Bell numbers, their relatives, and algebraic differential equations

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Abstract

We prove that the ordinary generating function of Bell numbers satisfies no algebraic differential equation over $\mathbf{C}(x)$ (in fact, over a larger field). We investigate related numbers counting various set partitions (the Uppuluri–Carpenter numbers, the numbers of partitions with j mod i blocks, the Bessel numbers, the numbers of connected partitions, and the numbers of crossing partitions) and prove for their ogf's analogous results. Recurrences, functional equations, and continued fraction expansions are derived.

1 Introduction

The Bell number B_n counts the partitions of $[n] = \{1, 2, \ldots, n\}$. The sequence of Bell numbers begins

$$(B_n)_{n\geq 1} = (1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, \ldots)$$

and is listed in EIS [28] as sequence A000110. It is well-known (Comtet [7, p. 210], Lovász [18, Problem 1.11], Stanley [30, p. 34]) that the exponential generating function of B_n is given by

$$B_e(x) = \sum_{n\geq 0} \frac{B_n x^n}{n!} = \mathrm{e}^{\mathrm{e}^x - 1}.$$

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From $\mathrm{e}^x = (B_e'/B_e)' = B_e'/B_e$ we obtain the algebraic differential equation

$$B_e''B_e - (B_e')^2 - B_e'B_e = 0.$$

In this article we prove that, on the other hand, the ordinary generating function (abbreviated *ogf*) of B_n,

$$B(x) = \sum_{n\geq 0} B_n x^n = 1 + x + 2x^2 + 5x^3 + 15x^4 + \cdots,$$

satisfies no algebraic differential equation (abbreviated *ADE*) over the field of rational functions $\mathbf{C}(x)$. Our proof uses the fact that $B(x)$ satisfies a simple "functional" equation which we show to be incompatible with any ADE.

The method of functional equations applies to several other combinatorial numbers related to B_n. In Section 2 we consider five more counting sequences besides B_n: the Uppuluri–Carpenter numbers $B_n^{\pm}$, the numbers $B_n^{j,i}$ of the partitions of $[n]$ having j mod i blocks, the Bessel numbers B_n^{B}, the numbers B_n^{co} of connected partitions, and the numbers B_n^{cr} of crossing partitions. While B_n, $B_n^{\pm}$, $B_n^{j,i}$, and B_n^{B} have been investigated before, B_n^{co} and B_n^{cr} seem new. In Propositions 2.1–2.4, 2.6, and 2.7 we give functional equations for the corresponding ogf's and/or relate them to $B(x)$. We give also recurrences for the counting sequences. Using the functional equations, we give two quick derivations of continued fraction expansions. While the expansion (6) of $B(x)$ differs from that found by Flajolet [9], the expansion (19) of $B^{\mathrm{B}}(x)$ coincides with that found by Flajolet and Schott [11]. (In [9] and [11] the expansions are derived by the general method of path diagrams due to Flajolet.) We also give continued fraction expansions of $B^{\pm}(x)$ and $B^{0,2}(x)$.

Proposition 3.3 in Section 3 says, roughly speaking, that in any ADE satisfied by the ogf's $B(x)$ and $B^{\pm}(x)$ all derivatives can be eliminated so that just an algebraic equation is obtained. This is used in Theorem 3.5 to prove the announced result and in fact a stronger one: $B(x)$ and $B^{\pm}(x)$ satisfy no ADE over the field $\mathbf{C}\{x\}$ of analytic Laurent series. Let $N(x) = \sum_{n\geq 0} n! \cdot x^n$ and $*$ be the Hadamard product of power series (the coefficientwise multiplication). Lipshitz and Rubel [17, Proposition 6.3 (ii) and Remark 5.3] gave an example of a power series $F(x)$ which satisfies an ADE over $\mathbf{C}(x)$ but $F(x) * N(x)$ does not. Theorem 3.5 provides another (somewhat simpler) example: since $B(x) = B_e(x) * N(x)$, one can take $F(x) = B_e(x)$. In Theorem 3.7 we show that no $B^{j,i}(x)$ satisfies an ADE over $\mathbf{C}\{x\}$ and that the ogf's $B^{\rm co}(x)$ and $B^{\rm cr}(x)$ satisfy no ADE over $\mathbf{C}(x)$. As for $B^{\rm B}(x)$, in Theorem 3.11 we prove a result weaker than the previous ones: $B^{\rm B}(x)$ satisfies no ADE over $\mathbf{C}\{x\}$ of order at most one. Our methods are mostly algebraic but Propositions 3.1 and 3.4 use analytic arguments. In Section 4 we give some concluding comments and open problems.

2 Bell numbers and their relatives

A *partition* P of a set X is a collection of nonempty and mutually disjoint subsets of X, called *blocks*, whose union is X. In all partitions that we consider here X is a finite subset of $\mathbf{N} = \{1, 2, \ldots\}$. For $a, b \in \mathbf{Z}$ with $a \le b$ and $n \in \mathbf{N}$, the symbols $[n]$ and $[a, b]$ denote the sets $\{1, 2, \ldots, n\}$ and $\{a, a+1, \ldots, b\}$, respectively. Let $X \subset \mathbf{N}$, $|X| = k \ge 0$, and the elements of X be $1 \le x_1 < x_2 < \ldots < x_k$. The *inner spaces* of X are the $k-1$ intervals $[x_1+1, x_2-1], [x_2+1, x_3-1], \ldots, [x_{k-1}+1, x_k-1]$. The two *outer spaces* are $[1, x_1-1]$ and $[x_k+1, \infty)$. Alltogether X has $k+1$ *spaces*.

The *Bell number* B_n is the number of all partitions of $[n]$ (or of any other n-element set). The *Stirling number* (of the second kind) $S(n,k)$ is the number of the partitions of $[n]$ with exactly k blocks. Clearly, $B_n = \sum_{k=1}^n S(n,k)$. For more information and references on B_n and $S(n,k)$ see [7, 28], Branson [4], and Pitman [20].

Proposition 2.1 *The ogf of Bell numbers $B(x) = \sum_{n\ge 0} B_n x^n = 1 + x + 2x^2 + \cdots$ satisfies the equations*

$$B(x) = 1 + \frac{x}{1-x} \cdot B(x/(1-x)) \quad (1)$$

$$B(x/(1+x)) = 1 + x \cdot B(x). \quad (2)$$

Proof. Any nonempty partition P of $[n]$, $n \in \mathbf{N}$, decomposes in the first block A, $1 \in A \in P$, and the possibly empty partition Q of $[n]\backslash A$ formed by the remaining blocks. Let $k = |\bigcup Q| = n - |A|$. The elements of A split into $k+1$ sequences, according to the spaces of $\bigcup Q$ in which they lie. The only restriction on the sequences is that the first one is nonempty. Thus

$$\begin{aligned} B(x) &= 1 + \frac{x}{1-x} \cdot \left(\sum_{k\ge 0} B_k x^k \cdot \frac{1}{(1-x)^k} \right) \\ &= 1 + \frac{x}{1-x} \cdot B(x/(1-x)). \end{aligned}$$

Eq. (2) follows from (1) by the substitution $x \to x/(1+x)$. $\square$

Iterating (1), we obtain the classical expansion

$$B(x) = \sum_{k \geq 0} \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}. \tag{3}$$

One can go in the other way and derive (1) from (3): (3) is invariant to the transformation given on the right hand side of (1). So (1) and (3) (and (2)) are equivalent. The third equivalent form of (1) and (3) is the recurrence

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k, \quad n \geq 1 \text{ and } B_0 = 1, \tag{4}$$

obtained by comparing in (1) the coefficients at x^n or by a direct combinatorial argument. It is well-known that the kth summand of (3) is just the ogf of Stirling numbers:

$$\sum_{n \geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}. \tag{5}$$

Iterating (1) in a different way, we derive a continued fraction expansion of $B(x)$. We start with the ogf $B^{\mathrm{ir}}(x)$ of the *irreducible partitions* which are the partitions P of $[n]$ such that for every $m \in [n-1]$ at least one block of P intersects both intervals $[m]$ and $[m+1,n]$. We have $B(x) = 1/(1-B^{\mathrm{ir}}(x))$. By (1),

$$B^{\mathrm{ir}}(x) = \frac{x}{1-(1-x)B^{\mathrm{ir}}(x/(1-x))}.$$

Iterating this equation and using again $B(x) = 1/(1-B^{\mathrm{ir}}(x))$, we obtain

$$B(x) = \frac{1}{1-B^{\mathrm{ir}}(x)} = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x-x^2}{1-x-\cfrac{x-2x^2}{1-2x-\cfrac{x-3x^2}{\vdots}}}}}. \tag{6}$$

But this is not as neat as the expansion in [9].

The *Uppuluri–Carpenter number* $B_n^{\pm}$ is the difference between the number of the partitions of $[n]$ with an even number of blocks and the number of the partitions with an odd number of blocks:

$$B_n^{\pm} = \sum_{k=1}^{n} (-1)^k S(n,k). \tag{7}$$

We have

$$(B_n^{\pm})_{n\geq 1} = (-1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180, -17731, \ldots)$$

which is sequence A000587 of EIS [28]. These numbers were investigated by Beard [3], Uppuluri and Carpenter [35], Kolokolnikova [15] (here appears the term "Uppuluri–Carpenter numbers"), Subbarao and Verma [34], and Y. Yang [38].

