Folding towards the infinite
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FOLDING

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Abstract. We define folding of a directed graph as a coloring (or a homomorphism) which is injective on all the *down sets* of a given depth. While in general foldings are as complicated as homomorphisms for some some classes they present an useful tool to study colorings and homomorphisms. Our main result yields for any proper minor closed class $\mathcal{K}$ a folding (of any prescribed depth) using a fixed number of colors. This in turn yields (for any $\mathcal{K}$) the existence of a K_k-free graph which bounds all K_k-free graphs belonging to $\mathcal{K}$. This has been conjectured in [9] and elsewhere and solved for $k = 3$ in [10]. Particularly, we prove (without using 4CT) the existence of a graph H which satisfies $\chi(H) \le 5$, $\omega(H) \le 4$ and such that any planar graph G is homomorphic to H. This is sandwiched between 4CT and 5CT for planar graphs and the general case has bearing to Hadwiger Conjecture.

1. Introduction

For a directed graph $\vec{G}, x \in V(\vec{G}), d$ a positive integer, we denote by $D^d_{\vec{G}}(x)$ the set of all those vertices y of $\vec{G}$ for which there exists a directed path of length $\le d$ from y to x; we also call $D^d_{\vec{G}}(x)$ *d-down set* in $\vec{G}$ (we are motivated by the terminology of the theory of partially ordered sets). Thus $D^1_{\vec{G}}(x)$ is just closed in-neighborhood of x in $\vec{G}$.

A homomorphism $f : \vec{G} \to \vec{H}$ is said to be *d-folding* if f restricted to any d-down set in G is injective. (Recall: A *homomorphism* $f : \vec{G} \to \vec{H}$ is a mapping $f : V(\vec{G}) \to V(\vec{H})$ satisfying $(f(x), f(y)) \in E(\vec{H})$ whenever $(x, y) \in E(\vec{G})$; see [9] for background and some related motivation). If $\vec{K}_k$ is an orientation of K_k, a d-folding $\vec{G} \to \vec{K}_k$ is then a d-folding which uses k colors. In this paper we study folding of minor closed classes and we establish the existence of a bounded folding for any such class. This result can be stated as follows:

Let $\mathcal{K}$ be a minor closed class of graphs. We say that $\mathcal{K}$ is *proper* if $\mathcal{K}$ does not contain all graphs. Alternatively, $\mathcal{K}$ is proper if there exists at least one forbidden minor. We have the following which may be seen as the main result of this paper:

Theorem 1.1. *Let $\mathcal{K}$ be a proper minor closed class of graphs, d a positive integer. Then there exists a positive integer $k = k(d, \mathcal{K})$ and an orientation $\vec{K}_k$ of K_k, such that for any graph $G \in \mathcal{K}$ there exists an acyclic orientation $\vec{G}$ and a d-folding $f : \vec{G} \to \vec{K}_k$ (i.e. a d-folding using at most k colors).*

It is well known that the graphs belonging to a proper minor closed class $\mathcal{K}$ have a bounded density and thus a bounded chromatic number. The celebrated Hadwiger's Conjecture asserts that this bound is identical to the maximal size of

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the complete graph in $\mathcal{K}$. See Section 5 below for a discussion of relevance of our results to Hadwiger's Conjecture.

Our Theorem 1.1 presents a structural refinement of this bound. This extension is not straightforward and as a consequence the present bound for $k(\mathcal{K})$ is very large (a tower function). We do not optimize at this point. Note that the minor closed property is used repeatedly in the proof and that Theorem 1.1 fails to be true for bipartite 2-degenerated graphs

We discovered Theorem 1.1 in the context of graphs and their homomorphisms. We write $G \leq H$ if there exists a homomorphism $G \to H$. This quasiorder is called *homomorphism* or *coloring order*. It is well known *Grötsch's Theorem* (see e.g. [7] and [14] for the best proof) that every triangle free planar graph is 3-colorable. Using homomorphism order this means that $G \leq K_3$ for any triangle free planar graph G. In yet another way we can also say that the class $\mathcal{P}_3$ of all K_3-free graphs is bounded by K_3. The following problem has been formulated in [9] and elsewhere:

Problem 1.1. Does there exists a triangle free graph H such that $G \leq H$ for any triangle free planar graph? In other words, is the class $\mathcal{P}_3$ bounded by a triangle free graph H?

