On flow and tension-continuous maps

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Abstract

A *cycle* of a graph G is a set $C \subseteq E(G)$ so that every vertex of the graph $(V(G), C)$ has even degree. If G, H are graphs, we define a map $\phi : E(G) \to E(H)$ to be *cycle-continuous* if the pre-image of every cycle of H is a cycle of G. A fascinating conjecture of Jaeger asserts that every bridgeless graph has a cycle-continuous mapping to the Petersen graph. Jaeger showed that if this conjecture is true, then so is the 5-cycle-double-cover conjecture and the Fulkerson conjecture.

Cycle continuous maps give rise to a natural quasi-order $\succ$ on the class of finite graphs. Namely, $G \succ H$ if there exists a cycle-continuous mapping from G to H. The goal of this paper is to study this and other related quasi-orders. In particular, we establish a number of connections between structural properties of these quasi-orders and traditional flow/coloring problems. Particulary we etablish the existence of arbitrarily large finite antichains with respect to the cycle continuous order. (Even the existence of antichains of size 2 was a problem). Our framework also leads to a variety of new questions. For instance, the

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following problem concerns a basic property of $\succ$ which we have been unable to resolve. Is there an infinite set of incomparable graphs under the order $\succ$?

1 Introduction

Some of the most striking conjectures in structural graph theory have an algebraic flavour. These include Tutte's conjectures on flows, a variety of polynomials associated with combinatorial phaenomena, the Hedetniemi product conjecture, and Ulam's reconstruction conjecture. In all these cases not only one can *formulate* these problems involving some familiar algebraic notions and constructions but in all of these cases some of the (currently) best results were obtained after the proper algebraic context was realized, see e.g. [6, 20, 22, 23, 24, 11, 13]. It is perhaps not surprising that many of these problems can be expressed as statements about partially ordered (or quasi-ordered) sets and classes. In some of the situation such a formulation is straightforward as the problem deals directly with the category of graphs and standart maps, such as homomorphisms. This is the case e.g. for the product conjecture, see e.g. [20, 14]. However in the other situations a *different* algebraic and order-theoretic formulation is far from obvious and the right definitions were sought for a long time. Sometimes strangely looking definitions are far from arbitrary as they reflect the experience gained with dealing with concrete problems (such as 4CC) and other algebraic concepts such as matroids, flows and tensions. The later notions are the subject of this paper.

The immediate motivation for this paper was provided by efforts to understand a paper of François Jaeger [7] in its relationship to Petersen flows and various maps between cycle spaces and to the later development. It appeared us soon that the fine distinctions in possible interpretations of Jaeger's insightfull approach leads to a variety of orders with interesting interplay and results.

We define nine orders for graphs. Although many of these orders look similar a small change in the definition has sometimes (well, in most cases) profound influence on the behaviour and problems related to the particular order. It is the essence of this approach that seemingly simple looking questions yield difficult problems.

We try to establish some of the basic properties of the orders we establish. For each order we try to determine the maximal/minimal elements, scales, atomes and similar order-theoretic concepts. In some cases, we have been unable to resolve even very basic properties about our orders. These open problems may be of some interest.

We do not aim for generalities for their own sake. Instead we want to put on solid ground conjectures (both small and grand) and all the observations and experience which was assembled and which in many instances is truly folkloristic. Perhaps surprisingly this relates many different areas by new links. Particularly, it displays prominently the role of the *critical problem* [18] (in many of its forms) as one of the only monotonne invariant which is at our disposal.

Our approach has some similarities to [10] where the authors are also interested in various maps between graphs (and mostly between edges). However despite of some formal similarities our approach is very different (although it is manifested in some subtle differencies): our mappings are defined by "continuous"-type condition (for example: by requiering that preimage of every cycle is a cycle), whereas mappings in [10] are mostly "open" (for example cycle preserving). Motivation of [10] is matroid theory (and strong maps are one of the classes considered). Our motivation is flow and coloring problems (following the original Tutte's approach). For these type of questions our "continuous" approach seems to be more suited. This is also indicated by the pleasing fact which we are going to prove in Section 4 Theorem 4.7 that homomorphisms (i.e. mappings of vertices which are edge preserving) are just "cut-tension continuous maps". (Homomorphisms, called color maps in [10], do not fit the matroidal scheme of [10].)

The structural richness of our approach is also indicated in the last two sections where we study "positive flows and tensions" and related this to digraph duality theorem.

2 Basic Definitions

All graphs considered in this paper are assumed to be finite unless it is explicitly stated otherwise. Graphs may have both loops and multiple edges. Frequently, we will have need to refer to both an oriented graph and the underlying undirected graph. If G is an undirected graph, then we may use $\vec{G}$ or $\check{G}$ to denote an orientation of G. If $\vec{G}$ or $\check{G}$ is defined to be an oriented graph, then it is understood that G is the underlying undirected graph.

Let G be a graph and let $C \subseteq E(G)$. We say that C is a *cycle* if every vertex of the graph $(V(G), C)$ has even degree. A *circuit* is a non-empty cycle which is minimal with respect to inclusion. We define the *odd-girth* $\gamma^o(G)$ of a graph G to be the size of the smallest circuit of G of odd cardinality (or ∞ if none exists). If $X \subseteq V(G)$, then we will let $\Delta(X)$ denote the set of edges with one end in X and one end in $V(G) \setminus X$. For a vertex $v \in V(G)$, we use $\Delta(v)$ to denote $\Delta(\{v\})$. Any set of edges of the form $\Delta(X)$ for some $X \subseteq V(G)$ is defined to be an *edge-cut*. A *bond* is a non-empty edge-cut which is minimal with respect to inclusion. We define $\lambda^o(G)$ to be the size of the smallest bond of G of odd cardinality (or ∞ if none exists). A single edge $e \in E(G)$ is a *cut-edge* if $\{e\}$ is an edge-cut.

If $\vec{G}$ is an oriented graph and $X \subseteq V(\vec{G})$, then we let $\Delta^+(X)$ denote the set of edges with tail in X and head in $V(\vec{G}) \setminus X$. We define $\Delta^-(X)$ to be $\Delta^+(V(\vec{G}) \setminus X)$ and as before, for a vertex $v \in V(\vec{G})$, we let $\Delta^+(v) = \Delta^+(\{v\})$ and $\Delta^-(v) = \Delta^-(\{v\})$. If $C \subseteq G$ is a circuit and $e, f \in E(\vec{G})$, then e and f are either given the same orientation relative to C or the opposite orientation relative to C. A *direction* of C is a pair (X, Y) of disjoint subsets of $E(C)$ with union $E(C)$ so that every $e \in X$ and $f \in Y$ have opposite orientation with respect to C. We call the edges in X *forward* edges and the edges in Y *backward* edges. We say that an edge $e \in E(\vec{G})$ is a *cut-edge* if the corresponding edge is a cut-edge of the underlying undirected graph G.

Let M be an abelian group, let $\vec{G}$ be an oriented graph, and let $\phi : E(\vec{G}) \to M$ be a map. We say that ϕ is a *flow* or an M*-flow* if

$$\sum_{e \in \Delta^+(v)} \phi(e) = \sum_{e \in \Delta^-(v)} \phi(e)$$

holds for every vertex $v \in V(\vec{G})$. We say that ϕ is a *tension* or an M*-tension* if

$$\sum_{e \in A} \phi(e) = \sum_{e \in B} \phi(e)$$

holds for every circuit $C \subseteq G$ where (A, B) is the direction of C.

We now follow the framework of Jaeger in [8] by defining a restricted class of flows and tensions. Let $B \subseteq M$ and assume that $-B = B$. If $\phi : E(\vec{G}) \to M$ is a flow (tension) and $\phi(E(\vec{G})) \subseteq B$, then we say that ϕ is a B*-flow* (B*-tension*). We say that a flow (tension) ϕ is *nowhere-zero* if it is a $(M \setminus \{0\})$-flow (tension). We say that $\phi : E(G) \to \mathbb{Z}$ is a k*-flow* (k*-tension*) for a positive integer k if ϕ is a B-flow (B-tension) where $B = \{-(k-1), \ldots, -1, 0, 1, \ldots, k-1\}$. If ϕ is a B-flow (B-tension) of $\vec{G}$ and we reverse the orientation of some edge $e \in E(\vec{G})$, then by replacing $\phi(e)$ by its additive inverse, we maintain that ϕ is a B-flow (B-tension). Thus, for an unoriented graph G, we have that some orientation of G has a B-flow (B-tension) if and only if every orientation of G has a B-flow (B-tension). In this case, we say that G *has a* B*-flow* (B*-tension*). Similarly, we say that G has a *nowhere-zero* M*-flow* or k*-flow* (M*-tension* or k*-tension*) if some (and thus every) orientation of G has such a flow (tension). Next we mention a famous conjecture of Tutte on nowhere-zero flows.

Conjecture 2.1 (The 5-flow conjecture (Tutte)) *Every graph with no cut-edge has a nowhere-zero 5-flow.*

In this introduction, we will focus most of our attention on B-flows. However, we wish to mention here that the theory of B-tensions is quite rich and is closely connected with graph coloring. Indeed, it is an easy fact that a graph has a B-tension if an only if it has a homomorphism to a certain Cayley graph. For completeness, this property is proved in the next section. Here we mention a corollary of this fact which gives evidence of the connection to graph coloring.

Proposition 2.2 *A graph G has a nowhere-zero k-tension if and only if it is k-colorable.*

Jaeger initiated the study of B-flows and B-tensions and observed that a number of important questions in graph theory may be phrased in terms of the existence of certain B-flows. Here we list three famous conjectures. For each of these problems we offer two equivalent formulations. The first is the traditional statement of the problem, the second is an equivalent statement in terms of B-flows.

Conjecture 2.3 (The five cycle double cover conjecture)

(1) *For every graph with no cut-edge, there is a list of five cycles so that every edge is contained in exactly two.*

(2) *Every graph with no cut-edge has a B-flow for the set $B \subseteq \mathbb{Z}_2^5$ consisting of those vectors with exactly two 1's.*

Conjecture 2.4 (The orientable five cycle double cover conjecture)

(1) *For every oriented graph with no cut-edge, there is a list of five 2-flows $\phi_1, \phi_2, \ldots, \phi_5$ with $\sum_{i=1}^{5} \phi_i = 0$ such that every edge is in the support of exactly two of these flows.*

(2) *Every graph with no cut-edge has a B-flow for the set $B \subseteq \mathbb{Z}^5$ consisting of those vectors with exactly three 0's, one 1, and one -1.*

Conjecture 2.5 (Fulkerson)

(1) *For every cubic graph with no cut-edge, there is a list of 6 perfect matchings so that every edge is contained in exactly two.*

(2) *Every graph with no cut-edge has a B-flow for the set $B \subseteq \mathbb{Z}_2^6$ consisting of those vectors with exactly four 1's.*

For the history of these conjectures see e.g. [2, 21, 8, 5] and also the original papers [4, 19, 3].

In addition to defining B-flows, Jaeger defined a type of mapping between graphs which is closely related to one we give here. We will discuss the relationship between our and Jaeger's definitions in Section 9 of this paper. Let $\vec{G}$ and $\vec{H}$ be oriented graphs, let M be an abelian group, and let $f : E(\vec{G}) \to E(\vec{H})$. We say that f is *M-flow-continuous* (*M-tension-continuous*) if $\phi \circ f$ is a M-flow (M-tension) of $\vec{G}$ for every M-flow (M-tension) ϕ of $\vec{H}$ (see Figure 1). The name flow-continuous (tension-continuous) is used here since in such a map every flow (tension) of $\vec{H}$ lifts to a flow (tension) of $\vec{G}$. Note that if f is a flow-continuous (tension-continuous) map from $\vec{G}$ to $\vec{H}$ and we reverse the direction of an arc $e \in E(\vec{H})$, then by reversing the arcs $f^{-1}(\{e\})$ in $\vec{G}$, we maintain that f is flow-continuous (tension-continuous). Also note that if f is M-flow-continuous (M-tension-continuous), then f is also M^n-flow-continuous (M^n-tension-continuous) for every positive integer n. Here we mention the key property of flows and continuous maps (this property is also satisfied by Jaeger's maps). This property may be viewed as the motivation for our study.

$$
\begin{tikzcd}
\mathbf{E(\vec{G})} \arrow[rr, "\mathbf{f}"] \arrow[dr, "\phi\circ\mathbf{f}"'] & & \mathbf{E(\vec{H})} \arrow[dl, "\phi"] \\
& \mathbf{M} &
\end{tikzcd}
$$

Figure 1: M-flow(tension)-continuous

Proposition 2.6 *If there is a M-flow-continuous (M-tension-continuous) map from $\vec{G}$ to $\vec{H}$ and H has a B-flow (B-tension) for some $B \subseteq M^n$, then G also has a B-flow (B-tension).*

Proof: We prove the proposition in the flow-continuous case. The tension-continuous case follows by the same argument. If $f : E(\vec{G}) \to E(\vec{H})$ is M-flow-continuous, then it is also M^n-flow-continuous. Thus, if $\phi : E(\vec{H}) \to B$ is a B-flow of H, then $\phi \circ f$ is a B-flow of G. $\square$

The above proposition is especially interesting because it suggests a different approach to showing the existence of a B-flow. To prove that G has a B-flow, it suffices to show that some orientation of G has an M-flow-continuous map to an orientation of a graph H which is known to have a B-flow. Based on this property, we now define for every abelian group M the relations $\succ_M^f$ and $\succ_M^t$ as follows. For any two unoriented graphs G, H, we write $G \succ_M^f H$ ($G \succ_M^t H$) if there exists an M-flow-continuous (M-tension-continuous) map between some orientation of G and some orientation of H. We write $G \not\succ_M^f H$ or $G \not\succ_M^t H$ if no such map exists. A relation is a *quasi-order* if it is reflexive and transitive. Next we prove that these relations are quasi-orders.

