Rainbow Arithmetic Progressions and Anti-Ramsey Results

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Abstract

The van der Waerden theorem in Ramsey theory states that for every k and t and sufficiently large N, every k-coloring of $[N]$ contains a monochromatic arithmetic progression of length t. Motivated by this result, Radoičić conjectured that every equinumerous 3-coloring of $[3n]$ contains a 3-term rainbow arithmetic progression, i.e., an arithmetic progression whose terms are colored with distinct colors. In this paper, we prove that every 3-coloring of the set of natural numbers for which each color class has density more than $1/6$, contains a 3-term rainbow arithmetic progression. We also prove similar results for colorings of $\mathbb{Z}_n$. Finally, we give a general perspective on other *anti-Ramsey-type* problems that can be considered.

1 Introduction

In 1916, Schur [24] proved that for every k, if n is sufficiently large, then every k-coloring of $[n] := \{1, \ldots, n\}$ contains a monochromatic solution of the equation $x + y = z$. More than seven decades later, Alekseev and Savchev [1] considered what Bill Sands calls an *un-Schur* problem [13]. They proved that for every equinumerous 3-coloring of $[3n]$ (i.e., a coloring in which different color classes have the same cardinality), equation $x + y = z$ has a solution with x, y and z belonging to different color classes. Such solutions will be called *rainbow* solutions. E. and G. Szekeres asked whether the condition of equal cardinalities for three color classes can be weakened [26]. Indeed, Schönheim [23] proved that for every 3-coloring of $[n]$, such that every color class has cardinality greater than $n/4$, equation $x + y = z$ has rainbow solutions. Moreover, he showed that $n/4$ is optimal.

Inspired by the problem above, Radoičić posed the following conjecture at the open problem session of the MIT Combinatorics Seminar.

Conjecture 1 *For every equinumerous* 3*-coloring of* $[3n]$ *there exists a rainbow* $AP(3)$*, i.e., a solution to the equation* $x + y = 2z$ *in which* x*,* y*, and* z *are colored with three different colors.*

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This conjecture can be considered as the counterpart of van der Waerden's theorem in Ramsey theory. Van der Waerden's theorem states that for every k and t, if N is sufficiently large, then every k-coloring of $[N]$ contains a monochromatic t-term arithmetic progression.

Backed by the computer evidence ($n \leq 56$), we pose the following stronger form of Conjecture 1.

Conjecture 2 *For every $n \geq 3$, every partition of $[n]$ into three color classes R, G, and B with $\min(|R|,|G|,|B|) > r(n)$, where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

Unable to settle the above conjectures, in this paper we prove the following *infinite* version of Conjecture 2.

Theorem 1 *Every 3-coloring of the set of natural numbers $\mathbb{N}$ with the upper density of each color greater than $1/6$ contains a rainbow $AP(3)$.*

A more precise statement of the above theorem and its proof will be presented in Section 2. We also show that there exist a 3-coloring of $[n]$ with $\min(|R|,|G|,|B|) = r(n)$, where r is the function defined in (1), that contains no rainbow $AP(3)$. This shows that Conjecture 2, if true, is the best possible.

An interesting corollary of Theorem 1 is the *modular* version of Conjecture 2, which states that if $\mathbb{Z}_n$ is colored with 3 colors such that the size of every color class is greater than $n/6$, then there exist x, y and z, each of a different color with $x+y \equiv 2z \pmod n$. It turns out that in this case $n/6$ is not the best possible. We will discuss further generalizations of the modular case of Conjecture 2 in Section 3.

Previous work regarding the existence of rainbow structures in a colored universe has been done in the context of *canonical* Ramsey theory (see [8, 7, 6, 20, 19, 15, 16, 17, 21] and references therein). However, the canonical theorems prove the existence of *either* a monochromatic structure *or* a rainbow structure. Our results are not "either-or" statements and, thus, are the first results in literature guaranteeing the sole existence of rainbow arithmetic progressions. In a sense, the conjectures and theorems above can be thought of as the first rainbow counterparts of classical theorems in Ramsey theory, such as van der Waerden's, Rado's and Szemerédi's theorems [12]. It is curious to note that anti-Ramsey problems have received great attention in the context of graph theory as well (see [9, 5, 2, 3, 22, 10, 4] and references therein).

In Section 4, we present a Rado-type theorem for colorings of $\mathbb{Z}_p$, using both classical and recent results from additive number theory. Finally, in Section 5, we give several open problems and a general perspective of various research problems in this area.