Is the class $\mathcal{P}_4$ of all K_4-free planar graphs bounded by a K_4-free graph?

Here we give an affirmative answer to this problem. In fact for $k = 3, 4$ one can prove that the classes $\mathcal{P}_k$ are bounded by a k-colorable K_k-free graph and we also prove an analogous result for any proper minor closed class $\mathcal{K}$:

Theorem 1.2. *Let $\mathcal{K}$ be a minor closed class of graphs all of which are t-colorable, let k be a positive integer. Then the class $\mathcal{K}_k$ of all K_k-free graphs in $\mathcal{K}$ is bounded by a k-colorable K_k-free graph.*

Similar result were obtained recently in [8] and in [10]. Particularly, [10] solved the above problem for $k = 3$ (triangle-free graphs). However the case of K_k-free graphs was left open and seems more difficult. Here we treat the problem in a more general context. All results may be seen as an evidence for the following general conjecture.

Let $\mathcal{A}, B$ be classes of graphs, $\mathcal{A} \subset \mathcal{B}$. We say that the class $\mathcal{A}$ is *bounded in $\mathcal{B}$* if there exists a graph $H \in \mathcal{B}$ such that $G \leq H$ for any $G \in \mathcal{A}$. Thus $\mathcal{A}$ is bounded in $\mathcal{A}$ iff $\mathcal{A}$ has the greatest element (with respect to $\leq$). The study of boundedness phenomena is one of the basic problems and we are pleased that in our setting it relates questions like Hadwiger conjecture to the mainstream mathematics (see Remarks).

Given a finite set $\mathcal{F}$ of graphs we denote by $\mathrm{Forb_h}(\mathcal{F})$ the class of all graphs G with no homomorphism $F \to G$ for an $F \in \mathcal{F}$. Equivalently and more formally, $\mathrm{Forb_h}(\mathcal{F}) = \{G; F \in \mathcal{F} \Rightarrow F \not\leq G\}$. As an example, note that $\mathrm{Forb_h}\{K_3\}$ is the class of all triangle free graphs.

Conjecture 1. Let $\mathcal{F}$ be any finite set of graphs. Then for any minor closed class of graphs $\mathcal{K}$ the class $\mathcal{K} \cap \mathrm{Forb_h}(\mathcal{F})$ is bounded in $\mathrm{Forb_h}(\mathcal{F})$.

It has been proved in [6] (see also [3] and [8] for a different proof) that for the class $\mathcal{K}_d$ of all graphs with all their vertices bounded by d and for any finite set of graphs $\mathcal{F}$ the analogous conjecture holds. The results of this paper and [10] verify the conjecture for classes $\mathrm{Forb_h}(\mathcal{F})$ for $\mathcal{F} = \{K_k\}$. Note that for graphs in general and even degenerated classes of graphs the analogous statement fails to be true: For $\mathcal{F} = \{K_3\}$ consider graphs $K_n^{\star\star}$ formed from K_n by subdividing each edge by two new vertices. All the graphs $K_n^{\star\star}$ are 2-degenerated yet they are not bounded by a finite triangle-free graph. Note also that this cannot be saved by (large) girth: Let G be a graph of girth ℓ with chromatic number k. Then the graph $G^{\star\star}$ has girth 3ℓ and there is no homomorphism of G into a triangle free graph with at most k vertices.

This paper is organized as follows: In Section 1 we prove Theorem 1.1. In Section 2 we prove the following result which is perhaps of an independent interest:

Theorem 1.3. *Let $t \geq 3$ be a positive integer. For any proper minor closed class $\mathcal{K}$ there exists $k = k(\mathcal{K}, t)$ such that any graph $G \in \mathcal{K}$ has a proper k-coloring with the property that any set of t color classes either contains a K_t or it induces a graph with chromatic number $< t$.*

Note that, again, an analogous statement fails to be true in general even for $k = 3$: In any k-coloring of the graph $K_n^{\star\star}$, n sufficiently large, there exists odd cycles of length ≥ 6 which are colored by at most 3 colors. Also the graphs $G^{\star\star}$ (see above) have girth 3ℓ and in any k-coloring contain odd cycles colored by at most 3 colors.(More complicated examples are provided by Ramsey theory.)