Proposition 2.7 *The relations $\succ_M^f$ and $\succ_M^t$ are quasi-orders on the class of finite graphs.*

Proof: For convenience, we give the proof only for $\succ_M^f$. The same argument also works for $\succ_M^t$. For any graph G, and any orientation $\vec{G}$ of G, the identity map from $E(\vec{G})$ to $E(\vec{G})$ is M-flow-continuous, so $G \succ_M^f G$. To prove that $\succ_M^f$ is transitive, let F, G, H be graphs with $F \succ_M^f G \succ_M^f H$. Then there exists a flow-continuous map f from an orientation $\vec{F}$ of F to an orientation $\vec{G}$ of G and a flow-continous map f' from an orientation $\check{G}$ of G to an orientation $\vec{H}$ of H. By possibly reversing the direction of some arcs in $\vec{G}$ and reversing the corresponding arcs in $\vec{F}$ (as described above), we may assume that $\vec{G} = \check{G}$. Now, for any M-flow ϕ of $\vec{H}$, the map $\phi \circ f'$ is a flow of $\vec{G} = \check{G}$ and the map $\phi \circ f' \circ f$ is a K-flow of F. Thus, the map $f' \circ f$ is a M-flow continuous map from $\vec{F}$ to $\vec{H}$ and we have that $F \succ_M^f H$ as required. $\square$

The main purpose of this paper is to investigate the structure of the quasi-orders $\succ_M^f$ and $\succ_M^t$. We will establish some basic properties and connections between these orders and raise some new open problems. Here we mention a fascinating conjecture of Jaeger concerning the order $\succ_{\mathbb{Z}_2}^f$ which we view as powerful motivation for the study of the flow-continuous quasi-orders. We use P_{10} to denote the Petersen graph (see Figure 2).

Conjecture 2.8 (Jaeger) *If G has no cut-edge, then $G \succ_{\mathbb{Z}_2}^f P_{10}$.*

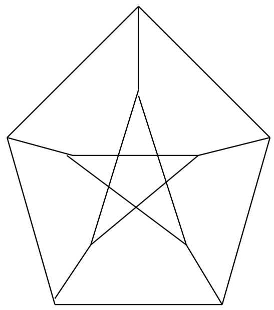

Figure 2: P_{10}

If this conjecture is true, then so is the five cycle double cover conjecture and Conjecture 2.5 of Fulkerson. This implication follows immediately from Proposition 2.6 and the second formulations of these conjectures given in the introduction and the fact that these conjectures hold for the Petersen graph.

In general, to prove that every graph in some set $\mathbf{X}$ of finite graphs has a B-flow, it suffices to establish a set $\mathbf{Y}$ of graphs with the property that every graph in $\mathbf{Y}$ has a B-flow and every graph in $\mathbf{X}$ has an M-flow-continuous map to some graph in $\mathbf{Y}$. Here we suggest another problem of this type which may be viewed as an oriented version of the above conjecture of Jaeger. An affirmative answer to this problem would imply the 5-flow conjecture, the orientable five cycle double cover conjecture, and conjecture 2.5 of Fulkerson. We let K_n denote the complete graph on n vertices.

Problem 2.9 *Does every graph G with no cut-edge satisfy $G \succ^f_{\mathbb{Z}} K_4$ or $G \succ^f_{\mathbb{Z}} P_{10}$?*

This paper is organized as follows. After establishing some general properties of flow and tension-continuous maps in the next two sections, we investigate each of the orders $\succ^f_{\mathbb{Z}_2}$, $\succ^f_{\mathbb{Z}}$, $\succ^t_{\mathbb{Z}_2}$, and $\succ^t_{\mathbb{Z}}$ in a separate section. Also a section is devoted to Jaeger order $\prec_J$ and its comparison to our approach (which we believe is more streamlined). The last two sections are devoted to "positive clone flows and tensios".

For the convenience we review here some simple characteristics of partial orders which we will investigate in all of these cases. Let $\succ$ be an order on S. Two elements $x, y \in S$ are *comparable* if either $x \succ y$ or $y \succ x$. We say that x *dominates* y if $x \succ y$ and we say that x and y are *equivalent* if $x \succ y$ and $y \succ x$. An element $x \in S$ is *maximal* (*minimal*) if x is equivalent to every element y for which $y \succ x$ ($x \succ y$). A set $Y \subseteq S$ is an *antichain* if no two elements in Y are comparable. A set $X \subseteq S$ is a *chain* if every two elements in X are comparable but not equivalent. An *increasing* (*decreasing*) *chain* is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_j \succ x_i$ if and only if $j \geq i$ ($j \leq i$). An increasing chain $\{x_n\}_{n=1}^{\infty}$ is said to be a *scaling* chain if every $y \in S$ which is not maximal is dominated by x_i for some $i \geq 1$. An element $y \in S$ is an *atom* if every $x \in S$ which is not minimal dominates y. Finally, we say that a function $f : S \to \mathbb{Z}$ is *monotone* if either $x \succ y$ implies $f(x) \geq f(y)$ or $x \succ y$ implies $f(x) \leq f(y)$.

3 Flow/Tension-continuous maps over rings

Before we study the orders generated by some particular groups, we wish to mention some general properties satisfied by all flow or tension-continuous maps over rings. This will shorten and unify some of our statements below. Our approach is the standard one.

Throughout this section, we will assume that K is a ring. For every oriented graph $\vec{G}$, we reguard $K^{E(\vec{G})}$ as a module over K. It follows from the definitions that the set of K-tensions and K-flows are both submodules of $K^{E(\vec{G})}$. If $\phi, \psi \in K^{E(\vec{G})}$, we say that ϕ and ψ are *orthogonal*, written $\phi \perp \psi$, if $\sum_{e \in E(\vec{G})} \phi(e)\psi(e) = 0$. It is an elementary fact that every K-flow is orthogonal to every K-tension. Furthermore, a map $\psi : E(\vec{G}) \to K$ is a K-flow (K-tension) if and only if it is orthogonal to every K-tension (K-flow).

Next we state a key equivalence. If $f : X \to Y$ and $\psi : X \to K$, then we let $\psi_f : Y \to K$ be given by the rule $\psi_f(y) = \sum_{x \in f^{-1}(\{y\})} \psi(x)$.

Theorem 3.1 *Let $\vec{G}$ and $\vec{H}$ be oriented graphs and let $f : E(\vec{G}) \to E(\vec{H})$. Then f is K-flow-continuous (K-tension-continuous) if and only if ψ_f is a tension (flow) of $\vec{H}$ for every tension (flow) ψ of $\vec{G}$.*

Proof: We prove the theorem only in the flow-continuous case. The tension continuous case follows by a similar argument. Let $f : E(\vec{G}) \to E(\vec{H})$ be a map, let ψ be a K-tension of G and let ϕ be a K-flow of H. Then we have the following equations :

$$\begin{aligned}
\sum_{e \in E(H)} \phi(e)\psi_f(e) &= \sum_{e \in E(H)} \phi(e) \sum_{s \in f^{-1}(\{e\})} \psi(s) \\
&= \sum_{e \in E(H)} \sum_{s \in f^{-1}(\{e\})} \phi(f(s))\psi(s) \\
&= \sum_{s \in E(G)} (\phi \circ f)(s)\psi(s)
\end{aligned}$$

If we assume that f is K-flow-continuous, then $(\phi \circ f)$ is a flow on G, so the last line in the above equation evaluates to zero. In this case, we have that $\psi_f \perp \phi$. Since ϕ was an arbitrary flow, it follows that ψ_f is orthogonal to every flow, so ψ_f is a tension as desired.

If we assume that ψ_f is a tension of H, then the first line in the above equation evaluates to zero. In this case, we have that $\psi \perp (\phi \circ f)$. Since ψ was an arbitrary tension, it follows that $\phi \circ f$ is orthogonal to every tension, so $\phi \circ f$ is a flow as desired. $\square$

Next we prove that for every graph H, there is a subspace $B \subseteq K^n$ with $-B = B$, such that $G \succ_K^f H$ ($G \succ_K^t H$) if and only if G has a B-flow (B-tension). This useful fact was first discovered by Jaeger. Although Jaeger only considered the ring $\mathbb{Z}_2$, his arguments extend naturally to arbitrary rings. Let A be a matrix with entries in K and columns indexed by $E(G)$, and let $a_1, a_2, \ldots, a_n$ denote the rows of A. We say that A *represents the cycle-space (cocycle-space) of* G if every a_i is a K-flow (K-tension) and for every K-flow (K-tension) ψ of G, there exist $x_1, x_2, \ldots, x_n \in K$ such that $\psi = \sum_{i=1}^n x_i a_i$.

Theorem 3.2 *Let $\vec{H}$ be an oriented graph, let K be a ring, let A be an $n \times m$ matrix over K which represents the cycle-space (cocycle-space) of $\vec{H}$, and let $B = \{x \in K^n \mid x$ or $-x$ is a column of $A\}$. Then for every graph G, we have that $G \succ_K^f H$ ($G \succ_K^t H$) if and only if G has a B-flow (B-tension).*

Proof: Again, we prove the statement only in the case when A represents the cycle-space of H. A similar argument proves the statement when A represents the cocycle-space of H. Let $a_1, a_2, \ldots, a_n$ denote the row vectors of A. We think of a_i as a map from $E(H)$ to K and we let $\phi : E(H) \to K^n$ be the map given by the rule $\phi(e) = (a_1(e), a_2(e), \ldots, a_n(e))$. Let $\vec{G}$ be an orientation of G, and let $f : E(\vec{G}) \to E(\vec{H})$ be a map. Next we establish the following claim.

Claim: f is K-flow-continuous if and only if $\phi \circ f$ is a flow.

Proof: If f is K-flow-continuous, then $a_i \circ f$ is a flow for $1 \leq i \leq n$, so ϕ is also a flow. On the other hand, if $\phi \circ f$ is a flow and $\psi : E(\vec{H}) \to K$ is any K-flow of H, then we may choose $x_i \in K$ for $1 \leq i \leq n$ so that $\psi = \sum_{i=1}^n x_i a_i$. Since $\phi \circ f$ is a flow, $a_i \circ f$ is a flow for $1 \leq i \leq n$ and we find that $\psi \circ f = \sum_{i=1}^n x_i a_i \circ f$ is also a flow. Since ψ was an arbitrary flow of $\vec{H}$, it follows that f is K-flow-continuous. This completes the proof of the claim.

Let B_0 denote the set of columns of the matrix A. It follows from the above claim that $G \succ_K^f H$ if and only if there exists an orientation $\vec{G}$ of G and a map $\psi : E(\vec{G}) \to B_0$ so that ψ is a flow. By reversing edges, the latter condition is equivalent to the statement that G has a B-flow. This completes the proof. $\square$

Let M be an abelian group and let $p : V(\vec{G}) \to M$ be a map (p for potential). We define the *coboundary* of p to be the map $\delta p : E(\vec{G}) \to M$ given by the rule $\delta p(e) = p(v) - p(u)$ if e is directed from u to v. It is easy to see that δp is always a tension. The following well known lemma shows that every tension arises in this manner from a potential.

Lemma 3.3 *For every tension $\phi : E(\vec{G}) \to M$, there exists a map $p : V(G) \to M$ so that $\delta p = \phi$. Further, if G is connected and $\delta p = \phi = \delta p'$, then there is a fixed $x \in M$ so that $p(v) - p'(v) = x$ for every $e \in E(\vec{G})$.*

Proof: Define the *height* of a walk W to be the sum of ϕ on the forward edges of W minus the sum of ϕ on the backward edges of W. Since ϕ is a tension, the height of every closed walk is zero. Now, choose a vertex u and define the map $p : V(\vec{G}) \to M$ by the rule $p(v) =$ the height of a walk from u to v. It follows from the fact that every closed walk has height zero that p is well defined. Furthermore, by construction $\delta p = \phi$. To prove the second statement in Lemma 3.3, let $p' : V(\vec{G}) \to K$ satisfy $\delta p' = \phi$. Now for any edge e directed from u to v, we have that $p'(v) - p'(u) = \phi(e) = p(v) - p(u)$. Thus, $p(v) - p'(v) = p(u) - p'(u)$ and Lemma 3.3 follows. $\square$

If G, H are undirected graphs. A *homomorphism* from G to H is a map $f : V(G) \to V(H)$ with the property that $f(u) \sim f(v)$ whenever $u \sim v$. It is easy to see that there is a homomorphism from G to K_n if and only if G is n-colorable. Thus, we may view homomorphisms as a generalization of graph coloring.

For any abelian group M and any subset $B \subseteq M$ with $B = -B$, we let $Cayley(M, B)$ denote the simple (undirected, but not necessarily loopless) graph with vertex set M in which two vertices $x, y \in M$ are adjacent if and only if $x - y \in B$. Note that $Cayley(M, B)$ is an infinite graph if M is infinite. The following proposition is a well-known equivalence which we sketch a proof of for completeness.

Proposition 3.4 *Let M be an abelian group and let $B \subseteq M$ with $-B = B$. Then a graph G has a B-tension if and only if there is a homomorphism from G to $Cayley(M, B)$.*

Proof: Let $\vec{G}$ be an orientation of G. If there is a homomorphism p from G to $Cayley(M, B)$, then the map $\psi = \delta p$ is a B-tension. If $\psi : E(\vec{G}) \to M$ is a B-tension, then by the above lemma, we may choose a map $p : V(\vec{G}) \to M$ so that $\delta p = \psi$. Since $V(\vec{G}) = V(G)$, the map p is a homomorphism from G to $Cayley(M, B)$ as desired. $\square$

Based on the above proposition and Theorem 3.2, we now have the following corollary.

Corollary 3.5 *Let $\vec{H}$ be an oriented graph, let K be a ring, let A be an $n \times m$ matrix over K which represents the cocycle-space of $\vec{H}$, and let $B = \{x \in K^n \mid x \text{ or } -x \text{ is a column of } A\}$. Then $G \succ_K^t H$ if and only if $G \succ_{hom} Cayley(K^n, B)$.*

Proof: By Theorem 3.2, $G \succ_K^t H$ if and only if G has a B-tension. By Proposition 3.4 this is equivalent to the existence of a graph homomorphism from G to $Cayley(K^n, B)$. $\square$

4 Comparing the quasi-orders

In this section we compare the quasi-orders induced by flow-continuous (tension-continuous) maps over different groups. At the end of this section, we introduce a quasi-order based on graph homomorphisms and we compare this to the tension-continuous orders. We begin with a definition of circuit-flows and cut-tensions followed by an easy (folkloristic) proposition which we prove for the sake of completeness.