Proposition 2.2 *The ogf* $B^{\pm}(x) = \sum_{n\geq 0} B_n^{\pm} x^n = 1 - x + x^3 + \cdots$ *of Uppuluri–Carpenter numbers satisfies the equations*

$$\begin{aligned} B^{\pm}(x) &= 1 - \frac{x}{1-x} \cdot B^{\pm}(x/(1-x)), & (8)\\ B^{\pm}(x/(1+x)) &= 1 - x \cdot B^{\pm}(x). & (9) \end{aligned}$$

Proof. By (5) and (7),

$$B^{\pm}(x) = \sum_{k\geq 0} \frac{(-x)^k}{(1-x)(1-2x)\ldots(1-kx)}.$$

This expansion is invariant to the transformation given on the right hand side of (8). Eq. (9) follows from (8) by the substitution $x \to x/(1+x)$. $\square$

We will need (9) also in the form solved for $B^{\pm}(x)$:

$$B^{\pm}(x) = \frac{1}{x}\Big(1 - B^{\pm}(x/(1+x))\Big). \tag{10}$$

It follows, completely analogously to the derivations of (4), that

$$B_n^{\pm} = -\sum_{k=0}^{n-1} \binom{n-1}{k} B_k^{\pm}, \quad n \geq 1 \text{ and } B_0^{\pm} = 1. \tag{11}$$

Using (11) it is straightforward to prove that $B_n^{\pm} < 0$ and $B_n^{\pm} > 0$ hold for infinitely many $n \in \mathbf{N}$. In [38, p. 4] this fact is derived from more complicated analytic considerations. It is open if ever $B_n^{\pm} = 0$ for $n > 2$. See Canfield and Pomerance [5] for a similar problem on $S(n,k)$. Analogously to (6) we obtain

$$B^{\pm}(x) = \cfrac{1}{1 + \cfrac{x}{1 - 2x + \cfrac{x - x^2}{1 - 3x + \cfrac{x - 2x^2}{1 - 4x + \cfrac{x - 3x^2}{\vdots}}}}}.$$

We define numbers related to $B_n^{\pm}$. Let $B_n^{j,i}$, where $j \in \mathbf{Z}$ and $i \in \mathbf{N}$, be the number of the partitions of $[n]$ whose number of blocks is congruent to j modulo i:

$$B_n^{j,i} = \sum_{\substack{k=1 \\ k \equiv j \bmod i}}^{n} S(n,k). \tag{12}$$

We set $B_0^{j,i}$ to be 1 if $j \equiv 0 \bmod i$ and 0 else. Obviously, $B_n^{\pm} = B_n^{0,2} - B_n^{1,2}$. For example,

$$(B_n^{3,4})_{n \geq 3} = (1, 6, 25, 90, 302, 994, 3487, 15210, 92489, \ldots).$$

For $i \in \mathbf{N}$ and $j \in \mathbf{Z}$ we set $B^{j,i}(x) = \sum_{n \geq 0} B_n^{j,i} x^n$. Numbers $B_n^{j,i}$ were investigated by Lehmer [16] who gave for them recurrent relations and identities.

Proposition 2.3 *Let, for $a \in \mathbf{N}$ and $b \in \mathbf{Z}$, $\delta_{b,a} = 1$ if b is divisible by a and $\delta_{b,a} = 0$ else. Let $i \in \mathbf{N}$, $j \in \mathbf{Z}$, $R \subset \mathbf{Z}$ with $|R| = i$ be any system of all i residues modulo i (e.g., $R = [0, i-1]$), and $l \in [0, i-1]$. Then we have the equations*

$$B(x) = \sum_{k \in R} B^{k,i}(x) \tag{13}$$

$$B^{j+1,i}(x) = \delta_{j+1,i} + \frac{x}{1-x} \cdot B^{j,i}(x/(1-x)) \tag{14}$$

$$B^{l,i}(x) = \frac{x^l}{(1-x)(1-2x)\ldots(1-lx)} + \frac{x^i}{(1-x)(1-2x)\ldots(1-ix)} \cdot B^{l,i}(x/(1-ix)) \tag{15}$$

where in (13) $B(x)$ is the ogf of Bell numbers and in (15) the first summand is 1 *for* $l=0$.

Proof. Eq. (13) follows immediately from the definitions. By (5) and (12),

$$B^{j,i}(x)=\sum_{\substack{k\ge 0\\ k\equiv j \bmod i}}\frac{x^k}{(1-x)(1-2x)\dots(1-kx)}.$$

Now (14) follows by considering the action of the substitution $x\to x/(1-x)$ on this expansion. Eq. (15) follows by the same way or it can be derived combinatorially. The combinatorial derivation is a refinement of the proof of (1). We take a partition P of $[n]$ with l modulo i blocks and order the blocks by their minima. The P's with l blocks are counted by $x^l/((1-x)(1-2x)\dots(1-lx))$ (by (5)). If P has more than l blocks (thus at least $i+l$), we decompose it in the partition $A=\{A_1,\dots,A_i\}$ consisting of the first i blocks and the partition Q consisting of the remaining blocks. $\bigcup A$ is split in $k+1$ sequences, where $k=|\bigcup Q|$, according to the spaces of $\bigcup Q$. The first sequence must be a partition with i blocks, which gives the factor $x^i/((1-x)(1-2x)\dots(1-ix))$. The other k sequences are represented by words over the alphabet $\{1,2,\dots,i\}$, which gives the factor $1/(1-ix)^k$. The Q's are counted by $B_k^{l,i}x^k$. Summing over $k\ge 0$, we get (15). $\Box$

Comparing in (15) the coefficients at x^n, one can obtain a recurrence for $B_n^{j,i}$ that is similar to (4) but more complicated. Analogously to (6), we can derive a continued fraction expansion for $B^{j,i}(x)$. For brevity we indicate only the case $j=0$, $i=2$ but the method is general. Let $C=C(x)=x^2+\cdots$ be given by $B^{0,2}(x)=1/(1-C)$ — notice that C is *not* the ogf of the irreducible partitions with even numbers of blocks — and T be the substitution $x\to x/(1-x)$. Note that T^j, $j\in\mathbf{Z}$, is the substitution $x\to x/(1-jx)$. Then (15) for $l=0$ and $i=2$ can be written as

$$B^{0,2}=1+(Tx)(T^2x)\cdot T^2B^{0,2}.$$

This is transformed by $B^{0,2}=1/(1-C)$ to

$$C=\frac{Tx\cdot T^2x}{1+Tx\cdot T^2x-T^2C}.$$

Iterating this equation, we obtain the continued fraction expansion

$$B^{0,2} = \frac{1}{1-C} = \cfrac{1}{1 - \cfrac{Tx \cdot T^2x}{1 + Tx \cdot Tx^2 - \cfrac{T^3x \cdot T^4x}{1 + T^3x \cdot T^4x - \cfrac{T^5x \cdot T^6x}{\vdots}}}}.$$

The *Bessel number* B_n^{B} is the number of the *non-overlapping partitions* of $[n]$, which are the partitions having no pair of blocks A, B such that $\min A < \min B < \max A < \max B$. We have

$$(B_n^{\mathrm{B}})_{n\geq 1} = (1, 2, 5, 14, 43, 143, 509, 1922, 7651, 31965, 139685, \ldots)$$

which is sequence A006789 of EIS [28]. Numbers B_n^{B} were introduced by Flajolet and Schott [11] who related their ogf to Bessel functions and coined their name. Recently they resurfaced in the work of Claesson [6] as counting permutations subject to both local and global restrictions.

Proposition 2.4 *The ogf of Bessel numbers* $B^{\mathrm{B}}(x) = \sum_{n\geq 0} B_n^{\mathrm{B}} x^n = 1 + x + 2x^2 + \cdots$ *satisfies the equations*

$$B^{\mathrm{B}}(x) = \frac{1}{1 - x - \frac{x^2}{1-x} \cdot B^{\mathrm{B}}(x/(1-x))} \quad (16)$$

$$B^{\mathrm{B}}(x/(1+x)) = \frac{1+x}{1 - x^2 \cdot B^{\mathrm{B}}(x)}. \quad (17)$$

Proof. Let P be a nonempty non-overlapping partition of $[n]$, $n \in \mathbf{N}$, and A be its first block, $1 \in A \in P$. If $B \in P$ is any other block, then either $\max B < \max A$ or $\min B > \max A$. Let the former blocks B form the partition P_1 and the latter blocks form the partition P_2. P_1 and P_2 are both non-overlapping and possibly empty. P decomposes uniquely in A, P_1, and P_2. Let $k = |\bigcup P_1|$. A is split into $k+1$ sequences according to the spaces of $\bigcup P_1$. If $k = 0$, there is only one nonempty sequence of the elements of A. If $k \geq 1$, the first and the last sequence must be nonempty and the remaining $k-1$ sequences may be empty. The set $\bigcup P_2$ follows after A and hence also after $\bigcup P_1$. Thus

$$\begin{aligned} B^{\mathrm{B}}(x) &= 1+\left(\frac{x}{1-x}+\frac{x}{1-x}\left(\sum_{k\ge 1} B^{\mathrm{B}}_k x^k \frac{1}{(1-x)^{k-1}}\right)\frac{x}{1-x}\right)B^{\mathrm{B}}(x) \\ &= 1+\left(x+\frac{x^2}{1-x}B^{\mathrm{B}}(x/(1-x))\right)B^{\mathrm{B}}(x). \end{aligned} \tag{18}$$

Solving this equation for $B^{\mathrm{B}}(x)$, we obtain (16). Eq. (17) follows from (16) by the substitution $x \to x/(1+x)$. □

Iterating (16), we obtain the continued fraction expansion of $B^{\mathrm{B}}(x)$ due to Flajolet and Schott [11, p. 424]:

$$B^{\mathrm{B}}(x) = \cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{x^2}{1-3x-\cfrac{x^2}{\vdots}}}}. \tag{19}$$

Comparing in (18) the coefficients at x^n, we obtain the recurrence

$$B^{\mathrm{B}}_n = B^{\mathrm{B}}_{n-1} + \sum_{\substack{i,\,j,\,k\ge 0\\ i+j+k=n-2}} \binom{i+j}{j} B^{\mathrm{B}}_i B^{\mathrm{B}}_k,\quad n\ge 2 \text{ and } B^{\mathrm{B}}_0 = B^{\mathrm{B}}_1 = 1.$$

We say that two subsets of $\mathbf{N}$ *cross* if there are four numbers $1 \le a < b < c < d$ such that a and c lie in one of the sets and b and d in the other. If $A, B \subset \mathbf{N}$ and A precedes B, we write $A < B$. If A lies in an inner space of B, we write $A \prec B$. Clearly, if $A, B \subset \mathbf{N}$ do not cross and are disjoint, then $A < B$ or $B < A$ or $A \prec B$ or $B \prec A$. The *crossing graph* $G(P)$ of a partition P has the blocks of P as its vertices and $\{A, B\}$ is an edge of $G(P)$ if and only if the blocks A and B cross. The *noncrossing partitions* P are those for which $G(P)$ has no edge. Note that every noncrossing partition is also non-overlapping but the opposite is not in general true. An interesting survey article on noncrossing partitions is Simion [27]. The number B^{nc}_n of the noncrossing partitions of $[n]$ is one of the many incarnations of the *Catalan numbers*:

$$B^{\mathrm{nc}}_n = \frac{1}{n+1}\binom{2n}{n},$$

see [27] or Stanley [31, Problem 6.19]. Here we are more interested in the numbers of the *connected partitions* and of the *crossing partitions*. The former are the partitions P with connected $G(P)$ and the latter are the partitions P whose $G(P)$ has no isolated vertex (i.e., every block of P crosses another block). We denote the number of the connected partitions of $[n]$ by B_n^{co} and the number of the crossing partitions of $[n]$ by B_n^{cr}. The following simple lemma helps to deal with crossing graphs of partitions.