2 The infinite form of our conjecture

Assume $c : \mathbb{N} \mapsto \{R, G, B\}$ is a 3-coloring of the set of natural numbers with colors Red, Green, and Blue. We can also think of c as an infinite sequence of the elements of $\{R, G, B\}$. Let

$R_c(n)$ be the number of integers less than or equal to n that are colored red. In other words, $R_c(n) := |[n] \cap \{i : c(i) = R\}|$. $G_c(n)$ and $B_c(n)$ are defined similarly. A rainbow $AP(3)$ is a sequence a_1, a_2, a_3 such that $a_1 + a_3 = 2a_2$ and $c(a_i) \neq c(a_j)$ for every $i \neq j$. We say that c is *rainbow-free*, if it does not contain any rainbow $AP(3)$.

Theorem 1 *Let c be a 3-coloring of $\mathbb{N}$ such that*

$$\limsup_{n\to\infty} \left(\min(R_c(n), G_c(n), B_c(n)) - n/6\right) = +\infty. \tag{2}$$

Then c contains a rainbow $AP(3)$.

Before proving Theorem 1 we define a few terms. We say that a string $s \in \{R, G, B, ?\}^k$ appears in c, if there exists an i such that for every $j = 1, \ldots, k$, either $s_j = c(i + j)$ or $s_j =?$. In this case, s appears in c at position i. For $x, y, z \in \{R, G, B\}, i_1, i_2 \in \mathbb{N}$ such that $\{x, y, z\} = \{R, G, B\}$ and $i_1 < i_2 - 1$, we say that c has a *color-change* of type xyz at positions (i_1, i_2), if $c(i_1) = x$, $c(i_2) = z$, and $c(j) = y$ for every $i_1 < j < i_2$.

Lemma 1 *Let c be a rainbow-free 3-coloring of $\mathbb{N}$. If there is a color-change of type xyz at position (i_1, i_2) for some $1 < i_1 < i_2$, then $c(i_1 - 1) = c(i_2 + 1) = y$.*

Proof: If $c(i_1 - 1) = z$, then $i_1 - 1, i_1, i_1 + 1$ is a rainbow $AP(3)$. Therefore, $c(i_1 - 1)$ is either y or x. Assume $c(i_1 - 1) = x$. One of the numbers $i_1 - 1$ and i_1 has the same parity as i_2. Let i'_1 denote this number. It is easy to see that $i'_1, (i'_1 + i_2)/2, i_2$ is a rainbow $AP(3)$. This contradiction show that $c(i_1 - 1) = y$. Similarly, $c(i_2 + 1) = y$. □

Corollary 1 *Let c be a rainbow-free 3-coloring of $\mathbb{N}$. If there is a color-change of type xyz at position (i_1, i_2) for some $1 < i_1 < i_2$, then both $yxyy?y$ and $y?yyzy$ appear in c at positions $i_1 - 1$ and $i_2 - 4$.*

Proof: It suffices to note that if c has a color-change at position (i_1, i_2), then $i_1 - i_2$ is odd, for otherwise $i_1, (i_1 + i_2)/2, i_2$ is a rainbow $AP(3)$. This, together with Lemma 1 imply that if there is a color-change of type xyz at position (i_1, i_2), then $c(i_1 + 4) = c(i_2 - 4) = y$. □

Lemma 2 *Every 3-coloring of $\mathbb{N}$ that contains both a color-change of type xyz and a color-change of type xzy contains a rainbow $AP(3)$.*

Proof: Assume c is a 3-coloring of $\mathbb{N}$ that contains a color-change of type xyz at position (i_1, i_2) and a color-change of type xzy at position (i'_1, i'_2). By Corollary 1, c contains $yxyy?y$ and $zxzz?z$ at positions $i_1 - 1$ and $i'_1 - 1$. Consider the following two cases:

- $i_1 \equiv i'_1 \pmod 2$: In this case, consider one of the following arithmetic progressions based on the value of $c((i_1 + i'_1 + 2)/2)$:

$$\begin{array}{rccl} i_1 + 1, & (i_1 + i'_1 + 2)/2, & i'_1 + 1 & \quad \text{if } c((i_1 + i'_1 + 2)/2) = x \\ i_1, & (i_1 + i'_1 + 2)/2, & i'_1 + 2 & \quad \text{if } c((i_1 + i'_1 + 2)/2) = y \\ i_1 + 2, & (i_1 + i'_1 + 2)/2, & i'_1 & \quad \text{if } c((i_1 + i'_1 + 2)/2) = z \end{array}$$

