Geometric Separation and Exact Solutions for the Parameterized Independent Set Problem on Disk Graphs[1]

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We consider the parameterized problem, whether for a given set $\mathcal{D}$ of n disks (of bounded radius ratio) in the Euclidean plane there exists a set of k non-intersecting disks. For this problem, we expose an algorithm running in time $n^{O(\sqrt{k})}$, that is—to our knowledge—the first algorithm with running time bounded by an exponential with a sublinear exponent. For λ-precision disk graphs of bounded radius ratio, we show that the problem is fixed parameter tractable with a running time $O(2^{O(\sqrt{k})} + n)$.

The results are based on problem kernelization and a new "geometric $\sqrt{\cdot}$-separator theorem" which holds for all disk graphs of bounded radius ratio. The presented algorithm then performs, in a first step, a "geometric problem kernelization" and, in a second step, uses divide-and-conquer based on our new "geometric separator theorem."

Keywords: disk graphs, independent set, parameterized complexity, graph separators, graph measure

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1. THE BACKGROUND

The problem and its motivation. In this paper, we study the parameterized INDEPENDENT SET problem on disk graphs, which takes as an input a set $\mathcal{D}$ of disks in the plane and an integer k and the task is to determine whether there are k mutually disjoint disks in $\mathcal{D}$. The problem is motivated by numerous applications, among which we want to highlight the area of frequency assignment problems in cellular networks [15, 22]. Here, one considers a set of antennas which transmit data on a given frequency to their local environment. Assuming that this environment can be modeled by a disk centered at the position of the antenna, the task to determine the maximum number of antennas which can operate simultaneously without any interferences using the same frequency becomes a maximum INDEPENDENT SET problem on a disk graph.[4]

Previous work. It is known [7] that the problem is NP-hard even for unit disk graphs (i.e., all disks have radius one) by a reduction from VERTEX COVER on planar graphs with maximum degree three. The reduction in [7] can be adapted easily to obtain NP-hardness for INDEPENDENT SET on the more restricted class of unit disk graphs where the centers of the disks are at mutual distance at least one (which we refer to as 1-precision disk graphs). A way to cope with this hardness was proposed by approximation theory [14, 16]. Very recently, Erlebach *et al.* [14] gave a PTAS for INDEPENDENT SET on disk graphs, which is based on a sophisticated use of so-called shifting techniques as they were introduced in Baker's work [4] for deriving polynomial time approximation schemes for various planar graph problems.

In this paper, however, we are interested in exact solutions for the given problem. Moreover, we take the viewpoint of parameterized complexity (see [12] for a detailed introduction to this subject). Formally, a parameterized problem is a two-dimensional language $\mathcal{L} \subseteq \Sigma^* \times \mathbb{N}$, where Σ is some alphabet. The input instances are objects of the form (I, k) and the integer k is called the *parameter*. We ask for an algorithm which decides if $(I, k) \in \mathcal{L}$ in time $2^{g(n,k)}$, where $n = |I|$. A parameterized problem is called *fixed parameter tractable*, if there is an algorithm with $g(n, k) = f(k)+h(n)$, where f is some arbitrary function and $h(n) \in O(\log n)$. The class of fixed parameterized tractable problems is denoted by FPT.

We want to briefly summarize various results on the parameterized and classical complexity of the INDEPENDENT SET problem for general graphs and for planar graphs (which are equivalent to the class of coin graphs [18],

[4]Note that we always consider disk graphs with given representation, i.e., with given set of disks in the plane. This makes sense, since most applications which are modeled by disk graphs already provide this representation in a very natural way.

TABLE 1.
Relating our results on INDEPENDENT SET on disk graphs to known results for general graphs and for planar graphs. (Lower bounds are under the assumption that 3SAT $\notin$ DTIME($2^{o(n)}$).)

graph class	*(classical) complexity*	*parameterized complexity*
general graphs	$2^{0.276\,n}$ [25] lower bound: $2^{\Omega(n)}$ [17]	W[1]-complete [11]
disk graphs DG_σ	$2^{O(\sqrt{n}\log(n))}$ [Rem. 5.1]	$2^{O(\sqrt{k}\log(n))}$ [Thm. 5.1] open: FPT or W[1]-hard ?
disk graphs $DG_{\sigma,\lambda}$ (with λ-precision)	$2^{O(\sqrt{n})}$ [Rem. 4.3]	$O(2^{O(\sqrt{k})}+n)$ [Cor. 4.1] hence: FPT
planar graphs	$2^{O(\sqrt{n})}$ [21], [Rem. 4.2] lower bound: $2^{\Omega(\sqrt{n})}$ [6]	$O(2^{O(\sqrt{k})}+n)$ [2, 3] hence: FPT

i.e., disk graphs where disks are not allowed to overlap); see Table 1 for an overview.

In parameterized complexity study, it is known [11] that INDEPENDENT SET is complete for the class $W[1]$, which captures parameterized *intractable* problems (see [12] for details). However, restricted to planar graphs, INDEPENDENT SET is in FPT, and for the (asymptotically) best known algorithm we get $h(n) = \log(n)$ and $f(k) = O(\sqrt{k})$ being sublinear in k (see [2, 3]). Here, n is the number of vertices in the graph and k is the size of the independent set we seek for.[5] Moreover, very recently, Cai and Juedes [6] showed that there is no fixed parameter algorithm with $f(k) = o(\sqrt{k})$ unless 3SAT$\in$ DTIME($2^{o(n)}$), which is generally consider to be very unlikely.

In the classical (one-dimensional) complexity study, the best known algorithm running in time $2^{e(n)}$ with $e(n) = 0.275n$ is due to Robson [25]. Moreover, $e(n) \in o(n)$ is impossible unless 3SAT$\in$ DTIME($2^{o(n)}$) (see [17]). If restricted to planar graphs, Lipton and Tarjan applied their well-known planar separator theorem [20] to get an algorithm with $e(n) = O(\sqrt{n})$. This is the best possible asymptotic behavior for e (unless 3SAT$\in$ DTIME($2^{o(n)}$)), since otherwise an algorithm with $e(n) \in o(n)$ in combination with a known

[5]In fact, since the algorithms in [2, 3] use linear time problem kernelization as a preprocessing, the running time even is $O(2^{O(\sqrt{k})} + n)$.

linear problem kernel, would lead to an algorithm for the parameterized problem better than the relative bound shown by Cai and Juedes.

Main results and methods used. It is an interesting phenomenon, that for planar graphs, in both cases (parameterized and classical complexity) the asymptotically best algorithms were derived based on separator theorems: in the case of classical complexity, by a direct divide and conquer approach using the planar separator theorem [21], and in the case of parameterized complexity, by a combination of so-called "reduction to a linear problem kernel" (see Section 3 for details) and divide and conquer [3]. Due to the large constants involved, the algorithms are considered to be impractical, but relevant from a theoretical point of view, since they match the corresponding relative lower bounds.

In this paper, for the case of disk graphs, we pursue a similar strategy of combining a geometric version of reduction to problem kernel with a divide and conquer approach based on an appropriate separator theorem. However, for disk graphs, so far such separator theorems are known only for so-called intersection graphs of τ-ply neighborhood systems [13, 23, 24], which are closely related to disk graphs with λ-precision, where all centers are at mutual distance of at least $\lambda > 0$.[6] With respect to general disk graphs, we quote from the introduction of Hunt *et al.* [16]:

> *"The [...] drawback is that problems such as maximum independent set and minimum dominating set [...]* **cannot** *be solved at all by the separator approach. This is because an arbitrary (unit) disk graph of n nodes can have a clique of size n."*

The key result in this paper is to show a way out of this dilemma by proving a new type of "geometric separator theorem" which holds for disk graphs with bounded radius ratio. Our geometric separator theorem can be seen as a generalization of (classical) separator theorems, where the guarantee is not on the size of the separator in terms of its number of vertices, but in terms of the space occupied by its disks.