Lemma 2.5 *Let P be a partition of $X \subset \mathbf{N}$ and C_1, C_2 be two distinct connected components of $G(P)$. Then (viewing C_1 and C_2 as sets of blocks of P) the disjoint subsets $\bigcup C_1$ and $\bigcup C_2$ of X do not cross. Thus $\bigcup C_1 < \bigcup C_2$ or $\bigcup C_2 < \bigcup C_1$ or $\bigcup C_1 \prec \bigcup C_2$ or $\bigcup C_2 \prec \bigcup C_1$.*

Proof. Let C_1 and C_2 be two nonempty disjoint sets of blocks of P such that each $G(C_i)$ is connected and $\bigcup C_1$ and $\bigcup C_2$ cross. This means that there are four numbers $a < b < c < d$ such that $a, c \in \bigcup C_1$ and $b, d \in \bigcup C_2$ (or C_1 and C_2 are switched). If every block of C_1 lay completely in $[b+1, d-1]$ or in the complement of the interval, $G(C_1)$ would be disconnected. Hence there is a block $A_1 \in C_1$ intersecting both $[b+1, d-1]$ and its complement: there exist $\alpha, \beta \in A_1$ such that $\alpha < b < \beta < d$ or $b < \alpha < d < \beta$. By the same argument, there is a block $A_2 \in C_2$ that intersects both the interval $[\alpha+1, \beta-1]$ and its complement. But this means that the sets A_1 and A_2 cross and that there is an edge in $G(P)$ between C_1 and C_2. So C_1 and C_2 cannot be two distinct components of $G(P)$. $\square$

In particular, if P, C_1, and C_2 are as stated in the lemma and $\min \bigcup C_1 < \min \bigcup C_2$, then $\bigcup C_1 < \bigcup C_2$ or $\bigcup C_2 \prec \bigcup C_1$.

Recall that for any power series $F(x) = a_1 x + a_2 x^2 + \cdots$ with $a_1 \neq 0$ there is a unique power series $G(x) = F(x)^{\langle -1 \rangle} = b_1 x + b_2 x^2 + \cdots$ with $b_1 \neq 0$, the *compositional inverse* of $F(x)$, such that $F(G(x)) = G(F(x)) = x$.

Proposition 2.6 *The ogf $B^{\mathrm{co}}(x) = \sum_{n \geq 0} B_n^{\mathrm{co}} x^n = 1 + x + x^2 + \cdots$ of numbers of connected partitions expresses in terms of the ogf $B(x)$ as*

$$B^{\mathrm{co}}(x) = \frac{x}{(xB(x))^{\langle -1 \rangle}} \tag{20}$$

where $^{\langle -1 \rangle}$ denotes the compositional inverse. It satisfies the functional equation

$$x \cdot B^{\mathrm{co}}(B^{\mathrm{co}}(x) - 1) - B^{\mathrm{co}}(x)^2 + (1 + x) \cdot B^{\mathrm{co}}(x) - x = 0. \tag{21}$$

Proof. Let P be any partition of $[n]$, with $n = 0$ and $P = \emptyset$ allowed, and C be the first component of $G(P)$, $1 \in \bigcup C$. Let $k = |\bigcup C|$. By the previous lemma and the remark after its proof, the remaining components of $G(P)$ split into k groups according to the spaces of $\bigcup C$ in which they lie (none of them lies in the first space). The components in one group may form an arbitrary partition and the groups are mutually independent. Thus, since C is a connected partition,

$$\begin{aligned} B(x) &= \sum_{k\geq 0} B_k^{\mathrm{co}} x^k \cdot B(x)^k \\ \frac{xB(x)}{x} = B(x) &= B^{\mathrm{co}}(xB(x)). \end{aligned} \tag{22}$$

Now (20) follows by substituting for x the power series $(xB(x))^{\langle -1\rangle}$.

We give an algebraic verification of (21) and then a combinatorial derivation. Let U be the substitution $x \to (x/(1+x)) \cdot B(x/(1+x))$. Then, by (2) and (22),

$$\begin{aligned} x \cdot B^{\mathrm{co}}(B^{\mathrm{co}}(x) - 1) &\overset{U}{\longrightarrow} \frac{x}{1+x} \cdot B(x/(1+x)) \cdot B(x) \\ B^{\mathrm{co}}(x)^2 - (1+x) \cdot B^{\mathrm{co}}(x) + x &\overset{U}{\longrightarrow} \frac{1}{1+x}\left(B(x/(1+x))^2 - B(x/(1+x))\right). \end{aligned}$$

By (2), the right hand sides are equal. Due to the inverse substitution, so are the left hand sides. This gives (21).

Now we derive (21) combinatorially. For a connected partition P of $X \subset \mathbf{N}$, let $a = \min \bigcup P$ denote the first element and $A \in P$ denote the first block: $a \in A$. We consider the class of all partitions, called the *D-partitions*, which arise from connected partitions P by deleting the first element a and marking the elements of $A_0 = A\backslash\{a\}$ by some label (so that they can be recognized). Note that $|A_0| \geq 1$ whenever $|\bigcup P| \geq 2$. If $D(x)$ is the ogf of D-partitions, then

$$B^{\mathrm{co}}(x) = 1 + xD(x). \tag{23}$$

To obtain another relation between $B^{\mathrm{co}}(x)$ and $D(x)$, consider a D-partition D and the graph G_0 that arises from $G(D)$ by deleting the vertex A_0. Let C be any innermost component of G_0, which means that $\bigcup C \succ \bigcup C'$ for no component C'. It follows, since D originated from a connected partition P, that at least one inner space of $\bigcup C$ must contain an element of A_0. We see that D-partitions are exactly the partitions D obtained by the following recursive construction: D is a concatenation of $k \geq 0$ partitions $R_1, R_2, \ldots, R_k$

where $\bigcup R_1 < \bigcup R_2 < \ldots < \bigcup R_k$ and for every $i = 1, \ldots, k$ either (i) R_i is one element of A_0 or (ii) R_i arises by taking a connected partition Q on $l = |\bigcup Q| \geq 2$ elements and inserting in the $l-1$ inner spaces of $\bigcup Q$ independently $l-1$ D-partitions, not all of them empty. Thus

$$D(x) = \sum_{k \geq 0} \left(x + \frac{B^{\mathrm{co}}(xD(x)) - xD(x) - 1}{D(x)} - (B^{\mathrm{co}}(x) - x - 1) \right)^k.$$

The first x in the summand accounts for the case (i) and the rest accounts for the case (ii). We sum the geometric series and replace, by (23), $xD(x)$ with $B^{\mathrm{co}}(x) - 1$. Further algebraic simplifications produce (21). $\Box$

Comparing in (22) the coefficients at x^n, we obtain the recurrence

$$B_n^{\mathrm{co}} = B_n - \sum_{r=1}^{n-1} B_r^{\mathrm{co}} \sum_{\substack{i_1, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n}} B_{i_1 - 1} \ldots B_{i_r - 1}$$

where $n \geq 1$. Besides numbers B_n^{co} it involves B_n, which is a certain aesthetical blemish. Using (21), we obtain a recurrence purely in terms of B_n^{co}:

$$B_n^{\mathrm{co}} = \sum_{r=2}^{n-1} B_r^{\mathrm{co}} \sum_{\substack{i_1, \ldots, i_r \geq 1 \\ i_1 + \cdots + i_r = n-1}} B_{i_1}^{\mathrm{co}} \ldots B_{i_r}^{\mathrm{co}} - \sum_{i=2}^{n-2} B_i^{\mathrm{co}} B_{n-i}^{\mathrm{co}}$$

where $n \geq 3$ and $B_1^{\mathrm{co}} = B_2^{\mathrm{co}} = 1$. We have

$$(B_n^{\mathrm{co}})_{n \geq 3} = (1, 2, 6, 21, 85, 385, 1907, 10205, 58455, 355884, 2290536, \ldots).$$

Proposition 2.7 *The ogf $B^{\mathrm{cr}}(x) = \sum_{n \geq 0} B_n^{\mathrm{cr}} x^n = 1 + x^4 + 5x^5 + \cdots$ of numbers of crossing partitions expresses in terms of the ogf $B(x)$ as*

$$B^{\mathrm{cr}}(x) = \frac{1}{x} \cdot \left(\frac{(1-x) \cdot (xB(x))^{\langle -1 \rangle}}{1 - x - (xB(x))^{\langle -1 \rangle}} \right)^{\langle -1 \rangle} \tag{24}$$

or

$$B^{\mathrm{cr}}(x) = \frac{1}{x} + \frac{1}{1 - x \cdot \left(\left(\frac{x - x^2 B(x)}{1 - x - xB(x)} \right)^{\langle -1 \rangle} \right)^{-1}} \tag{25}$$

where $^{\langle -1 \rangle}$ denotes the compositional inverse.

