A Discrete Non-Pfaffian Approach to the Ising Problem.

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Abstract

We describe a method developed in fifties by Kac, Ward, Potts, Feynman and Sherman to solve the 2-dimensional Ising problem using methods of discrete mathematics. This approach is older and not so well known to discrete mathematics community as the Pfaffian approach. Using the results of Sherman and the theory of Pfaffian orientations developed by Galluccio and Loebl we generalise the results to arbitrary (non-planar) graphs.

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1 Introduction

Let $G = (V, E)$ be a finite graph. V is the set of *vertices* and E is the set of unordered pairs of vertices called *edges*. We allow multiple edges and loops. Moreover we will assume that a variable x_e is associated with each edge e of G. If $A \subset E$ then we let $x_A = \prod_{e \in A} x_e$. The *degree* of a vertex is the number of edges incident with it. Each loop contributes 2 to the degree.

A (spanning) subgraph $H = (V, E')$ of G is called *even* if each vertex of H has an even degree, possibly zero. The *generating function of even subgraphs* is a polynomial $\mathcal{E}(G, x) = \sum x_A$ over all even subgraphs (V, A) of G.

This polynomial is extensively studied in discrete mathematics in matching theory and also in statistical physics since it is equivalent via a theorem of van der Waerden [16] to the Ising problem partition function of G.

Since the solution of the 2-dimensional (planar) Ising problem by Onsager [9], the physisists were trying to reproduce his solution by more understandable methods. In fifties and in the beginning of sixties two discrete solutions appeared: the Pfaffian method of Fisher and Kasteleyn [2, 7] and the 'paths method' of Kac, Ward, Potts, Feynman and Sherman [6, 10, 13, 14]. The Pfaffian method seems to be better known to discrete mathematicians. Recently it has been further developed in [3, 4, 5, 8] and independently in [15] and in [11, 12] to express $\mathcal{E}(G, x)$ or, equivalently, the Ising partition function of an arbitrary graph G as a linear combination of Pfaffians of matrices associated with G.

The aim of this article is to describe the path method and to prove an analogous general theorem. All the results apart of Lemma 3.1, Theorem 3.5 and most of the proofs essentially appeared in Sherman's paper [13].

Section 2 contains a theorem conjectured by Feynman and proved by Sherman. Section 3 contains the main result: a generalisation of the theorem of Feynman and Sherman to arbitrary graphs which is analogous to the theory of Pfaffian orientations. The motivation for the Feynman's conjecture was a work of Kac, Ward and Potts [6, 10]. This together with concrete formulas for the 3-dimensional Ising problem will be described in a continuation of this paper.

2 A Theorem of Feynman and Sherman

Let $G = (V, E)$ be a graph and $D = (V, A(G))$ an arbitrary orientation of G. If $e \in E$ then a_e will denote the orientation of e in $A(G)$ and a^{-1} will be the reversed directed edge to a. We let $x_{a_e} = x_{a_e^{-1}} = x_e$. A circular sequence $p = v_1, a_1, v_2, a_2, ..., a_n, v_{n+1} = v_1$ is called *non-periodic closed walk* if the following conditions are satisfied: $a_i \in \{a_e, a_e^{-1} : e \in E\}$, $a_i \neq a_{i+1}^{-1}$ and $(a_1, ..., a_n) \neq Z^m$ for some sequence Z and $m > 1$. We let $X(p) = \prod_{i=1}^{n} x_{a_i}$ and $sign(p) = (-1)^{1+n(p)}$, where $n(p)$ is the *winding number* of p, i.e. the number of integral revolutions of the tangent vector of p. Finally let $W(p) = sign(p)X(p)$.

There is a natural equivalence on non-periodic closed walks: p is equivalent with reversed p. Each equivalence class has two elements and will be denoted by $[p]$. We let $W([p]) = W(p)$ and note that this definition is correct since equivalent walks have the same sign. The following theorem was conjectured by Feynman and proved by Sherman [13].