Let $\vec{G}$ be an oriented graph and let M be an abelian group. For every $X \subseteq V(\vec{G})$ and every $z \in M$, define the map $\gamma_X^z : E(\vec{G}) \to M$ by the rule

$$\gamma_X^z(e) = \begin{cases} z & \text{if } e \in \Delta^+(X) \\ -z & \text{if } e \in \Delta^-(X) \\ 0 & \text{otherwise} \end{cases}$$

For any map $\phi : E(\vec{G}) \to M$, we say that ϕ is a *cut-tension* if there exist $X \subseteq V(G)$ and $z \in M$ so that $\phi = \gamma_X^z$. If such an X, z exist with the added property that $\Delta(X)$ is a bond, then we say that ϕ is a *bond-tension.* The following observation follows from the definitions.

Observation 4.1 *If $\vec{G}$ is connected and ϕ is a cut-tension of $\vec{G}$, then every $p : V(\vec{G}) \to M$ which satisfies $\delta p = \phi$ must have $|p(V(\vec{G})| \leq 2$.*

If $C \subseteq G$ is a circuit and (A, B) is a direction of C, and then $z \in M$,the map $\phi_C^z : E(\vec{G}) \to M$ given by the rule

$$\psi_C^z(e) = \begin{cases} z & \text{if } e \in A \\ -z & \text{if } e \in B \\ 0 & \text{otherwise} \end{cases}$$

is a flow. We define any flow of this form to be a *circuit-flow*. Circuit-flows are dual to bond-tensions. Since we will not need the flow analogue of a cut-tension, we will not define it.

Proposition 4.2 *For every flow (tension) ϕ of $\vec{G}$, there exist circuit-flows (bond-tensions) $\phi_1, \phi_2, \ldots, \phi_n$ such that $\phi = \sum_{i=1}^n \phi_i$.*

Proof: We prove the proposition in the case that ϕ is a flow. The case when ϕ is a tension follows by a similar argument. We proceed by induction on $Supp(\phi)$. The proposition is trivially true if $Supp(\phi) = \emptyset$, so we may assume that this is not so and choose an edge $e \in E(\vec{G})$ with $\phi(e) = x \neq 0$. It follows immediately from the definitions that there is no edge-cut C containing e with the property that $\phi(f) = 0$ for every $f \in C \setminus \{e\}$. Thus, we may choose a circuit D with $e \in D$ such that $D \subseteq Supp(\phi)$. Let $\phi_1 : E(\vec{G}) \to M$ be a circuit-flow with $Supp(\phi_1) = D$ and with $\phi_1(e) = x$. By induction, we may choose a list of circuit flows $\phi_2, \phi_3, \ldots, \phi_n$ with $\sum_{i=2}^n \phi_i = \phi - \phi_1$. By construction, $\phi_1, \phi_2, \ldots, \phi_n$ is a list of circuit flows with the required properties. $\square$

All this looks very easy, however on this level of generality one cannot expect many results. Yet we can characterize the minimal and maximal elements in the flow and tension continuous orders over every group. In particular, the following shows that the minimal elements in the flow (tension) continuous order are independent of the group.

Theorem 4.3 *A graph is minimal in $\succ_M^f$ ($\succ_M^t$) if an only if it contains a cut-edge (loop). A graph G is maximal in $\succ_M^f$ ($\succ_M^t$) if and only if there is an orientation $\vec{G}$ of G such that every constant map from $E(\vec{G})$ to M is a flow (tension).*

Proof: We prove the proposition only for the flow order $\succ_M^f$. The same argument works for $\succ_M^t$ if we replace every occurance of "flow" with "tension" and interchange the use of the words "loop" and "cut-edge".

Let $\vec{H}$ be an oriented graph with a cut-edge s and let $\vec{G}$ be any oriented graph. We claim that the map $f : E(\vec{G}) \to E(\vec{H})$ given by the rule $f(e) = s$

for every $e \in E(\vec{G})$ is M-flow-continuous. To see this, let ϕ be a flow of $\vec{H}$. Then $\phi(s) = 0$, so $\phi \circ f$ is identically zero and we have that it is a flow.

To see that these are the only minimal graphs, let $\vec{G}$ be an oriented graph without a cut-edge and let $\vec{H}$ be an oriented graph with a single edge s which is a cut-edge. By the above argument $G \succ_M^f H$. We claim that $H \not\succ_M^f G$. To see this, let $f : E(\vec{H}) \to E(\vec{G})$ be a map and let $e = f(s)$. Since e is not a cut-edge of $\vec{G}$, there is a circuit containing e, so we may choose a circuit-flow $\phi : E(\vec{G}) \to M$ with $e \in Supp(\phi)$. Now $\phi \circ f$ is not a flow of $\vec{H}$.

Let G be a graph with an orientation $\vec{G}$ such that every constant map from $\vec{G}$ to M is a flow. Let $\vec{H}$ be an oriented graph, let $s \in E(\vec{H})$ be an edge and let $f : E(\vec{G}) \to E(\vec{H})$ be the map given by the rule $f(e) = s$ for every $e \in E(\vec{G})$. Then for every flow $\phi : E(\vec{H}) \to M$, the map $\phi \circ f$ is constant, so by assumption it is a flow.

To see that these are the only maximal graphs, let H be a graph with no orientation satisfying the property above and let $\vec{H}$ be an orientation of H. Let $\vec{G}$ be an orientation of a graph with a single edge s which is a loop. By the above argument $G \succ_M^f H$. We claim that $H \not\succ_M^f G$. To see this, let $f : E(\vec{H}) \to E(\vec{G})$ be a map and choose $x \in M$ such that the function on $E(\vec{H})$ which is constantly x is not a flow. Then the map $\phi : E(\vec{G}) \to M$ given by the rule $\phi(s) = x$ is a flow, but $\phi \circ f$ is not. The claim follows. $\square$

We can also prove that the order $\succ_{\mathbb{Z}}^f$ ($\succ_{\mathbb{Z}}^t$) is the most restrictive among the flow (tension) continuous orders.

Theorem 4.4 *If $G \succ_{\mathbb{Z}}^f H$ ($G \succ_{\mathbb{Z}}^t H$), then $G \succ_M^f H$ ($G \succ_M^t H$) for every abelian group M.*

Proof: We prove the proposition only for the case $G \succ_{\mathbb{Z}}^t H$. The flow-continuous case follows by a similar argument. Let $f : E(\vec{G}) \to E(\vec{H})$ be a $\mathbb{Z}$-tension continuous map from an orientation of G to an orientation of H and let $\phi : E(\vec{H}) \to M$ be a tension. By Proposition 4.2, we may choose bond-tensions $\phi_{X_1}^{z_1}, \phi_{X_2}^{z_2}, \ldots, \phi_{X_n}^{z_n}$ of $\vec{H}$ such that $\phi = \sum_{i=1}^n \phi_{X_i}^{z_i}$. Since $\phi \circ f = \sum_{i=1}^n \phi_{X_i}^{z_i} \circ f$, it suffices to show that $\phi_{X_i}^{z_i} \circ f$ is a tension for $1 \leq i \leq n$. Let $i \in \{1, 2, \ldots, n\}$ and consider the bond-tension $\gamma_{X_i}^1 : E(\vec{G}) \to \mathbb{Z}$. By assumption, $\gamma_{X_i}^1 \circ f$ is a $\mathbb{Z}$-tension of $\vec{H}$, but it follows immediately from this that $\gamma_{X_i}^{z_i} \circ f$ is an M-tension of $\vec{H}$. Since $1 \leq i \leq n$ was arbitrary, we have that $\phi \circ f$ is M-tension-continuous as required. $\square$

Next we turn our attention to the quasi-order related to graph homomorphisms: we define the relation $\succ_{hom}$ by the rule $G \succ_{hom} H$ if there exists a homomorphism from G to H. Since the identity map is a homomorphism and the composition of two homomorphisms is a homomorphism, $\succ_{hom}$ is a quasi-order.

We also define homomorphisms between oriented graphs as mappings peserving direction of arcs. Each homomorphism $f : \vec{G} \to \vec{H}$ induces a mapping $f^\sharp : E(\vec{G}) \to E(\vec{H})$ defined by $f^\sharp(x,y) = (f(x), f(y))$. This induced mapping will be called chromatic mapping $E(\vec{G}) \to E(\vec{H})$ (induced by f). It is easy to see that for every homomorphism $p : V(G) \to V(H)$ and every orientation $\vec{H}$ of H, there exists an orientation $\vec{G}$ of G and a map $f : E(\vec{G}) \to E(\vec{H})$ which is chromatic with respect to p. So in particular, $G \succ_{hom} H$ if and only if there is a chromatic map from some orientation of G to some orientation of H. The following proposition gives a key property of chromatic maps.

Proposition 4.5 *Let M be an abelian group and let $f : E(\vec{G}) \to E(\vec{H})$ be a chromatic map. Then $\phi \circ f$ is a cut-tension of $\vec{G}$ for every cut-tension $\phi : E(\vec{H}) \to M$ of $\vec{H}$.*

Proof: Let $p : V(G) \to V(H)$ be a homomorphism so that f is chromatic with respect to p and choose $z \in M$ and $X \subseteq V(\vec{G})$ so that $\phi = \gamma^z_X$. It follows from the definitions that $\phi \circ f = \gamma^z_{p^{-1}(X)}$, so $\phi \circ f$ is an M-cut-tension as required.

Our next theorem applies the above proposition to show that every chromatic map is M-tension continuous.

Theorem 4.6 *If $f : E(\vec{G}) \to E(\vec{H})$ is chromatic, then f is M-tension continuous for every abelian group M.*

Proof: If $\phi : E(\vec{H}) \to M$ is a tension, then by Proposition 4.2 we may choose cut-tensions $\phi_1, \phi_2, \ldots, \phi_n$ such that $\sum_{i=1}^n \phi_i = \phi$. By the above proposition, $\phi_i \circ f$ is a cut-tension, so in particular it is a tension. Thus, $\phi \circ f = \sum_{i=1}^n \phi_i \circ f$ is a tension of $\vec{G}$. Since ϕ was arbitrary, it follows that f is M-tension continuous as required. $\square$

The following proposition proves perhaps a surprizing converse of Proposition 4.5, thus giving an equivalent formulation of graph homomorphisms in terms of "cut-tension-continuous maps".

Theorem 4.7 *Let $\vec{G}$ and $\vec{H}$ be connected oriented graphs and let $f : E(\vec{G}) \to E(\vec{H})$. Then f is chromatic if and only if $\phi \circ f$ is a cut-tension of $\vec{G}$ for every cut-tension $\phi : E(\vec{H}) \to \mathbb{Z}$ of $\vec{H}$.*

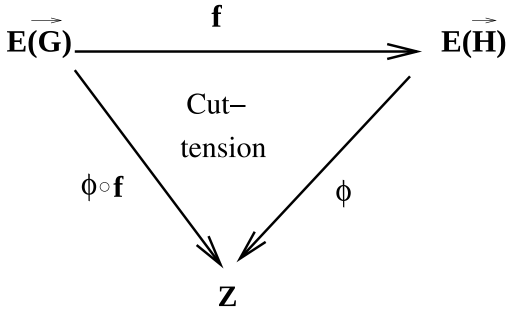

Figure 3: Cut-tension

Proof: The "only if" part of the proof is an immediate consequence of Proposition 4.5. To prove the "if" part, we will assume that $\phi \circ f$ is a cut-tension of $\vec{G}$ for every cut-tension $\phi : E(\vec{H}) \to \mathbb{Z}$ of $\vec{H}$. For every vertex $v \in V(\vec{H})$ we have that $\gamma^1_{\{v\}} : E(\vec{H}) \to \mathbb{Z}$ is a cut-tension, so by our assumption, we may there exists $Y_v \subseteq V(\vec{G})$ so that $\gamma^1_{\{v\}} \circ f = \gamma^1_{Y_v}$.

Claim 1: If $u, v \in V(\vec{H})$ and $u \neq v$, then either $Y_u \cap Y_v = \emptyset$ or $Y_u \cup Y_v = V(\vec{G})$.
Proof of Claim 1: For every $A \subseteq V(\vec{G})$, let $\chi_A : V(\vec{G}) \to \{0,1\}$ be the characteristic map given by the rule $\chi_A(v) = 1$ if $v \in A$ and $\chi_A(v) = 0$ otherwise. Clearly $\delta_{\chi_A} = -\gamma^1_A = \gamma^1_{V(G)\setminus A} = \gamma^1_{\bar{A}}$ (by $\bar{A}$ we denoted the complement of A). Now, $\gamma^1_{\{u,v\}} = \gamma^1_u + \gamma^1_v$, so we have that

$$\begin{aligned}
\gamma^1_{\{u,v\}} \circ f &= \gamma^1_{\{u\}} \circ f + \gamma^1_{\{v\}} \circ f \\
&= \gamma^1_{Y_u} + \gamma^1_{Y_v} \\
&= \delta\chi_{\bar{Y}_u} + \delta\chi_{\bar{Y}_v} \\
&= \delta(\chi_{\bar{Y}_u} + \chi_{\bar{Y}_v})
\end{aligned}$$

Since $\gamma^1_{\{u,v\}} \circ f$ is a cut-tension, and $\vec{G}$ is connected, we have by Observation 4.1 that $p = \chi_{Y_u} + \chi_{Y_v}$ takes on at most two distinct values. If p does take on two distinct values, then these values must differ by exactly one since $\delta p = \gamma^1_{\{u,v\}} \circ f$. By definition every $w \in \bar{Y}_u \cap \bar{Y}_v$ must satisfy $p(w) = 2$ and every $w \in V(\vec{G}) \setminus (\bar{Y}_v \cup \bar{Y}_u)$ must satisfy $p(w) = 0$, so it follows that at least one of these sets must be empty as required.

Fix distinct vertices $u, v \in V(\vec{G})$. By possibly switching the orientations of every edge in $\vec{G}$ and then replacing Y_w with $V(\vec{G}) \setminus Y_w$ for every $w \in V(G)$, we may assume that $Y_u \cap Y_v = \emptyset$. The next claim shows that now, Y_w and $Y_{w'}$ are disjoint whenever $w \neq w'$.

Claim 2: $Y_w \cap Y_{w'} = \emptyset$ if $w \neq w'$.