This result is used to optimally solve the parameterized INDEPENDENT SET problem on disk graphs of bounded radius ratio in time $2^{g(n,k)}$ with $g(n,k) = O(\sqrt{k}\log(n))$, which is — to our knowledge — the first algorithm with running time bounded by a function with a the exponent sublinear in k. In the worst case (i.e., when k equals n) this turns into an algorithm of running time $2^{e(n)}$ with the sublinear term $e(n) = \sqrt{n}\log(n)$; a running time which cannot be achieved for general graphs (unless 3SAT $\in$ DTIME($2^{o}(n)$)).

[6] From an application point of view this is a realistic scenario, since, e.g., for the frequency assignment problem, we may assume that all antennas have a certain width.

Moreover, in the case of disk graphs with λ-precision, we can even show that the INDEPENDENT SET problem is in FPT with $g(n,k) = f(k) + \log(n)$ and $f(k) = O(\sqrt{k})$. The results are summarized in Table 1.

Finally, we want to mention that completely independent of our work, Lev-Tov and Peleg [19] gave (among others) exact algorithms for the INDEPENDENT SET problem on unit disk graphs with similar running times. Namely, they gave a $2^{O(\sqrt{n}\log(n))}$ algorithm for unit disk graphs and a $2^{O(\sqrt{n})}$ time algorithm if in addition the centers of the unit disks are required to sit on a given grid of span δ. It is clear that our results outperform these results with respect to various aspects: Firstly, our algorithms are stated in the parameterized complexity setting and our results for the classical complexity setting are derived as sideproduct. It might be — as in the case of a large clique — that even though n could be huge, the size of the largest independent set of a disk graph (i.e., the parameter k) can be very small. In this sense, an $n^{O(\sqrt{n})}$ is completely inferior to an $n^{O(\sqrt{k})}$ algorithm.

Secondly, our results are given for a much broader class of graphs than the ones in [19]. Lev-Tov and Peleg only consider the case of *unit* disk graphs, whereas our setting allows disks of *distinct* radii (as long as the ratio of largest and smallest radii is bounded). Their $2^{O(\sqrt{n})}$ only holds if the centers of the unit disks are assumed to sit on a given grid of span δ. This class of disk graphs clearly is more restrictive than the class of λ-precision graphs considered in our setting.

2. PRELIMINARIES AND NOTATION

Our subject to explore are intersection graphs of geometric objects in the plane, namely of disks. We assume that each point z of the plane is determined by its x any y coordinates and the plane is equipped by the standard distance $d(z,z') = \sqrt{(x-x')^2 + (y-y')^2}$. The plane distance of two geometric objects S and S' is determined as the minimum distance among all pairs of $z \in S$ and $z' \in S'$. If $\mathcal{S} = \{S_1, \ldots, S_n\}$, $S_i \subseteq \mathbb{R}^2$ is a collection of geometric objects, we denote by $\bigcup \mathcal{S} = \bigcup_{i=1}^{n} S_i$ the union of $\mathcal{S}$. For a collection $\mathcal{S}$, let $G_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$ denote the *intersection graph* of $\mathcal{S}$, i.e., $V_{\mathcal{S}} = \{v_1, \ldots, v_n\}$ and $E_{\mathcal{S}} = \{(v_i, v_j) \mid S_i \cap S_j \neq \emptyset\}$. The collection $\mathcal{S}$ is called the *representation* of $G_{\mathcal{S}}$. Moreover, for a subset $\mathcal{S}' \subseteq \mathcal{S}$, we denote by $V_{\mathcal{S}'} \subseteq V_{\mathcal{S}}$ the subset of vertices induced by $\mathcal{S}'$, i.e., $V_{\mathcal{S}'} = \{v_i \mid S_i \in \mathcal{S}'\}$. In this setting $G_{\mathcal{S}'} = G_{\mathcal{S}}[V_{\mathcal{S}'}] = G_{\mathcal{S}}[\mathcal{S}']$ is the subgraph of $G_{\mathcal{S}}$ induced by the set of vertices $V_{\mathcal{S}'}$.

Disk graphs. A disk $D \subseteq \mathbb{R}^2$ is specified by a triple $(r,x,y) \in \mathbb{R}^3$, where (x,y) are coordinates of the center of the disk in the Euclidean plane and r is its radius. The graph class of *disk graphs*, denoted by DG is the set of all graphs G, for which we find a collection of disks $\mathcal{D} = \{D_1, \dots, D_n\}$ such that $G = G_{\mathcal{D}}$. Note, that for given collection $\mathcal{D}$, the graph $G_{\mathcal{D}}$ is given with a natural embedding in the plane, where v_i sits in the position of the center of D_i.

The class of disk graphs of *bounded radius ratio* σ is the subclass $DG_\sigma \subset DG$ of all graphs $G \in DG$ which admit a representation $\mathcal{D} = \{D_1, \dots, D_n\}$, such that $(\max_{i=1,\dots n} r_i)/(\min_{i=1,\dots n} r_i) \leq \sigma$, where r_i denotes the radius of disk D_i. The parameter σ is called *radius ratio*. By a rescaling argument, for a graph $G \in DG_\sigma$ with representation $\mathcal{D}$, we can always achieve, that the smallest disk in $\mathcal{D}$ has radius one and, hence, all radii being upper bounded by σ. We will use this as a general assumption throughout the paper.

Finally, a collection $\mathcal{D}$ is said to have λ*-precision* if all centers of disks are pairwise at least λ apart (see [16, Definition 3.2]). Again, by a rescaling argument, all disk graphs have a representation with λ-precision, however only some graphs of DG_σ allow a representation with radii in $[1, \sigma]$ and precision λ. We denote such class of graphs by $DG_{\sigma,\lambda}$

Throughout the paper, we assume that a disk graph G is given together with its representation witnessing its membership in DG, DG_σ or $DG_{\sigma,\lambda}$, respectively. This make sense from an application point of view, since the graph is usually derived from the placement of real objects in the space.

Grid graphs. Fix an arbitrary constant $\delta > 0$, and consider the infinite grid of span δ as the planar graph $H^\delta = (W^\delta, E^\delta)$ with vertices $W^\delta = \{w_{i,j} \mid i, j \in \mathbb{Z}\}$ and edges $E^\delta = \{(w_{i,j}, w_{k,l}) \mid |i-j| + |k-l| = 1\}$. The canonical (straight-line) embedding of H^δ is given by putting vertex $w_{i,j}$ at the coordinates $(i\delta, j\delta)$. The set of faces $\mathcal{F}^\delta$ of H^δ contains all closed squares $F^\delta_{i,j} = [i\delta, (i+1)\delta] \times [j\delta, (j+1)\delta] \subseteq \mathbb{R}^2$. For a grid vertex $w \in W^\delta$, we define the *face neighborhood* $\hat{N}(w) := \{F \in \mathcal{F}^\delta \mid w \in F\}$. Similarly for $W' \subseteq W^\delta$, we let $\hat{N}(W') = \bigcup_{w \in W'} \hat{N}(w)$.