Theorem 2.1 *If $G = (V, E)$ is a planar graph then*

$$\mathcal{E}(G, x) = \prod(1 + W([p])$$

where $\prod$ is the formal infinite product over all equivalence classes of non-periodic closed walks of G.

Note that the product is infinite even for a very simple graph consisting of one vertex and two loops. In fact, Sherman proved a generalisation of this theorem which will be used in section 3. In order to state this generalisation we need to introduce the *crossover condition*: assume graph G is properly drawn in the plane and let v be a vertex of degree 4 of G and let p be a non-periodic closed walk of G. We say that p satisfies the *crossover condition* at v if the way p passes through v is consistent with the crossover pairing of the four edges incident with v.

Remark. The following Theorem 2.2 is formulated for planar graphs such that each degree is even and at most four. Hence in order to show that Theorem 2.1 follows from it, we must reduce Theorem 2.1 to the case of planar graphs with each degree even and at most four. This may be done easily as follows: first double each edge and let the variables associated with the new edges equal to zero. This makes each degree even. Then replace each vertex v with incident edges $e_1, ..., e_{2k}$, $k > 2$, listed in a circular order given by a fixed drawing of G in the plane, by a path of $2k - 2$ vertices. Let the variables of the edges of the path equal to one. Next double each edge of the perfect matching of this path and let the variables of these new edges equal to zero. Finally join the edges $e_1, ..., e_{2k}$ to the vertices of the auxiliary path so that the order is preserved along the path and each degree is four: there is a unique way to do that. Observe that Theorem 2.1 holds for G if and only if it holds for the graph obtained from G by the above construction.

Theorem 2.2 *Let $G = (V, E)$ be a planar graph properly drawn in the plane and such that each degree is even and at most four. Let $U = \{v_1, ..., v_k\}$ be a subset of vertices of G such that each v_i has degree 4. Let $\prod'_{G,U}(1 + W([p])$ denote the formal infinite product over all equivalence classes of non-periodic closed walks of G which satisfy the crossover condition at each v_i, $i = 1, ..., k$. An even subgraph H of G is called acceptable if for each v_i and two edges incident with v_i and joined by the crossover pairing at v_i, if H contains one of them then it contains also the other. If H is acceptable then we let $c(H)$ equal the number of vertices of U such that H contains all four edges incident with it. Then*

$$\prod{}'_{G,U}(1 + W([p]) = \sum(-1)^{c(H)} x_H$$

where the sum is over all acceptable even subgraphs of G.

Proof. We proceed in two steps. First we show that when the infinite product is expanded as a sum of monomials of variables, the coefficient corresponding to x_H, H acceptable subgraph, equals $(-1)^{c(H)}$. In the second step we show that all the remaining coefficients are zero.

Claim 1. Let H be an admissible subgraph of G. If $\prod'_{G,U}(1+W([p])$ is expanded as a sum of monomials of variables then the coefficient of x_H equals $(-1)^{c(H)}$.

Proof of Claim 1. By induction on the number of vertices of non-zero degree in H. If H has just one vertex then it consists of one loop e or two loops e, f and $c(H)$ equals zero or one. If H consists of one loop only then $\prod'_{G,U}(1+W([p]) = (1+x_e)\times$ product of terms which cannot influence the coefficient at x_H. If H consists of two loops and $c(H) = 0$ then $\prod'_{G,U}(1+W([p])$ equals $(1+x_e)(1+x_f)(1+x_e x_f)(1-x_e x_f)\times$ product of terms which cannot influence the coefficient at x_H.

Next let $c(H) = 1$ and H consist of two loops. $\prod'_{G,U}(1+W([p])$ equals $(1-x_e x_f)\times$ product of terms which cannot influence the coefficient at x_H. Hence the base of the induction is verified.

