldots, v_n\}$ and $S_0 = S$. For $i = 1, \ldots, n$ we define S_i as follows: $X_{S_i}(v_j) = X_{S'}(v_j)$ whenever $i \geq j$, and $X_{S_i}(v_j) = X_S(v_j)$ whenever $i < j$. It is clear that $S_i \in \mathcal{S}(G, F)$, $i = 0, \ldots, n$, because every transition of X_{S_i} is either in X_S or in $X_{S'}$. Moreover, $X_{S_{i-1}}(v_j) = X_{S_i}(v_j)$ whenever $j \neq i$ and thus, by Lemma 7, S_i arises from S_{i-1} by an MSJ-transformation at vertex v_i. $\square$

Remark

1. Note that in Theorem 2 performing first a multi-split and then a multi-join is considered as one operation. We cannot control the F-compatibility of the intermediate product between multi-split and multi-join (compare Theorem 1).
2. By Lemma 4, S' arises from S by a split, by a join, or by a segment-reversal (at v) if and only if $X_S(u) = X_{S'}(u)$ for $u \neq v$ and $|X_S(v) \setminus X_{S'}(v)| = 2$. This together with Lemma 7 implies that operations split, join and segment-reversal can be also considered as special cases of MSJ-transformations (for $|\tilde{V}| = 2$).

4 Compatible eulerian trails

Compatible eulerian trails and transformations between them were extensively studied by many authors (see e.g. [1, 3, 10, 11, 16, 24]). Here we consider eulerian trails as a special case of decompositions of a graph into closed trails. The results of Section 3 are applied then to eulerian trails. In particular, as a consequence of Theorem 1 for $k = \ell = 1$, we obtain Theorem of Fleischner et al. [11] concerning transformations between F-compatible eulerian trails for F partitions system (Corollary 1) and Kotzig's theorem [16] concerning transformations between arbitrary eulerian trails in a graph, (Corollary 2):

Corollary 1 *Let F be a forbidden partition system in a graph G and let $T, T' \in \mathcal{S}_1(G, F)$. Then there is a sequence $S_0 = T, S_1, \ldots, S_n = T'$ such that*

1. *for every $i = 0, \ldots, n$, $S_i \in \mathcal{S}_1(G, F) \cup \mathcal{S}_2(G, F)$*
2. *for every $i = 1, \ldots, n$, $|X_{S_i} \setminus X_{S_{i-1}}| = 2$ (i.e. S_i arises from S_{i-1} by a segment reversal, by a split or by a join),*
3. *if for some i, $S_i \in \mathcal{S}_2(G, F)$ then $S_{i+1} \in \mathcal{S}_1(G, F)$*

Proof: The statement follows immediately from Theorem 1 for $k, \ell = 1$. $\square$

Corollary 2 *Let $T, T' \in \mathcal{S}_1(G)$. Then there is a sequence $T_0 = T, T_1, \ldots, T_n = T'$ such that for every $i = 1, \ldots, n$, T_i arises from T_{i-1} by a segment reversal.*

Proof: By Corollary 1 applied to $F = \emptyset$ there is a sequence $S_0 = T, S_1, \ldots,$ $S_n = T'$ such that

1. for every $i = 0, \ldots, n$, $S_i \in \mathcal{S}_1(G) \cup \mathcal{S}_2(G)$
2. for every $i = 1, \ldots, n$, S_i arises from S_{i-1} by a segment reversal, by a split or by a join,
3. if for some i, $S_i \in \mathcal{S}_2(G)$ then $S_{i+1} \in \mathcal{S}_1(G)$, or in other words if S_i arises from S_{i-1} by a split, then S_{i+1} arises from S_i by a join.