Proof. The derivation of (24) uses the same decomposition as that of (20). Only $B(x)$ is replaced with $B^{\mathrm{cr}}(x)$ because now P is a possibly empty crossing partition, and $B^{\mathrm{co}}(x)$ is replaced with $B^{\mathrm{co}}(x) - x/(1-x)$ because now the first component C must not be a single vertex. Thus

$$\frac{xB^{\mathrm{cr}}(x)}{x} = B^{\mathrm{cr}}(x) = (B^{\mathrm{co}}(x) - x/(1-x)) \circ (xB^{\mathrm{cr}}(x)). \tag{26}$$

Substituting for x the power series $(xB^{\mathrm{cr}}(x))^{\langle -1\rangle}$, we have

$$(xB^{\mathrm{cr}}(x))^{\langle -1\rangle} = \frac{x}{B^{\mathrm{co}}(x) - x/(1-x)}.$$

Taking the inverse again and replacing $B^{\mathrm{co}}(x)$ according to (20), we obtain (24).

To derive (25), we employ another decomposition. We call a partition P *sequential* if it is empty or if there are $k \geq 1$ blocks $A_1, \ldots, A_k$ in P such that $A_1 < A_2 < \ldots < A_k$ and for every other block B of P we have $B \prec A_i$ for some i. Let P be any partition. Consider the induced subgraph H of $G(P)$ formed by the components C such that $|C| \geq 2$ and $\bigcup C \not\prec A$ for every one-vertex component A. It follows that H is a crossing partition and for every space of $\bigcup H$ the partition lying in it is sequential. Also, this decomposition of P into a crossing partition H on l elements and $l+1$ sequential partitions is unique. Thus

$$B(x) = \sum_{l\geq 0} B_l^{\mathrm{cr}} x^l F_1^{l+1}, \;\; F_1 = \frac{1}{1-F_2}, \;\; F_2 = \frac{x}{1-xB(x)}.$$

Here $F_1(x)$ counts the sequential partitions lying in a space of $\bigcup H$. We obtain the relation

$$xB(x) = F(x) \cdot B^{\mathrm{cr}}(F(x)) \;\; \text{where} \;\; F(x) = xF_1(x) = x \cdot \frac{1-xB(x)}{1-x-xB(x)}.$$

Combining both equations, we get

$$F(x) = x \cdot \frac{1 - F(x) \cdot B^{\mathrm{cr}}(F(x))}{1 - x - F(x) \cdot B^{\mathrm{cr}}(F(x))}.$$

Substituting for x the power series $F(x)^{\langle -1\rangle}$ and solving the result for $B^{\mathrm{cr}}(x)$, we get (25). □

We rewrite (26) as

$$xB^{\mathrm{cr}}(x)^2 - (1+x)B^{\mathrm{cr}}(x) - (xB^{\mathrm{cr}}(x) - 1)\cdot B^{\mathrm{co}}(xB^{\mathrm{cr}}(x)) = 0.$$

Comparing the coefficients at x^n, we obtain the recurrence

$$B_n^{\mathrm{cr}} = B_{n-1}^{\mathrm{cr}} + \sum_{i=1}^{n-2} B_i^{\mathrm{cr}} B_{n-1-i}^{\mathrm{cr}} + \sum_{r=1}^{n}(B_r^{\mathrm{co}} - B_{r-1}^{\mathrm{co}}) \sum_{\substack{i_1,\ldots,i_r\geq 1\\ i_1+\cdots+i_r=n}} B_{i_1-1}^{\mathrm{cr}} \ldots B_{i_r-1}^{\mathrm{cr}}$$

where $n \geq 2$, $B_0^{\mathrm{cr}} = 1$ and $B_1^{\mathrm{cr}} = 0$. We have $B_1^{\mathrm{cr}} = B_2^{\mathrm{cr}} = B_3^{\mathrm{cr}} = 0$ and

$$(B_n^{\mathrm{cr}})_{n\geq 4} \quad = \quad (1, 5, 20, 84, 388, 1951, 10529, 60478, 367953, \ldots).$$

3 Power series and algebraic differential equations

The ring of power series with complex coefficients $\mathbf{C}[[x]]$ contains the subring $\mathbf{C}\{x\}$ of *analytic* (or *convergent*) power series; $\mathbf{C}\{x\}$ consists of all power series $F(x) = a_0 + a_1x + a_1x^2 + \cdots$ which absolutely converge in a neighborhood of zero. Besides the ring operations, $\mathbf{C}\{x\}$ is closed also to division (if defined), differentiation and to substitutions. The next proposition follows from the standard results of algebraic geometry on local parametrizations of plane curves by the Puiseux series. We found the exposition in Fischer [8, chapters 6 and 7] very readable. Ruiz [25] contains further information on analytic approximations (M. Artin's approximation theorem).

Proposition 3.1 *Suppose that $F \in \mathbf{C}[[x]]$ is algebraic over $\mathbf{C}\{x\}$, that is,*

$$A_0F^n + \cdots + A_{n-1}F + A_n = 0$$

holds for some $n \in \mathbf{N}$ and some analytic coefficients $A_i \in \mathbf{C}\{x\}$, $A_0 \neq 0$. Then F is analytic too. $\square$

The field of fractions of $\mathbf{C}[[x]]$ is the field of *Laurent series* $\mathbf{C}((x))$ consisting of all formal sums $F(x) = a_kx^k + a_{k+1}x^{k+1} + \cdots$ where $a_i \in \mathbf{C}$ and $k \in \mathbf{Z}$. For $F \neq 0$ the requirement $a_k \neq 0$ makes the representation unique and we denote this k as $\mathrm{ord}(F)$. We set $\mathrm{ord}(0) = \infty$. We write $[x^n]F$ to denote the coefficient of x^n in the Laurent series F and use similar notation for coefficients in polynomials in two variables. The field of fractions of $\mathbf{C}\{x\}$ consists of all $F \in \mathbf{C}((x))$ such that $x^k F \in \mathbf{C}\{x\}$ for some $k \in \mathbf{N}$. For simplicity we denote this field by $\mathbf{C}\{x\}$ as well. Proposition 3.1 holds also for $F \in \mathbf{C}((x))$ and this broader understanding of $\mathbf{C}\{x\}$.

An ADE (algebraic differential equation) over K of order k, where K is a subfield of $\mathbf{C}((x))$ and $k \geq 0$ is an integer, is the differential equation

$$P(X, X', \ldots, X^{(k)}) = 0$$

where $P(y_0, y_1, \ldots, y_k)$ is a polynomial over K in $k+1$ variables and $X \in \mathbf{C}((x))$ is an unknown. If $F \in \mathbf{C}((x))$ satisfies such an equation for some k, we say that F is *differentially algebraic over* K. For $k = 0$ this simply means that F is algebraic over K. A power series $F \in \mathbf{C}[[x]]$ is called *D-finite* (or *holonomic*) if it satisfies a linear differential equation over $\mathbf{C}(x)$, that is, it satisfies an ADE with $P = \alpha_0 y_0 + \cdots + \alpha_k y_k$, $\alpha_i \in \mathbf{C}(x)$. See Stanley [29] and [31, chapter 6] for the importance of this class of power series for combinatorial enumeration.

It is clear from the preceding that the substitutions $x \to x/(1-x)$ and $x \to x/(1+x)$ will play an important role. We denote the latter by S. For $F(x) \in \mathbf{C}((x))$ we write SF for $F(x/(1+x))$. S is an automorphism of the field $\mathbf{C}((x))$ and $\mathrm{ord}(SF) = \mathrm{ord}(F)$ for every F. As for the differentiation, by the chain rule $SF' = (1+x)^2 \cdot (SF)'$. Also, for $j \in \mathbf{Z}$ the iteration S^j is just the substitution $x \to x/(1+jx)$.

Lemma 3.2 *Let $r \in \mathbf{Z}$ and $G, H \in \mathbf{C}((x))$ with $G = 1 + (r-1)x + \cdots$ and $\mathrm{ord}(H) = r$. Then the equation*

$$F = G \cdot SF + H \tag{27}$$

has no solution $F \in \mathbf{C}((x))$.

Proof. Let us try a generic solution $F = \sum_{n \geq k} a_n x^n$, $k = \mathrm{ord}(F)$. We have to satisfy the equation

$$\sum_{n \geq k} a_n x^n = (1 + (r-1)x + \cdots) \cdot \sum_{n \geq k} a_n \left(\frac{x}{1+x}\right)^n + b_r x^r + \cdots$$

where $b_r, a_k \neq 0$. Thus

$$\begin{aligned} a_k x^k + a_{k+1} x_{k+1} + R_1 &= a_k x^k + (-k a_k + (r-1) a_k + a_{k+1}) x^{k+1} + R_2 \\ &\quad + b_r x^r + R_3 \\ R_1 &= (-k + r - 1) a_k x^{k+1} + R_2 + b_r x^r + R_3 \end{aligned}$$

where $\mathrm{ord}(R_1), \mathrm{ord}(R_2) \geq k+2$ and $\mathrm{ord}(R_3) \geq r+1$. No F satisfies the equation because for no $k \in \mathbf{Z}$ the orders in the last equation match. The order on the left is always $\geq k+2$. For $k < r-1$ the order on the right is $k+1$. For $k = r-1$ the coefficient in the bracket vanishes and the order is again $r = k+1$. For $k > r-1$ the order is $r \leq k$. $\square$

Now we consider the ogf's of Bell numbers and of Uppuluri–Carpenter numbers.