Proof of Claim 2: If there is a vertex w so that $Y_w \cap Y_u \neq \emptyset$, then by Claim 1, $Y_w \cup Y_u = V(\vec{G})$, so in particular $Y_v \subseteq Y_w$. But then by Claim 1 we must have either $Y_v = \emptyset$ or $Y_w = V(\vec{G})$ and either possibility contradicts our assumption. Thus, we find that Y_w is disjoint from Y_u for every $w \in V(\vec{G}) \setminus \{u\}$. If there exist vertices $w, w' \in V(\vec{G}) \setminus \{u\}$ so that $Y_w \cap Y'_w \neq \emptyset$, then $Y_w \cup Y_{w'} = V(\vec{G})$, by Claim 1 so either $Y_w \cap Y_u \neq \emptyset$ or $Y_{w'} \cap Y_u \neq \emptyset$. This contradiction implies that $Y_w \cap Y_{w'} = \emptyset$ whenever $w \neq w'$ as required.

The following observation follows immediately from our construction.

Observation: If $e \in E(\vec{H})$ is directed from u to v, then $f^{-1}(\{e\}) \subseteq \Delta^+(Y_u) \cap \Delta^-(Y_v)$.

Since H does not have any isolated vertices, Claim 2 and the above observation imply that $\{Y_w \mid w \in V(\vec{H})\}$ is a partition of $V(\vec{G})$. Now, let $p : V(\vec{G}) \to V(\vec{H})$ be given by the rule $p(v) = u$ if $v \in Y_u$. It follows from the above observation that f is chromatic with respect to p. This completes the proof. $\square$

Let us rephrase this Theorem in terms of homomorphisms:

Corollary 4.8 *Given two connected oriented graphs $\vec{G}$ and $\vec{H}$, a mapping $f : V(\vec{G}) \to V(\vec{H})$ is a homomorphism $\vec{G} \to \vec{H}$ if and only if $\phi \circ f^\sharp$ is a cut-tension of $\vec{G}$ whenever ϕ is a cut-tension of $\vec{H}$.*

This is indicated by Figure 4.

The cut-tension continuous mappings form a quasi-order on their own which will be denoted by $\succ_{ct}$. The above Theorem 4.7 then means that $\succ_{ct} = \succ_{hom}$. It follows that (despite of its similarity to $\succ_t$) the cut-tension order is very rich (and indeed countable universal) quasi-order, see [14]. However note that $\succ_{ct}$ is a proper subset of $\succ_t$. Examples are abundant. For example for any oriented bipartite graph $\vec{G}$ we have $\vec{G} \succ_t \vec{K}_2$ however the oriented bipartite graphs $\vec{G}$ satisfaying $\vec{G} \not\succ_{hom} \vec{K}_2$ induce a universal poset on theire own (for example $\vec{P}_2 \not\succ_{hom} \vec{K}_2$).

5 The order $\succ^f_{\mathbb{Z}_2}$

We start our investigation of particular orders by the order which has perhaps the most intuitive appeal.

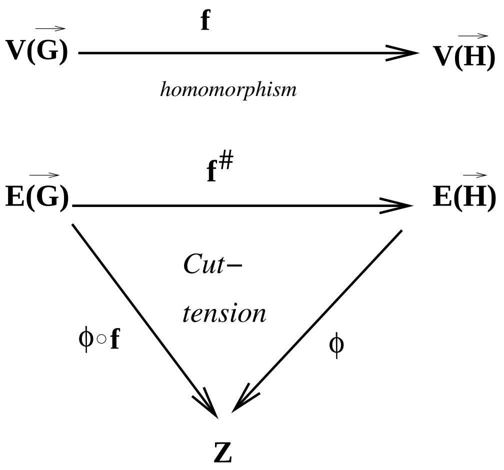

Figure 4: Homomorphism and cut-tension

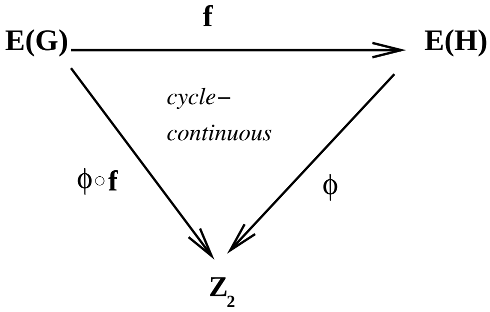

Figure 5: Cycle-continuous

In the ring $\mathbb{Z}_2$, addition and subtraction are the same operation. As such, the orientation of the graph does not play a role, and we define a map $\phi : E(G) \to \mathbb{Z}_2$ to be a *flow* of the unoriented graph G if $\sum_{e\in\Delta(v)} \phi(e) = 0$ for every $v \in V(G)$. Note that this definition is consistent with our earlier definitions since a map $\phi : E(G) \to \mathbb{Z}_2$ is a flow if and only if it is a flow of some (and thus every) orientation of G. A set $X \subseteq E(G)$ is a cycle if and only if it is the support of a $\mathbb{Z}_2$-flow. Therefore, a map $f : E(G) \to E(H)$ is $\mathbb{Z}_2$-flow-continuous if and only if $f^{-1}(C)$ is a cycle of G for every cycle $C \subseteq E(H)$. Based on this link, we call such a map *cycle-continuous*. We start with a corollary of Theorem 4.3 which establishes the maximal and minimal elements in this order.

Corollary 5.1 *A graph G is maximal in $\succ^f_{\mathbb{Z}_2}$ if and only if every vertex of G has even degree. A graph G is minimal in $\succ^f_{\mathbb{Z}_2}$ if and only if it has a* *cut-edge.*

Based on the above corollary, we can now restate Jaeger's conjecture from the introduction as follows.

Conjecture 5.2 (Jaeger) *P_{10} is the only atom in the cycle-continuous order.*

In the first paragraph of this section, we observed that a set of edges is a cycle if and only if it is the support of a $\mathbb{Z}_2$-flow. Similarly, a set of edges $D \subseteq E(G)$ is an edge-cut if and only if it is the support of a $\mathbb{Z}_2$-tension. With the help of this observation, Theorem 3.1 now gives us a monotone invariant of $\succ^f_{\mathbb{Z}_2}$.

Proposition 5.3 *If $G \succ^f_{\mathbb{Z}_2} H$ then $\lambda^o(G) \geq \lambda^o(H)$.*

Proof: If $\lambda^o(G) = \infty$ then there is nothing to prove, so we may assume that $\lambda^o(G)$ is finite and choose an edge-cut $C \subseteq E(G)$ of size $\lambda^o(G)$. Let $\psi : E(G) \to \mathbb{Z}_2$ be given by the rule $\psi(e) = 1$ if $e \in C$ and $\psi(e) = 0$ otherwise. Now, ψ is a tension of G, so by Theorem 3.1, ψ_f is a tension of H. Let D be the support of ψ_f. Now, $|D|$ is odd since $|C|$ was odd, and D is an edge-cut of H. Thus, we have that $\lambda^o(H) \leq \lambda^o(G)$ as desired. $\square$

For every positive integer h, we let K_2^h denote the graph on two vertices consisting of h edges in parallel. For any graph G, we say that a set of edges $J \subseteq E(G)$ is a *postman join* if $E(G) \setminus J$ is a cycle. The following proposition gives a characterization of when $G \succ^f_{\mathbb{Z}_2} K_2^{2a+1}$ and when $K_2^{2a+1} \succ^f_{\mathbb{Z}_2} G$.

Proposition 5.4
1. *$K_2^{2a+1} \succ^f_{\mathbb{Z}_2} G$ if and only if $\lambda^o(G) \leq 2a+1$.*
2. *$G \succ^f_{\mathbb{Z}_2} K_2^{2a+1}$ if and only if $E(G)$ can be partitioned into $2a+1$ postman joins.*

Proof: Since $\lambda^o(K_2^{2a+1}) = 2a+1$, Proposition 5.3 gives us the "only if" direction of (1). To prove the "if" direction, let G be a graph with $\lambda^o(G) \leq 2a+1$ and choose an odd edge-cut C of G of size $\leq 2a+1$. Next choose a map $f : E(K_2^{2a+1}) \to E(G)$ with the property that $|f^{-1}(\{e\})|$ is odd for every $e \in C$ and $f^{-1}(E(G) \setminus C) = \emptyset$. It follows easily that this map is cycle-continuous.

To see the "only if" direction of (2) let $f : E(G) \to E(K_2^{2a+1})$ be cycle-continuous, and note that $f^{-1}(\{e\})$ is a postman join for every $e \in$

$E(K_2^{2a+1})$ since $f^{-1}(E(K_2^{2a+1}) \setminus \{e\})$ is a cycle. To see the "if" direction, let $\{J_1, J_2, \ldots, J_{2a+1}\}$ be a partition of $E(G)$ into postman joins, let $E(K_2^{2a+1}) = \{e_1, e_2, \ldots, e_{2a+1}\}$, and consider the map $f : E(G) \to E(K_2^{2a+1})$ given by the rule $f(s) = e_i$ if $s \in J_i$. It is easily verified that this map is cycle-continuous. This completes the proof. $\square$

Based on this proposition, we have the following scaling chain.

Proposition 5.5 *The graphs $K_2^1, K_2^3, K_2^5, \ldots$ form a scaling chain in $\succ^f_{\mathbb{Z}_2}$.*

Proof: It follows immediately from (1) of the previous proposition that $K_2^1, K_2^3, K_2^5, \ldots$ is an increasing chain. If G is any non maximal graph in $\succ^f_{\mathbb{Z}_2}$, then $\lambda^o(G) < \infty$ and we have by (1) of the previous proposition that there exists a positive integer $2a+1$ such that $K_2^{2a+1} \succ^f_{\mathbb{Z}_2} G$. $\square$

Proposition 5.4 above shows a connection between the cycle-continuous order and the problem of partitioning the edge set of a graph into postman joins. We now state a special case of a conjecture of Rizzi [17] concerning postman joins. In the language of the cycle-continuous order, his conjecture asserts that the graph K_2^{2a+1} is comparable with every other graph.

Conjecture 5.6 (Rizzi) *If $K_2^{2a+1} \not\succ^f_{\mathbb{Z}_2} G$, then $G \succ^f_{\mathbb{Z}_2} K_2^{2a+1}$.*

Since every graph dominates K_2^1, this conjecture obviously holds for $a = 0$. A graph can be partitioned into 3 postman joins if and only if it has a nowhere-zero 4-flow. With this, the above conjecture for $a = 1$ follows from Jaeger's 4-flow theorem. A recent result of DeVos and Seymour asserts that Rizzi's conjecture holds with an added factor of two.

Theorem 5.7 (DeVos, Seymour) *If $K_2^{4a-1} \not\succ^f_{\mathbb{Z}_2} G$ then $G \succ^f_{\mathbb{Z}_2} K_2^{2a+1}$.*

If Jaeger's conjecture 2.8 is correct, then every graph is comparable with P_{10}. If Rizzi's conjecture 5.6 is correct, then every graph is comparable with K_2^{2a+1} for every nonnegative integer a. In light of these conjectures it may not be suprising that it is tricky to construct antichains in this order. In particular, we cannot solve the following problem.

Problem 5.8 *Does $\succ^f_{\mathbb{Z}_2}$ contain an infinite antichain?*

We can prove that the order $\succ^f_{\mathbb{Z}_2}$ does contain finite antichains of arbitrary size. We have two closely related families of graphs which demonstrate this fact. One of these families comes from a clever construction of Xuding

Zhu and will be given below. The second family will be described here, but we will postpone some of the proofs to an appendix.

Let G be a r-regular graph with $\lambda^o(G) = r < \infty$ (so in particular, r is odd). If every edge-cut of G of size r is of the form $\Delta(x)$ for some $x \in V(G)$ and $G \not\succ^f_{\mathbb{Z}_2} K_2^r$, then we say that G is an *r-snark*. If $G \setminus e \succ^f_{\mathbb{Z}_2} K_2^r$ for every edge $e \in E(G)$, then we say that G is *critical*. Both of our families of antichains are based on the following proposition.

Proposition 5.9 *If G, H are non-isomorphic critical r-snarks with $|E(G)| = |E(H)|$, then G and H are incomparable in the order $\succ^f_{\mathbb{Z}_2}$.*

Proof: Suppose the proposition is false and let $\phi : E(G) \to E(H)$ be a cycle-continuous map. First we establish the following claim.

Claim: the map ϕ is a bijection.

Proof of Claim: Since G and H have the same number of edges by assumption, it will suffice to prove that ϕ is onto. Suppose (for a contradiction) that ϕ is not onto and choose an edge $e \in E(H)$ which is not in the image of $E(G)$. It follows that ϕ is a cycle-continuous map from G to $H \setminus e$. But then we have that $G \succ^f_{\mathbb{Z}_2} H \setminus e \succ^f_{\mathbb{Z}_2} K_2^r$ which contradicts the assumption that G is an r-snark.

Let $v \in V(G)$ and consider the edge-cut $\Delta_G(v)$. The image of $\Delta_G(v)$ is another edge-cut of odd size, so by assumption, it must be equal to $\Delta_H(v')$ for some vertex $v' \in V(H)$. If u and v are distinct vertices of G and $\phi(\Delta_G(u)) = \Delta_H(u')$ and $\phi(\Delta_G(v)) = \Delta_H(v')$ for $u', v' \in V(H)$, then it follows from the fact that ϕ is a bijection that $u' \neq v'$. Thus the map which sends every $v \in V(G)$ to the corresponding $v' \in V(H)$ is an isomorphism between G and H and we have a contradiction. $\square$

Let M be a perfect matching of the Petersen graph P_{10}. Define the graph $P(a, b)$ to be the graph obtained from P_{10} by adding $a - 1$ parallel edges to every edge not in M and adding $b - 1$ parallel edges to every edge in M. Our first family of antichains is as follows.

Theorem 5.10 (Zhu) *For every nonnegative integer k, the set $\{P(2j + 1, 6k - 4j + 1) \mid 0 \leq j \leq k\}$ is an antichain in $\succ^f_{\mathbb{Z}_2}$ of size $k + 1$.*

Proof: Let $\mathcal{F}_k = \{P(2j + 1, 6k - 4j + 1) \mid 0 \leq j \leq k\}$ and let $G \in \mathcal{F}_k$. It follows from our construction that G is $(6k+3)$-regular, $\lambda^o(G) = 6k+3$, and every odd edge-cut of G of size $6k+3$ is of the form $\Delta(x)$ for some $x \in V(G)$.