Now assume Claim 1 is true for all subgraphs H with n vertices and we will prove it for the graphs with $n+1$ vertices. Hence let H be an acceptable subgraph with $n+1$ vertices. A vertex v of H will be called *free* if it doesnot contribute to $c(H)$, i.e. if it has degree 2 in H or if the crossover condition is not imposed at it. Let $k = n+1-c(H)$ be the number of free vertices. We continue by induction on k. First let $k = 0$, i.e. each vertex of H has degree four and there is a crossover condition imposed at it. The crossover conditions cause that there is a unique decomposition of H into non-periodic closed walks $p_1, ..., p_r$ such that $x_H = \prod_{i=1}^r X(p_i)$. If $r = 1$ then observe that $sign(p_1) = (-1)^{c(H)}$. If $r > 1$ then $\prod_{i=1}^r sign(p_i) = (-1)^{c(H)}$ since any two of the p_i's mutually intersect in an even number of vertices (and each vertex contributs to $c(H)$).

Hence let $k > 0$ and Claim 1 holds for all acceptable subgraphs with less than k free vertices. If all free vertices have degree 2 in H then we may proceed as in the case $k = 0$. Hence let v be a free vertex of H of degree four in H. Denote the edges incident with v by north, east, south and west according to the cyclic order induced by the planar drawing.

Partition the non-periodic closed walks of G which satisfy the crossover conditions at the vertices of U into four classes. Classes I,II,III contain walks that have an edge incident with v and:

class I contains the walks that are consistent with west-north and east-south pairing,

class II contains the walks that are consistent with west-south and east-north pairing,

class III contains the walks that are consistent with north-south and east-west pairing (i.e. consistent with the crossover condition at v), and finally

class IV contains the walks that do not contain any edge incident with v.

Suppose $p \in I$ and $q \in II$. Then the product $W[p]W[q]$ can make no contribution to x_H and the same is true for II, III and I, II. Hence if $\prod'_{G,U}(1+W([p])$ is expanded as a sum, the coefficient of x_H is the sum of the corresponding coefficients in $I \times IV$,

$II \times IV$ and $III \times IV$.

The contribution to $I \times IV$ can be regarded as the coefficient of $x_{H'}$ in $\prod'_{G',U}(1+W([p])$ where G' and H' are obtained from G and H by deleting vertex v and by identifying the west, north edges into one edge, and the east, south edges into one edge. Analogously we can treat the case $II \times IV$. Hence by the induction assumption the sum of the contributions from $I \times IV$ and $II \times IV$ is $2(-1)^{c(H)}$. The contribution to $III \times IV$ can be regarded as coming from $\prod'_{G,U\cup\{v\}}(1+W([p])$, i.e. one additional cross-over condition is imposed on vertex v. Using the induction assumption again (this time for k) we get that this contribution equals $(-1)^{c(H)+1}$.

Summarising when the product $\prod'(1+W([p])$ is expanded as a sum, the coefficient of x_H equals $2(-1)^{c(H)} + (-1)^{c(H)+1}$ which we wanted to show.

End of proof of Claim 1.

To finish the proof of Theorem 2.2 we need to show that the remaining coefficients, i.e. the coefficients corresponding to the products of variables where at least one of the exponents is greater than one, are all equal to zero. To that end, temporarily consider $\prod'_{G,U}(1+W(p))$ where now the product is over non-periodic closed walks and so it is the square of the product we are considering. Let $a_1 > a_1^{-1} > ... > ...$ be a linear order of orientations of the edges of graph G.

Let A_1 be the set of all non-periodic closed walks p such that a_1 appears in p. Each $p \in A_1$ has a unique factorisation into words $(W_1, ..., W_k)$ each of which starts with a_1 and has no other appearance of a_1. Some of these words contain a_1^{-1} and some do not. We will need a curious lemma on coin arrangements stated below. The Lemma is proved in [13] and Sherman remarks in [13] that another proof has been devised by A. Selberg.