The key notion for the proof of Theorem 1.3 is the above notion of d-folding and using that we prove in Section 4 Theorem 1.2 by an universal construction given in [8] (we sketch a proof for completeness). Section 5 contains concluding remarks and open problems.

2. Folding of Minor Closed Classes

In this section we prove Theorem 1.1. The proof proceeds by double induction and towards this end we find it useful to refine the notion of folding:

Let $a, b, a > 0, b \geq 0$ be positive integers. An (a, b)-path from x to y is a path of length $a+b$ which has the first a of its edges oriented from x to y and the remaining b edges oriented in the reverse direction from y to x.

The following definitions refine the notion of d-folding:

Definition 2.1. Let a, b be integers ($a \geq b \geq 0$ and $a \neq 0$) and let $\vec{G}$ be a directed graph. Then, a vertex-coloring of $\vec{G}$ (respectively. an edge-coloring) is (a, b)-*deep* if any (k, l)-path in $\vec{G}$, $0 < k \leq a$ and $0 \leq l \leq b$ has all its vertices (respectively. edges) of different colors.

Note that (as opposed to vertex-coloring) an (a, b)-deep edge-coloring need not be a homomorphism. However every (a, b)-deep vertex-coloring induces (a, b)-deep edge-coloring. Note also that a (d, d)-deep vertex coloring does not coincide with a d-folding: in a (d, d)-deep coloring, some edges may be oriented from a vertex with color i to a vertex with color j and some others may be oriented from a vertex with color j to a vertex with color i, though this cannot be the case in a folding. However, if a 1-folding is a (d, d)-deep coloring it is clearly a d-folding.

Our proof of Theorem 1.1 follows this scheme:

$i.$$(a, 0)$-deep vertex coloring exists for every a;
$ii.$$(a, 1)$-deep edge coloring exists for every a;
$iii.$$(a, 1)$-deep vertex coloring exists for every a;

iv.(a,b)-deep vertex coloring and (a,b)-deep edge coloring implies existence of a $(a,b+1)$-deep edge coloring;

v.(a,b)-deep vertex coloring, $b>0$, and $(a,b+1)$-deep edge coloring implies existence of a $(a,b+1)$-deep vertex coloring;

vi. $(2,0)$-deep vertex coloring implies existence of a 1-folding;

vii. (d,d)-deep vertex coloring and 1-folding implies existence of a d-folding.

(Clearly $iv. \Rightarrow ii.$ but it is instructive to prove $ii.$ separately.)

These statements are stated precisely in the seven Lemmas 2.1, 2.2, 2.5, 2.6, 2.9, 2.10 and 2.11 below. Together these statements constitute proof of Theorem 1.1.

Lemma 2.1. *Let $\vec{G}$ be a directed acyclic graph, such that each vertex has indegree at most k. Then, $\vec{G}$ has a $(p,0)$-deep vertex coloring using $\frac{k^{p+1}-1}{k-1}$ colors.*

Proof. Consider a topological sort $v_1,\dots,v_n$ of the vertices. For each vertex v_i, there are at most $\frac{k^{p+1}-1}{k-1}-1$ vertices in the graph $D^p_{\vec{G}}(v_i)-v_i$. Hence, we may choose an unused colors to color v_i. □

Lemma 2.2. *Let $\vec{G}$ be a directed acyclic graph, such that each vertex has indegree at most k. Then, $\vec{G}$ has a $(p,1)$-deep edge coloring using $\frac{k^{p+1}-1}{k-1}$ colors.*

Proof. Consider a topological sort $v_1,\dots,v_n$ of the vertices. For each vertex v_i, there are at most $\frac{k^{p+1}-1}{k-1}-k-1$ edges in the graph $D^p_{\vec{G}}(v_i)-v_i$. Hence, we may choose k unused colors to color the edges incoming v_i. □

Before proving $iii.$ we take time out for the definition and a Lemma.