Now, for an odd integer r, and r-regular graph G satisfies $G \succ^f_{\mathbb{Z}_2} K_2^r$ if and only if G is r-edge-colorable. It is well known that none of the graphs in $\mathcal{F}_k$ are $(6k{+}3)$-edge-colorable. Thus, we find that every graph in $\mathcal{F}_k$ is a $(6k{+}3)$-snark with the same number of edges. Now, it follows from a theorem of Rizzi [17] that for every $e \in E(G)$ the graph $G \setminus e$ contains $(6k+3)$-disjoint postman joins, so $G \setminus e \succ^f_{\mathbb{Z}_2} K_2^{(6k+3)}$. It follows from this and Proposition 5.9 that $\mathcal{F}_k$ is an antichain as desired. $\square$

The above construction gives us antichains of arbitrary size, but requires graphs with λ^o large. Our second construction provides antichains of arbitrary size with λ^o bounded (and thus bounded in $\succ^f_{\mathbb{Z}_2}$ by K_2^3).

Let G, H be cubic graphs, let $st \in E(G)$ and let x_1, x_2, t and x_3, x_4, s be the neighbors of s and t respectively. Let $y_1y_2, y_3y_4 \in E(H)$ be nonadjacent edges. Let F be a graph obtained from the disjoint union of $G \setminus \{s,t\}$ and $H \setminus \{y_1y_2, y_3y_4\}$ by adding new edges with ends x_i, y_i for $1 \le i \le 4$, F is again a cubic graph. We say that F is a *dot product* of G with H. In an appendix, we prove the following theorem.

Theorem 5.11 *If G is a critical 3-snark, then every dot product of G with P_{10} is a critical 3-snark.*

Since the Petersen graph P_{10} is a critical 3-snark, any dot product of P_{10} with itself is critical. There are two nonisomorphic graphs on 18 vertices known as Blanusa's snarks [1] which are obtained as dot products of P_{10} with itself. By Proposition 5.9, these two graphs are incomparable in the order $\succ^f_{\mathbb{Z}_2}$. It is straightforward to iterate this operation to create large families of nonisomorphic 3-snarks on the same number of edges, which by Proposition 5.9 are antichains. Equivalentely, there are arbitrarily large antichains of graphs under K_2^3 (in $\succ^f_{\mathbb{Z}_2}$).

6 The order $\succ^t_{\mathbb{Z}_2}$

As was the case in the order $\succ^f_{\mathbb{Z}_2}$, the orientation of the edges does not play any role in the order $\succ^t_{\mathbb{Z}_2}$,and thus $\succ^t_{\mathbb{Z}_2}$ relates to undirected graphs. Similarly as in the prevoius Section we define a map $\phi : E(G) \to \mathbb{Z}_2$ to be a *tension* of the undirected graph G if $\sum_{e \in E(C)} \phi(e) = 0$ for every circuit C of G. As observed earlier, a set of edges is an edge-cut if and only if it is the support of a $\mathbb{Z}_2$-tension. Thus, a map $f : E(G) \to E(H)$ is $\mathbb{Z}_2$-tension-continuous if and only if $f^{-1}(C)$ is an edge-cut of G for every edge-cut C of H. Based on this property, we call such a map *cut-continuous*. We begin by

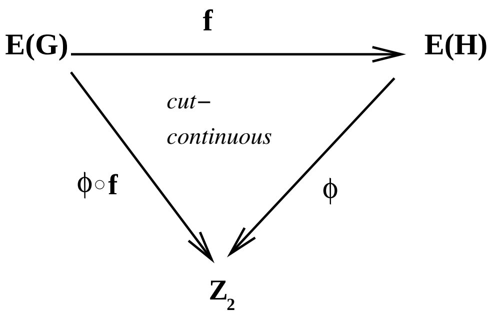

Figure 6: Cut-continuous

stating a corollary of Theorem 4.3 which gives us the maximal and minimal elements in this order.

Corollary 6.1 *The maximal elements in $\succ^t_{\mathbb{Z}_2}$ are the bipartite graphs. The minimal elements are the graphs which contain loops.*

As was the case with the cycle-continuous order, Theorem 3.1 implies that odd girth is a monotone invariant:

Proposition 6.2 *If $G \succ^t_{\mathbb{Z}_2} H$ then $\gamma^o(G) \geq \gamma^o(H)$.*

Proof: If $\gamma^o(G) = \infty$ then there is nothing to prove, so we may assume that $\gamma^o(G)$ is finite and choose a cycle $C \subseteq E(G)$ of size $\gamma^o(G)$. Let $\psi : E(G) \rightarrow \mathbb{Z}_2$ be given by the rule $\psi(e) = 1$ if $e \in C$ and $\psi(e) = 0$ otherwise. Now, ψ is a flow of G, so by Theorem 3.1, ψ_f is a flow of H. Let D be the support of ψ_f. Now, $|D|$ is odd since $|C|$ was odd, and D is a cycle of H. Thus, we have that $\gamma^o(H) \leq \gamma^o(G)$ as desired. $\square$

We let C_n denote the circuit of length n for every $n \geq 1$. Let Q_n denote the graph of the n-cube for $n \geq 1$. The vertex set of Q_n is $\{0,1\}^n$ and two vertices are adjacent if and only if they differ in a single coordinate. Let Q^+_{2n} denote the graph obtained from the $2n$-cube by adding all edges between vertices which differ in every coordinate. Now, we have the following proposition which characterizes when a graph dominates and is dominated by C_{2a+1}.

Proposition 6.3

1. $C_{2a+1} \succ^t_{\mathbb{Z}_2} G$ *if and only if* $\gamma^o(G) \leq 2a+1$
2. $G \succ^t_{\mathbb{Z}_2} C_{2a+1}$ *if and only if* $G \succ_{hom} Q^+_{2a}$.

Proof: Since $\gamma^o(C_{2a+1}) = 2a+1$, Proposition 6.2 gives us the "only if" direction of (1). To prove the "if" direction, let G be a graph with $\gamma^o(G) \leq 2a+1$ and choose an odd cycle C of G of size $\leq 2a+1$. Next choose a map $f : E(C_{2a+1}) \to E(G)$ with the property that $|f^{-1}(\{e\})|$ is odd for every $e \in C$ and $f^{-1}(E(G) \setminus C) = \emptyset$. It follows easily that this map is tension-continuous.

For (2), let $B \subseteq \mathbb{Z}_2^{2a}$ be the subset consisting of all vectors with exactly one 1 together with the vector $(1,1,1,\ldots,1)$. Now, $Q_{2a}^+ \cong Cayley(\mathbb{Z}_2^{2a}, B)$, so by Corollary 3.5, it suffices to see that the following $2a \times (2a+1)$ matrix represents the $\mathbb{Z}_2$-cocyle-space of C_{2a+1}.

$$\begin{bmatrix} 1 & & & 0 & 1 \\ & 1 & & & 1 \\ & & \ddots & & \vdots \\ 0 & & & 1 & 1 \end{bmatrix}$$

□

As was the case with the graphs K_2^{2a+1} in the cycle-continuous order, the odd circuits C_{2a+1} form a scaling chain in the cut-continuous order.

Proposition 6.4 *The odd circuits $C_3, C_5, C_7, \ldots$ form a scaling chain in the cut-continuous order $\succ^t_{\mathbb{Z}_2}$.*

Proof: It follows immediately from (1) of the previous proposition that $C_3, C_5, \ldots$ is an increasing chain. If G is any non maximal graph in $\succ^t_{\mathbb{Z}_2}$, then $\gamma^o(G) < \infty$ and we have by (1) of the previous proposition that there exists a positive integer $2a+1$ such that $C_{2a+1} \succ^t_{\mathbb{Z}_2} G$. □

For any graph G, we let G^n denote the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are distance n in G. The following proposition gives a condition similar to that in part 2 of Proposition 6.3 for graphs dominating a complete graph. Our setting provides a short proof :

Proposition 6.5 (Linial, Meshulam, Tarsi [10]) *$G \succ^t_{\mathbb{Z}_2} K_n$ if and only if $G \succ_{hom} Q_n^2$.*

Proof: Let $B \subseteq \mathbb{Z}_2^n$ be the set of all vectors with exactly two 1's. Then $Q_n^2 \cong Cayley(\mathbb{Z}_2^n, B)$. Let A be the incidence matrix of K_n considered as a matrix over $\mathbb{Z}_2$. Then A represents the $\mathbb{Z}_2$-cocycle-space of K_n and

$\{x \in \mathbb{Z}_2^n \mid x \text{ is a column of } A\} = B$. Thus, the proposition now follows from Corollary 3.5 $\square$

The above proposition 6.3 demonstrates that $G \succ^t_{\mathbb{Z}_2} K_3$ if and only if $G \succ_{hom} K_4$. We have a similar relation for the complete graphs with 2^n vertices.

Proposition 6.6 *$G \succ^t_{\mathbb{Z}_2} K_{2^n}$ if and only if $G \succ_{hom} K_{2^n}$.*

Proof: Let f be a cut-continuous map from G to K_{2^n} and choose edge-cuts $D_1, D_2, \ldots, D_n$ of K_{2^n} so that $\bigcup_{i=1}^n D_i = E(K_{2^n})$. Now $f^{-1}(D_i)$ for $1 \leq i \leq n$ is a list of n edge-cuts of G containing every edge. It follows that $\chi(G) \leq 2^n$, so $G \succ_{hom} K_{2^n}$. If $G \succ_{hom} K_{2^n}$, then by Theorem 4.6 we have that $G \succ^t_{\mathbb{Z}_2} K_{2^n}$. $\square$

Thus, we find that K_3 and K_4 are equivalent and that the sequence $K_4, K_8, K_{16}, \ldots$ is a strict descending chain in ${\mathbb{Z}_2}^t$. For every graph G we define the *log-chromatic number* $\chi_{\log}(G) = \lceil \log_2 \chi(G) \rceil = \min\{n : G \succ_{hom} K_{2^n}\}$. Our next proposition shows that $\chi_{\log}$ is a monotone invariant with respect to the cut continuous order.

Proposition 6.7 *If $G \succ^t_{\mathbb{Z}_2} H$, then $\chi_{\log}(G) \leq \chi_{\log}(H)$.*

Proof: Let $\chi_{\log}(H) = n$. Then we have that $G \succ^t_{\mathbb{Z}_2} H \succ^t_{\mathbb{Z}_2} K_{2^n}$ so by Proposition 6.6 $G \succ_{hom} K_{2^n}$ so $\chi_{\log}(G) \leq \chi_{\log}(H)$ as desired. $\square$

With this last proposition, we are ready to construct an infinite antichain in the cut-continuous order.

Proposition 6.8 *The order $\succ^t_{\mathbb{Z}_2}$ contains an infinite antichain.*

Proof: Let G_0 be an arbitrary graph. To create G_{i+1} given $G_0, G_1, \ldots, G_i$, choose G_{i+1} to be a graph with $\gamma^o(G_{i+1}) > \gamma^o(G_i)$ and with $\mu(G_{i+1}) > \mu(G_i)$ (such a graph always exists due to the existence of graphs with arbitrarily high girth and chromatic number). It now follows from Proposition 6.2 that there is no cut-continuous map from G_j to G_{i+1} for every $1 \leq j \leq i$ and from proposition 6.7 that there is no cut-continuous map from G_{i+1} to G_j for every $1 \leq j \leq i$. $\square$

We have two monotone invariants in the cut-continuous order, namely γ^o and μ. The above construction uses these two invariants to build an infinite antichain. It would be interesting to know if graphs of high chromatic number are essential for this construction. In particular, we offer the following problem.

Problem 6.9 *Does there exist an infinite antichain of graphs in the cut-continuous order which have a bounded chromatic number?*

7 The order $\succ^f_{\mathbb{Z}}$

The order $\succ^f_{\mathbb{Z}}$ is similar to the flow order $\succ^f_{\mathbb{Z}_2}$ except that the orientations of edges begin to play a strong role.

$$\begin{tikzcd}
\mathbf{E}(\vec{\mathbf{G}}) \arrow[rr, "\mathbf{f}"] \arrow[dr, "\phi\circ\mathbf{f}"'] \arrow[rr, phantom, shift right=6ex, "\text{Z–cycle}"] & & \mathbf{E}(\vec{\mathbf{H}}) \arrow[dl, "\phi"] \\
& \mathbf{Z} &
\end{tikzcd}$$

Figure 7: Z-cycle

We begin with a corollary of Theorem 4.3 which gives us the maximal and minimal elements in this order.

Corollary 7.1 *The maximal elements of* $\succ^f_{\mathbb{Z}}$ *are the graphs in which every vertex has even degree. The minimal elements are the graphs with cut-edges.*

We say that an oriented graph $\vec{G}$ is a *(mod p)* orientation (of G) for a positive integer p if $deg^+(v) - deg^-(v) \equiv 0$ (modulo p) for every $v \in V(G)$. We say that an undirected graph G has a *mod* p orientation if there is an orientation of G which is a mod p orientation. Note that if G has a mod p orientation for an even integer p, then every vertex of G has even degree. With this definition, we are ready to characterize when $K_2^{2a+1} \succ^f_{\mathbb{Z}} G$ and when $G \succ^f_{\mathbb{Z}} K_2^{2a+1}$.

Proposition 7.2
1. $K_2^{2a+1} \succ^f_{\mathbb{Z}} G$ *if and only if* $\lambda^o(G) \leq 2a+1$.
2. $G \succ^f_{\mathbb{Z}} K_2^{2a+1}$ *if and only if* G *has a mod* $(2a+1)$ *orientation.*

Proof: If $K_2^{2a+1} \succ^f_{\mathbb{Z}} G$, then $K_2^{2a+1} \succ^f_{\mathbb{Z}_2} G$, so by Proposition 5.3, $\lambda^o(G) \leq 2a+1$. If $\lambda^o(G) \leq 2a+1$, then choose an odd edge-cut $\Delta(X)$ of G of size $2b+1 \leq 2a+1$. Let $\vec{G}$ be an orientation of G such that $\Delta^-(X) = \emptyset$. Let u, v be the vertices of ${K_2}^{2a+1}$, let $e_1, e_2, \ldots, e_{2a+1}$ be the edges of ${K_2}^{2a+1}$

and let $\vec{K_2}^{2a+1}$ be an orientation with edges $e_1, e_2, \ldots, e_{a+b+1}$ directed from u to v and with edges $e_{a+b+2}, \ldots, e_{2a+1}$ directed from v to u. Choose a map $f : E(\vec{K_2}^{2a+1}) \to E(\vec{G})$ such that f maps $\{e_1, e_2, \ldots, e_{2b+1}$ injectively onto the set $\Delta^+(X)$ and such that $f(\{e_{2b+2}, e_{2b+3}, \ldots, e_{2a+1}\}) = \{e\}$ for some $e \in E(\vec{G})$. For every flow $\phi : E(\vec{G}) \to \mathbb{Z}$ the map $\phi \circ f$ is a flow of $\vec{K_2}^{2a+1}$, so we have that $K_2^{2a+1} \succ^f_{\mathbb{Z}} G$ as desired.