Definition 2.1. For a collection of disks $\mathcal{D} = \{D_1, \dots, D_n\}$, we define $H^\delta_{\mathcal{D}}$ to be the smallest subgraph of the infinite grid H^δ induced by a set of grid points which completely covers all disks in $\mathcal{D}$. We call $H^\delta_{\mathcal{D}}$ the *covering grid (of span δ) for $\mathcal{D}$*. In other words, if we define the set of faces hit by $\mathcal{D}$ as $\mathcal{F}^\delta_{\mathcal{D}} = \{F \in \mathcal{F}^\delta \mid F \cap \bigcup \mathcal{D} \neq \emptyset\}$, and if $W^\delta_{\mathcal{D}}$ is the set of all grid points of $\mathcal{F}^\delta_{\mathcal{D}}$, then the covering grid $H^\delta_{\mathcal{D}}$ is the subgraph of H^δ induced by $W^\delta_{\mathcal{D}}$.

An example which illustrates the construction of $H^\delta_{\mathcal{D}}$ is given in Fig. 1.

Let $\mathcal{D}$ be a collection of disks and consider a set $S \subseteq \mathbb{R}^2$ (e.g., a set of grid vertices or a set of faces). Then $\mathcal{D}[S] = \{D \in \mathcal{D} \mid D \cap S \neq \emptyset\}$ is called the set of disks induced by S.

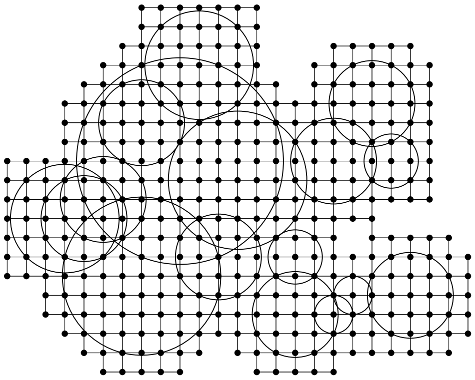

FIG. 1. The covering grid $H^{\delta}_{\mathcal{D}}$ induced by a collection of disks $\mathcal{D}$.

Measures. We use the standard Lebesgue measure μ in $\mathbb{R}^2$ as follows: For a Lebesgue measurable set $S \subseteq \mathbb{R}^2$, $\mu(S)$ denotes the *size* of S, i.e., the space in $\mathbb{R}^2$ occupied by S. In particular, for a collection of disks $\mathcal{D} = \{D_1, \ldots, D_n\}$, let $\mu(\mathcal{D}) = \mu(\bigcup \mathcal{D})$ be the space covered by the union of the disks $D_1, \ldots, D_n$.

Definition 2.2. Let $\mathbb{G}$ be a graph class closed under taking subgraphs. A function $\xi : \mathbb{G} \to \mathbb{R}^+$, that is monotonous with respect to the subgraph ordering, i.e., $\xi(G) \leq \xi(G')$ if $G \subseteq G'$, and for which $\xi((\emptyset, \emptyset)) = 0$, is called a *graph measure*.

Example 2.1. We specify two graph measures which play a decisive role throughout the paper.

1. The usual *counting measure* $|\cdot|$ assigning to a graph G the size of its vertex set $|V_G|$ is clearly a graph measure.
2. The *Lebesgue measure* $\mu(\cdot)$ assigning to a disk graph $G_{\mathcal{D}}$ with representation $\mathcal{D}$ the value $\mu(G_{\mathcal{D}}) = \mu(\mathcal{D})$ is a graph measure for DG, when we restrict the subgraph ordering to $G_{\mathcal{D}} \subseteq G_{\mathcal{D}'} \Leftrightarrow \mathcal{D} \subseteq \mathcal{D}'$.

Remark 2. 1. The value $\mu(G_{\mathcal{D}})$ can be computed in time polynomial in $n := |\mathcal{D}|$. In a naive approach[7], we could compute all (at most $2n^2$) intersection points of the n disks. Then, one forms a grid consisting of

[7]A sweep-line algorithm could perform much better. For our purposes, however, it is sufficient that we can compute $\mu(G_{\mathcal{D}})$ in polynomial time.

$O(n^4)$ rectangles, by drawing a horizontal and a vertical line through each of the intersection points. Now each of the rectangles either is completely covered by the disks, does not intersect with any disk, or is partially covered by disks that do not intersect in this rectangle. Hence, we can compute the space covered by the disks for each rectangle separately (by a simple formula) in linear time.

3. A GEOMETRIC PROBLEM KERNELIZATION

Reduction to problem kernel is a core technique in the design of fixed parameter algorithms.

DEFINITION 3.1. Let $\mathcal{L}$ be a parameterized problem, i.e., $\mathcal{L}$ consists of pairs (I, k), where the problem instance I has a solution of size k (the parameter).

A map $\Phi : (I, k) \mapsto (I', k')$, is called a *problem kernel reduction*, if there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that

1. Φ is computable in time polynomial in $|I|$ and k.
2. $(I, k) \in \mathcal{L}$ if and only if $(I', k') \in \mathcal{L}$.
3. $|I'| \leq g(k')$ and $k' \leq k$.

The instance (I', k') is called the *reduced instance* or the *problem kernel* and $g(k')$ is called the *size* of the problem kernel.

It is a well-known fact, that a parameterized problem is in FPT if and only if it admits a reduction to a problem kernel.

As an example, we mention that, due to the 4-color theorem it is easy to derive a problem kernel of size $4k$ for INDEPENDENT SET on planar graphs. For disk graphs, we can prove a geometric version of a problem kernel. By this we mean, that the size of the reduced instance is upper bounded by $O(k)$, when measured by the (Lebesgue) measure $\mu(\cdot)$ instead of the counting measure $|\cdot|$.

PROPOSITION 3.1. *For the parameterized* INDEPENDENT SET *(IS) problem on DG_σ there exists a "geometric" problem kernel, i.e., there is a procedure that in polynomial time transforms an instance $(G_\mathcal{D}, k)$ to an instance $(G_{\mathcal{D}'}, k)$, such that $(G_\mathcal{D}, k) \in IS$ iff $(G_{\mathcal{D}'}, k) \in IS$ and*

$$\pi k \;\leq\; \mu(G_{\mathcal{D}'}) \;\leq\; 9\pi\sigma^2 k.$$

Proof. Let be given an instance $(G_{\mathcal{D}}, k)$, together with the representation $\mathcal{D}$. Recall, that due to our assumptions all disks have radius in the range $[1, \sigma]$ and $G_{\mathcal{D}}$ is naturally embedded in the plane with respect to $\mathcal{D}$.

Observe first, that $\mu(G_{\mathcal{D}}) > 9\pi\sigma^2 k$ implies that $(G_{\mathcal{D}}, k) \in$ IS. We use the fact that $\mu(\mathcal{D}[N(v)]) \leq (3\sigma)^2\pi$ for any vertex $v \in V$, i.e., the neighborhood of any vertex may occupy the space at most $9\pi\sigma^2$.

And, secondly, if $\mu(G_{\mathcal{D}}) < \pi k$, then $(G_{\mathcal{D}}, k) \notin$ IS, since the representation of any independent set of k vertices needs space at least πk.

Hence,

$$\Phi((G_{\mathcal{D}}, k)) := \begin{cases} (G_{\mathcal{D}'}, k) & \text{if } \mu(G_{\mathcal{D}}) > 9\pi\sigma^2 k \\ (G_{\emptyset}, k) & \text{if } \mu(G_{\mathcal{D}}) < \pi k \\ (G_{\mathcal{D}}, k) & \text{otherwise,} \end{cases}$$

where $\mathcal{D}'$ is the set of k disjoint disks of radius one yields the desired transformation. The time needed to compute the transformation Φ is determined by the time needed to determine $\mu(G_{\mathcal{D}})$, which is polynomial by Remark 2.1. ▮

Again, note that this is not a problem kernel according to Definition 3.1, since the size of $G_{\mathcal{D}}$ is measured by the (Lebesgue) measure $\mu(\cdot)$, which, in general, is not related to the (instance) size of G. However, for disk graphs with λ-precision we can upper bound the counting measure by the Lebesgue measure.