A Lemma on Coin Arrangements. Suppose we have a fixed collection of N objects of which m_1 are of one kind, m_2 are of second kind,...,m_n are of nth kind. Let b_k be the number of exhaustive unordered arrangements of these symbols into k disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. For example let us have 10 coins of which 3 are pennies, 4 are nickles and 3 are quarters. Then $\{(p,n),(n,p),(p,n,n,q,q,q)\}$ is not a correct arrangement since (p,n) and (n,p) represent the same circular order. If $N>1$ then $\sum_{i=1}^{N}(-1)^{i+1}b_i = 0$.

Proof of the Lemma. The Lemma follows immediately if we expand the LHS of the Witt equality and collect terms where the sums of the exponents of the z_i's are the same.

Witt Identity (see [1]): let $z_1, ..., z_k$ be commuting variables. Then

$$\prod_{m_1,...,m_k \geq 0} (1 - z^{m_1}...z^{m_k})^{M(m_1,...,m_k)} = 1 - z_1 - z_2 - ... - z_k,$$

where $M(m_1, ..., m_k)$ is the number of different nonperiodic sequences of z_i's taken with respect to circular order.

End of Proof of the Lemma.

Claim 2. $\prod_{p\in A_1}(1+W(p)) = 1 + x_{a_1}d_{11}$ where d_{11} is formal (possibly infinite) sum of monomials none of which has x_{a_1} as a factor.

Proof of Claim 2.

First note the following simple fact: if p_1, p_2 are two non-periodic closed walks such that p_1p_2 is also non-periodic then $sign(p_1p_2) = -sign(p_1)sign(p_2)$.

Let D be a monomial summand in the expansion of $\prod_{p\in A_1}(1+W(p))$. Hence D is a product of finitely many $W(p), p \in A_1$.

Each $p \in A_1$ has a unique factorisation into words defined above. Each word may appear several times in the factorisation of p and also in the factorisation of different non-periodic closed walks. Let $B(D)$ be the set-system of all the words (with repetition) appearing in the factorisations of the aperiodic closed walks of D.

It directly follows from the Lemma on Coin Arrangements that the sum of all monomial summands D in the expansion of $\prod_{p\in A_1}(1+W(p))$, which have the same $B(D)$ of more than one element is zero.Hence the monomial summands D which survive in the expansion of $\prod_{p\in A_1}(1+W(p)$ all have $B(D)$ consisting of exactly one word. This word may but neednot contain a_1^{-1}. However only the summands with their word NOT containng a_1^{-1} survive since if $b, c_1, ..., c_k$ contain neither a_1 nor a_1^{-1} then

$$W(a_1ba_1^{-1}c_1a_1^{-1}c_2...a_1^{-1}c_k) + W(a_1b^{-1}a_1^{-1}c_1a_1^{-1}c_2...a_1^{-1}c_k) +$$
$$W(a_1ba_1^{-1}c_1a_1^{-1}c_2...a_1^{-1}c_k^{-1}) + W(a_1b^{-1}a_1^{-1}c_1a_1^{-1}c_2...a_1^{-1}c_k^{-1}) = 0.$$

End of Proof of Claim 2.

Analogously let A_2 be the set of all non-periodic closed walks p such that a_1^{-1} appears in p. Note that possibly $A_1 \cap A_2 \neq \emptyset$. Each $p \in A_2$ has a unique factorisation into words $(W_1, ..., W_k)$ each of which starts with a_1^{-1} and has no other appearance of a_1^{-1}. Some of these words contain a_1 and some do not. The following claim may be proved in exactly the same way as Claim 2.

Claim 3. $\prod_{p\in A_2}(1+W(p)) = \prod_{p\in A_1-A_2}(1+W(p)) = \prod_{p\in A_2-A_1}(1+W(p)) = \prod_{p\in A_1}(1+W(p))$.

Let B be the set of non-periodic closed walks in which neither a_1 nor a_1^{-1} appear. We may assume (by some minimality assumption) that

Claim 4. $\prod_{p\in B}(1+W(p)) = (1+d_{12})^2$, where d_{12} is a formal sum of monomials, none of which has x_{a_1} as a factor.