Proposition 3.3 *Let K be a subfield of $\mathbf{C}((x))$ that contains $\mathbf{C}(x)$ and is closed on the substitution S. If $B(x)$ is differentially algebraic over K, then $B(x)$ is algebraic over K. The same holds for $B^{\pm}(x)$.*

Proof. We proceed by induction on the order k of the ADE satisfied by $B = B(x)$. If $k = 0$, B is algebraic over K by the definition. We suppose that the statement holds for all orders $\leq k$ and that B satisfies an ADE over K of order $k+1$. This can be written as

$$P(C, C') = 0$$

where $C = B^{(k)}$, $P \in L[y, z]$ is a nonzero bivariate polynomial, and L is the field $K(B, B', \ldots, B^{(k-1)})$ (for $k = 0$ we set $L = K$). We deduce from this that $C = B^{(k)}$ is algebraic over L. Then B satisfies an ADE over K of order at most k and we are done by the inductive assumption.

We look first at the action of S on C and C'. By (2),

$$SB = 1 + xB \quad \text{and} \quad SB' = (1+x)^2(SB)' = (1+x)^2B + x(1+x)^2B'.$$

It follows easily by the differentiation that for $l \geq 0$

$$SB^{(l)} = \alpha_l + \beta_l B^{(l)} \quad \text{and} \quad SB^{(l+1)} = \gamma_l + \delta_l B^{(l)} + \varepsilon_l B^{(l+1)}$$

where the coefficients $\alpha_l, \gamma_l \in K(B, B', \ldots, B^{(l-1)})$ and $\beta_l, \delta_l, \varepsilon_l \in K$ satisfy $\alpha_0 = 1, \beta_0 = x, \gamma_0 = 0, \delta_0 = (1+x)^2, \varepsilon_0 = x(1+x)^2$ and

$$\begin{aligned}
\alpha_{l+1} &= \gamma_l + \delta_l B^{(l)}, \\
\beta_{l+1} &= \varepsilon_l, \\
\gamma_{l+1} &= (1+x)^2(\gamma_l' + \delta_l' B^{(l)}), \\
\delta_{l+1} &= (1+x)^2(\delta_l + \varepsilon_l'), \\
\varepsilon_{l+1} &= (1+x)^2\varepsilon_l.
\end{aligned}$$

Thus

$$SC = \alpha + \beta C \quad \text{and} \quad SC' = \gamma + \delta C + \varepsilon C'$$

where $\alpha, \gamma \in L$ and $\beta, \delta, \varepsilon \in \mathbf{Z}[x]$. The field L is closed to S. The recurrences show that $\beta, \delta, \varepsilon \neq 0$, $\varepsilon\beta^{-1} = (1+x)^2$, $\beta = x + \cdots$, and $\mathrm{ord}(\delta) = 0, \delta(0) = k+1$.

We take the polynomial $P(y,z)$, which vanishes on C, C', to be minimal in the sense that $b = \deg_z(P)$ is smallest among all such P and also the degree a of $[z^b]P \in L[y]$ is smallest among all such P with z-degree b. If $b = 0$, C is algebraic over L and we are done. We assume that $b \geq 1$ and derive a contradiction. The action of S on $P(C, C') = 0$ produces the identity

$$Q(C, C') = 0$$

where $Q \in L[y,z]$ is given by

$$Q(y,z) = (SP)(\alpha + \beta y, \gamma + \delta y + \varepsilon z).$$

It follows, crucially, that Q is also minimal in our sense and $[y^a z^b]Q = \beta^a \varepsilon^b (S\tau)$ where $\tau = [y^a z^b]P$. Let $\rho = \beta^a \varepsilon^b (S\tau)\tau^{-1}$. Consider the polynomial

$$R(y,z) = \rho P(y,z) - Q(y,z).$$

$R(C, C') = 0$ and hence R must be identically zero; otherwise it would contradict the minimality of P (and Q). So P must satisfy the identity

$$\rho P(y,z) = (SP)(\alpha + \beta y, \gamma + \delta y + \varepsilon z). \tag{28}$$

We show that it is contradictory.

Let z^d be the second largest power of z that appears in P with a nonzero coefficient. It follows from (28) that the case $d < b-1$ is impossible. Hence $d = b-1$. Let $c = \deg_y([z^{b-1}]P)$ and $\sigma = [y^c z^{b-1}]P$. Comparing in (28) the coefficients at $y^c z^{b-1}$, we obtain the equation

$$\rho\sigma = (S\sigma)\beta^c \varepsilon^{b-1} \quad \text{if} \quad a+1 < c \tag{29}$$

$$\rho\sigma = (S\sigma)\beta^c \varepsilon^{b-1} + (S\tau)\beta^a b \delta \varepsilon^{b-1} \quad \text{if} \quad a+1 = c. \tag{30}$$

(It follows from (28) that we cannot have $a+1 > c$.) Dividing eq. (29) by $\rho\tau = \beta^a \varepsilon^b (S\tau)$, we obtain the identity

$$\sigma\tau^{-1} = S(\sigma\tau^{-1}) \cdot \beta^{c-a} \varepsilon^{-1}.$$

But $\mathrm{ord}(\sigma\tau^{-1}) = \mathrm{ord}(S(\sigma\tau^{-1}))$ (S preserves orders) and $\mathrm{ord}(\beta^{c-a}\varepsilon^{-1}) = \mathrm{ord}(\beta^{c-a-1}\cdot\beta\varepsilon^{-1}) > 0$ because $\mathrm{ord}(\beta) = 1$ and $\beta\varepsilon^{-1}$ is a unit. We have a contradiction.

Manipulating (30) in the same way, we obtain the identity

$$\sigma\tau^{-1} = S(\sigma\tau^{-1})\cdot\beta\varepsilon^{-1} + b\delta\varepsilon^{-1}.$$

We have $\beta\varepsilon^{-1} = (1+x)^{-2} = 1 - 2x + \cdots$ and $b\delta\beta^{-1} = b(k+1)x^{-1} + \cdots$. Applying Lemma 3.2 with $r = -1$, $F = \sigma\tau^{-1}$, $G = \beta\varepsilon^{-1}$, and $H = b\delta\beta^{-1}$, we see that this identity is also contradictory. We have shown that $b \geq 1$ always leads to a contradiction.

For $B^{\pm}(x)$ the previous proof needs only minor adjustments: now $\beta_0 = -x$, $\delta_0 = -(1+x)^2$ and $\varepsilon_0 = -x(1+x)^2$. Thus $\beta = -x + \cdots$ and $\delta(0) = -(k+1)$. Everything else is as before, in particular $\varepsilon\beta^{-1} = (1+x)^2$ and $\mathrm{ord}(\delta) = 0$. $\square$

To apply this proposition in combination with Proposition 3.1, we need to establish that $B(x) \notin \mathbf{C}\{x\}$ and $B^{\pm}(x) \notin \mathbf{C}\{x\}$.

Proposition 3.4 *The power series $B(x)$ and $B^{\pm}(x)$ are nonanalytic.*

Proof. We begin with $B^{\pm}(x)$ and approach it by means of (8) and (10). Suppose, for the contradiction, that $B^{\pm}(x)$ is analytic and has the radius of convergence $r > 0$. By (8) (or by (9)), $B^{\pm}(x)$ cannot be a polynomial (take $x \to \infty$) and therefore $|B^{\pm}_n| \geq 1$ for infinitely many $n \in \mathbf{N}$. So $r \leq 1$. $B^{\pm}(x)$ defines in the disc $|z| < r$ an analytic function $B^{\pm}(z)$. Let $\alpha \in \mathbf{C}$, $|\alpha| = r$, be a singularity of $B^{\pm}(z)$ on the circle of convergence $|z| = r$. A simple calculation shows that

$$\{z \in \mathbf{C}:\ |z| = r\ \&\ |\tfrac{z}{1-z}| < r\} = \{z \in \mathbf{C}:\ |z| = r\ \&\ \mathrm{Re}(z) < \tfrac{1}{2}r^2\}$$

and

$$\{z \in \mathbf{C}:\ |z| = r\ \&\ |\tfrac{z}{1+z}| < r\} = \{z \in \mathbf{C}:\ |z| = r\ \&\ \mathrm{Re}(z) > -\tfrac{1}{2}r^2\}.$$

If $\mathrm{Re}(\alpha) < 0$, we use (8) to continue analytically $B^{\pm}(z)$ to a neighborhood of α, which contradicts the definition of α. For $\mathrm{Re}(\alpha) \geq 0$ we use (10) to obtain the same contradiction. Since $\alpha \neq 1$ in the former case, $\alpha \neq -1$ in the latter case, and never $\alpha = 0$, we need not be worried by the poles $z = -1, 0, 1$ in (8) and (10). For every location of α one of (8) and (10) leads to a contradiction. Thus $r = 0$ and $B^{\pm}(x)$ is nonanalytic.

The same argument, using (1) and (2), applies to $B(x)$. In fact, now we need only (2) because $\alpha = r > 0$ would be a singularity of $B(x)$ if it were analytic (by Pringsheim theorem). Alternatively, we can use Stirling numbers $S(n, k)$ and the set of words $W(n, k) = \{w \in [k]^* : |w| = n \ \& \ w \text{ uses every } i \in [k]\}$ to give a lower bound to B_n. For every $n \ge k \ge 1$,

$$B_n \ge S(n,k) = \frac{|W(n,k)|}{k!} \ge \frac{k^n - k \cdot (k-1)^n}{k!} = \frac{k^n}{k!} - \frac{(k-1)^n}{(k-1)!}.$$

This shows again that $B(x)$ is nonanalytic. □

For $|B_n^{\pm}|$ no simple combinatorial lower bound seems available. Our proof gives $\limsup_{n \to \infty} \log |B_n^{\pm}|/n = +\infty$. Subbarao and Verma [34] used the exponential generating function $B_e^{\pm}(x) = \mathrm{e}^{1-\mathrm{e}^x}$ and proved, among other results, the stronger bound $\limsup_{n \to \infty} \log |B_n^{\pm}|/(n \log n) = 1$ (in fact, they proved it for more general numbers). Y. Yang [38] proved, among other results, that the sequence $(|B_n^{\pm}|)_{n \ge 1}$ is not eventually monotone, $\log |B_n^{\pm}| \le \log B_n - \pi^2 n/(2 \log^2 n) + O(n (\log \log n)^2 / \log^3 n)$, and $\#\{n \le x : B_n^{\pm} = 0\} = O(x^{2/3})$ (but perhaps this set has always just one element).