Thus it suffices to prove that split and join (performed consecutively) can be replaced by a finite sequence of segment reversals. More formally, if $S_{i-1}, S_{i+1} \in \mathcal{S}_1(G)$, $S_i \in \mathcal{S}_2(G)$ such that S_i arises from S_{i-1} by a split and S_{i+1} arises from S_i by a join, then there are T_i^1, T_i^2 such that T_i^1 arises from S_{i-1} by a segment reversal, and also T_i^2 from T_i^1 and S_{i+1} from T_i^2.

Indeed, if S_i arises from S_{i-1} by a split at u and S_{i+1} arises from S_i by a join at v, $u \neq v$, then $S_{i-1} = u, P_1, v, P_2, u, P_3, v, P_4, u$ for some subtrails P_i, $i = 1, \ldots, 4$. Then $S_i = \{S_i^1 = u, P_1, v, P_2, u;\ S_i^2 = u, P_3, v, P_4, u\}$ and $S_{i+1} = u, P_3, v, P_2, u, P_1, v, P_4, u$. We define T_i^1, T_i^2:

$$T_i^1 = u, P_2^{-1}, v, P_1^{-1}, u, P_3, v, P_4, u,$$

$$T_i^2 = u, P_2^{-1}, v, P_3^{-1}, u, P_1, v, P_4, u.$$

It is straightforward to check that T_i^1 arises from S_{i-1} by a segment reversal at u, T_i^2 from T_i^1 by a segment reversal at v, and S_{i+1} from T_i^2 by a segment reversal at u.

The proof is the same also for $u = v$, only some of the subtrails P_i may be empty. $\Box$

About the existence of a compatible eulerian trail in (G, F), it is also interesting to note that for F a forbidden partition system, $\mathcal{S}_1(G, F) \neq \emptyset$ if and only if for every v and every equivalence class C of $F(v)$, $|C| \leq 1/2 \deg(v)$. (This was observed already in [16]). Note that this is exactly the same condition as in Lemma 2. Thus, for a partition system F, it follows that $\mathcal{S}_1(G, F) \neq \emptyset \Leftrightarrow \mathcal{S}(G, F) \neq \emptyset$. More generally we have:

Corollary 3 (Interpolation for partition systems) *Let F be a partition system in a graph G. Assume $\mathcal{S}_k(G, F) \neq \emptyset$ for some k. Then $\mathcal{S}_{k'}(G, F) \neq \emptyset$ for every $1 \leq k' \leq k$.*

Proof: As $\mathcal{S}_k(G, F) \neq \emptyset$ we have also $\mathcal{S}_1(G, F) \neq \emptyset$. Then we can apply Lemma 6 for $\ell = 1$ and the statement immediately follows. $\Box$

So for partition systems, the existence of an element in the lower part of the spectrum of $\mathcal{S}(G, F)$ (i.e. in $\mathcal{S}_k(G, F)$ for small k) is easy. Note that important problems (such as the Cycle double cover conjecture) are at the opposite (upper) end of the spectrum and their existence is hard even for partition systems (see Subsection 6.2). For general transition systems (other than partition systems) interpolation of course does not hold and for example deciding whether $\mathcal{S}_1(G,F) \neq \emptyset$ is an NP-complete problem (see Section 6). Let us remark that formally similar questions for 2-factors were solved completely in [18].

5 Compatible decompositions of digraphs

In this section we consider similar questions as in Sections 3 and 4, but for digraphs instead of undirected graphs. As we will see, some questions can be solved similarly as for undirected graphs, sometimes our situation in the directed case is essentially harder. First, we modify the definitions given in Section 2 for digraphs.

A directed graph (or shortly a digraph) with the vertex-set V and the arc-set A will be always denoted by $D=(V,A)$. $A^+(v)$ (resp. $A^-(v)$) denotes the set of all arcs outgoing from (resp. incoming to) v. We put $A(v) = A^+(v) \cup A^-(v)$

A *transition* through v in D is any two-element subset of $A(v)$ with one element in $A^+(v)$ and the other one in $A^-(v)$. $\mathcal{T}(v)$ again denotes the set of all transitions through v and $\mathcal{T}(D) = \bigcup_{v\in V} \mathcal{T}(v)$ all transitions of D.