In $\prod_{p\in A_1}(1+W(p)) \times \prod_{p\in A_2}(1+W(p)) = (1+x_{a_1}d_{11})^2$, the non-periodic closed walks from $A_1 \cap A_2$ have been counted doubly, while the non-periodic closed walks from $A_1 - A_2$ and $A_2 - A_1$ have been counted singly. Hence

$$[\prod_{p\in(A_1\cup A_2)}(1+W(p))]^2 =$$

$$\prod_{p\in A_1}(1+W(p)) \times \prod_{p\in A_2}(1+W(p)) \times \prod_{p\in A_1-A_2}(1+W(p)) \times \prod_{p\in A_2-A_1}(1+W(p)) =$$

$$(1+x_{a_1}d_{11})^4.$$

This means that

$$[\prod{}'_{G,U}(1+W([p])]^2 = \prod{}'_{G,U}(1+W(p)) =$$

$$\prod_{p\in(A_1\cup A_2)}(1+W(p)) \times \prod_{p\in B}(1+W(p)) =$$

$$(1+x_{a_1}d_{11})^2(1+d_{12})^2,$$

and

$$\prod{}'_{G,U}(1+W([p]) = (1+x_{a_1}d_{11})(1+d_{12}).$$

Thus, there are no monomial summands having factors $x^n_{a_1}$, $n \geq 2$. Analogous arguments dispose of summands with factors $x^n_{a_i}$, $i \neq 1, n \geq 2$. Hence the proof of the Theorem is finished.

□

3 A Formula For General Graphs

As Sherman have noticed, Theorem 2.2 may be used to express $\mathcal{E}(G,x)$ for general graphs as a linear combination of infinite products. Let us consider toroidal graphs first, and we will again assume that each degree is even and at most four: by the remark before Theorem 2.2 this may be done without loss of generality. Let us take a representation of the torus as a rectangle with identified edges. We will assume that a graph G is drawn there properly and so that all the vertices belong to the interior of the rectangle. If p is a non-periodic closed walk of G then let $h(p)$ denote the number of horisontal rectangle edge crossings of p and let $v(p)$ analogously denote the number of vertical rectangle edge crossings of p. The notation $h(H)$ and $v(H)$ is also used for even subgraphs H of G. Finally we let $W_h(p) = (-1)^{h(p)}W(p)$, $W_v(p) = (-1)^{v(p)}W(p)$ and $W_{h,v}(p) = (-1)^{h(p)+v(p)}W(p)$.

The following Theorem 3.2 and in particular Theorem 3.5 are based on a curious lemma.

Lemma 3.1 *Let R be the set of all $0,1$-vectors of length $2n$ and let a be an arbitrary integer vector of length $2n$. Then*

$$2^{-n}(-1)^{\sum_{i=1}^n a_{2i-1}a_{2i}}[\sum_{r\in R}(-1)^{ra}(-1)^{s(r)}] = 1,$$

where $s(r)$ denotes the number of i such that $r_{2i-1} = r_{2i} = 1$.

Proof. We proceed by induction on n. The initial case $n=1$ may be easily checked by hand. Next assume that Lemma 3.1 is true for n and we want to prove it for $n+1$. Let R' be the set of all $0,1$-vectors of length $2(n+1)$ and let a' be an arbitrary integer vector of length $2(n+1)$. Let a denote the initial part of a' of length $2n$. Then

$$2^{-n-1}(-1)^{\sum_{i=1}^{n+1} a'_{2i-1}a'_{2i}}[\sum_{r\in R'}(-1)^{ra'}(-1)^{s(r)}] =$$

$$2^{-1}(-1)^{a'_{2n+1}a'_{2n+2}}\alpha[(-1)^{a'_{2n+1}} + (-1)^{a'_{2n+2}} - (-1)^{a'_{2n+1}+a'_{2n+2}} + 1],$$

where

$$\alpha = 2^{-n}(-1)^{\sum_{i=1}^{n} a_{2i-1}a_{2i}}[\sum_{r\in R}(-1)^{ra}(-1)^{s(r)}].$$