If $G \succ^f_{\mathbb{Z}} K_2^{2a+1}$, then let $\vec{H}$ be an orientation of K_2^{2a+1} such that every edge has the same head and choose an orientation $\vec{G}$ of G and a $\mathbb{Z}$-flow-continuous map $f : E(\vec{G}) \to E(\vec{H})$. We claim that $\vec{G}$ is a mod $2a+1$ orientation of G. Let $\{e_1, e_2, \ldots, e_{2a+1}\} = E(\vec{H})$ and let $X_i = f^{-1}(\{e_i\})$ for $1 \leq i \leq 2a+1$. Now for every $2 \leq i \leq 2a+1$, the map $\phi : E(\vec{H}) \to \mathbb{Z}$ given by the rule

$$\phi_i(e) = \begin{cases} 1 & \text{if } e = e_1 \\ -1 & \text{if } e = e_i \\ 0 & \text{otherwise} \end{cases}$$

is a flow. Thus $\phi_i \circ f$ is a 2-flow of $\vec{G}$ and for every $v \in V(G)$ we have that $\sum_{e \in \Delta^+(v)} \phi_i(e) - \sum_{e \in \Delta^-(v)} \phi_i(e)$. It follows from this that

$$|X_1 \cap \Delta^+(v)| - |X_1 \cap \Delta^-(v)| = |X_i \cap \Delta^+(v)| - |X_i \cap \Delta^-(v)|$$

holds for every $v \in V(G)$. Since the above equation holds for every $1 \leq i \leq 2a+1$ we have that $|\Delta^+(v)| - |\Delta^-(v)| = (2a+1)(|X_i \cap \Delta^+(v)| - |X_i \cap \Delta^-(v)|)$. Thus $\vec{G}$ is a mod $2a+1$ orientation of G as desired.

Let $\vec{G}$ be a mod $2a+1$ orientation of G. Suppose that there is a vertex $v \in V(G)$ with $\Delta^+(v) \neq \emptyset$ and $\Delta^-(v) \neq \emptyset$. Choose an edge $e \in \Delta^-(v)$ with tail u and an edge $e' \in \Delta^+(v)$ with head w and form a new oriented graph $\vec{G_1}$ by deleting the edges e, e' and adding a new edge directed from u to w. Now $\vec{G_1}$ is still a mod $2a+1$ orientation. Further, any M-flow continuous map from G_1 to H naturally extends to a M-flow continuous map from G to H. Thus, by repeating this operation, we may assume that either $\Delta^+(v)$ or $\Delta^-(v)$ is empty for every $v \in V(G)$. Suppose that $v \in V(G)$ with $|\Delta^+(v)| > 2a+1$. Then we may form a new oriented graph $\vec{G_2}$ by replacing v by two new vertices v_1, v_2 so that every edge incident with v now attaches to one of v_1, v_2 and such that $|\Delta^+_{G_2}(v_1)|$ and $|\Delta^+_{G_2}(v_2)|$ are both positive multiples of $(2a+1)$. As before, $\vec{G_2}$ is a mod $2a+1$ orientation. Further, any M-continuous map from G_2 to H can easily be extended to a M-continuous map from G to H. Thus, by repeating this operation, we may assume that $|\Delta^+(v)| = 2a+1$ and $\Delta^-(v) = \emptyset$ or $|\Delta^-(v)| = 2a+1$ and

$\Delta^+(v) = \emptyset$ for every $v \in V(G)$. Let $X = \{v \in V(G) \mid \Delta^-(v) = \emptyset\}$ and let $Y = V(G) \setminus X$. Then G is a $(2a+1)$-regular bipartite graph with bipartition (X, Y). By König's theorem, there exists a partition of $E(G)$ into perfect matchings $\{Z_1, Z_2, \ldots, Z_{2a+1}\}$. Let $\vec{H}$ be an orientation of K_2^{2a+1} so that every edge has the same head and let $\{e_1, e_2, \ldots, e_{2a+1}\}$ be arcs of $\vec{H}$. Define the map $f : E(\vec{G}) \to E(\vec{H})$ by the rule $f(e) = e_i$ if $e \in Z_i$. It follows easily that f is $\mathbb{Z}$-flow continuous. This completes the proof. □

Based on this proposition, we find an infinite chain of graphs of the form K_2^{2a+1} as before.

Proposition 7.3 *The graphs $K_2^1, K_2^3, K_2^5, \ldots$ form an scaling chain.*

Proof: This follows immediately from part 1 of the preceeding proposition. □

Jaeger [9] has conjectured that every $4p$-edge-connected graph has a mod $(2p+1)$ orientation. He proved that if this conjecture is true, then both Tutte's 3-Flow and 5-Flow conjectures are also true. The following is a slight extension of this conjecture which seems quite reasonable and is suggested by some work of Zhang (see [25]) and Zhu (see [26]).

Conjecture 7.4 (An extension of Jaeger's modular orientation conjecture) *If $K_2^{4p-1} \not\succ_{\mathbb{Z}}^{f} G$, then $G \succ_{\mathbb{Z}}^{f} K_2^{2p+1}$.*

A graph G has a nowhere-zero 3-flow if and only if it has a mod 3 orientation which exists if and only if $G \succ_{\mathbb{Z}}^{f} K_2^3$. The following proposition shows that G has a nowhere-zero 4-flow if and only if $G \succ_{\mathbb{Z}}^{f} K_4$.

Proposition 7.5 *A graph G has a nowhere-zero 4-flow if and only if $G \succ_{\mathbb{Z}}^{f} K_4$.*

Proof: Tutte proved that G has a nowhere-zero 4-flow if and only if it has a B-flow where $B \subseteq \mathbb{Z}^4$ is the set of all vectors with two 0's, one 1, and one -1. Thus, the proposition follows from Theorem 3.2 and the observation that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

represents the $\mathbb{Z}$-cycle-space of K_4. □

Based on the above propositions, we have that the Petersen graph does not dominate K_2^3 or K_4 and that K_2^3 does dominate the Petersen. It is not difficult to verify the the Petersen graph and K_4 are $\succ_{\mathbb{Z}}^f$-incomparable. In analogy with Jaeger's conjecture on cycle-continuous mappings to the Petersen graph, we offer the following problem.

Problem 7.6 *Are P_{10} and K_4 the atoms of $\succ_{\mathbb{Z}}^f$?*

Problem 2.9 above may be out of reach yet, but the ordering $\succ_{\mathbb{Z}}^f$ does suggest some more approachable questions. For instance, there is a 3 elements chain consisting of $K_2^3 \succ_{\mathbb{Z}}^f V_8 \succ_{\mathbb{Z}}^f K_4$ (one can prove that K_4 and V_8, see figure 8, are not equivalent in the order $\succ_{\mathbb{Z}}^f$). Jaeger's 4-flow theorem asserts that every 4-edge-connected graph dominates K_4, while a conjecture of Jaeger (actually a weak version of Tutte's 3-flow conjecture) asserts that for some k every k-edge-connected graph dominates K_2^3. The following problem is a natural weakening of that conjecture.

Problem 7.7 *Does there exist a fixed integer k so that every k-edge-connected graph dominates V_8?*

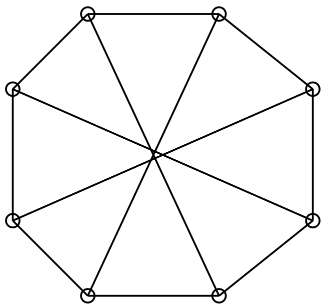

Figure 8: V_8

8 The order $\succ_{\mathbb{Z}}^t$

The order $\succ_{\mathbb{Z}}^t$ is similar to the order $\succ_{\mathbb{Z}_2}^t$ except that the orientations of edges begin to play a strong role and it is more related to the homomorphism order $\succ_{hom}$.

We begin by applying Theorem 4.3 to establish the maximal and minimal elements.

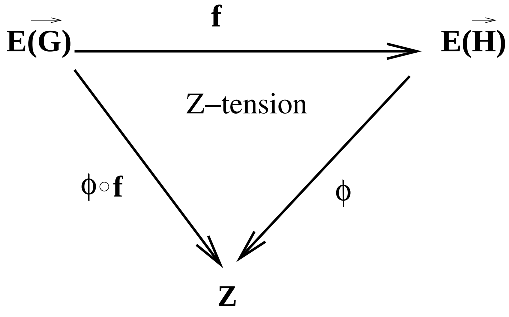

Figure 9: Z-tension

Corollary 8.1 *The maximal graphs in the order* $\succ^t_{\mathbb{Z}}$ *are the bipartite graphs. The minimal graphs are the graphs which contain loop edges.*

As before, odd-girth γ^o is a monotone invariant.

Proposition 8.2 *If* $G \succ^t_{\mathbb{Z}} H$ *then* $\gamma^o(G) \geq \gamma^o(H)$

Proof: If $G \succ^t_{\mathbb{Z}} H$, then $G \succ^t_{\mathbb{Z}_2} H$, so by Proposition 6.2 $\gamma^o(G) \geq \gamma^o(H)$ as required. $\square$

Proposition 6.6 showed that $G \succ^t_{\mathbb{Z}_2} K_{2^n}$ if and only if $G \succ_{hom} K_{2^n}$. The following proposition gives a similar equivalence for the order $\succ^t_{\mathbb{Z}}$ but for a much richer class of graphs.

Theorem 8.3 $G \succ^t_{\mathbb{Z}} Cayley(\mathbb{Z}_n, B)$ *if and only if* $G \succ_{hom} Cayley(\mathbb{Z}_n, B)$.

Proof: The "if" direction is an immediate consequence of Theorem 4.6. To prove the "only if" direction, let G be a graph with $G \succ^t_{\mathbb{Z}} Cayley(\mathbb{Z}_n, B)$. Then $G \succ^t_{\mathbb{Z}_n} Cayley(\mathbb{Z}_n, B)$. Now, $Cayley(\mathbb{Z}_n, B)$ has a B-tension, so G also has an B-tension. But then by Proposition 3.4 we have that $G \succ_{hom} Cayley(\mathbb{Z}_n, B)$. $\square$

Based on this theorem, we have the following corollary.

Corollary 8.4
1. $G \succ^t_{\mathbb{Z}} K_n$ *if and only if* $G \succ_{hom} K_n$.
2. $G \succ^t_{\mathbb{Z}} C_n$ *if and only if* $G \succ_{hom} C_n$.

Proof: 1. follows from $K_n \cong Cayley(\mathbb{Z}_n, \mathbb{Z}_n \setminus \{0\})$. 2. follows from $C_n \cong Cayley(\mathbb{Z}_n, \{-1, 1\})$. $\square$

Based on this proposition, we have the following infinite chains.

Proposition 8.5 *In the order $\succ^f_{\mathbb{Z}}$, the graphs $C_3, C_5, C_7, \ldots$ form a scaling chain and $K_3, K_4, K_5, \ldots$ form a decreasing chain.*

Proof: This follows immediately from the above corollary. $\square$

We proved the existence of an infinite antichain $\mathcal{X}$ in the order $\succ^t_{\mathbb{Z}_2}$. It follows from Theorem 4.4 that $\mathcal{X}$ is also an infinite antichain in the order $\succ^t_{\mathbb{Z}}$. However even for the more restrictive order $\succ^t_{\mathbb{Z}}$ we do not know whether there exists an infinite antichain in $\succ^t_{\mathbb{Z}}$ which consists from graphs of bounded chromatic number (this is analogy of the Problem 6.9).

9 Jaeger's Order

As we stated in the introduction one of our main motivations for this paper was provided by Jaeger's work. However rather than following his actual definitions we tried to follow his ideas and it is in this section where we compare our approach (which we believe is more streamlined one) to the Jaeger's original definitions.

In [8] Jaeger defined the following relation: Let $G_1 = (E_1, V_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We say that $G_1 \prec_J G_2$ if and only if there exits a subdivision $G'_1 = (V'_1, E'_1)$ of G_1 and a bijective mapping β from E_2 to E'_1 such that, for each $\mathbb{Z}_2$-flow ϕ of G'_1, $\phi \circ \beta$ is a $\mathbb{Z}_2$-flow of G_2. We write $G_1 \simeq_J G_2$ if $G_1 \prec_J G_2$ and $G_2 \prec_J G_1$. It is easy to see that $\prec_J$ is a quasi-order. We now have the following proposition relating $\prec_J$ with $\succ^f_{\mathbb{Z}_2}$.

Proposition 9.1 *If G_1, G_2 are graphs then $G_1 \succ_J G_2$ if and only if there exists a cycle-continuous map $\phi : E(G_1) \to E(G_2)$ which is onto. In particular, $G_1 \succ_J G_2$ implies that $G_1 \succ^f_{\mathbb{Z}_2} G_2$.*

Proof: This is easy to see. The "if" direction is clear. For the "only if" direction observe that it suffices to subdivide every $e \in E(G_2)$ by $|f^{-1}(e)| - 1$ vertices. $\square$

It follows from this proposition that the order $\succ_J$ is a subset of $\succ^f_{\mathbb{Z}_2}$. The following proposition is an easy consequence of this fact.