LEMMA 3.1. *Let the collection of disks $\mathcal{D}$ have λ-precision. Then,*

$$|\mathcal{D}| \;\leq\; \frac{4}{\pi}\lambda^{-2}\mu(\mathcal{D}).$$

Proof. Observe that, for given set of n disks $\mathcal{D}$ with radius at least one and centers of mutual distance at least λ, the smallest value $\mu(\mathcal{D})$ is obtained by optimally placing all centers in the interior of a disk. However, such a disk must have radius at least $\frac{\lambda\sqrt{n}}{2}$. Hence, since every disk has radius at least one, we have

$$\mu(\mathcal{D}) \geq \left(\frac{\lambda\sqrt{n}}{2} + 1\right)^2 \pi \geq \frac{\pi}{4}\lambda^2 n.$$

▮

COROLLARY 3.1. *The parameterized* INDEPENDENT SET *problem on disk graphs $DG_{\sigma,\lambda}$ (with λ-precision) admits a problem kernel of size $36(\frac{\sigma}{\lambda})^2 k$, hence, it is in FPT.*

Proof. This follows immediately by Proposition 3.1 and Lemma 3.1. ▌

Remark 3. 1. A linear problem kernel for INDEPENDENT SET on $DG_{\sigma,\lambda}$ can even be found in *linear* time if the computation of $\mu(G_{\mathcal{D}})$ is done in an approximate way: Recall the notion from Definition 2.1. For some constant $\delta > 0$, let $H^{\delta}_{\mathcal{D}}$ be the covering grid (of span δ) for $\mathcal{D}$ and let $\mathcal{F}^{\delta}_{\mathcal{D}}$ be its faces, i.e., the set of faces of the infinite grid H^{δ} hit by $\mathcal{D}$. Observe that the set $\mathcal{F}' := \{F \in \mathcal{F}^{\delta}_{\mathcal{D}} \mid F \subseteq D\}$ can be constructed in time $O(|\mathcal{D}|)$ since, for each disk $D \in \mathcal{D}$, we need to check at most $O((\pi\sigma^2)/\delta^2)$ many faces of the covering grid. Using $|\mathcal{F}'|\delta^2 \leq \mu(G_{\mathcal{D}})$ and the argument given in the proof of Proposition 3.1 we know that $(G_{\mathcal{D}}, k)$ is a "yes"-instance, if $|\mathcal{F}'|\delta^2 > 9\pi\sigma^2 k$, which means that we only have to deal with the case that $|\mathcal{F}'|\delta^2 \leq 9\pi\sigma^2 k$. An easy computation shows that for a disk of radius at least one, the ratio between fully and partially covered grid squares of span δ is at most $c_\delta := (1+\sqrt{2}\delta)^2/(1-\sqrt{2}\delta)^2$. Hence, in the remaining case where $|\mathcal{F}'|\delta^2 \leq 9\pi\sigma^2 k$ we get

$$\mu(G_{\mathcal{D}'}) \leq |\mathcal{F}^{\delta}_{\mathcal{D}}|\delta^2 \leq c_\delta |\mathcal{F}'|\delta^2 \leq c_\delta 9\pi\sigma^2 k.$$

(Observe that $c_\delta \searrow 1$ as $\delta \to 0$.)

4. A GEOMETRIC SEPARATOR THEOREM

In the following, we prove the key result — a new geometric $\sqrt{\cdot}$-space separator theorem — that makes our divide and conquer strategy work. For this purpose, in a first subsection, we briefly introduce the notion of vertex separators and give a short overview on known $\sqrt{\cdot}$-separator theorems for planar graphs and disk graphs with λ-precision. In a second subsection, we prove our key result, a geometric $\sqrt{\cdot}$-separator theorem for general disk graphs of bounded radius ratio.

4.1. Classical $\sqrt{\cdot}$-vertex separator theorems

We briefly discuss (classical) vertex separator theorems. For this purpose we give a somewhat generalized notion of separator theorems.

DEFINITION 4.1. Let $G = (V, E)$ be an undirected graph. A *separator* $V_S \subseteq V$ of G partitions V into two *parts* V_A and V_B such that

- $V_A \,\dot\cup\, V_S \,\dot\cup\, V_B = V$, and
- no edge joins a vertex of V_A to V_B.

The triple (V_A, V_S, V_B) is also called a *separation* of G.

In order to provide a quantitative approach to separators, we need the notion of "measure" as introduced in Section 2.

DEFINITION 4.2. Let ξ be a graph measure. An $f(\cdot)$-*vertex separator theorem for the measure* ξ *(and constants* $\alpha < 1$, $\beta > 0$*)* on a class of graphs $\mathbb{G}$ which is closed under taking subgraphs is a theorem of the following form:
For any $G \in \mathbb{G}$ there exists a separation (V_A, V_S, V_B) of G such that

1. $\xi(G[V_S]) \leq \beta \cdot f(\xi(G))$
2. $\xi(G[V_A]), \xi(G[V_B]) \leq \alpha \cdot \xi(G)$

Separator theorems are a useful tool in the design of fixed parameter algorithms.

Remark 4. 1. In [3, Theorem 1], the authors proved that whenever a graph class $\mathbb{G}$ admits a $\sqrt{\cdot}$-separator for the measure $|\cdot|$, and if a so-called "glueable" graph problem $\mathcal{G}$ (such as parameterized INDEPENDENT SET) admits a linear problem kernel on $\mathbb{G}$ that can be constructed in time $T_{\text{kernel}}(n, k)$, then $\mathcal{G}$ can be solved in time $O(2^{O(\sqrt{k})} + T_K(n, k))$.

$\sqrt{\cdot}$-separator theorems on planar graphs. Stated in this framework, the planar separator theorem due to Lipton and Tarjan [20] can be formulated as follows.

THEOREM 4.1. *On the class of planar graphs, there exists a* $\sqrt{\cdot}$*-separator theorem for the counting measure* $|\cdot|$ *with constants* $\alpha = \frac{2}{3}$ *and* $\beta = 2\sqrt{2}$*. Moreover, the corresponding separation can be found in linear time.*

Later, Djidjev [8] improved the constants to $\alpha = \frac{2}{3}$ and $\beta = \sqrt{6}$. The current record for $\alpha = \frac{2}{3}$ is $\beta \approx 1.97$ [10]. Djidjev has also shown a lower bound of $\beta \approx 1.55$ for $\alpha = \frac{2}{3}$ [8]. Similar $\sqrt{\cdot}$-separator theorems are also known for other graph classes, e.g., for the class of graphs of bounded genus, see [9].

Remark 4. 2. Note that Theorem 4.1 can directly be used to obtain a $2^{O(\sqrt{n})}$-algorithm for INDEPENDENT SET on planar graphs (see [21], or [3, Prop.1]).

$\sqrt{\cdot}$-separator theorems on disk graphs with λ-precision. In terms of geometric graphs, a $\sqrt{\cdot}$-separator theorem for the counting measure was proven on the class of intersection graphs of so-called *τ-ply neighborhood systems* as studied in a series of papers by Miller, Teng, Thurston and Vavasis (see [24] for an overview).