By induction assumption we have that $\alpha = 1$ and applying again the first step of the induction, we get that the lemma holds. □

Theorem 3.2 *If $G = (V, E)$ is a toroidal graph where each degree is even and at most four then*

$$\mathcal{E}(G, x) = 1/2[\prod(1 + W_h([p]) + \prod(1 + W_v([p]) + \prod(1 + W_{h,v}([p]) - \prod(1 + W([p])],$$

where $\prod$ is the formal infinite product over all equivalence classes of non-periodic closed walks of G.

Proof. 'Unglue' the edges of the rectangle. Hence each rectangle edge crossing now corresponds to 'leaving' the rectangle and 'coming back' to the rectangle by the oposite rectangle edge. If we draw all this to the plane, we get $h(G)v(G)$ crossings of the curves representing the edges of G. Let G' be the graph obtained from G by introducing a vertex to each such intersection. Note that G' is properly drawn in the plane and each degree of G' is even and at most four. Let us call the new vertices *special* and note that each special vertex has degree four in G'. Further note that each non-periodic closed walk p of G corresponds to the non-periodic closed walk p' of G' which satisfies the crossover condition at each special vertex. Moreover $sign(p') = (-1)^{h(p)+v(p)}sign(p)$ and thus $W([p']) = W_{h,v}([p])$.

Using Theorem 2.2 we get that

$$\prod(1 + W_{h,v}([p]) = \prod{}'(1 + W([p']) = \sum(-1)^{h(H)v(H)}x_H,$$

where the sum goes over all acceptable subgraphs H of G', i.e. over all even subgraphs of G. Hence also

$$\prod(1 + W_v([p]) = \sum(-1)^{h(H)v(H)+h(H)}x_H,$$

$$\prod(1 + W_h([p]) = \sum(-1)^{h(H)v(H)+v(H)}x_H$$

and

$$\prod(1 + W([p]) = \sum(-1)^{h(H)v(H)+h(H)+v(H)}x_H.$$

Let H be an arbitrary even subgraph of G. Then the coefficient of x_H in

$$1/2[\prod(1 + W_h([p]) + \prod(1 + W_v([p]) + \prod(1 + W_{h,v}([p]) - \prod(1 + W([p])]$$

equals

$$1/2(-1)^{h(H)v(H)}[(-1)^{h(H)} + (-1)^{v(H)} - (-1)^{h(H)+v(H)} + 1].$$

This equals 1 by the previous Lemma 3.1.

□

Using the machinery of the theory of Pfaffian orientations we can write down a formula for general graphs. The machinery is based on considering graphs embedded on orientable surfaces of arbitrary genus.

Definition 3.3 *A surface S_g consists of a base B_0 and $2g$ bridges B_j^i, $i = 1, ..., g$ and $j = 1, 2$, where*

i) B_0 is a convex $4g$-gon with vertices $a_1, ..., a_{4g}$ numbered clockwise;

ii) B_1^i, $i = 1, \ldots, g$, is a 4-gon with vertices $x_1^i, x_2^i, x_3^i, x_4^i$ numbered clockwise. It is glued with B_0 so that the edge $[x_1^i, x_2^i]$ of B_1^i is identified with the edge $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ of B_0 and the edge $[x_3^i, x_4^i]$ of B_1^i is identified with the edge $[a_{4(i-1)+3}, a_{4(i-1)+4}]$ of B_0;

iii) B_2^i, $i = 1, \ldots, g$, is a 4-gon with vertices $y_1^i, y_2^i, y_3^i, y_4^i$ numbered clockwise. It is glued with B_0 so that the edge $[y_1^i, y_2^i]$ of B_2^i is identified with the edge $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ of B_0 and the edge $[y_3^i, y_4^i]$ of B_2^i is identified with the edge $[a_{4(i-1)+4}, a_{4(i-1)+5(mod 4g)}]$ of B_0.