Theorem 3.5 *The ogf $B(x)$ of Bell numbers and the ogf $B^{\pm}(x)$ of Uppuluri–Carpenter numbers satisfy no algebraic differential equation over $\mathbf{C}\{x\}$.*

Proof. $\mathbf{C}\{x\}$ contains $\mathbf{C}(x)$ and is closed to the substitution S. Suppose, for the contradiction, that $B(x)$ satisfies an ADE over $\mathbf{C}\{x\}$. By Proposition 3.3, $B(x)$ is algebraic over $\mathbf{C}\{x\}$. By Proposition 3.1, it is even analytic. But this contradicts Proposition 3.4. The same for $B^{\pm}(x)$. □

Proposition 3.1 is appealing and nice in its generality but it has a lengthy proof. One can avoid it and prove the transcendence of $B = B(x)$ and $B^{\pm}(x)$ over $\mathbf{C}\{x\}$ directly as follows. Suppose, for the contradiction, that $B^k + A_1 B^{k-1} + \cdots + A_k = 0$ where $A_i \in \mathbf{C}\{x\}$ and $k \ge 1$ is minimum. We can assume that $k \ge 2$. The substitution S and (2) give us another equation $B^k + \frac{k + SA_1}{x} B^{k-1} + \cdots = 0$. If $\frac{k+SA_1}{x} \ne A_1$, we subtract both equations and obtain a contradiction with $k \ge 2$ or with the minimality of k. Else we have the equation $SA_1 = xA_1 - k$. It has a unique solution $A_1 \in \mathbf{C}((x))$. The analytic continuation argument from the proof of Proposition 3.4 shows that the solution is nonanalytic. So $A_1 \notin \mathbf{C}\{x\}$, which contradicts the assumption $A_1 \in \mathbf{C}\{x\}$. The same for $B^{\pm}(x)$.

If K is a subset of $\mathbf{C}((x))$, we denote

$$\mathcal{A}^K = \{F \in \mathbf{C}((x)) : \ F \text{ satisfies an ADE over } K\}.$$

In the next proposition we collect the closure properties of $\mathcal{A}^K$ needed to handle the ogf's $B^{j,i}(x)$, $B^{\mathrm{co}}(x)$, and $B^{\mathrm{cr}}(x)$. These are standard results of differential algebra (see, for example, Ostrowski [19, §5]) but for the reader's convenience we prove them here. We say that $M \subset \mathbf{C}((x))$ is *closed to substitutions* if $F, G \in M$, $G(0) = 0$, always implies $F(G) \in M$. Similarly, M is *closed to compositional inverses* if $F^{\langle -1 \rangle} \in M$ whenever $F \in M$ and $F^{\langle -1 \rangle}$ exists.

Proposition 3.6 *1. For every differential subfield K of $\mathbf{C}((x))$, $\mathcal{A}^K$ is a (differential) subfield of $\mathbf{C}((x))$. 2. Let K be a differential subfield of $\mathbf{C}((x))$ that is closed to substitutions. Then $F(G) \in \mathcal{A}^K$ whenever $F \in \mathcal{A}^K$ and $G \in K$, $G(0) = 0$. 3. $\mathcal{A}^{\mathbf{C}(x)}$ is closed to compositional inverses.*

Proof. 1. First note that if $F \in \mathbf{C}((x))$ satisfies an ADE over K of order at most n, then every derivative $F^{(n+1)}, F^{(n+2)}, \ldots$ expresses rationally over K in terms of $F, F', \ldots, F^{(n)}$. Second, any $n+1$ rational functions in n variables $A_1, A_2, \ldots, A_{n+1} \in L(y_1, \ldots, y_n)$ (L is an arbitrary field) must be algebraicly dependent over L: $P(A_1, \ldots, A_{n+1}) = 0$ for some nonzero polynomial P with coefficients in L. (The vanishing of $P(A_1, \ldots, A_{n+1})$ translates to a homogeneous linear system whose unknowns are the coefficients of P. If P has large degree, the system has more unknowns than the equations.) Now we suppose that F and G satisfy an ADE over K of order at most n and want to show that also $FG \in \mathcal{A}^K$. By the Leibniz formula and the first remark, $(FG)^{(m)} \in K(F, F', \ldots, F^{(n)}, G, G', \ldots, G^{(n)})$ for every $m \geq 0$. By the second remark, $FG, (FG)', \ldots, (FG)^{(2n+2)}$ are algebraicly dependent over K and thus $FG \in \mathcal{A}^K$. Similarly for $F + G$ and F/G.

2. We suppose that F satisfies an ADE over K of order at most n, take a $G \in K$ with $G(0) = 0$, and we want to show that $F(G) \in \mathcal{A}^K$. Since $F^{(m)} \in K(F, F', \ldots, F^{(n)})$ for every $m \geq 0$, by the assumption on K we have that for every $m \geq 0$ also $F^{(m)}(G) \in K(F(G), F'(G), \ldots, F^{(n)}(G))$. By the chain rule, $F(G)^{(m)} \in K(F(G), F'(G), \ldots, F^{(n)}(G))$ for every $m \geq 0$. So $F(G), F(G)', \ldots, F(G)^{(n+1)}$ are algebraicly dependent over K.

3. Suppose that F satisfies an ADE over $\mathbf{C}(x)$ of order at most n and $F^{\langle -1\rangle}$ exists. Thus $F^{(m)} \in \mathbf{C}(x, F, F', \dots, F^{(n)})$ for every $m \geq 0$. Differentiating $(F^{\langle -1\rangle})' = 1/F'(F^{\langle -1\rangle})$, we express every $(F^{\langle -1\rangle})^{(m)}$, $m \geq 1$, rationally over $\mathbf{C}$ in terms of $F'(F^{\langle -1\rangle}), \dots, F^{(m)}(F^{\langle -1\rangle})$. So $(F^{\langle -1\rangle})^{(m)} \in \mathbf{C}(F^{\langle -1\rangle}, x, F'(F^{\langle -1\rangle}), \dots, F^{(n)}(F^{\langle -1\rangle}))$ for every $m \geq 0$. It follows that the derivatives $F^{\langle -1\rangle}$, $(F^{\langle -1\rangle})', \dots$, $(F^{\langle -1\rangle})^{(n+1)}$ are algebraicly dependent over $\mathbf{C}(x)$. $\Box$

Theorem 3.7 *For every $i \in \mathbf{N}$ and $j \in \mathbf{Z}$, the ogf $B^{j,i}(x)$ satisfies no ADE over $\mathbf{C}\{x\}$. The ogf's $B^{\mathrm{co}}(x)$ and $B^{\mathrm{cr}}(x)$ satisfy no ADE over $\mathbf{C}(x)$.*

Proof. Let $i \in \mathbf{N}$ be fixed. Suppose that for some $j, 0 \leq j < i$, $B^{j,i}(x)$ satisfies an ADE over $\mathbf{C}\{x\}$. $\mathbf{C}\{x\}$ meets the hypotheses of 1 and 2 of Proposition 3.6 and, of course, $x/(1+x)$ belongs to $\mathbf{C}\{x\}$. Using (14) and 2 of Proposition 3.6, we get that $B^{j,i} \in \mathcal{A}^{\mathbf{C}\{x\}}$ for every $j \in [0, i-1]$. By (13) and 1 of Proposition 3.6, we get $B \in \mathcal{A}^{\mathbf{C}\{x\}}$. But this contradicts Theorem 3.5. As for $B^{\mathrm{co}}(x)$, by (20) we have $B(x) \in \mathbf{C}(x, (x/B^{\mathrm{co}})^{\langle -1\rangle})$. By 1 and 3 of Proposition 3.6, $B^{\mathrm{co}} \in \mathcal{A}^{\mathbf{C}(x)}$ would imply $B \in \mathcal{A}^{\mathbf{C}(x)}$, which is impossible. For $B^{\mathrm{cr}}(x)$ we argue similarly, using (24) or (25). $\Box$

It is a natural question if in the last theorem the result for $B^{\mathrm{co}}(x)$ and $B^{\mathrm{cr}}(x)$ holds also for the wider field of analytic coefficients $\mathbf{C}\{x\}$.

We proceed to the last ogf, $B^{\mathrm{B}}(x)$. The difficulty is that (16) and (17) are, in contrast to the equations for $B(x)$ and $B^{\pm}(x)$, nonlinear.

Proposition 3.8 *The power series $B^{\mathrm{B}}(x)$ is nonanalytic.*

Proof. Let $n \in \mathbf{N}$ and $Q = \{C_1, C_2, \dots, C_m\}$, $m \leq n$, be any partition of $[n+1, 2n]$; we have ordered the blocks so that $\min C_1 = n+1 < \min C_2 < \dots < \min C_m$. We set $C_{m+1} = \dots = C_n = \emptyset$ and $A_i = \{i, 3n-i+1\} \cup C_i$, $i = 1, 2, \dots, n$. $P = \{A_1, \dots, A_n\}$ is a non-overlapping partition of $[3n]$ and different partitions Q give different partitions P. Thus $B^{\mathrm{B}}_{3n} \geq B_n$ for every $n \in \mathbf{N}$ and $B^{\mathrm{B}}(x)$ is nonanalytic by Proposition 3.4. $\Box$

We need a slight generalization of Lemma 3.2. Its proof is very similar and is left to the interested reader.