Definition 2.2. Let $S=(c_1,\dots,c_a)$ be a sequence of colors of an edge-coloring of a directed graph $\vec{G}$. Then, S is *feasible* if there exists some monotone path $\vec{P}$ of $\vec{G}$ (that is: an $(a,0)$-path) *realizing* S, in the sense that its edges have color $c_1,\dots,c_a$ (according to the forward order). An edge e of $\vec{G}$ is *reached* by S if it belongs to some path realizing S.

Lemma 2.3. *Let $S=(c_1,\dots,c_a)$ be any feasible sequence of $\vec{G}$, let $\vec{G}$ has an $(a,1)$-deep edge-coloring. Then, the union of all the paths in $\vec{G}$ realizing S is a forest, the components of which are branchings oriented from some roots, with edges colored $c_1,\dots,c_a$ depending on the height of the edge on the tree.*

Proof. Assume there exists a vertex v having two incoming edges e,f reached by S, having respectively colors c_i and c_j $(i<j)$. Then, there exists a monotone path $\vec{P}$ of length j ending with f, having edges of color $c_1,\dots,c_j$ and thus including an edge of color c_i. Hence, $D^p_{\vec{G}}(v)-v$ includes an edge of color c_i, a contradiction. □

Lemma 2.4. *Assume $\vec{G}$ has a $(a,1)$-deep edge-coloring with β colors and an acyclic orientation with indegrees bounded by k, assume S is a sequence of a colors and c is a color. Then, there exists a vertex coloring using $f(k)$ colors such that any path, with signature $(a,1)$ with edges colored according to S for its monotone part of length a and c for its last edge, has endpoints with different colors.*

Proof. First remark that there exists no path with signature $(a,1)$ colored as requested if $c\in S$, as the edge coloring is $(a,1)$-deep.

Consider the graph $\vec{G}^+$ obtained from $\vec{G}$ by adding an edge between vertices x and y whenever x and y are neither identical or adjacent and there exists some path with signature $(a, 1)$ having x and y as endpoints and colored as requested.

Let A be a subset of $V(\vec{G}) = V(\vec{G}^+)$ and let $B = V(\vec{G}) \setminus A$. The union of all the monotone paths of $\vec{G}$ realizing S is a forest. Consider the graph H which is the union of $\vec{G}_A$ and all the $(a, 1)$ paths properly colored with endpoints in A. Contract in H all the edges with color in S having at least one incidence outside A. Simplify the resulting graph, thus obtaining H'. As H' is a minor of G, it belongs to $\mathcal{C}$ and has density at most $k/2$. It is easily checked that $\vec{G}_A^+$ as density at most $k^a(k-1) + k + k/2$ and thus chromatic number at most $2k^{a+1} - 2k^a + 3k + 1$ □

These result implies *iii.* in the following form:

Lemma 2.5. *Let $\mathcal{C}$ be a proper minor closed class of graphs, let k be such that each graph is $\mathcal{C}$ has density at most $k/2$ and let $p \geq 1$ be an integer. Then any graph $G \in \mathcal{C}$ with an acyclic orientation $\vec{G}$ with indegree at most k has a $(p, 1)$-deep vertex-coloring using $g(k)$ colors.*

Proof. This results from the previous lemma, applied to all the possible color sequences and all $a \leq p$. □

It remains to prove the inductive steps *iv.* and *v.*:

Lemma 2.6. *Let $a > b \geq 0$ be integers. If $\vec{G}$ has a (a, b)-deep vertex-coloring (with α colors) and a (a, b)-deep edge-coloring (with β colors), it has a $(a, b+1)$ edge-coloring with $\alpha\beta$ colors.*

Proof. Color each edge $e = (u, v)$ by the pair $(c_e(e), c_v(v))$, where c_e is a (a, b)-deep edge-coloring with β colors and c_v a (a, b)-deep vertex-coloring with α colors . Consider any (k, l)-path $\vec{P} = (u_1, \dots, u_k, u_{k+1} = v_{l+1}, v_l, \dots, v_1)$ such that $k \leq a$ and $l \leq b$. If $l < b$, the result follows from the fact c_e is (a, b)-deep. If $l = b$, we get that all the vertices and edges in the subpath $\vec{P}' = (u_1, \dots, u_{k+1} = v_{b+1}, \dots, v_2)$ have different colors. Hence, no edge (x, y) in P' is such that $c_v(y) = c_v(v_2)$ and thus no edge in $\vec{P}'$ has the same color as (v_1, v_2). □

Definition 2.3. Let $\vec{G}$ be a directed graph and let $e = (u, v)$. The *semi-contraction* of e consists in removing all the incoming edges of v but e, and then contracting e. The resulting graph is noted $\vec{G} \oslash e$.