Observe that in Definition 3.3 we denote by $[a, b]$ edges of polygons and not edges of graphs. The usual representation in the space of an orientable surface $\mathcal{S}$ of genus g may be then obtained from S_g by the following operation: for each bridge B, glue together the two segments which B shares with the boundary of B_0, and delete B.

Definition 3.4 *A graph G is called a g-graph if it may be embedded on S_g so that all the vertices belong to the base B_0, and if the embedding of an edge uses a bridge, then it crosses the bridge.*

This is analogous to the situation described above for the torus: we can imagine that we contract all the bridges (and get a usual representation of an orientable surface of genus g), draw our graph there, and then split the bridges back. The resulting drawing is a g-graph with its embedding on S_g. From now on, we shall consider g-graphs together with a fixed embedding on S_g. If G is a g-graph and p is a non-periodic closed walk of G then we denote by $a(p)$ the vector of length $2g$ such that $a(p)_{2(i-1)+j}$ equals the number of times p crosses bridge B_j^i, $i = 1, ..., g$, $j = 1, 2$. Similarly we will use the notation $a(H)$ where H is an even subgraph of G.

Next we present a formula for general g-graphs. Note that any graph G may be embedded as a a g-graph where g is genus of G and that we only need to consider g-graphs that have all degrees even and at most four (by a remark before theorem 2.2).

Let $R(g)$ denote the set of all $0, 1$-vectors of length $2g$. If G is a g-graph (with all degrees even and at most four), p a non-periodic closed walk of G and $r \in R(g)$, then let $W_r([p]) = (-1)^{ra(p)} W([p])$.

Theorem 3.5 *If $G = (V, E)$ is a g-graph where each degree is even and at most four then*

$$\mathcal{E}(G, x) = 2^{-g} \sum_{r \in R(g)} (-1)^{s(I-r)} \prod (1 + W_r([p])),$$

where $\prod$ is the formal infinite product over all equivalence classes of non-periodic closed walks of G and I denotes the vector of all ones.

Proof. We proceed analogously as in the proof of Theorem 3.2. We consider G embedded in the plane by the projection of the bridges B^i_j outside B_0. We get $\sum_{i=1}^{g} a(G)_{2i-1}a(G)_{2i}$ crossings of the curves representing the edges of G. Let G' be the graph obtained from G by introducing a vertex to each such intersection. Note that G' is properly drawn to the plane, and each degree of G' is even and at most four. Let us call the new vertices *special* and note that each special vertex has degree four in G'. Further note that each non-periodic closed walk p of G corresponds to the non-periodic closed walk p' of G' which satisfies the crossover condition at each special vertex. Moreover $sign(p') = (-1)^{Ia(p)} sign(p)$ and thus $W([p']) = W_I([p])$.

Using Theorem 2.2 we get that

$$\prod (1 + W_I([p]) = \prod{}' (1 + W([p']) = \sum (-1)^{\sum_{i=1}^{g} a(H)_{2i-1}a(H)_{2i}} x_H,$$

where the sum is over all acceptable subgraphs H of G', i.e. over all even subgraphs of G. Hence for $r \in R(g)$ we have

$$\prod (1 + W_r([p]) = \sum (-1)^{\sum_{i=1}^{g} a(H)_{2i-1}a(H)_{2i} + (I-r)a(H)} x_H,$$

where the sum is over all even subgraphs H of G.

Let H be an arbitrary even subgraph of G. Then the coefficient of x_H in

$$2^{-g} \sum_{r \in R(g)} (-1)^{s(I-r)} \prod (1 + W_r([p])$$

equals

$$2^{-g} (-1)^{\sum_{i=1}^{g} a(H)_{2i-1}a(H)_{2i}} \sum_{r \in R(g)} (-1)^{s(I-r)} (-1)^{(I-r)a(H)}.$$

This equals 1 by the previous Lemma 3.1, since we can replace r by $I - r$ in the summation.

□

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