If $X(v)$ is a partition of $A(v)$ into two-element classes such that every class of $X(v)$ is a transition through v (i.e. for every class C of $X(v)$, $|C \cap A^+(v)| = |C \cap A^+(v)| = 1$), then $X(v)$ is called a *transition system through* v and $X = \bigcup_{v\in V} X(v)$ is called a *transition system of* D.

$\mathcal{S}_k(D)$ denotes the set of all decompositions of D into exactly k (edge-disjoint) directed closed trails. $\mathcal{S}(D) = \bigcup_{k=1}^{\infty} \mathcal{S}_k(D)$. A transition system X_S of $S \in \mathcal{S}(D)$ is defined analogously as above. Again, there is a $1-1$ correspondence between elements of $\mathcal{S}(D)$ and transition systems of D.

If F denotes the set of forbidden transitions in D, then $S \in \mathcal{S}(D)$ is said to be F-*compatible* in (D,F) if $X_S \cap F = \emptyset$. $\mathcal{S}(D,F)$ (resp. $\mathcal{S}_k(D,F)$ denotes the set of all F-compatible decompositions of D into directed closed trails (resp. k directed closed trails).

For any system $F(v)$ of forbidden transitions through v, we can define a bipartite graph $G_{F(v)}$ with the bipartition $A^-(v), A^+(v)$ by $\{e,e'\} \in E(G_{F(v)}) \Leftrightarrow \{e,e'\} \in F(v)$. We call F a forbidden *partition system* in D if $G_{F(v)}$ is a vertex-disjoint union of complete bipartite graphs with both color classes of equal size, i.e. $G_{F(v)} = \bigcup_{i\in I} K_{m_i,m_i}$ (compare with the definition of a forbidden partition system in an undirected graph).

The following characterize the existence of F-compatible decompositions of (D,F) into closed directed trails.

Lemma 8 *Let (D, F) be an eulerian graph with an arbitrary forbidden system F. $\mathcal{S}(D, F) \neq \emptyset$ if and only if for every v there is a transition system $X(v)$ through v such that $X(v) \subseteq \mathcal{T}(v) \setminus F(v)$.*

Proof: We proceed similarly to the case of undirected graphs: If there is some $S \in \mathcal{S}(D, F)$ then X_S satisfies $X_S(v) \subseteq \mathcal{T}(v) \setminus F(v)$ for every v. Conversely, if for every v there is some $X(v)$ satisfying the condition then $X = \bigcup_{v \in V} X(v)$ is a transition system of D which defines $S \in \mathcal{S}(D, F)$. $\square$

In fact one can express the size $|\mathcal{S}(D, F)|$ (for a fixed balanced digraph D and a forbidden system F) by the following formula:

$$|\mathcal{S}(D, F)| = \prod_{v \in V(D)} \mathrm{per}(A_{F(v)}),$$

where for a vertex v, the matrix $A_{F(v)} = (a_{i,j})$ is a (square) $\deg^-(v) \times \deg^+(v)$ matrix defined by $a_{i,j} = 1 \Leftrightarrow \{e_i, e_j\} \notin F(v)$, otherwise $a_{i,j} = 0$ (e_i is an arc incoming to v and e_j is an arc outgoing from v).

It follows that in full generality the evaluation of $|\mathcal{S}(D, F)|$ is equivalent to the evaluation of $\{0, 1\}-$permanents (number of matchings of bipartite graphs).

It is also interesting to note that for any balanced orientation D of a $2k$-regular graph G on n vertices and for any fixed transition system X, $|\mathcal{S}(D, X)|$ is exponentially smaller that $|\mathcal{S}(D)|$ (because $|\mathcal{S}(D)| \sim (k!)^n$, and $|\mathcal{S}(D, X)| \sim (\frac{k!}{e})^n$). Whether a similar phenomenon holds for $\mathcal{S}_i(D)$ and $\mathcal{S}_i(D, X)$ is not clear even for $i = 1$ ($|\mathcal{S}_1(D)|$ is of course counted by the classical BEST-formula [7, 28]).