If A is a set of pairwise non-adjacent edges, then the *semi-contraction* of an edge in A does not remove an edge in A and does not make two edges in A adjacent. Thus, all the edges in A may be successively *semi-contracted* (in any order, according to the commutativity of contractions and deletions). The resulting graph is noted $\vec{G} \oslash A$.

Remark 2.7. Obviously, if $\vec{G}$ has indegrees bounded by k, so is $\vec{G} \oslash e$; if the orientation of $\vec{G}$ is acyclic, so is the one of $\vec{G} \oslash e$.

Lemma 2.8. *Let $\vec{G}$ be a directed graph with a (a, b)-depth edge-coloring ($b \geq 2$) and let c be any edge color. Let $R(c)$ be the set of all the edges e of color c, such that there exists some path $\vec{P}_e$ with signature (a, b) ending with e.*

Then, in the graph $\vec{G} \oslash R(c)$ obtained from $\vec{G}$ by semi-contracting all the edges in $R(c)$, any (a, b)-path $\vec{P}$ ending with an edge of color c is mapped into an $(a, b-1)$-path.

Proof. First remark that no two edges in $R(c)$ may be adjacent has they would form either $(1,1)$-path or $(2,0)$-path and thus should not have the same color.

Consider two edge $e \neq e' \in R(c)$ and two (a,b)-paths $\vec{P}, \vec{P}'$ ending respectively with e and e'. Let $\vec{P} = (u_1, \dots, u_{a+1} = v_{b+1}, \dots, v_1)$ and $e' = (x, y)$.

Then, the edge e' is not incoming to an internal vertex of $\vec{P}$: if $y = u_i$ (for some $2 \leq i \leq a$), then the path $(x, y, u_{i+1}, \dots, u_{a+1} = v_{b+1}, \dots, v_1)$ is an $(a-i+2, b)$-path although it includes two edges of color c, a contradiction; if $y = v_i$ (for some $2 \leq i \leq b+1$), the path $(v_1, \dots, v_i = y, x)$ is an $(i-1, 1)$-path although it includes two edges of color c, a contradiction.

Thus, the semi-contraction of e' does not affect $\vec{P}$ and $\vec{P}$ remains unaltered by the semi-contraction of $R(e) - e$. The semi-contraction of e then transforms $\vec{P}$ into an $(a, b-1)$-path (as e is not incident to the sink of $\vec{P}$). □

Finally we obtain v.:

Lemma 2.9. *Let $\mathcal{C}$ be a proper minor closed class of graphs with density bounded by $k/2$. Let $a > b \geq 1$ be integers. Assume any graph $G \in \mathcal{C}$ with acyclic orientation $\vec{G}$ with indegree bounded by k has a (a,b)-deep vertex-coloring (with α colors) and a $(a, b+1)$-deep edge-coloring (with β colors). Then, any graph $G \in \mathcal{C}$ with acyclic orientation $\vec{G}$ with indegree bounded by k has a $(a, b+1)$ vertex-coloring with $\alpha^{\beta+1}$ colors.*

Proof. Consider any acyclic orientation $\vec{G}$ of G with indegree bounded by k. Any of the graphs $\vec{G} \oslash R(c)$ belongs to $\mathcal{C}$, is acyclicly oriented with indegree bounded by k. Take the product of all the vertex-coloring induced by (a,b)-deep vertex-coloration of $\vec{G} \oslash R(c)$ when c ranges over all the colors of a $(a, b+1)$-deep edge-coloration with a (a,b)-deep vertex-coloring of $\vec{G}$. Then, the obtained vertex coloring is clearly at least (a,b)-deep. Moreover, for any path $\vec{P}$ with signature $(a, b+1)$ with last edge of color c, $\vec{P}$ becomes a path with signature (a,b) in $\vec{G} \oslash R(c)$. Thus, the only two vertices which may have the same color in $\vec{P}$ are those which are identified in $\vec{G} \oslash R(c)$. As these are adjacent and hence colored differently in the (a,b)-deep vertex-coloring of $\vec{G}$, $\vec{P}$ is rainbow. □

