# Borodin's Conjecture on Diagonal Coloring is False


Daniel Král'[1,*] Riste Škrekovski[2,†]

[1]Institute for Theoretical Computer Science, Charles University,
Malostranské náměstí 25, 118 00 Prague, Czech Republic.
E-mail: kral@kam.mff.cuni.cz

[2]Department of Mathematics, University of Ljubljana
Jadranska 19, 1111 Ljubljana, Slovenia.
E-mail: skreko@kam.mff.cuni.cz



**Abstract**

In a 1-diagonal coloring, vertices of any face and vertices of any two faces sharing an edge have to get different colors. Borodin proved that any triangulation of a surface of Euler genus $g \geq 2$ can be 1-diagonally colored by $\lfloor \frac{13+\sqrt{73+48g}}{2} \rfloor$ colors. The bound is conjectured to be sharp for all surfaces except for the plane. We disprove this conjecture.


In a 1-diagonal coloring of an embedded graph, two vertices have to get different colors if they are incident with a same face or a removal of a single edge make them incident with the same face. This kind of coloring was first studied by Bouched, Fouquet, Jolivet, and Riviere [1]. Borodin proved in [2] that any (simple) triangulation of a surface of Euler genus $g$ can be 1-diagonally colored by $B(g) := \lfloor \frac{13+\sqrt{73+48g}}{2} \rfloor$ colors and he conjectured that this bound is sharp. This conjecture also appears in [3] as Problem 3.10. In this note, we show that the conjecture is false for infinitely many values of $g$.


*Institute for Theoretical Computer Science is supported as project LN00A056 by Ministry of Education of the Czech Republic.

†Supported in part by the Ministry of Science and Technology of Slovenia, Research Project Z1-3129.

If $G$ is a triangulation, denote by $G^+$ the graph obtained from $G$ by adding edges between those vertices $a$ and $b$ such that there are faces $axy$ and $bxy$ in $G$ if such an edge $ab$ is not already present in $G$; such edges are considered to be drawn on the surface passing through the faces $axy$ and $bxy$ and crossing the edge $xy$. Note that a 1-diagonal coloring of $G$ is a proper coloring of $G^+$ and vice versa. We first extend Theorem 6 of [1]:

**Theorem 1** *Let $G$ be a triangulation of a surface of genus $g \geq 2$. Let $H$ be an induced subgraph of $G^+$, and let $n$ and $m$ be the numbers of its vertices and edges. Then, $m \leq 6(n + g - 2)$. Moreover, if $H$ is a proper subgraph of $G^+$, then the inequality is strict.*

**Proof:** Let $P$ be the set of edges of $H$ which are also edges of $G$ and let $D$ be the set of the remaining edges of $H$. Set $D'' \subseteq D$ to be a maximum-cardinality subset of $D$ such that no two edges of $P \cup D''$ cross. Let $P^+ = P \cup D''$ and $D^- = D \setminus D''$. The following claim was proven in [1]:

**Claim 1** *Each edge of $D''$ is crossed by at most two edges of $D^-$.*

Let $P_i$ for $i = 0, 1$ ($D''_i$ for $i = 0, 1, 2$) be edges of $P$ ($D''$) crossed by exactly $i$ edges of $D^-$. Claim 1 immediately yields:

$$P^+ = P_0 \cup P_1 \cup D''_0 \cup D''_1 \cup D''_2 \tag{1}$$

Definitions of $P_i$ and $D''_i$ give:

$$|D^-| \leq 2|D''_2| + |D''_1| + |P_1| \tag{2}$$

The following inequality was proven in [1]:

$$|D''_2| \leq |P_0| \tag{3}$$

Inequalities (2) and (3) combine together with (1) to:

$$|D^-| \leq |P_0| + |P_1| + |D''_1| + |D''_2| \leq |P^+| \tag{4}$$

Since the edges of $P^+$ form a graph embeddable on a surface of genus $g$, by the Euler formula, we get:

$$m = |P| + |D| = |P^+| + |D^-| \leq 2|P^+| \leq 6(n + g - 2) \tag{5}$$

Assume that $H$ is a proper subgraph of $G^+$ and $m = 6(n+g-2)$. Then, inequalities (2), (3), (4) and (5) turn into equalities: From (1) and (4), we have that $D''_0 = \emptyset$. And, from (2), each edge of $D^-$ is crossed by precisely one edge of $P^+$.