For undirected graphs and F a partition system, we were able to characterize all pairs (G, F) such that G has an F-compatible eulerian trail. We recall that $\mathcal{S}_1(G, F) \neq \emptyset \Leftrightarrow \mathcal{S}(G, F) \neq \emptyset$. For digraphs the situation is less clear as can be seen from the following example.

Example: Let C_n be a circuit on n vertices and let $S(C_n)$ be the symmetric orientation of C_n (that is $V(S(C_n)) = V(C_n)$, $A(S(C_n)) = \{(u, v), (v, u) \mid \{u, v\} \in E(C_n)\}$). We set $F(v) = \{\{(u, v), (v, u)\} \mid \{u, v\} \in E(v)\}$. Clearly F is a partition (even transition) system in $S(C_n)$. Moreover, $\mathcal{S}_1(S(C_n), F) = \emptyset$ while $\mathcal{S}(S(C_n), F) \neq \emptyset$.

We have some further examples which indicate that the complexity of deciding whether $\mathcal{S}_1(D, F) \neq \emptyset$ (for a digraph D and a partition system F in D) is difficult (compare with Section 6).

We recall that F being a forbidden partition system means that $G_{F(v)}$ is a vertex-disjoint union of complete bipartite graphs, i.e. $G_{F(v)} = \bigcup_{i \in I} K_{m_i, m_i}$. We put $m(v) = \max\{m_i \mid i \in I\}$. The existence of an F-compatible eulerian trail depends on $m(v)$ in the following way:

Lemma 9 *Let (D, F) be an eulerian digraph with a forbidden partition system F.*

1. *If there is an F-compatible eulerian trail in D, then $m(v) \leq \frac{1}{2}(\deg_D^+(v))$ for every v,*

2. *If $m(v) < \frac{1}{3}(\deg_D^+(v))$ for every v, then D has an F-compatible eulerian trail.*

Proof: The first part of the statement is obvious. To prove the second part, we choose $T \in \mathcal{S}_1(D)$ such that $|X_T \cap F|$ is minimal possible. We suppose (for contradiction) that $|X_T \cap F| \neq 0$ and we find $T' \in \mathcal{S}_1(D)$ such that $|X'_T \cap F| < |X_T \cap F|$. This will yield a contradiction with the choice of T.

There is at least one forbidden transition in X_T, say $\{e, e'\} \in X_T(v) \cap F$. Let v^1 denote the occurrence of v incident to the transition $\{e, e'\}$. As $m(v) < \frac{1}{3}(\deg_D^+(v))$, there are at most $\lceil \frac{2}{3}(\deg_D^+(v)) \rceil - 3$ occurrences of v (including v^1) which are incident to some arc f such that $\{e, f\} \in F(v)$ or $\{f, e'\} \in F(v)$. We call them *bad* occurrences. Hence, there are at least $\lfloor \frac{1}{3}(\deg_D^+(v)) \rfloor + 3$ occurrences of v which are incident to none of them. We call them *good* occurrences. Let v^2 be the first good occurrence of v after v^1 and let g be the arc incoming to v^2 on T. Since there are at least $\lfloor \frac{1}{3}(\deg_D^+(v)) \rfloor + 2$ good occurrences of v other than v^2, there must be some good occurrence v^3 and an arc h outgoing from v^3 on T such that $\{g, h\} \notin F(v)$. We can split T in v^2 and v^3 and then join the segment from v^2 to v^3 in between the arcs e and e'. In this way we obtain $T' \in \mathcal{S}_1(D)$. It follows from the choice of v^2 and v^3 that $|X'_T \cap F| < |X_T \cap F|$. □

When F is a forbidden transition system, then $m(v) = 1$ for every v and the previous lemma can be restated in the following way:

Corollary 4 *Let (D, F) be an eulerian digraph with a forbidden transition system F.*

1. *If there is an F-compatible eulerian trail in D, then $\deg_D^+(v) \geq 2$ for every v,*

2. *If $\deg_D^+(v) \geq 4$ for every v, then D has an F-compatible eulerian trail.*

Next we concentrate on the possibility of transforming decompositions of D into (directed) closed trails one to another. *Split* and *join* make a good sense even for digraphs (while *segment reversal* does not).