Lemma 2.10. *Let $\vec{G}$ be a directed graph. If $\vec{G}$ has a $(2,0)$-deep vertex coloring with k colors, then there exists an orientation $\vec{K}_N$ of K_N (with $N = k2^{k-1}$), such that $\vec{G}$ has a 1-folding to $\vec{K}_N$.*

Proof. For any $1 \leq i \neq j \leq k$, there exists no monotone path of length 2 with vertices colored i, j. Thus, for any vertex v of color i, either v has an incoming edge from a vertex colored j and no outgoing edge to a vertex colored j, or it has no incoming edge from a vertex colored j. Let $\mu_{i,j}$ be the mapping from the subset V_i of $V(\vec{G})$ with vertices colored i to $\{0,1\}$ be defined as follows: $\mu_{i,j}(v) = 0$ if v has no incoming edge from a vertex colored j, and $\mu_{i,j}(v) = 1$ otherwise. Then, for any vertex v with color i, the k-uple $(i, \mu_{i,1}(v), \dots, \mu_{i,i-1}(v), \mu_{i,i+1}(v), \dots, \mu_{i,k}(v))$ clearly defines a 1-folding to K_N. □

Lemma 2.11. *Let $\vec{G}$ be a directed graph. If $\vec{G}$ has a 1-folding to $\vec{K}_N$ and a (d,d)-deep vertex coloring with k colors, then it has a d-folding to some orientation of K_{kN}.*

Proof. Clearly, the product of the coloring will work. □

As a consequence of all this we obtain Theorem 1.1 in the following form:

Theorem 2.12. *Let $\mathcal{C}$ be a proper minor closed class of graphs, let k be such that each graph is $\mathcal{C}$ has density at most $k/2$ and let $a \geq b \geq 0$ be integers ($a > 0$). Then any graph $\vec{G} \in \mathcal{C}$ with an acyclic orientation with indegree at most k has a (a,b)-deep vertex-coloring using $F_{a,b}(k)$ colors.*

3. K_k-free Colorings

Theorem 3.1. *Let a directed graph $\vec{G}$ have a 1-folding to $\vec{K}_N$ inducing coloring which is both a $(3,0)$- and $(2,1)$-deep vertex coloring. Then, for any subgraph H in G which is colored by p colors, either $\chi(H) < p$ or H includes a clique of size p.*

Proof. We proceede by induction on p. Assume the statement is proved for $p-1$. Let $\vec{G}$ be a given graph with stated 1-folding into $\vec{K}_N$. Consider the homomorphic image of $\vec{G}$ as a subgraph $\vec{K}'$ of $\vec{K}_N$ and consider the subgraph $\vec{K}''$ of $\vec{K}'$ induced by any p colors. Then, we can assume without loss of generality that $\vec{K}''$ contains a monotone path with p vertices (for otherwise $\chi(K'') \leq p-1$ and we are done). We number the colors according to this monotone path from 1 to p for convenience. Consider now any subgraph $\vec{H}$ of $\vec{G}$ with all its vertices colored by $1,\ldots,p$. If the subgraph of $\vec{H}$ induced by vertices with colors $1,\ldots,p-1$ has no clique of size $p-1$, then it has chromatic number at most $p-2$. Hence, in this case we have $\chi(\vec{H}) \leq p-1$. Otherwise, we may assume $\vec{H}$ is p -color critical. Let $\vec{K}$ be a clique of size $p-1$ of this subgraph. Then, as $\vec{G}$ is acyclic oriented, so is $\vec{K}$. According to the existence of a monotone path $1,2,\ldots,p-1$ the orientation of $\vec{K}$ follows. Let u be the vertex with color $p-1$. Thus u has an incoming edge from the vertices of $\vec{K}$ colored $1,\ldots,p-2$. As the degree of u in $\vec{H}$ is at least $p-1$ (because $\vec{H}$ is p-color-critical), and as u may not have other incoming edges from vertices colored $1,...,p-2$ other than those in K (here we use $(2,1)$-deep vertex coloring) and as the edge of $\vec{K}'$ linking $p-1$ to p is oriented from $p-1$ to p (this is the last edge of the monotone path), $\vec{H}$ contains an edge (u,v) with the vertex v is colored by color p. As our coloring is also a $(3,0)$-deep vertex coloring, v may not have an outgoing edge (for otherwise we would get an oriented cycle of form $i,p-1,p,i$) and, again using $(2,1)$-deep vertex coloring, the vertex v may not have incoming edges from vertices outside of $\vec{K}$. Thus, as the degree of v is at least $p-1$ (as $\vec{H}$ is p-color-critical), the vertex v is adjacent to all the vertices of the clique $\vec{K}$ and thus $\vec{K}$ together with the vertex u form a clique of size p. □