Lemma 3.9 *Let $r \in \mathbf{Z}$ and $G, H \in \mathbf{C}((x))$ with $G = 1 + (r-1)x + \cdots$ and* $\mathrm{ord}(H) = r$. *Then the equation*

$$F_1 = G \cdot SF_1 + H + F_2$$

has no solution $F_1, F_2 \in \mathbf{C}((x))$ with $\mathrm{ord}(F_2) \geq \mathrm{ord}(F_1) + 2$. $\square$

$\mathbf{C}((x))$ is a subfield of the field of Puiseux series

$$\mathbf{C}((x))^{\mathrm{P}} = \{\sum_{n \geq k} a_n x^{n/r} : \ k \in \mathbf{Z}, \ r \in \mathbf{N}, \text{ and } a_n \in \mathbf{C}\}.$$

$\mathbf{C}((x))^{\mathrm{P}}$ is the algebraic closure of $\mathbf{C}((x))$ (see, for example, Fischer [8, Theorem 7.2] or Walker [36]). A Puiseux series $\rho(x)$ is analytic if $\rho(x^r) \in \mathbf{C}\{x\}$ for some $r \in \mathbf{N}$. Analytic Puiseux series form the field $\mathbf{C}\{x\}^{\mathrm{P}}$. In the proof of Theorem 3.11 we need the result that $\mathbf{C}\{x\}^{\mathrm{P}}$ is the algebraic closure of $\mathbf{C}\{x\}$ ([8, Complement of Theorem 7.2]). This is a strengthening of Proposition 3.1. The substitution S is an automorphism of the fields $\mathbf{C}((x))^{\mathrm{P}}$ and $\mathbf{C}\{x\}^{\mathrm{P}}$ and it preserves order.

Equations of the type $P(F, SF) = 0$, where $P \in \mathbf{C}((x))[y, z]$ and $F \in \mathbf{C}((x))$ is an unknown, play in our approach an important role. In (2) P is linear and $B(x)$ is the unique solution of (2) in $\mathbf{C}((x))$; similarly for (9) and (15). In (27) P is linear too but Lemma 3.2 tells us that in some situations there is no solution. In (17) P is nonlinear and (17) has in $\mathbf{C}((x))$ two solutions: $B^{\mathrm{B}}(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ and $\overline{B^{\mathrm{B}}}(x) = x^{-2} - B^{\mathrm{B}}(-x) = x^{-2} - 1 + x - 2x^2 + 5x^3 - \cdots$. One easily checks that $\overline{B^{\mathrm{B}}}(x)$ solves (17) by writting $\overline{B^{\mathrm{B}}}(x) = x^{-2} - MB^{\mathrm{B}}$, where M is the substitution $x \to -x$, and using the relation $SM = MS^{-1}$.

It is convenient to replace $B^{\mathrm{B}}(x)$ with $F(x) \in \mathbf{C}((x))$ given by

$$F(x) = \frac{1}{xB^{\mathrm{B}}(x)} \tag{31}$$

because this change of variables turns (17) into the simpler equation

$$SF = x^{-1} - F^{-1}. \tag{32}$$

Eq. (32) can be written equivalently as $F = UF$ where

$$UF = \frac{1}{x^{-1} - SF}. \tag{33}$$

We extend the transformation U to $\mathbf{C}((x))^{\mathrm{P}}$. $U\rho$ is defined for every $\rho \in \mathbf{C}((x))^{\mathrm{P}}$, except for $\rho^* = S^{-1}x^{-1} = x^{-1} - 1$, and U is a bijection between $\mathbf{C}((x))^{\mathrm{P}}\backslash\{\rho^*\}$ and $\mathbf{C}((x))^{\mathrm{P}}\backslash\{0\}$. For the proof of Theorem 3.11 we need to know that the equations $\rho = U^j\rho$, $j \in \mathbf{N}$, have in $\mathbf{C}((x))^{\mathrm{P}}$ only nonanalytic solutions ρ.

Lemma 3.10 *For every $j \in \mathbf{N}$, the equation $\rho = U^j\rho$ has in $\mathbf{C}((x))^{\mathrm{P}}$ exactly two solutions: $\rho = F(x) = 1/(xB^{\mathrm{B}}(x)) = x^{-1} - 1 - x - 2x^2 - 5x^3 - 15x^4 - \cdots$ and $\rho = \overline{F}(x) = 1/(x\overline{B^{\mathrm{B}}}(x)) = 1/(x^{-1} - xB^{\mathrm{B}}(-x)) = x + x^3 - x^4 + 3x^5 - 7x^6 + 20x^7 - \cdots$. Both are nonanalytic.*

Proof. By the above discussion, F and $\overline{F}$ are solutions of (32) and of $\rho = U\rho$. Thus they solve also $\rho = U^j\rho$ for every $j \in \mathbf{N}$. F and $\overline{F}$ are nonanalytic since B^{B} is (by Proposition 3.8). It remains to be shown that $\rho = U^j\rho$ has in $\mathbf{C}((x))^{\mathrm{P}}$ at most two solutions. This equation is equivalent to the system

$$S\rho_i = x^{-1} - \rho_{i+1}^{-1},\ i = 1, 2, \ldots, j$$

where $\rho_1 = \rho$ and $\rho_{j+1} = \rho_1 = \rho$. Let $k_i = \mathrm{ord}(\rho_i) \in \mathbf{Q}$. If $k_r > 1$ for some r, then the r-th equation of the system implies that $k_{r+1} = 1$ and the remaining equations imply that all k_i are equal to 1, which is a contradiction. If $k_r < 1$ for some r, then the $r-1$-th equation implies that $k_{r-1} = -1$ and the remaining equations imply that all k_i are equal to -1. Thus either (i) all k_i are -1 or (ii) all k_i are 1. For a Puiseux series η we let $\mathrm{ford}(\eta)$ denote the smalest $e \in \mathbf{Q}\backslash\mathbf{Z}$ such that x^e has in η a nonzero coefficient; we set $\mathrm{ford}(\eta) = \infty$ if $\eta \in \mathbf{C}((x))$. Clearly, $\mathrm{ford}(S\eta) = \mathrm{ford}(\eta)$ and $\mathrm{ford}(\eta^{-1}) = \mathrm{ford}(\eta) - 2 \cdot \mathrm{ord}(\eta)$ (for $\eta \neq 0$). Suppose that we have the case (i) and $\mathrm{ford}(\rho_i) < \infty$ for some i. We take the largest fractional order $\mathrm{ford}(\rho_r) < \infty$. The $r-1$-th equation gives us $\mathrm{ford}(\rho_{r-1}) = \mathrm{ford}(\rho_r) + 2$, which contradicts the maximality of $\mathrm{ford}(\rho_r)$. In the case (ii) we get a similar contradiction taking the smallest $\mathrm{ford}(\rho_i) < \infty$. Thus in the solution all ρ_i must be Laurent series. We denote, for $n \in \mathbf{Z}$ and $1 \leq i \leq j$, $a_{n,i} = [x^n]\rho_i$. In the case (i) the equations imply $a_{-1,i} = 1$ for every i. For $n > -1$ the comparison of the coefficients at x^n in the i-th equation gives us a relation $P(a_{-1,i}, \ldots, a_{n,i}, a_{-1,i+1}, \ldots, a_{n-2,i+1}) = 0$ where P is an integral polynomial (depending on n but not on i) in which $a_{n,i}$ appears only as the monomial $a_{n,i}$. Thus all j $a_{n,i}$'s are uniquely determined by the previously computed $a_{m,i}$'s, $m < n$ and $1 \leq i \leq j$, and in the case (i) there is a unique solution in $\mathbf{C}((x))$ (which must lie in $\mathbf{Z}((x))$), namely $\rho_i = \overline{F}$. A similar argument shows that in the case (ii) there is a unique solution in $\mathbf{C}((x))$, $\rho_i = F$. $\square$

Theorem 3.11 *The ogf $B^{\mathrm{B}}(x)$ of Bessel numbers satisfies no ADE over $\mathbf{C}\{x\}$ of order at most one.*

Proof. We replace $B^{\mathrm{B}}(x)$ with the $F(x)$ given by (31). It is clear that $B^{\mathrm{B}}(x)$ satisfies an ADE over $\mathbf{C}\{x\}$ of order at most one if and only if $F(x)$ does. We assume that $P(F, F') = 0$ for a nonzero polynomial $P \in \mathbf{C}\{x\}[y, z]$ and derive a contradiction. Let $b = \deg_z(P)$ be minimum. By Propositions 3.1 and 3.8, $B^{\mathrm{B}}(x)$ is transcendental over $\mathbf{C}\{x\}$. Thus $F(x)$ is also transcendental over $\mathbf{C}\{x\}$ and $b \geq 1$. We make P monic in z: $P(y, z) = z^b + R(y)z^{b-1} + \cdots$ where $R \in \mathbf{C}\{x\}(y)$; now $P \in \mathbf{C}\{x\}(y)[z]$. ¿From (32) we have $SF' = (1 + x)^2 \cdot (-x^{-2} + F'/F^2)$. So the action of S on $P(F, F') = 0$ yields the identity $Q(F, F') = 0$, where $Q \in \mathbf{C}\{x\}(y)[z]$ is given by

$$Q(y, z) = (SP)(\alpha - 1/y, \beta + \gamma z/y^2)$$

with

$$\alpha = x^{-1}, \ \beta = -(1 + x)^2/x^2, \ \text{ and } \ \gamma = (1 + x)^2. \tag{34}$$

Eliminating the power z^b, we obtain the identity $W(F, F') = 0$ where $W = P - \gamma^{-b}y^{2b}Q$. Since $\deg_z(W) \leq b - 1$, W must be identically zero. In particular, $[z^{b-1}]W = 0$, which means that

$$\frac{\gamma R(y)}{y^2} - b\beta - (SR)(\alpha - 1/y) = 0 \tag{35}$$

where $\alpha, \beta, \gamma \in \mathbf{C}(x)$ are given in (34), $b \in \mathbf{N}$, and $R = [z^{b-1}]P \in \mathbf{C}\{x\}(y)$. Our task is now to show that no R satisfies (35).