Definition 3 *Let $S = \{S_1, \ldots, S_k\} \in \mathcal{S}_k(D)$*

1. *Let $S_k = v, x_1, \ldots, x_p, v, y_1, \ldots y_r, v$. We say that $S' = \{S'_1, \ldots, S'_{k+1}\} \in \mathcal{S}_{k+1}(D)$ arises from S by a* split at v *if $S_i = S'_i$, for $i = 1, \ldots k-1$, $S'_k = v, x_1, \ldots, x_p, v$ and $S'_{k+1} = v, y_1, \ldots, y_r, v$.*

2. *Let* $S_{k-1} = v, x_1, \ldots, x_p, v$ *and* $S_k = v, y_1, \ldots, y_r, v$. *We say that* $S' = \{S'_1, \ldots, S'_{k-1}\} \in \mathcal{S}_{k-1}(D)$ *arises from* S *by a* join at v *if* $S_i = S'_i$, *for* $i = 1, \ldots k-2$ *and* $S'_{k-1} = v, x_1, \ldots, x_p, v, y_1, \ldots y_r, v$.

Lemma 10 *Let* X, Y *be two transition systems of* D *such that* $X \setminus Y = \{t_1, t_2\}$ *for some* $t_1, t_2 \in X$. $X = X_S$ *for* $S \in \mathcal{S}_k(D)$ *Then one of the following condition holds:*

1. $Y = X_{S'}$ *for some* $S' \in \mathcal{S}_{k+1}(D)$ *and* S' *can be obtained from* S *by a split.*

2. $Y = X_{S'}$ *for some* $S' \in \mathcal{S}_{k-1}(D)$ *and* S' *can be obtained from* S *by a join.*

Proof: Let $t_1 = \{e, e'\}, t_2 = \{f, f'\}$. The only possibility for Y is $Y = X_S \setminus \{t_1, t_2\} \cup \{\{e, f'\}, \{e', f\}\}$.

If t_1, t_2 occur in the same closed trail of S, then S' arises from S by a split. If t_1, t_2 occur in different elements of S, then S' arises from S by a join. $\square$

Example: We show that statements which would correspond to Theorem 1 and Corollary 1 do not hold for digraphs even if F is a forbidden transition system in D.

Let D be a directed 3-cycle such that every arc is trippled, i.e. $V(D) = \{u, v, w\}$ and $A(D) = \{e_i, f_i, g_i \mid e_i = \{u, v\}, f_i = \{v, w\}, g_i = \{w, u\}, i = 1, 2, 3, \}$. Let $F = \{\{e_i, f_i\}, \{f_i, g_i\}, \{g_i, e_i\} \mid i = 1, 2, 3\}$ be a forbidden transition system in D. Let

$$T = e_1, f_2, g_3, e_2, f_3, g_1, e_3, f_1, g_2,$$

$$T' = e_1, f_3, g_2, e_3, f_2, g_1, e_2, f_1, g_3.$$

Clearly, $T, T' \in \mathcal{S}_1(D, F)$. It is easy to see that when performing any split on T or T' we always obtain $S \in \mathcal{S}_2(D)$ which is not F-compatible.

Only if $F = \emptyset$ we can prove an analog of Kotzig's theorem for digraphs (Corollary 5). This was already proved in [11] and independently also in [24]. We prove it easily as a consequence of Theorem 1.