As a corollary, we obtain the following:

Corollary 3.2. *Let $\vec{G}$ have a 1-folding to $\vec{K}_N$ inducing a $(3,0)$- and $(2,1)$-deep vertex coloring. Then, for any K_k-free subgraph H in G geting p colors, we have $\chi(H) \leq \lfloor p\frac{k-1}{k} \rfloor$.*

Proof. Partition the p colors in $\lfloor \frac{p}{k} \rfloor$ classes of size p and one with the remaining $(p-k.\lfloor \frac{p}{k} \rfloor)$ elements (if any). Then, any of the classes of size p induces a graph of chromatic number at most $p-1$. The chromatic number of the whole graph has then the requested bound. □

4. K_k-free Bounds

It follows from the main result (Theorem 5) of [8] that our Theorem 3.1 implies the existence of a K_k-free bound. (Note that Theorem 5 of [8] is more general as it deals with *locally U-colorable graphs*; but presently we do not have analogy of Theorem 3.1 in this more general case.) For the sake of completeness of this paper we include a direct proof of our particular case.

Let $N, k, N > k > 1$ be positive integers. We denote by $[N]$ the set $\{1, 2, \ldots, N\}$ and by $[N]^k$ the set of all k-element subsets of $[N]$. We define the graph H_k^N as follows: The vertices $V(H_k^N)$ are all pairs (i, c) where $i \in [N]$ and c is a coloring of $[N]^k$ by $k-1$ colors (i.e. $c : [N]^k \to [k-1]$). The edges of H_k^N are pairs $\{(i, c), (i', c')\}$ where $i \neq i'$ and every set $S \in [N]^k$ which contains both i and i' gets different colors in c and c' (i.e. $c(S) \neq c'(S)$).

It is clear that for any choice of N and k H_k^N does not contain K_k (for if we would denote its vertices by $(i_1, c_1), (i_2, c_2), \ldots, (i_k, c_k)$ then the k-set $S = \{i_1, i_2, \ldots, i_k\}$ would have to get k distinct colors contrary to our assumption.) On the other hand given a K_k-free graph G with a coloring $f : G \to K_N$ as in Theorem 3.1 we can define a homomorphism $g : G \to H_k^N$ by $g(v) = (f(v), c)$ where for any k-tuple S of colors we consider a $k-1$-coloring c_S of the subgraph G_S onduced by the vertices of colors from S and we define $c(S) = c_S(v)$ (with $c(S)$ beeing arbitrary if $f(v) \notin S$). One can check easily that this mapping g is indeed a homomorphism $G \to H_k^N$.

This together with Theorem 3.1 finishes the proof of Theorem 1.2.

5. Remarks

1. The acyclic chromatic number $\chi_a(G)$ is a minimal number of colors needed to color vertices of G such that the subgraph induced by any two colors is a forest. d-folding is a structural strengthening of this notion. It has been proved in [10] that there is a function f such that any graph G with $\chi_a(G) \leq k$ has a 1-folding which uses at most $f(k)$ colors. This result is best possible as for every k there exists a graph G_k with acyclic chromatic number 3 for which any 2-folding uses at least k colors. One such example we can obtain as follows:

Denote by K the complete graph K_N of order N, with all the edges subdivided twice. Such a graph has acyclic chromatic number 3 (all the original vertices are colored 1 and the two vertices on an edge are colored 2 and 3). Consider any orientation of K and assume there exists a 2-folding with x colors. Let A be the original set of vertices K_N. If there is a sink in the subdivision vertices between two vertices a, b in A, then a and b have different colors. Otherwise, at least one of a and b gets an incoming edge from the subdivided vertices. As any vertex has at most $x-1$ incoming edges (because of the folding), there are at most $(x-1)N$ subdivided edges having a source. Hence, the number of colors in A is at least equal to the minimum chromatic number of a graph on N vertices with $((N-1)/2 - (x-1)N$ edges. But, by Turan's theorem, N is at most equal to $2x(x-1)+1$.