Proposition 9.2 *Let $B \subseteq \mathbb{Z}_2^k$ for some positive integer k. If $G \succ_J H$ and H has a B-flow, then G also has a B-flow.*

Proof: This follows immediately from Propositions 9.1 and 2.6. $\square$

If G is a graph we will denote by $\mu(G)$ the dimension of its cycle space and we denote by $\tau(G)$ the maximum number of edge-disjoint circuits of G. The following proposition proved by Jaeger now follows from the definitions:

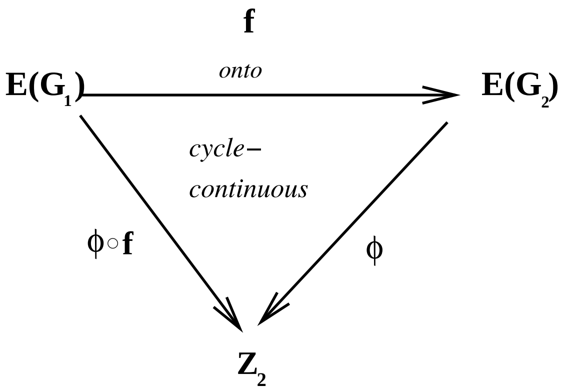

Figure 10: $G_1 \succ_J G_2$

Proposition 9.3 *If $G_2 \succ_J G_1$ then the following holds:*
i. $|E(G_1)| \leq |E(G_2)|$*;*
ii. $\mu(G_1) \leq \mu(G_2)$*;*
iii. $\tau(G_1) \leq \tau(G_2)$

The monotone parameters appearing in the above proposition make the construction of arbitrarily large antichains for $\succ_J$ quite simple. For instance, to form an antichain of size k, choose graphs $G_1, G_2, \ldots, G_k$ so that $\lambda^o(G_1) > \lambda^o(G_2) > \ldots > \lambda^o(G_k)$ and then subdivide edges of these graphs so that $|E(G_1)| < |E(G_2)| < \ldots < |E(G_k)|$. Despite these simple constructions we do not know if there exists an infinite antichain in Jaeger's order. Next we establish a scaling chain in $\succ_J$.

Proposition 9.4 *The graphs $K_2^1, K_2^3, K_2^5, \ldots$ form a scaling chain in $\succ_J$.*

Proof: To show that $K_2^1, K_2^3, K_2^5, \ldots$ is a chain we will show that $K_2^{2a+1} \succ_J K_2^{2a-1}$ for every $a \geq 1$. To see this, replace one edge of K_2^{2a-1} by a path of length three to form the graph $(K_2^{2a-1})'$. Now, let $f : E(K_2^{2a+1}) \rightarrow E((K_2^{2a-1})')$ be a bijection. It follows easily that f satisfies Jaeger's condition.

Let G be a graph with a vertex v of degree $2k+1$ and let $m = |E(G \setminus v)|$. We claim that $K_2^{2m+2k+1} \succ_J G$. To see this, subdivide every edge $e \in E(G)$ not incident with v to form the graph G' and let ϕ be a bijection from $E(K_2^{2m+2k+1})$ to $E(G')$. By construction, every cycle of G' has an even number of edges. Thus ϕ demonstrates that $K_2^{2m+2k+1} \succ_J G$ and we conclude that $K_2^1, K_2^3, K_2^5, \ldots$ is a scaling chain. $\square$

If A is a matrix over $\mathbb{Z}_2$ which represents the cycle-space of P_{10} and B is the set of columns in A, then we say that a B-flow of a graph G is a *Petersen*-flow of G. In Jaeger's original article, he conjectured that every bridgeless graph G must satisfy $G \succ_J P_{10}$, $G \succ_J K_2^3$, or $G \succ_J K_1^1$. This is equivalent to Conjecture 2.8, and Jaeger showed that it is also equivalent to the conjecture that every bridgeless graph has a Petersen-flow. These equivalent conjectures are collected in the following theorem.

Theorem 9.5 *For every graph G, the following statements are equivalent.*
(i) $G \succ_J P_{10}$ or $G \succ_J K_2^3$ or $G \succ_J K_1^1$.
(ii) $G \succ_{\mathbb{Z}_2}^f P_{10}$
(iii) G has Petersen-flow.

Proof: The equivalence between (ii) and (iii) is an immediate consequence of Theorem 3.2. Since $K_1^1 \succ_{\mathbb{Z}_2}^f K_2^3 \succ_{\mathbb{Z}_2}^f P_{10}$, it follows that (i) implies (ii). To see that (ii) implies (i), let $\phi : E(G) \to E(P_{10})$ be cycle-continuous. If G is onto, then $G \geq_J P_{10}$ and we are finished. Otherwise, there is an edge $e \in E(P_{10})$ not in the image of ϕ and we find that $G \succ_{\mathbb{Z}_2}^f P_{10} \setminus e \succ_{\mathbb{Z}_2}^f K_2^3$. If ψ is a cycle-continuous map from G to K_2^3, then either ψ is onto and $G \geq_J K_2^3$ or there is an edge $f \in E(K_2^3)$ not in the image of ψ and we find that $G \succ_{\mathbb{Z}_2}^f K_2^3 \setminus f \succ_{\mathbb{Z}_2}^f K_1^1$. In this case we must have $G \succ_J K_1^1$ so we are done. $\Box$

If G is a cubic graph, then a *Petersen edge-coloring* of G is a coloring of the edges of G using edges of P_{10} so that any three adjacent edges of G map to three adjacent edges of P_{10}.

Proposition 9.6 *If G is a cubic graph, then the following statements are equivalent.*
(i) $G \succ_{\mathbb{Z}_2}^f P_{10}$
(ii) G has a Petersen edge-coloring

Proof: It follows immediately that (ii) implies (i). To see the reverse direction, note that by Proposition 5.3, in every cycle-continuous mapping from G to P_{10}, the image of every vertex star must be a vertex star. $\Box$

In Jaeger's original article, he showed that Conjecture 2.8 could be reduced to cubic graphs, thus establishing another form of his conjecture :

Conjecture 9.7 (Jaeger) *Every bridgeless cubic graph has a Petersen edge-coloring.*

10 Directed Cycle Continuous Maps

In this and the following section, we turn our attention to directed graphs. In this section we define a directed analogue of cycle continuous maps, and in the next section we define a directed analogue of cut continuous maps. Since we will not need to consider different orientations of undirected graphs, we shall use G, H to denote directed graphs.

If G is a directed graph and $C \subseteq E(G)$, then we say that C is a *directed cycle* if every vertex of the graph $(V(G), E(G) \setminus C)$ has indegree equal to outdegree. If C is a nonempty directed cycle which is minimal with respect to inclusion, then we say that C is a *directed circuit*. We say that a map $f : E(G) \to E(H)$ is *directed-cycle-continuous* if $f^{-1}(C)$ is a directed cycle of G whenever C is a directed cycle of H.

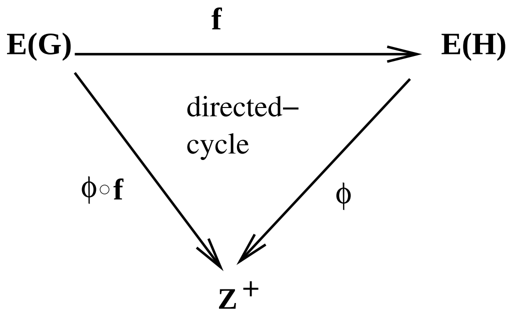

Figure 11: Directed-cycle

We define the relation $\succ_{dcycle}$ by the rule $G \succ_{dcycle} H$ if there exists a directed-cycle-continuous map from G to H.

We say that a $\mathbb{Z}$-flow $\phi : E(G) \to \mathbb{Z}$ is nonnegative if $\phi(e) \geq 0$ for every $e \in E(G)$. In this case, we say that ϕ is a $\mathbb{Z}^+$-*flow*. The following theorem gives an equivalent formulation of directed cycle continuous maps in terms of $\mathbb{Z}^+$-flows.

Theorem 10.1 *Let G, H be directed graphs. For every map $f : E(G) \to E(H)$, the following statement are equivalent:*

1. *f is directed cycle continuous*
2. *For every $\mathbb{Z}^+$-flow ϕ of H, $\phi \circ f$ is a $\mathbb{Z}^+$-flow of G.*

Proof: 1 $\Rightarrow$ 2: Let ϕ be a $\mathbb{Z}^+$-flow of H. Choose nonnegative 2-flows $\phi_1, \phi_2, \ldots, \phi_k$ so that $\sum_{i=1}^{k} \phi_i = \phi$. Now, the support of each ϕ_i is a directed cyle C_i. By hypothesis $f^{-1}(C_i)$ is a directed cycle. Let ψ_i the nonnegative

2-flow of G with support $f^{-1}(C_i)$ (so $\psi_i(e) = 1$ if $e \in f^{-1}(C_i)$ and $\psi_i(e) = 0$ otherwise) and let $\psi = \sum_{i=1}^{k} \psi_i$. Now $\phi \circ f = \psi$ is a $\mathbb{Z}^+$-flow of G as required.

$2 \Rightarrow 1$ Let C be a directed cycle of H and let ϕ be the $\mathbb{Z}^+$-flow of H given by the rule $\phi(e) = 1$ if $e \in C$ and $\phi(e) = 0$ otherwise. By hypothesis, $\psi = \phi \circ f$ is a $\mathbb{Z}^+$-flow. Now, $\psi(e) = 1$ if $e \in f^{-1}(C)$ and $\psi_i(e) = 0$ otherwise, so $f^{-1}(C)$ is a directed cycle as required. □

Corollary 10.2 *If G, H are digraph, H has a nonnegative nowhere-zero k-flow ϕ, and $G \succ_{dcycle} H$, then G has a nonnegative nowhere-zero k-flow.*

Proof: If $f : E(G) \to E(H)$ is directed cycle continuous, then $\phi \circ f$ is a nonnegative nowhere-zero k-flow of G. □

We say that G is *balanced* if $E(G)$ is a directed cycle. The following proposition establishes that the maximal digraphs in the order $\succ_{dcycle}$ are precisely the balanced digraphs.

Proposition 10.3 *A digraph is maximal in $\succ_{dcycle}$ if and only if it is balanced. A digraph is minimal in $\succ_{dcycle}$ if and only if it is not strongly connected.*

Proof: If G is balanced, then for any digraph H and any edge $e \in E(H)$, the map which sends every edge of G to e is directed cycle countinuous. If G is a directed graph with a vertex for which indegree is not equal to outdegree, then there is no directed cycle continuous map from G to the directed graph with one vertex and one loop. Thus the maximal elements in $\succ_{dcycle}$ are the balanced digraphs as claimed.

If G is not strongly connected, then for any edge $e \in E(G)$ which is not in a directed circuit and for any digraph H, the map which sends every edge of H to e is directed cycle continuous. If G is a digraph which is strongly connected, then there is no directed cycle continuous map from G to the digraph consisting of a single arc joining two vertices. It follows that the minimal elements of $\succ_{dcycle}$ are the digraphs which are not strongly connected as claimed. □

We define the directed graph $D_{a,b}$ to be the loopless digraph with vertex set $\{u, v\}$ and with a edges from u to v and b edges from v to u. We call the edges from u to v *up* edges and the edges from v to u *down* edges. The following proposition shows a scale in the order $\succ_{dcycle}$ using these digraphs.

Proposition 10.4 *The set $\{D_{a,b} \mid a > b\}$ is a scale in the directed cycle order.*

Proof: Let G be a digraph which is not balanced and choose $X \subseteq V(G)$ so that $|\Delta^+(X)| = a > b = |\Delta^-(X)|$. Choose a map $\phi : E(D_{a,b}) \to E(G)$ so that the a up edges map bijectively to $\Delta^+(X)$ and so that the b down edges map bijectively to $\Delta^-(X)$. It follows easily that ϕ is directed cycle continuous. $\square$

The next proposition characterizes when $D_{a,b} \succ_{dcycle} D_{c,d}$.

Proposition 10.5 *If a, b, c, d are positive integers with $a > b$ and $c > d$, then $D_{a,b} \geq_{dcycle} D_{c,d}$ if and only if $s = (a-b)/(c-d)$ is an integer, $a \geq sc$ and $b \geq sd$.*

Proof: We begin with the following auxilliary lemma.

Lemma 10.6 *If $f : E(G) \to E(D_{a,b})$ is directed cycle continuous, then for every $X \subseteq V(G)$ there is a nonnegative integer s to that for every up edge $e \in E(D_{a,b})$ and every down edge $e' \in E(D_{a,b})$ we have the following equalities.*

$$\begin{aligned} s &= |f^{-1}(e) \cap \Delta^+(X)| - |f^{-1}(e) \cap \Delta^-(X)| \\ s &= |f^{-1}(e') \cap \Delta^-(X)| - |f^{-1}(e') \cap \Delta^+(X)| \end{aligned}$$

Proof of Lemma: If e is an up edge of $D_{a,b}$ and e' is a down edge of $D_{a,b}$ then $f^{-1}(\{e,e'\})$ is a directed cycle so $|f^{-}(\{e,e'\}) \cap \Delta^+(X)| = |f^{-1}(\{e,e'\}) \cap \Delta^-(X)|$. It follows from this that $|f^{-1}(e) \cap \Delta^+(X)| - |f^{-1}(e) \cap \Delta^-(X)| = |f^{-1}(e') \cap \Delta^-(X)| - |f^{-1}(e') \cap \Delta^+(X)|$. Since this equality must hold for every up edge and every down edge, there must exist a fixed s with the properties claimed above. $\square$

It follows from the lemma that $s = (a-b)/(c-d)$ must be an integer and it is clear that $a \geq sc$ and $c \geq sd$. If a, b, c, d are positive integers satisfying the properties above, then it is possible to construct a directed cycle continuous map from $D_{a,b}$ to $D_{c,d}$ by sending s up edges of $D_{a,b}$ to every up edge of $D_{c,d}$, sending s down edges of $D_{a,b}$ to every down edge of $D_{c,d}$ and then sending all remaining unassigned edges of $D_{a,b}$ to the same edge of $D_{c,d}$. $\square$

The following proposition is an immediate consequence of the above result and establishes the existence of an infinite antichain.

Proposition 10.7 *The set* $\{D_{1,b} \mid b \geq 0\}$ *is an antichain.*

Proof: This follows immediately from Proposition 10.5. $\square$

Next we have the following simple lemma.

Lemma 10.8 *Let* G *be a strongly connected digraph which is not balanced. Then* $D_{1,k}$ *is incomparable with* G *whenever* $k > |E(G)|$.