Theorem 3 *Let* D *be a digraph and let* $S \in \mathcal{S}_k(D)$ *and* $S' \in \mathcal{S}_\ell(D)$, $k \leq \ell$. *Then there is a sequence* $S_0 = S, S_1, \ldots, S_n = S'$ *such that*

1. *for every* $i = 0, \ldots, n$, $S_i \in \mathcal{S}_j(D)$ *and* $j \leq \ell + 1$,

2. *for every* $i = 1, \ldots, n$, S_i *arises from* S_{i-1} *by a split or by a join,*

3. *if for some* i, $S_i \in \mathcal{S}_{\ell+1}(D)$ *then* $S_{i+1} \in \mathcal{S}_\ell(D)$

Proof: We are going to apply Theorem 1. Let G be the corresponding underlying graph of D and let $E^+(v)$ (resp. $E^-(v)$) denote the edges of G corresponding to $A^+(v)$ (resp. $A^-(v)$). Further let $F(v)$ consists of two classes, $E^+(v)$ and $E^-(v)$. $F = \bigcup_{v \in V} F(v)$ is a forbidden partition system in G and clearly, there is a 1-1 correspondence between elements of $\mathcal{S}_1(G, F)$ and $\mathcal{S}_1(D)$. Furthermore, the choice of F guarantees that segment reversals are never used. $\square$

Corollary 5 *Let $T, T' \in \mathcal{S}_1(D)$. Then there is a sequence $S_0 = T, S_1, \ldots, S_n = T'$ such that*

1. *for every $i = 0, \ldots, n$, $S_i \in \mathcal{S}_1(D) \cup \mathcal{S}_2(D)$*
2. *for every $i = 1, \ldots, n$, S_i arises from S_{i-1} by a split or by a join,*
3. *if for some i, $S_i \in \mathcal{S}_2(D)$ then $S_{i+1} \in \mathcal{S}_1(D)$*

Proof: The statement follows immediately from Theorem 3 for $k, \ell = 1$. $\square$

6 Final remarks

6.1 Complexity

It follows from Lemmas 1 and 8 that $\mathcal{S}(G, F) \neq \emptyset$ and $\mathcal{S}(D, F) \neq \emptyset$ can be tested in polynomial time.

The question of deciding whether (for general F) $\mathcal{S}_1(G, F) \neq \emptyset$ (resp. $\mathcal{S}_1(D, F) \neq \emptyset$) is harder.

Theorem 4 *Deciding whether $\mathcal{S}_1(D, F) \neq \emptyset$ (for general forbidden system F in a digraph D) is an NP-complete problem.*

Proof: It is quite obvious that the problem is in NP. Next we will show that testing hamiltonicity in a digraph can be reduced to our problem, see [15].

Let D be a given digraph, we construct a digraph D'' and a forbidden system F in D'' such that D has a hamiltonian cycle if an only if (D'', F) has an F-compatible eulerian trail. This will finish the proof.

First of all we define a digraph D' by $V(D') = \{v^1, v^2 \mid v \in V(D)\}$ and $A(D') = A^1(D') \cup A^2(D')$, where $A^1(D') = \{(v^1, v^2) \mid v \in V(D)\}$ and $A^2(D') = \{(u^2, v^1) \mid (u, v) \in A(D)\}$. Further, let D'' arises from D' by contracting all arcs in $A^2(D')$.

Note that $|A(D'')| = |V(D)|$ (arcs in $A^1(D')$ correspond to vertices of D) and if $(u, v) \in A(D)$ then $u^2 = v^1$ in D''. Hence if D is hamiltonian, then D'' is eulerian. But not conversely. For the other implication we need to define a forbidden system F in D''. We set $F = \{\{(u^1, u^2), (v^1, v^2)\} \mid u^2 = v^1$ in D'' and $(u, v) \notin A(D)\}$. It follows from the choice of F that (D'', F) has an F-compatible eulerian trail if and only if D has a hamiltonian cycle.