DEFINITION 4.3. A *(2-dimensional) τ-ply neighborhood system* is a collection $\mathcal{D}$ of disks such that the intersection of any $(\tau+1)$ distinct disks in $\mathcal{D}$ is empty. By $DG_{\tau\text{-ply}}$ we denote the subclass of DG of disk graphs $G_{\mathcal{D}}$ which have a representation by a τ-ply neighborhood system $\mathcal{D}$.

The following result was proven in [23, Theorem 5.1] and [13, Theorems 4.3 and 5.1].[8]

THEOREM 4.2. *On the class $DG_{\tau\text{-ply}}$ there exists a $\sqrt{\cdot}$-separator theorem for the measure $|\cdot|$ with constants $\alpha = \frac{3}{4}$ and $\beta = O(\sqrt{\tau})$. The corresponding separation can be found in linear time.*

There is a close relation of τ-ply neighborhood systems to disks of bounded radius ratio with λ-precision. This relation will then establish a separator theorem on $DG_{\sigma,\lambda}$.

LEMMA 4.1. *For every $\sigma \geq 1$ and $\lambda > 0$ there exists constant τ with $\tau = O((\frac{\sigma}{\lambda})^2)$ such that $DG_{\sigma,\lambda} \subseteq DG_{\tau\text{-ply}}$.*

Proof. Let $\mathcal{D}$ be a collection of disks with radii in $[1,\sigma]$ and λ-precision. A given disk $D \in \mathcal{D}$ of radius σ and center z can be intersected only by other disks from $\mathcal{D}$ having their center in a cycle of radius 2σ centered around z. Since $\mathcal{D}$ has λ-precision there can be at most τ many centers inside this cycle of radius 2σ where τ is some constant with $\tau = O((\frac{\sigma}{\lambda})^2)$. ∎

Theorem 4.2 then directly implies the following.

THEOREM 4.3. *On the class $DG_{\sigma,\lambda}$ of disk graphs with bounded radius ratio and λ-precision there exists a $\sqrt{\cdot}$-separator theorem for the measure $|\cdot|$ with constants $\alpha = \frac{3}{4}$ and $\beta = O(\frac{\sigma}{\lambda})$.*

By Remark 4.1, combining the linear problem kernel for INDEPENDENT SET on $DG_{\sigma,\lambda}$ (see Corollary 3.1 and Remark 3.1) with this separator theorem, we obtain the following.

COROLLARY 4.1. *The parameterized* INDEPENDENT SET *problem on $DG_{\sigma,\lambda}$ can be solved in time $O(2^{O(\sqrt{k})} + n)$.*

[8] The result in [23, 13] was stated for the d-dimensional case.

Remark 4. 3. In the worst case $k = n$, we have an $O(2^{O(\sqrt{n})})$-algorithm for INDEPENDENT SET on $DG_{\sigma,\lambda}$, which means that we get a similar result as in the planar case.

4.2. A new geometric $\sqrt{\cdot}$-space separator theorem

In this subsection, we prove an analogue to the classical $\sqrt{\cdot}$-separator theorems for the class DG_σ of disk graphs with bounded radius ratio. Note that graphs in DG_σ may contain arbitrary large cliques, which means that a $\sqrt{\cdot}$-separator theorem does not hold for the counting measure $|\cdot|$. However, if we use the (Lebesgue) measure $\mu(G_\mathcal{D})$, where, $\mu(G_\mathcal{D})$ denotes the space occupied by $\bigcup \mathcal{D}$ (see Section 2), we obtain the following result.

THEOREM 4.4. *On the class DG_σ of disk graphs with bounded radius ratio, there exists a $\sqrt{\cdot}$-separator theorem for the Lebesgue graph measure $\mu(\cdot)$.*

More precisely, there exist constants $\alpha < 1$ and β such that, for every graph $G_\mathcal{D} \in DG_\sigma$ with representation $\mathcal{D}$, we find three sets $\mathcal{D}_A, \mathcal{D}_S, \mathcal{D}_B \subseteq \mathcal{D}$, such that $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ is a separation for $G_\mathcal{D}$, satisfying

1.$\mu(\mathcal{D}_S) \leq \sigma^2 \beta \sqrt{\mu(\mathcal{D})}$,
2.$\mu(\mathcal{D}_A), \mu(\mathcal{D}_B) \leq \alpha \mu(\mathcal{D})$.

Moreover, this separation can be found in time polynomial in $|V_\mathcal{D}|$.

Recall the notion from Section 2. In order to prove the Theorem we need the following result.

PROPOSITION 4.1. *For any ε there exists δ such that for any set of disks $\mathcal{D}$, each of radius at least one, it holds*[9]

$$|W_\mathcal{D}| \leq \frac{1+\varepsilon}{\delta^2} \mu(\mathcal{D}).$$

Proof. We first of all claim that if we multiply all radii of disks in $\mathcal{D}$ by an arbitrary factor $\eta \geq 1$, the total area of the new set is at most $\eta^2 \mu(\mathcal{D})$. To see this, we use the theorem of Bern and Sahai [5] stating that if any set of disks is shifted in the plane by a continuous motion such that the center-to-center distance does not increase in time, then the total area of the union of disks is also non-increasing.

[9]Note that this is a non-trivial result since $|W^\delta_\mathcal{D}|$ depends on δ.

geometric_separator(disks $\mathcal{D}$)
Returns sets $\mathcal{D}_S$, $\mathcal{D}_A$ and $\mathcal{D}_B$ corresponding to a separation $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ of $G_{\mathcal{D}}$ with respect to the measure $\mu(G_{\mathcal{D}})$.

- scale $\mathcal{D}$ such that the smallest disk has unit radius.
- fix any $\varepsilon < \frac{1}{2}$ and select δ according to Proposition 4.1 and construct the graph $H^{\delta}_{\mathcal{D}}$ (e.g., $\varepsilon = \frac{1}{4}, \delta = \frac{1}{20}$)
- run the algorithm of Lipton and Tarjan (see Theorem 4.1) on the graph $H^{\delta}_{\mathcal{D}}$ to obtain a separation (W_A, W_S, W_B) with

  a) $|W_S| \leq \beta' \sqrt{|W^{\delta}_{\mathcal{D}}|}$ and

  b) $|W_A|, |W_B| \leq \alpha' |W^{\delta}_{\mathcal{D}}|$,

for the constants $\beta' = 2\sqrt{2}$ and $\alpha' = \frac{2}{3}$.
- return the three sets

$$\begin{aligned} \mathcal{D}_S &:= \mathcal{D}[\hat{N}(W_S)] \\ \mathcal{D}_A &:= \mathcal{D}[W_A] \setminus \mathcal{D}_S, \\ \mathcal{D}_B &:= \mathcal{D}[W_B] \setminus \mathcal{D}_S. \end{aligned} \tag{1}$$

FIG. 2. Separator algorithm corresponding to Theorem 4.4.