Suppose, for the contradiction, that $R \in \mathbf{C}\{x\}(y)$ satisfies (35). Then $R \neq 0$ and R can be written as

$$R(y) = \delta \cdot \frac{\prod_{i=1}^{k}(y - \rho_i)^{n_i}}{\prod_{i=1}^{l}(y - \rho_{k+i})^{n_{k+i}}} \tag{36}$$

where $\delta \in \mathbf{C}\{x\}$ is nonzero, $n_i \in \mathbf{N}$, and ρ_i are $k+l$ mutually distinct analytic Puiseux series. First we show that $l = 0$, that is, the denominator of R is 1. The substitution $y \to \alpha - 1/y$ and the action of S in (35) transform the factor

$y-\rho$ to $\alpha - y^{-1} - S\rho = -y^{-1}$ if $\rho = \rho^* = x^{-1} - 1$ and to $(U\rho)^{-1}y^{-1}(y - U\rho)$ if $\rho \neq \rho^*$, where U is defined in (33). The factorization (36) is transformed to

$$(SR)(\alpha - 1/y) = \varepsilon \cdot \frac{\prod_{i=1}^{k}(y - U\rho_i)^{n_i}}{\prod_{i=1}^{l}(y - U\rho_{k+i})^{n_{k+i}}} \cdot y^{n_{k+1}+\cdots+n_{k+l}-n_1-\cdots-n_k} \tag{37}$$

where $\varepsilon \in \mathbf{C}\{x\}$ is nonzero and we use the convention that $y - U\rho_i = 1$ for $\rho_i = \rho^*$.

Let M_1, M_2, and M_3 be the sets of poles of $R(y), y^{-2}R(y)$, and $(SR)(\alpha - 1/y)$, respectively. Since the left hand side of (35) is identically zero, we must have $M_2 = M_3$, including the multiplicities. M_2 is M_1 with possibly added 0 and, by (37), $M_3 = \{U\rho : \ \rho \in M_1 \ \& \ \rho \neq \rho^*\}$ with possibly added 0. If $\rho^* \in M_1$, then $M_1 \supset M = \{U^{-1}\rho^*, U^{-2}\rho^*, \ldots\}$. The set M is infinite because $U^{-j}\rho^* \neq 0$ for every $j \in \mathbf{N}$ (clearly, $\mathrm{ord}(U^j 0) = 1$ for every $j \in \mathbf{N}$). This contradicts the finiteness of M_1 and thus $\rho^* \notin M_1$. Suppose that $M_1 \neq \emptyset$ and take an arbitrary $\rho \in M_1$. Since $UM_1 \subset M_1$, U is injective, and M_1 is finite, we have for some $j \in \mathbf{N}$ the cycle $U^j\rho = \rho$. By Lemma 3.10, $\rho = F$ or $\rho = \overline{F}$ and ρ is nonanalytic. But this contradicts the assumption $\rho \in \mathbf{C}\{x\}^{\mathrm{P}}$ ($\rho = \rho_i$ for some i, $k+1 \leq i \leq k+l$). Hence $M_1 = \emptyset$.

We have derived so far that $R \in \mathbf{C}\{x\}[y]$. The multiplicity of $0 \in M_2$ is at most 2 and the multiplicity of $0 \in M_3$ is, by (37) and $U\rho_i \neq 0$, $\deg(R)$. Thus $\deg(R) \leq 2$ and

$$R(y) = A_2y^2 + A_1y + A_0$$

for some $A_i \in \mathbf{C}\{x\}$. Substituting this to (35), we obtain the system

$$\begin{aligned} \gamma \cdot A_2 - \alpha^2 \cdot SA_2 - \alpha \cdot SA_1 - SA_0 - \beta b &= 0 \\ \gamma \cdot A_1 + 2\alpha \cdot SA_2 + SA_1 &= 0 \\ \gamma \cdot A_0 - SA_2 &= 0. \end{aligned}$$

We replace in the first two equations A_2 with the expression $A_2 = S^{-1}\gamma A_0$ obtained from the third equation. Then, applying S^{-1} again, we express from the first equation A_1 in terms of A^0 and substitute the expression in the second equation. Using (34), we obtain this equation for A_0:

$$B_1 \cdot S^{-2}A_0 + B_2 \cdot S^{-1}A_0 + B_3 \cdot A_0 + B_4 \cdot SA_0 + B_5 = 0$$

where

$$B_1 = \frac{x^2}{(1-2x)^2},\ B_2 = \frac{x^2+x-1}{1-x},\ B_3 = -x^2-x+1,$$
$$B_4 = -\frac{x^2(1-x)}{(1+x)^2},\ \text{ and }\ B_5 = b(2-x).$$

We recast the equation as

$$S^{-1}A_0 = \frac{-B_3}{B_2}\cdot A_0 - \frac{B_5}{B_2} - \frac{B_1\cdot S^{-2}A_0 + B_4\cdot SA_0}{B_2}.$$

Lemma 3.9, applied with $r=0$, $G=-B_3/B_2=1-x$ and $H=-B_5/B_2=2b+\cdots$, tells us that the equation has no solution $F_1=S^{-1}A_0\in\mathbf{C}((x))$. We have arrived at a contradiction. $\square$

4 Concluding remarks

The identity $B_e(x)=B(x)*\mathrm{e}^x$ ($*$ is the Hadamard product), whose modification $B(x)=B_e(x)*N(x)$ we mentioned in the introduction, leads to a quick proof that $B(x)$ is not D-finite. Differentiating $B_e(x)=\mathrm{e}^{\mathrm{e}^x-1}$, one sees easily that $B_e(x)$ is not D-finite (this is stated in [31, p. 191] as an example). Clearly, $\mathrm{e}^x=\sum_{n\ge0}x^n/n!$ is D-finite. Since the class of D-finite power series is closed to the Hadamard product ([31, Theorem 6.4.12]), it follows that $B(x)$ is not D-finite. The same argument applies to $B^{\pm}(x)$ and $B^{j,i}(x)$.

The example of Lipshitz and Rubel [17] that we also mentioned in the introduction is quite interesting: $\sum_{n\ge0}x^{n^2}\in\mathcal{A}^{\mathbf{C}(x)}$ as proved by Jacobi in 1847 but $\sum_{n\ge0}n^2!\cdot x^{n^2}\notin\mathcal{A}^{\mathbf{C}(x)}$ since the growth of the coefficients violates a bound derived by Mahler. For more information and problems on ADE's see Rubel [23, 24].

We conjecture that the ogf $B^{\mathrm{B}}(x)$ of Bessel numbers is not differentially algebraic over $\mathbf{C}\{x\}$. Banderier et al. [1, Example 20] noted that $B^{\mathrm{B}}(x)$ is not D-finite because the asymptotics of B_n^{B}, found in [11], is incompatible with the possible growths of the coefficients of D-finite power series as determined in Wimp and Zeilberger [37]. Is there an algebraic proof of this fact?

We mention two more applications of $B(x)$. 1. Reducing (3) modulo any given $m\in\mathbf{N}$, we get an expansion of a rational function. It follows from this that the sequence $(B_n \bmod m)_{n\geq 0}$ is eventually periodic. But much more is known on the modular behaviour of B_n, see Kahale [12] and Shparlinskiy [26] and the references they give. 2. Let us call, for a given $k \in \mathbf{N}$, a partition P of $[n]$ *k-sparse* if $x - y \geq k$ whenever $x > y$ lie in the same block of P. Of the $B_4 = 15$ partitions of [4] only 5 are 2-sparse: 1/2/3/4, 13/2/4, 14/2/3, 1/24/3, and 13/24. That $5 = B_3$ is not an accident. If $B_n^{(k|}$ denotes the number of k-sparse partitions of $[n]$, then for $n \geq k$ always $B_n^{(k|} = B_{n-k+1}$. This was proved by W. Yang [39] and earlier by Prodinger [21]; see also [30, Problem 1.4.29] and Prodinger [22]. One can easily prove $B_n^{(2|} = B_{n-1}$ by means of (2): since $B(x) = B^{(2|}(x/(1-x))$ (every partition of $[n]$ is uniquely obtained by "blowing up" the elements of a 2-sparse partition of $[m], m \leq n$), by (2) indeed $B^{(2|}(x) = 1 + xB(x)$. Similarly, denoting $B_n^{\mathrm{B},(k|}$ the number of k-sparse non-overlapping partitions of $[n]$, the equations $B^{\mathrm{B}}(x) = B^{\mathrm{B},(2|}(x/(1-x))$ and (17) imply the identity

$$B_n^{\mathrm{B},(2|} = \sum_{i=0}^{n-2} B_i^{\mathrm{B},(2|} \cdot B_{n-i-2}^{\mathrm{B}} \quad (n \geq 2).$$

So $(B_n^{\mathrm{B},(2|})_{n\geq 1} = (1, 1, 2, 4, 10, 27, 80, 255, 870, \ldots)$.

While connected and crossing partitions seem new, connected and crossing *matchings* have been considered before. These are the corresponding partitions in which all blocks have two elements. Connected matchings were investigated and enumerated by Stein [32] and others; see Klazar [14] or Flajolet and Noy [10] for more references and results. Crossing matchings appear briefly in Stoimenow [33, p. 217] (the condition of crossing — every chord crosses another chord — is important in the investigation of Vassiliev knot invariants by chord diagrams, see Bar-Natan [2]) and are enumerated also in [14]. In [14] we prove that the ogf's of connected and crossing matchings are not D-finite. It is a consequence of the fact that these ogf's, in contrast to the partition case, satisfy an ADE over $\mathbf{C}(x)$, in fact of order 1. Except for the class of noncrossing partitions, not much seems to be known about enumeration of partition classes defined by forbidden substructures. For example, let us call a partition P of $X \subset \mathbf{N}$ 3-*noncrossing* if $G(P)$ has no triangle. In other words, P has no three mutually crossing blocks. What can be said about the numbers of 3-noncrossing partitions of $[n]$? What is their asymptotics? Similarly one can consider the numbers of k-noncrossing partitions, $k \geq 3$. It follows from the more general bounds in Klazar [13] that these numbers grow only exponentially. However, the exact asymptotics or enumeration seem not to be known even for the case of matchings.

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