It may happen (in fact it mostly happens) that D'' is a directed multigraph. If we insist on D'' being a simple digraph, we may subdivide each arc of D'' with a vertex of degree two. $\square$

Let us remark that this proof may be used to establish $\sharp P$-hardness of the counting $|\mathcal{S}_1(D,F)|$ and $|\mathcal{S}_1(G,F)|$. Note that the evaluation of $|\mathcal{S}_1(D)|$ is polynomial and the complexity status of $|\mathcal{S}_1(G)|$ open, see [21].

Corollary 6 *Deciding whether $\mathcal{S}_1(G,F) \neq \emptyset$ (for general forbidden system F in an undirected graph G) is an NP-complete problem.*

Proof: Again, it is obvious that the problem is in NP. We will show that testing $\mathcal{S}_1(D,F) \neq \emptyset$ in a digraph can be reduced to our problem. This together with the previous Theorem 4 will finish the proof.

Suppose that (D,F) (a digraph with an arbitrary forbidden system F) is given. We will construct an undirected graph G and a forbidden system F' in G such that $\mathcal{S}_1(D,F) \neq \emptyset \Leftrightarrow \mathcal{S}_1(G,F') \neq \emptyset$.

We set G to be the corresponding underlying graph of D and let $E^+(v)$ (resp. $E^-(v)$) denote the edges of G corresponding to $A^+(v)$ (resp. $A^-(v)$). If $e \in E(G)$, then the corresponding arc in D is denoted by $\overrightarrow{e}$. We set:

$$F_1(v) = \{\{e,f\} \mid \{e,f\} \subseteq E^-(v) \text{ or } \{e,f\} \subseteq E^+(v)\},$$

$$F_2(v) = \{\{e,f\} \mid \{\overrightarrow{e}, \overrightarrow{f}\} \in F(v)\},$$

$$F' = F_1 \cup F_2.$$

It is easy to see that there is a 1-1 correspondence between F-compatible eulerian trails in D and F'-compatible eulerian trails in G. Indeed, if T is an F-compatible eulerian trail in D, then it is also an F'-compatible eulerian trail in G (since $F_2 \subseteq F'$). Conversely, if T is an F'-compatible eulerian trail in G, then the choice of F_1 guarantees that it is also an eulerian trail in D and the choice of F_2 that it is F-compatible. Hence, $\mathcal{S}_1(D,F) \neq \emptyset \Leftrightarrow \mathcal{S}_1(G,F') \neq \emptyset$. $\square$

Note that if we restrict the input to (G,F), F is a forbidden partition system, then testing whether $\mathcal{S}_1(G,F) \neq \emptyset$ can be done in polynomial time (for undirected graphs). This follows immediately from the fact that $\mathcal{S}_1(G,F) \neq \emptyset \Leftrightarrow \mathcal{S}(G,F) \neq \emptyset$, if F is a partition system. Moreover, assuming $\mathcal{S}(G,F) \neq \emptyset$, one can find an eulerian trail by a polynomial algorithm (see [5]).

6.2 Compatible circuit decompositions

Let again (G,F) be an eulerian graph with a forbidden system and let $S = \{S_1,\ldots,S_n\} \in \mathcal{S}(G,F)$. S is called a *compatible circuit decomposition* of (G,F), if S_i is a simple circuit for every $i = 1,\ldots,n$.

Obviously a necessary condition on (G, F) to have a compatible circuit decomposition is $|C \cap B| \leq 1/2|B|$ for any equivalence class C of the partition $F(v)$ and any edge-cut B in G. But this condition is not sufficient:

Example: Take $G = K_5$ with the vertex set $V = \{1, 2, 3, 4, 5\}$. The edge set $E(K_5)$ is an edge-disjoint union of two 5-cycles, $C_1 = 1, 2, 3, 4, 5, 1$ and $C_2 = 1, 3, 5, 2, 4, 1$. We set F (the forbidden partition system of G) to be $F = X_{C_1} \cup X_{C_2}$. Then $(G = K_5, F)$ satisfies $|C \cap B| \leq 1/2|B|$ for any equivalence class C of the partition $F(v)$ and any edge-cut B in G but it has no compatible circuit decomposition (which is easy to check). The following strong theorem is due to Fan and Zhang [9]