If we first multiply the coordinates of the centers of $\mathcal{D}$ as well as their radii by η, the new set of disks covers the space $\eta^2\mu(\mathcal{D})$. We now shift all disk centers to the original position by the continuous mapping corresponding to the continuous application of magnification factor of the range $\eta \to 1$. The claim follows by applying Bern and Sahai's theorem to this motion.[10]

Now, fix an arbitrary positive $\delta \leq \frac{1}{3}(\sqrt{2(1+\varepsilon)} - \sqrt{2})$ and consider the set $W^{\delta}_{\mathcal{D}}$. Each point w of $W^{\delta}_{\mathcal{D}}$, could be represented as a unique square of side size δ with w placed at the center. (These squares correspond to the grid squares shifted by $\frac{\delta}{2}$ in both coordinates.) All these new squares could be covered by the original disks if we enlarge all radii in the ratio $(1 + \frac{3\sqrt{2}}{2}\delta)$. Then, due to the selection of δ,

$$|W^{\delta}_{\mathcal{D}}| \leq \left(1 + \frac{3\sqrt{2}}{2}\delta\right)^2 \frac{\mu(\mathcal{D})}{\delta^2} \leq \frac{1+\varepsilon}{\delta^2}\mu(\mathcal{D}).$$

∎

Proof. (of Theorem 4.4) The sets $\mathcal{D}_S$, $\mathcal{D}_A$ and $\mathcal{D}_B$ will be determined according to (1) in the procedure given in Fig. 2. We now prove that,

[10]We thank Jiří Matoušek for suggesting us this elegant argument for this proof.

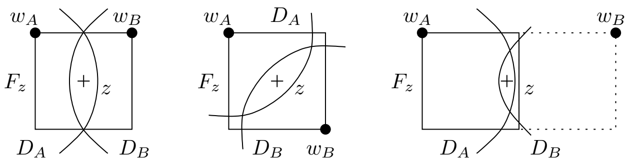

FIG. 3. Intersecting disks select from the grid two vertices w_A and w_B at distance at most two.

indeed $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ is a separation of G and that properties (1.) and (2.) of the theorem for the computed sets $\mathcal{D}_S, \mathcal{D}_A$, and $\mathcal{D}_B$ hold:

$(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ *is a separation of* G*:* Showing that $(V_{\mathcal{D}_A}, V_{\mathcal{D}_S}, V_{\mathcal{D}_B})$ is a separation of $G_\mathcal{D}$ is equivalent to proving that $\bigcup \mathcal{D}_A \cap \bigcup \mathcal{D}_B = \emptyset$. Recall that (W_A, W_S, W_B) is the separation of $H^\delta_\mathcal{D}$ obtained by the algorithm of Lipton and Tarjan. First of all we claim that

$$\bigcup \mathcal{D}_A \cap W^\delta_\mathcal{D} \subseteq W_A, \quad \text{and} \quad \bigcup \mathcal{D}_B \cap W^\delta_\mathcal{D} \subseteq W_B. \tag{2}$$

To see this, note that $\bigcup \mathcal{D}_A \cap W_S = \emptyset$, since if there is a disk $D \in \mathcal{D}_A$ containing a point $w \in W_S$ — by construction of $\mathcal{D}_S$ — we had $D \in \mathcal{D}_S$. Suppose now that there is a vertex $w_B \in W_B$ which lies in a disk $D \in \mathcal{D}_A$. By definition of the set $\mathcal{D}_A$, we find a vertex $w_A \in W_A$ inside D as well. Then, there exists a path P in $H^\delta_\mathcal{D}$ which connects w_A and w_B and is completely placed inside the disk D. Since (W_A, W_S, W_B) is a separation of $H^\delta_\mathcal{D}$ there exists a vertex of W_S on P contradicting $\bigcup \mathcal{D}_A \cap W_S = \emptyset$. Since $W^\delta_\mathcal{D} = W_A \cup W_S \cup W_B$, we get $\bigcup \mathcal{D}_A \cap W^\delta_\mathcal{D} \subseteq W_A$. The property $\bigcup \mathcal{D}_B \cap W^\delta_\mathcal{D} \subseteq W_B$ follows similarly.

Assume now, for contradiction, that vertices $v_A \in V_{\mathcal{D}_A}$ and $v_B \in V_{\mathcal{D}_B}$ form an edge of $E_\mathcal{D}$, that means there exists some point z in $D_A \cap D_B$. Let $F_z \in \mathcal{F}^\delta$ be any of the (at most four) squares containing z, and let W_z be the four grid vertices adjacent to F_z in $H^\delta_\mathcal{D}$.

We claim that we can find two grid points $w_A \in D_A$, $w_B \in D_B$, that are in $H^\delta_\mathcal{D}$ at distance at most two. W.l.o.g. we may assume that D_A intersects at least one corner vertex w_A of F_z (otherwise we symmetrically exchange subscripts A and B). We now consider three cases (all cases are depicted in Fig. 3). Either D_B also intersects a corner vertex w_B of F_z or D_B does not intersect any of the corner vertices of F_z. In case D_B intersects a corner vertex w_B, this vertex either is adjacent to w_A (see the left-hand diagram in Fig. 3) or it lies opposite to w_A (see the middle diagram in Fig. 3). In any case, w_A and w_B are at distance at most two. Now consider the case, where D_B does not intersect any of the corner vertices of F_z. In this case, it intersects one side of F_z, and since $\delta \ll 1$, it contains two grid vertices of the square sharing this side (see the last case depicted in Fig. 3). In particular, there is one grid vertex $w_B \in D_B$ which has distance at most two to $w_A \in D_A$.

By equation (2) we have $w_A \in W_A$ and $w_B \in W_B$, and, hence, they must be separated by W_S. If w_A and w_B are neighbors this is a contradiction. If they have distance two in $H^\delta_{\mathcal{D}}$, the vertex w connecting w_A and w_B must lie in W_S. Since w lies on the boundary of F_z, we have $F_z \in \hat{N}(W_S)$. Thus, since by construction $\mathcal{D}_S = \mathcal{D}[\hat{N}(W_S)]$, we get $D_A, D_B \in \mathcal{D}_S$, a contradiction.

ad property (1.): Consider a vertex $w \in W_S$ and the neighborhood $\hat{N}(w)$. As shown in Fig. 4, all disks in $\mathcal{D}$ which intersect $\hat{N}(w)$ must lie inside a cycle of radius $(2\sigma + \sqrt{2}\delta)$ centered at w. This is clear, since disks in $\mathcal{D}$ have radius bounded by σ and since the grid has span δ. More formally, we get

$$\mu(\mathcal{D}[\hat{N}(w)]) \leq (2\sigma + \sqrt{2}\delta)^2\pi.$$

Moreover, this implies that

$$\begin{aligned} \mu(\mathcal{D}_S) \;&=\; \mu(\mathcal{D}[\hat{N}(W_S)]) \;=\; \mu\Big(\bigcup_{w \in W_S} \mathcal{D}[\hat{N}(w)]\Big) \\ &\leq \sum_{w \in W_S} \mu(\mathcal{D}[\hat{N}(w)]) \;\leq\; \Big((2\sigma + \sqrt{2}\delta)^2\pi\Big)\,|W_S| \;\leq\; 5\sigma^2\pi|W_S|. \end{aligned} \tag{3}$$

If we chose $\varepsilon < \frac{1}{2}$, by the choice of δ according to Proposition 4.1, we can use in the last step $\sqrt{2}\delta < \frac{1}{6} \leq \frac{\sigma}{6}$. Using property (a) of the third step of the algorithm in Fig. 2 and Proposition 4.1, we have $|W_S| \leq \beta'\sqrt{|W_{\mathcal{D}}|} \leq \beta'\sqrt{\frac{1+\varepsilon}{\delta^2}\mu(\mathcal{D})}$ which together with the estimate (3) establishes

$$\mu(\mathcal{D}_S) \leq \sigma^2\beta\sqrt{\mu(\mathcal{D})} \quad \text{for} \quad \beta = \frac{5\pi\beta'}{\delta}\sqrt{(1+\varepsilon)}. \tag{4}$$

ad property (2.): First of all, observe that the set $\bigcup \mathcal{D}_A$ is completely covered by the square faces of the subgraph $H^\delta_{\mathcal{D}}[W_A]$, induced by the vertices of W_A. To see this, suppose there is a point $z \in D$ (for some $D \in \mathcal{D}_A$) which lies in some square $F_z \in \mathcal{F}^\delta$ of $H^\delta_{\mathcal{D}}$ but not of $H^\delta_{\mathcal{D}}[W_A]$. As above, if the four vertices adjacent to F_z host a vertex of W_B or W_S, we get $D \cap \hat{N}(W_S) \neq \emptyset$, a contradiction.