If $D''_2 \neq \emptyset$, choose any edge $ab \in D''_2$ and let $axy$ and $bxy$ be the two neighboring faces in $G$. Precisely one of vertices of $x, y$ is missing in $H$, say $y$. Let $xx'$ and $xx''$ be the edges which cross $ab$. Note that $ab$ is the only edge which crosses $xx'$. The same holds for $xx''$. Therefore, we may replace the edge $ab$ with the two edges $xx'$ and $xx''$ in $D''$. But this is a contradiction by the maximality of $D''$. Hence, $D''_2 = \emptyset$ and also $P_0 = \emptyset$ due to (3).

If $P \neq \emptyset$, there exists a vertex $a$ such that some of its neighbors are in $H$ and some are not. In such case, there would be a face $abc$ of $G$ such that the edge $ab$ is present in $H$ and the vertex $c$ is missing. But then $ab$ has to be in $P_0$ ($c$ is not in $H$) which is impossible. One may conclude $P = \emptyset$ and so $P^+ = D''_1$.

Now, choose an edge $ab \in D''_1$ and let $axy$ and $bxy$ be the two neighboring faces in $G$. Since $P$ is empty both $x, y$ are missing in $H$. This implies that $ab$ is not crossed by any edge of $D$ and it cannot be in $D''_1$. Thus $D''_1 = \emptyset$, $P^+ = \emptyset$ and $D^- = \emptyset$ from (4). Then, $P \cup D = \emptyset$ and $m = 0$ which is a contradiction.

■

**Theorem 2** *Let $G$ be a triangulation of a surface of genus $g = 48k^2+38k+6$ for integer $k \geq 0$. Then, the minimum degree of any induced subgraph of $G^+$ is at most $B(g) - 2$.*

**Proof:** A straightforward calculation yields that $B(g)$ is an even integer equal to $24k + 16$ satisfying the following equality:

$$B(g)(B(g) - 1) = 12(B(g) + g - 2) \tag{6}$$

Let $H$ be an induced subgraph of $G^+$ with $n$ vertices and $m$ edges. Also, let $e$ be the number of edges of $G$. If $n \leq B(g) - 1$, the statement of the theorem is clearly true. Otherwise, we distinguish several possibilities:

1. $n > B(g)$: In this case we bound the average degree of $H$ using Theorem 1 and (6):

$$\frac{2m}{n} \leq \frac{12(n+g-2)}{n} < \frac{12(B(g)+g-2)}{B(g)} = B(g) - 1$$

2. $n = B(g)$ *and* $H = G^+$*:* Assume that minimum degree of $G^+$ is at least $B(g)-1$. Then, the minimum degree of $G$ is at least $\lceil (B(g)-1)/2 \rceil = B(g)/2$ and $e \geq n\,B(g)/4 = B(g)^2/4$. Since $G$ is a triangulation of a surface of Euler genus $g$, it holds that $e \leq 3(n+g-2) = 3(B(g)+g-2)$ which contradicts (6).

3. $n = B(g)$ *and* $H$ *is a proper subgraph of* $G^+$*:* We bound the average degree of $H$ again using Theorem 1 and (6):

$$\frac{2m}{n} < \frac{12(n+g-2)}{n} = \frac{12(B(g)+g-2)}{B(g)} = B(g)-1$$

Thus the theorem is established.

■

An immediate consequence of Theorem 2 is the following corollary which implies that Borodin's conjecture is false.

**Corollary 1** *Any triangulation of a surface of genus* $g = 48k^2+38k+6$ *can be 1-diagonally colored using* $B(g)-1$ *colors.*

# Remark

A little more careful analysis in the proofs of the theorems could provide an evidence that the conjecture is false also for some other genera. We do not so in order to keep the paper short.

# References


[1] A. Bouchet, J.-L. Fouquet, J.-L. Jolivet, M. Riviere: On a Special Face Colouring of Cubic Graphs, Ars Combinatoria **24** (1987), 67–76.

[2] O. V. Borodin: Diagonal coloring of the vertices of triangulations, Discrete Mathematics **102** (1992), 95–96.

[3] T. R. Jensen, B. Toft: Graph Coloring Problems, Wiley (1995), New York.