Proof: Let e' be the unique up edge of $D_{1,k}$ and let $e_1, e_2, \ldots, e_k$ be the down edges of $D_{1,k}$. First suppose that there is a directed cycle continuous map $f : E(G) \to E(D_{1,k})$. By assumption, there is a down edge e_j such that $f^{-1}(\{e_j\}) = \emptyset$. Thus $f^{-1}(\{e'\}) = f^{-1}(\{e', e_j\})$ is a directed cycle. Since $f^{-1}(\{e', e_i\})$ must be a directed cycle for every $1 \leq i \leq k$, we find that $f^{-1}(\{e_i\})$ is directed cycle for every $1 \leq i \leq k$ and we conclude that $E(G) = f^{-1}(E(D_{1,k})$ is a directed cycle. This contradicts our assumption.

Next suppose that there is a directed cycle continuous map $f : E(D_{1,k}) \to E(G)$. By assumption, there is an edge $s \in E(G)$ so that $|f^{-1}(\{s\}) \cap \{e_1, e_2, \ldots, e_k\}| \geq 2$. Since G is strongly connected, we may choose a directed circuit C of G so that $s \in C$. Now $f^{-1}(C)$ is a directed cycle of $D_{1,k}$ by assumption, but $f^{-1}(C)$ contains at least two down edges of $D_{1,k}$. Since this is impossible, there must not exist such a map f. $\square$

We finish the study of this order with the following proposition.

Proposition 10.9 *The order* $\succ_{dcycle}$ *does not have any atoms.*

Proof: Let G be an atom of $\succ_{dcycle}$. It is $G \succ_{dcycle} \vec{K}_2$ and G is strongly connected (by proposition 10.3). Let $D_{1,k}$ be incomparable with G. Then the disjoint union $G + D_{1,k} = G'$ satisfies : $G \succ_{dcycle} G'$ and $G' \not\succ_{dcycle} G$ which contradicts the assumption. (We could also define G' as $1 - sum$ of G and $D_{1,k}$; thus G' can be assumed to be connected).
$\square$

11 Directed Cut Order

In this section, we define and study a directed cut order. We have omitted several of the proofs from this section which are quite similar to arguments appearing in the previous section. If G is a directed graph and $X \subseteq V(G)$ then we say that $\Delta(X)$ is a *directed cut* if either $\Delta^+(X) = \emptyset$ or $\Delta^-(X) = \emptyset$. If H is a directed graph, and $f : E(G) \to E(H)$ then we say that f is *directed*

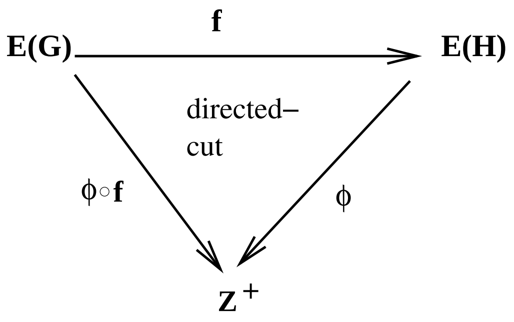

Figure 12: Directed-cut

cut continuous if $f^{-1}(C)$ is a directed cut of G whenever C is a directed cut of H.

We define the relation $\succ_{dcut}$ by the rule $G \succ_{dcut} H$ if there exists a directed cut continuous map from G to H. We call the order $\succ_{dcut}$ the *directed cut order.* This order is dual to the directed cycle order and is related to the orders for undirected graphs based on tensions. We say that a $\mathbb{Z}$-tension $f : E(G) \to \mathbb{Z}$ is nonnegative if $f(e) \geq 0$ for every $e \in E(G)$. In analogy with Theorem 10.1 we have the following basic result. Since the proof of this theorem is quite similar to that of Theroem 10.1, we omit it.

Theorem 11.1 *Let G, H be directed graphs. For a mapping $f : E(G) \to E(H)$ the following statement are equivalent:*

1. *f is directed cut continuous*
2. *For any $\mathbb{Z}^+$-tension ϕ of H, $\phi \circ f$ is a $\mathbb{Z}^+$-tension of G.*

The following proposition establishes the maximal and minimal elements in this order. In particular it shows that the graphs of interest in the order $\succ_{dcut}$ are the acyclic digraphs (herafter referred to as DAG's). Since this proof is quite similar to the proof of Proposition 10.3, we omit it.

Proposition 11.2 *A digraph G is minimal in the order $\succ_{dcut}$ if and only if it contains a directed circuit. G is maximal in $\succ_{dcut}$ if and only if $E(G)$ is a directed cut.*

It is now easy to see that the order $\leq_{dcut}$ has no atoms as under every DAG H we find a sufficiently large tournament T_n. However the upper end of the directed cut order is quite interesting: the single arc (i.e. the path P_1) covers the directed path P_2 of length 2 (by a case analysis), however the structure of the graphs under P_2 seems to be already very complicated.

This is indicated by the following which relates these cases to combinatorial optimization:

Denote by $\tau(G)$ the maximal number of arcs of G no two of which are contained in same directed cut of G. Similarly denote by $\kappa(G)$ the minimal number of directed cuts in G which cover all the arcs of G (we define this for DAG's only). Let T_n denote the transitive tournament on n vertices.

Proposition 11.3 *Let G be a directed graph. Then*

i. $\tau(G)$ is equal to the maximal length of a directed path P_k for which $P_k \succ_{dcut} G$;

ii. $\kappa(G)$ is bounded from above by the minimal k for which it holds $G \succ_{dcut} T_k$

iii. $\kappa(G) \leq \kappa(H)$ whenever $G \prec_{dcut} H$.

Proof: We compare the definitions. All directed cuts of P_k are formed by single edges and *ii.* follows from *iii.* which in turn is yet another version of the critical problem argument. $\square$

The number $\kappa(G)$ is easily computable and is equal to the largest length of an directed path in G (recall that we are only considering acyclic digraphs), this length is also called *height* of G and it is equal to the minimal k such that $G \prec T_k$ (recall, $\prec_{hom}$ is the homomorphism order introcued in Section 4). Summarizing we have: For every acyclic digraph G the following holds:

$$G \not\prec_{hom} P_k \Longleftrightarrow T_k \prec_{hom} G.$$

In this homomorphism setting this statement is one of the simplest *singleton homomorphism dualities* which were completely characterized for graphs and relational structures in [15] (all these dualities relate to DAG's (and their generalization for relational structures). Several (easier) types of these dualities we tested for the directed cut order and they remain valid. The simplest two such examples are the following:

$$G \not\prec_{dcut} P_2 \Longleftrightarrow T_2 \prec_{dcut} G.$$
$$G \not\prec_{dcut} P_3 \Longleftrightarrow T_3 \prec_{dcut} G.$$

We do not know any other examples of such dualities for the directed cut order.

We shall finish this paper with another relationship of the orders $\leq_{dcut}$ and $\prec_{hom}$ which points to yet another direction:

Proposition 11.4 *For every undirected graph G one can associate a DAG $\Phi(G)$ such that for any two graphs G, H there exists a homomorphism $H \to$*

G if and only if there is a directed cut continuous mapping $\Phi(H) \to \Phi(G)$. Formally,

$$\Phi(G) \prec_{dcut} \Phi(H) \iff H \prec_{hom} G.$$

Proof: (a sketch) We describe $\Phi(G) = (V', E')$ for a graph $G = (V, E)$:

Put $V' = V \times \{0, 1, 2, 3, 4\}$ and let E' be the set of the following arcs:
$((v, i), (v, i + 1))$ for $i = 0, 1, 2, 3, v \in V$;
$((v, 1), (v', 3))$ for every edge $\{v, v'\} \in E$.

Note that any directed cut continuous mapping maps the set of edges of any directed path to a set of edges no two of which belong to a common directed cut. In the graph $\Phi(G)$ the only such a set of edges of size 4 corresponds to a directed path of length 4. Analyzing the directed paths in $\Phi(G)$ further we see that there are two types of directed paths: of length 4 and these paths correspond to the vertices of G and of length 3 and these paths contain arcs corresponding to the edges of G. It is then not difficult to conclude that any directed cut continuous mapping $\Phi(H) \to \Phi(G)$ implies the existence of a homomoprhism $H \to G$. $\square$

It follows the that the directed cut order $\prec_{dcut}$ is indeed very rich: It contains every countable partial order as induced suborder. See [14] for more on the homomorphism order.

Appendix

The purpose of this appendix is to prove Theorem 5.11. In this section, it will be helpful for us to work with maps which are not necessarily flows. Let G be a graph, let k be a positive integer, and let $\phi : E(G) \to \mathbb{Z}_2^k$. The *boundary* of ϕ is the map $\partial\phi : V(G) \to \mathbb{Z}_2^k$ given by the rule

$$\partial\phi(v) = \sum_{e \in \Delta(v)} \phi(e)$$

Thus, ϕ is a flow if and only if $\partial\phi$ is identically zero. The following identity will be quite useful.

$$\sum_{v \in V(G)} \partial\phi(v) = \sum_{v \in V(G)} \sum_{e \in \Delta(v)} \phi(e) = \sum_{e \in E(G)} 2\phi(e) = 0 \tag{1}$$

Next we record three properties of critical 3-graphs.

Proposition 11.5 *Let G be a critical 3-graph, let s, t be adjacent vertices in G, and let x_1, x_2, t and x_3, x_4, s be the neighbors of s and t respectively.*

Let $G' = G \setminus \{s, t\}$ and let $X = \{x_1, x_2, x_3, x_4\}$. Then we have:

1. *There exists a nowhere-zero map $\phi : E(G') \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $supp(\partial\phi) = X$ and $\partial\phi(x_1) = \partial\phi(x_2)$*
2. *For every $e \in E(G')$, there exists a nowhere-zero map $\phi : E(G' \setminus e) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $supp(\partial\phi) = X$ and $\partial\phi(x_1) \neq \partial\phi(x_2)$*
3. *For every $1 \le i \le 4$ there exists a nowhere-zero map $\phi : E(G') \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $supp(\partial\phi) = X \setminus \{x_i\}$.*

Proof: To prove 1, choose a nowhere-zero flow ϕ of $G \setminus \{st\}$. Then $\phi' = \phi|_{E(G')}$ has $supp(\partial\phi') = X$ and $\partial\phi'(x_1) = \partial\phi'(x_2)$ as desired. For 2, choose a nowhere-zero flow ϕ of $G \setminus e$. Then $\phi' = \phi|_{E(G' \setminus e)}$ has $supp(\partial\phi') = X$ and $\partial\phi'(x_1) \neq \partial\phi'(x_2)$ as desired. For 3, we may assume without loss that $i = 1$. In this case, we may choose a nowhere-zero flow ϕ of $G \setminus sx_1$. Now $\phi' = \phi_{E(G')}$ has $supp(\partial\phi') = X \setminus \{x_1\}$ as required. $\square$

The following proposition records some similar properties of the Petersen graph. Since these properties are quite routine to verify, we state this proposition without proof.

Proposition 11.6 *Let y_1y_2, y_3y_4 be nonadjacent edges of P_{10} let $P = P_{10} \setminus \{y_1y_2, y_3y_4\}$ and let $Y = \{y_1, y_2, y_3, y_4\}$. Then we have:*

1. *For every $a_1, a_2, a_3, a_4 \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$ with $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1 \neq a_2$, there exists a nowhere-zero map $\phi : E(P) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $supp(\partial\phi) = Y$ and such that $\partial\phi(y_i) = a_i$ for $1 \le i \le 4$.*
2. *For every $e \in E(P)$ and every $a_1, a_2, a_3, a_4 \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{0\}$ with $a_1 + a_2 + a_3 + a_4 = 0$ and $a_1 = a_2$, there exists a nowhere-zero map $\phi : E(P \setminus \{e\}) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $supp(\partial\phi) = Y$ and such that $\partial\phi(y_i) = a_i$ for $1 \le i \le 4$.*
3. *Let $a_1, a_2, a_3, a_4 \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and assume that $a_j = 0$ for exactly one $1 \le j \le 4$. Then there exists a nowhere-zero map $\phi : E(P) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $supp(\partial\phi) \subseteq Y$ and such that $\partial\phi(y_i) = a_i$ for $1 \le i \le 4$.*

With these propositions in place, we are ready to prove Theorem 5.11.

Theorem 5.11 *For every critical snark G, the dot product of G with the Petersen graph is critical.*

Proof: Let $st \in E(G)$, let x_1, x_2, t and x_3, x_4, s be the neighbors of s and t respectively. Let $G' = G \setminus \{s, t\}$ and let $X = \{x_1, x_2, x_3, x_4\}$. Let $y_1y_2, y_3y_4 \in E(P_{10})$ be nonadjacent edges, let $Y = \{y_1, y_2, y_3, y_4\}$ and let $P = P_{10} \setminus \{y_1y_2, y_3y_4\}$. Let F be the graph obtained from G' and P by adding edges with ends x_i, y_i for $1 \leq i \leq 4$ and let $f \in E(F)$. It suffices to show that $F \setminus f$ has a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. The following observation reduces this task to that of finding suitable nowhere-zero maps on $G' \setminus f$ and $P \setminus f$.

Let $\phi : E(G' \setminus f) \to \mathbb{Z}_2 \times \mathbb{Z}_2$, $\psi : E(P \setminus f) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be maps. We say that (ϕ, ψ) is *good* if ϕ and ψ satisfy the following properties.

(i) ϕ, ψ are nowhere-zero.

(ii) $supp(\partial\phi) \subseteq X$ and $supp(\partial\psi) \subseteq y$.

(iii) $\partial\phi(x_i) = \partial\phi(y_i)$ for $1 \leq i \leq 4$.

(iv) $\partial\phi(x_i) = 0$ if and only if $f = x_i y_i$.

If (ϕ, ψ) is good then the map $\nu : E(F) \setminus f \to \mathbb{Z}_2 \times \mathbb{Z}_2$ given by the rule

$$\nu(e) = \begin{cases} \phi(e) & \text{if } e \in E(G') \setminus f \\ \psi(e) & \text{if } e \in E(P) \setminus f \\ \partial\phi(x_i) & \text{if } e = x_i y_i \neq f \end{cases}$$

is a nowhere-zero flow of $F \setminus f$. Thus, to complete the proof it suffices to find a good pair of maps (ϕ, ψ).

If $f = x_i y_i$ for some $1 \leq i \leq 4$, then the existence of a good pair follows from part 3 of Proposition 11.5 and part 3 of Proposition 11.6. If $f \in P$, then the existence of a good pair follows from part 1 of Proposition 11.5 and part 2 of Proposition 11.6. If $f \in G'$, then the existence of a good pair follows from part 2 of Proposition 11.5 and part 1 of Proposition 11.6. □

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