By this observation and by the fact that $|W_A| \leq \alpha'|W_{\mathcal{D}}|$, we get

$$\mu(\mathcal{D}_A) \;\leq\; \mu(H^\delta_{\mathcal{D}}[W_A]) \;\leq\; \delta^2|W_A| \;\leq\; \delta^2\alpha'|W^\delta_{\mathcal{D}}| \;\leq\; \alpha'(1+\varepsilon)\,\mu(\mathcal{D}),$$

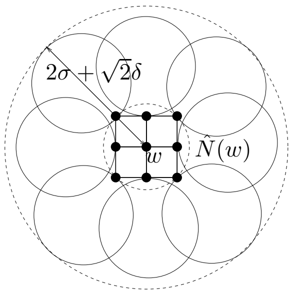

FIG. 4. For the proof of Theorem 4.4: $\mu(\mathcal{D}_S) \le (2\sigma + \sqrt{2}\delta)^2 \pi |W_S|$.

where Proposition 4.1 was used in the last step.

Similarly, one proves $\mu(\mathcal{D}_B) \;\le\; \alpha'(1+\varepsilon)\,\mu(\mathcal{D})$.

running time: The running time of the algorithm is determined by the time needed to apply Lipton and Tarjan's algorithm to the graph $H^{\delta}_{\mathcal{D}}$ which is $O(W^{\delta}_{\mathcal{D}})$. Using Proposition 4.1, we get

$$|W^{\delta}_{\mathcal{D}}| \;\le\; (1+\varepsilon)\delta^{-2}\mu(\mathcal{D}) \;\le\; (1+\varepsilon)\delta^{-2}\pi\sigma^2|\mathcal{D}| \;=\; O(\sigma^2|\mathcal{D}|).$$

∎

Observe that the choice of ε affects the tradeoff of getting small α on the one hand (it may be arbitrary close to α' as $\varepsilon \to 0$) which on the other hand causes the growth of β according to equation (4) as well as enlarging $|W^{\delta}_{\mathcal{D}}|$. For example by our choice of $\varepsilon = \frac{1}{4}$ and $\delta = \frac{1}{20}$ we get $\alpha = \alpha'(1+\varepsilon) = \frac{5}{6}$, but $\beta > 460$ and any disk of unit radius intersects at least 1250 grid points.

5. THE ALGORITHM AND ITS ANALYSIS

We use the geometric kernelization of Section 3 and a divide and conquer approach based on the new geometric separator theorem from Section 4.2 to derive an algorithm for the INDEPENDENT SET problem on disk graphs of bounded radius ratio.

THEOREM 5.1. *The parameterized* INDEPENDENT SET *problem on DG_σ can be solved in time $n^{O(\sqrt{k})}$.*

Proof. On input instance $(G_{\mathcal{D}}, k)$, in a first step, perform the geometric kernelization explained in Section 3. After this step, without loss of generality, we may assume that $\mu(\mathcal{D}) \le ck$ for some constant c.

In a second step, the divide-and-conquer procedure `indep_set` shown in Fig. 5 is applied to the kernel $\mathcal{D}$. Denote by $T(n, s)$ the time needed to

`disks indep_set(disks` $\mathcal{D}$`)`
Finds the maximal independent set of $G_{\mathcal{D}}$.

- `if` $(\mathcal{D} = \emptyset)$ `then` return $\emptyset$ `else`
- run `geometric_separator(`$\mathcal{D}$`)` to obtain sets $\mathcal{D}_S$, $\mathcal{D}_A$, and $\mathcal{D}_B$.
- `for` all independent sets I_S of vertices in $G_{\mathcal{D}_S}$ `do`

$$\begin{aligned} \mathcal{D}'_A &:= \mathcal{D}_A \setminus \{D \in \mathcal{D}_A \mid \exists D' \in \mathcal{D}[I_S] : D \cap D' \neq \emptyset\} \\ \mathcal{D}'_B &:= \mathcal{D}_B \setminus \{D \in \mathcal{D}_B \mid \exists D' \in \mathcal{D}[I_S] : D \cap D' \neq \emptyset\} \\ I'_S &:= I_S \cup \texttt{indep\_set(}\mathcal{D}'_A\texttt{)} \cup \texttt{indep\_set(}\mathcal{D}'_B\texttt{)} \end{aligned}$$

- return $(I'_S)_{\text{opt}}$, for which

$$|(I'_S)_{\text{opt}}| = \min\{\ |I'_S| \mid I_S \text{ is an independent set in } G_{\mathcal{D}_S}\ \}$$

FIG. 5. Divide and conquer algorithm for INDEPENDENT SET for disk graphs of bounded radius ratio based on the new geometric separator theorem. Note that for a fixed independent set I_S of $G_{\mathcal{D}_S}$, no further disk from $\mathcal{D}_A$ and $\mathcal{D}_B$ intersecting disks from $\mathcal{D}[I_S]$ can be chosen. This explains the construction of $\mathcal{D}'_A$ and $\mathcal{D}'_B$.

compute `indep_set(`$\mathcal{D}$`)` of a collection of n disks $\mathcal{D}$ with $\mu(\mathcal{D}) \leq s$. Let $p(|\mathcal{D}|)$ be the polynomial time needed to compute the sets $\mathcal{D}_S, \mathcal{D}_A$, and $\mathcal{D}_B$ according to Theorem 4.4, and $q(|\mathcal{D}|)$ be the polynomial time needed to perform the constructions of $\mathcal{D}'_A$ and $\mathcal{D}'_B$. Note that in $\mathcal{D}_S$ at most $\lfloor \frac{\beta\sqrt{s}}{\pi} \rfloor$ many disks can be independent, since $\mu(\mathcal{D}_S) \leq \beta\sqrt{s}$ and every disk has radius at least one. Hence, the total number of independent sets in $G_{\mathcal{D}_S}$ is upper bounded by

$$\sum_{i=0}^{\lfloor \frac{\beta\sqrt{s}}{\pi} \rfloor} \binom{n}{i} \leq n^{\widehat{\beta}\sqrt{s}},$$

where $\widehat{\beta}$ is some constant. Then, the recursion we have to solve in order to compute an upper bound on $T(n,s)$ reads as follows:

$$T(n,s) \leq p(n) + n^{\widehat{\beta}\sqrt{s}} \cdot q(n) \cdot 2\,T(n, \alpha s).$$

Hence, for n large enough, and a suitable constant $\widetilde{\beta}$ we have

$$\begin{aligned} T(n,s) &\leq n^{\widetilde{\beta}\sqrt{s}} \cdot T(n, \alpha s) \leq \prod_{i=0}^{\log_{\frac{1}{\alpha}}(s)} n^{\widetilde{\beta}\sqrt{\alpha^i s}} \cdot T(n,1) \\ &\leq n^{\widetilde{\beta}\sqrt{s}(\sum_{i=0}^{\infty} \alpha^{\frac{i}{2}})} \cdot T(n,1) = n^{\frac{\widetilde{\beta}\sqrt{s}}{1-\sqrt{\alpha}}} \cdot T(n,1). \end{aligned}$$

Note that $T(n,1)$ is constant, since $\mu(\mathcal{D}) \leq 1$ implies $\mathcal{D} = \emptyset$, because for every disk D we have $\mu(D) \geq \pi$. By plugging in the values $n = |\mathcal{D}|$ and $s = ck$, we obtain the running time as we have claimed. ▌

Remark 5. 1. In the worst case $k = n$, we have an $2^{O(\sqrt{n}\log(n))}$-algorithm, a running time with a sublinear exponent that cannot be achieved for general graphs (unless 3SAT$\in$ DTIME$(2^{o(n)})$) [17].