Thus, for any x, the clique of order $2x(x-1)+1$ with edges subdivided twice has acyclic chromatic number 3 and 2-folding "number" is at least x.

2. Our Conjecture 1 may be seen as a finitary approximation to Hadwiger conjecture, see e.g. [7]. In our language Hadwiger conjecture may be expressed as follows:

Conjecture 2. (Hadwiger)

Any minor closed class $\mathcal{K}$ with bounded chromatic number has greatest element which is a complete graph.

In the other words, if a minor closed class $\mathcal{K}$ is bounded (by a finite graph, for example by a large complete graph) then it has the greatest element which is a complete graph. In this context one may see our Conjecture 1 as an approximation to Hadwiger conjecture: instead of asking for the greatest element of class $\mathcal{K}$ we ask for a bound with local properties similar to those in $\mathcal{K}$ (such as not containing a given complete graph). On the other hand the following naturally arises as a weaker form of Hadwiger conjecture:

Conjecture 3. Any minor closed class $\mathcal{K}$ with bounded chromatic number has greatest element.

In our setting the Hadwiger conjecture is not an isolated statement. For example the following problem is a weakening of it in another direction. We know that the coloring (or homomorphism) order $\leq$ is a quasiorder (which may be turned into a partial order by a suitable canonical factorization). Given a class $\mathcal{K}$ we say that a graph H is *supremum of* $\mathcal{K}$ if

i. $G \leq H$ for any $G \in \mathcal{K}$ (i.e. H is a bound for $\mathcal{K}$);

ii. If $G < H$ then there exists $G' \in \mathcal{K}$ such that $G \leq G' \leq H$ (i.e. H is the smallest bound of $\mathcal{K}$).

It follows from the *density theorem* [13] that for any positive integer k the class of all K_k-free k-colorable graphs has supremum K_k (and this class fails to contain a graph which is its greatest element). The following conjectures that this density phenomena cannot happen for geometrically restricted classes of graphs:

Conjecture 4. If H is a supremum of proper minor closed class $\mathcal{K}$ then $H \in \mathcal{K}$ (i.e. the supremum of $\mathcal{K}$, if it exists, is also the greatest element of $\mathcal{K}$).

3. The following is a consequence of various results of this paper and of [10]. It shows the diversity and fine interplay of very different and seemingly unrelated coloring characteristics when restricted to minor closed classes.

Theorem 5.1. *(Characterization of proper minor closed classes)*

Let $\mathcal{C}$ be a a minor closed class of graphs. The following statements are equivalent

i. the acyclic chromatic number $\chi_a(G)$ is bounded for $G \in \mathcal{C}$;

ii. the oriented chromatic number $\vec{\chi}(G)$ is bounded for $G \in \mathcal{C}$, see [11]*;*

iii. the star chromatic number $\chi_{st}(G)$ is bounded for $G \in \mathcal{C}$, see [10]*;*

iv. colored mixed graphs in $\mathcal{C}$ may be colored by a fixed number of colors (in the sense of [1, 12]*);*

v. the chromatic number $\chi(G)$ is bounded for all $G \in \mathcal{C}$;

vi. the clique number $\omega(G)$ is bounded for all $G \in \mathcal{C}$;

vii. the edge density of all graphs G is bounded for all $G \in \mathcal{C}$.

vii. $\mathcal{C}$ is proper *minor closed class of graphs (i.e. $\mathcal{C}$ is not the class of all graphs).*

viii. For any positive integer k the class $\mathcal{K} \cap Forb_{\mathrm{h}}(\mathcal{K}_k)$ is bounded by a K_k-free graph;

ix. There exists $k \geq 3$ such that the class $\mathcal{K} \cap Forb_{\mathrm{h}}(\mathcal{F})$ is bounded by a K_k-free graph.

We decided to stop here.

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