Theorem 5 *Let (G, F) be an eulerian graph with a forbidden partition system F satisfying $|C \cap B| \leq 1/2|B|$ for any class C of the partition $F(v)$ and any edge-cut B in G. (G, F) has a compatible circuit decomposition whenever G does not contain K_5 as a minor.*

A similar result does not hold in our more general setting. First of all, we should assume that for every edge $e \in E(G)$ there is an F-compatible simple circuit which contains e. Nevertheless, even then this condition is not sufficient:

Example: We put $G = K_{2,4}$ with the vertex-set $V = \{a_1, a_2, a_3, a_4, b_1, b_2\}$. For simpler notation let $e_{i,j}$ denotes the edge $\{a_i, b_j\}$. Let F consists of the following four transitions in G, $F = \{\{e_{1,2}, e_{1,4}\}, \{e_{1,3}, e_{1,4}\}, \{e_{2,2}, e_{2,3}\}, \{e_{2,2}, e_{2,4}\}\}$. It is not hard to check that G has no F-compatible circuit decomposition, G does not contain K_5 as a minor and for every edge $e \in E(G)$ there is an F-compatible simple circuit containing e (e.g the circuit containing e and $e_{1,1}$).

Intuitively, the decomposition into simple circuits lies on the other side of the spectra of $\mathcal{S}(G, F)$ than $\mathcal{S}_1(G, F)$. This can be made more precise by means of the notion of *defect* of S:

For $S = \{S_1, \ldots, S_t\} \in \mathcal{S}(G, F)$, a compatible decomposition of G into closed trails, we define a number $def(S)$ (*defect* of S) which will measure how far is S from being a compatible circuit decomposition.

$$def(S) = \sum_{i=1}^{t} \sum_{v \in V(G)} (\deg_{S_i}(v) - 2)$$

Note that if all S_i are simple circuits, then $\deg_{S_i}(v) = 2$ for all i and thus $def(S) = 0$ whenever S is a compatible circuit decomposition of (G, F).

We further define

$$def(G, F) = \min_{S \in \mathcal{S}(G,F)} def(S)$$

$def(G,F)$ is well-defined whenever $\mathcal{S}(G,F) \neq \emptyset$. This is according to Lemma 1 whenever for every vertex v, $\mathcal{T}(v) \setminus F(v)$ contains a transition system $X(v)$ through v.

Clearly, if $\mathcal{S}(G,F) \neq \emptyset$, we have that $def(G,F) \leq \sum_{v \in V}(\deg(v) - 2) = 2|E| - 2|V|$ and this bound is achieved for any F-compatible eulerian trail. On the other hand we have:

Lemma 11 *$def(G,F) = 0$ if and only if (G,F) has a compatible circuit decomposition.*

We do not know the complexity status of deciding whether $def(G,F) = 0$ for general forbidden system F.

The notion of compatible circuit decomposition is closely related to the well-known Cycle double cover conjecture (see e.g. [27]): Let G be an arbitrary connected graph (not necessarily Eulerian). Let G_2 be a graph obtained from G by replacing each edge of G by two parallel edges e_1, e_2. Obviously, G_2 is Eulerian. Let $F_2(v) = \{\{e_1, e_2\} \mid e \in E(v)\}$ be a forbidden (transition) system in G_2. Then every compatible circuit decomposition of (G_2, F_2) forms a cycle double cover of G. Thus the following conjecture is another reformulation of the Cycle double cover conjecture.

Conjecture 1 *Let G be a bridgeless graph and let G_2 and F_2 be defined as above. Then $def(G_2, F_2) = 0$.*

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