6. CONCLUSION AND OPEN PROBLEMS

In this paper, we prove a geometric separator theorem which in some sense extends the idea of common separator theorems for planar graphs and disk graphs of λ-precision to arbitrary disk graphs of bounded radius ratio.

The geometric separator theorem together with a geometric problem kernelization is applied to design an algorithm for the parameterized INDEPENDENT SET problem for disk graphs of bounded radius ratio. The running time of this algorithm is $n^{O(\sqrt{k})}$. If, in addition, the disks have λ-precision (for a fixed λ), the problem is in FPT with a running time $O(2^{O(\sqrt{k})} + n)$.

We mention here that the use of our algorithm also improves the running time of the PTAS for the INDEPENDENT SET problem on unit disk graphs [16]. For given precision ε, the known time bound is $n^{O(\frac{1}{\varepsilon})}$ and using our separator technique we get running time $n^{O(\frac{1}{\sqrt{\varepsilon}})}$ Even though our method can be used for disk graphs of bounded radius ratio, the key step for the PTAS on disk graphs [14] requires enlisting *all* independent sets from a collection of disks of bounded measure which cannot be captured straightforwardly from the existing separation. We leave it as an open problem whether our separation approach can be useful also in such situation.

Besides, we remark that the methods which yields the $O(2^{O(\sqrt{k})}+n)$ time algorithm on disk graphs with bounded radius ratio and λ-precision can be carried over to all problems which, on the one hand, admit a geometric problem kernel and which, on the other hand, can be solved by a divide and conquer approach based on a separator theorem. This latter property was characterized in [3] using the notion of "weak glueability." As an example, we mention the weakly glueable DOMINATING SET problem, for which we can prove a (linear) geometric problem kernel similarly to Proposition 3.1.

In terms of the methods used in this paper, it is an interesting question, whether a similar geometric separation theorem also holds for disk graphs with arbitrary radius ratio or for intersection graphs of other geometric objects.

As to parameterized complexity, we leave it as an open problem, whether INDEPENDENT SET on disk graphs (of bounded radius ratio) is in FPT or complete for the classes $W[1]$, respectively. We want to emphasize, that we are not aware of a (non-artificial) $W[1]$-complete problem which allows for an algorithm of running time $n^{o(k)}$, i.e., with running time bounded by an exponential with a sublinear exponent.

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REFERENCES

1. P. K. Agarwal, M. van Kreveld, and S. Suri. Label placement by maximum independent set in rectangles. In *Computational Geometry: Theory and Applications*, **11(3–4):** 209–218, 1998.
2. J. Alber, H. Fernau, and R. Niedermeier. Parameterized complexity: exponential speed-up for planar graph problems. In *Proceedings of the 28th ICALP 2001*, Springer-Verlag LNCS 2076, pp. 261–272, 2001. Long version available as Technical Report TR01-023, Electronic Colloquium on Computational Complexity (ECCC), Trier, March 2001.
3. J. Alber, H. Fernau, and R. Niedermeier. Graph separators: a parameterized view. In *Proceedings of the 7th COCOON 2001*, Springer-Verlag LNCS 2108, pp. 318–327, 2001.
Long Version available as Technical Report WSI-2001-8, Wilhelm-Schickard-Institut für Informatik, Universität Tübingen.
4. B. S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *Journal of the ACM*, **41(1)**:153–180, 1994.
5. M. Bern and A. Sahai. Pushing disks together — The continuous-motion case. *Discrete Comput. Geom.*, **20(4)**:499–514, 1998.
6. L. Cai and D. W. Juedes. Subexponential parameterized algorithms collapse the W-hierarchy. In *Proceedings of the 28th ICALP 2001*, Springer-Verlag LNCS 2076, pp. 273–284, 2001.
7. B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Mathematics*, **86**:165–177, 1990.
8. H. N. Djidjev. On the problem of partitioning planar graphs. *SIAM J. Algebraic Discrete Methods*, **3(2)**:229–240, 1982.
9. H. N. Djidjev. A separator theorem for graphs of fixed genus. *Serdica*, **11**:319–329, 1985.
10. H. N. Djidjev and S. M. Venkatesan. Reduced constants for simple cycle graph separation. *Acta Informatica*, **34**:231–243, 1987.

11. R. G. Downey and M. R. Fellows. Fixed Parameter Tractability and Completeness II: On completeness for W[1]. *Theoret. Comp. Sci.*, **189**:109–131, 1995.
12. R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Monographs in Computer Science. Springer-Verlag, 1999.
13. D. Eppstein, G. L. Miller, and S. H. Teng. A deterministic linear time algorithm for geometric separators and its applications. In *Proceedings of the 9th ACM Symp. on Computational Geometry*, pp. 99–108, 1993.
14. T. Erlebach, K. Jansen, and E. Seidel. Polynomial-Time Approximation Schemes for Geometric Graphs. In *Proceedings of the 12th ACM-SIAM SODA*, pp. 671–679, 2001.
15. C. Glasser, S. Reith, and H. Vollmer. The complexity of base station positioning in cellular networks. In *Proceedings of the ICALP 2000 Workshops*, Carleton Press, pp. 167–177, 2000.
16. H. B. Hunt, M. V. Marathe, V. Radhakrishnan, S. S. Ravi, D. J. Rosenkrantz, and R. E. Stearns. NC-approximation schemes for NP– and PSPACE–hard problems for geometric graphs. In *Journal of Algorithms*, **26**:238–274, 1998.
17. R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? In *Proceedings of the 39th FOCS*, pp. 653–664, 1998.
18. P. Koebe. Kontaktprobleme der konformen Abbildung. *Ber. Verh. Saechs. Akad. Leipzig*, **88**:141–164, 1936.
19. N. Lev-Tov and D. Peleg. Exact algorithms and approximation schemes for base station placement problems. In *Proceedings of the 8th SWAT*, pp.90–99, 2002.
20. R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. In *SIAM Journal of Applied Mathenatics*, **36(2)**:177–189, 1979.
21. R. J. Lipton and R. E. Tarjan. Applications of a planar separator theorem. In *SIAM Journal on Computing*, **9(3)**:615–627, 1980.
22. E. Malesińska. *Graph-Theoretical Models for Frequency Assignment Problems.* PhD thesis, Technische Univerität Berlin, 1997.
23. G. L. Miller, S. H. Teng, and S. A. Vavasis. A unified geometric approach to graph separators. In *Proceedings of the 32nd FOCS*, pp. 538–547, 1991.
24. G. L. Miller, S. H. Teng, W. Thurston, and S. A. Vavasis. Separators for sphere packings and nearest neighbor graphs. In *Journal of the ACM.*, **44(1)**:1–29, 1997.
25. J. M. Robson. Algorithms for maximum independent sets. *Journal of Algorithms*, **7**:425–440, 1986.