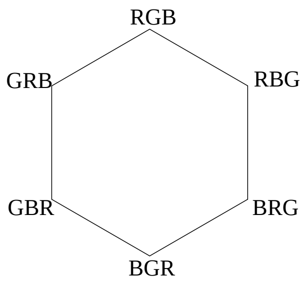

Figure 1: Different types of color-changes

- $i_1 \not\equiv i'_1 \pmod 2$: In this case, consider one of the following arithmetic progressions based on the value of $c((i_1 + i'_1 + 1)/2)$:

$$\begin{array}{cccl} i_1 - 1, & (i_1 + i'_1 + 1)/2, & i'_1 + 2 & \quad \text{if } c((i_1 + i'_1 + 1)/2) = x \\ i_1, & (i_1 + i'_1 + 1)/2, & i'_1 + 1 & \quad \text{if } c((i_1 + i'_1 + 1)/2) = y \\ i_1 + 1, & (i_1 + i'_1 + 1)/2, & i'_1 & \quad \text{if } c((i_1 + i'_1 + 1)/2) = z \end{array}$$

It is easy to see that in each case the arithmetic progression that we considered is a rainbow arithmetic progression. □

Similarly, we can prove that a rainbow-free 3-coloring of $\mathbb{N}$ cannot contain color-changes of type xyz and yxz at the same time. Therefore, we get the following corollary.

Corollary 2 *Let c be a rainbow-free 3-coloring of $\mathbb{N}$. Then for every two types of color-changes that are connected in Figure 1 by an edge, c cannot contain both of them at the same time.*

The following lemma shows an important property of rainbow-free 3-colorings of $\mathbb{N}$. Note that we don't need any assumption about the density of colors here. In fact, it is possible to prove the conclusion of this lemma even without the assumption that each color is used infinitely many times.

Lemma 3 *Let c be a rainbow-free 3-coloring of $\mathbb{N}$. Assume each color is used for coloring infinitely many numbers in c. Then there are two distinct colors $x, y \in \{R, G, B\}$ that never appear next to each other in c.*

Proof: Assume, for contradiction, that every two distinct colors appear next to each other somewhere in c. In other words, for any two distinct colors x and y, there is an i such that one of i and $i+1$ is colored with x and the other is colored with y. Consider the smallest number j greater than i that is colored with the third color, z. Such a number exists, since by assumption each color is used infinitely often in c. There must be a color-change of type xyz or yxz at position (j', j), for some $j' < j$. This shows that for every three distinct colors $x, y, z \in \{R, G, B\}$, either a color-change of type xyz, or a color-change of type yxz must appear in c. A similar argument shows that either a color-change of type xyz or a color-change of type xzy must appear in c. This together with Corollary 2 imply that for every two types of color-changes that are connected in Figure 1 by an edge, c contain exactly one of them. Therefore, either c contains color-changes of

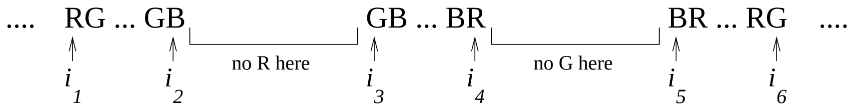

Figure 2: Lemma 3

types RGB, BRG, and GBR, and no color-change of type RBG, BGR, or GRB, or vice versa. We assume, without loss of generality, that c contains color-changes of types RGB, BRG, and GBR, and does not contain any color-change of type RBG, BGR, or GRB.

Consider a color-change of type RGB at position (i_1, i_2), let i_4 be the smallest number greater than i_2 that is colored red, and i_6 be the smallest number greater than i_4 that is colored green. Since c does not contain any color-change of type BGR or RBG, there must be a color-change of type GBR at position (i_3, i_4) for some $i_2 < i_3 < i_4$, and a color-change of type BRG at position (i_5, i_6) for some $i_4 < i_5 < i_6$. Notice that all numbers between i_2 and i_3 are colored blue or green, and all numbers between i_4 and i_5 are colored blue or red. (See Figure 2). One important observation is that R and G do not appear next to each other after i_1 and before i_6.

By Corollary 1, c contains $G?GGBG$ and $RBRR?R$ at positions $i_2 - 4$ and $i_5 - 1$. We consider two cases based on the parity of $i_2 + i_5$.

- $i_2 + i_5$ **is odd:** Consider the number $(i_2 + i_5 - 1)/2$. In c, this number cannot be colored red, for otherwise we have a rainbow $AP(3)$: $i_2 - 1, (i_2 + i_5 - 1)/2, i_5$. Also, it cannot be colored blue because of the arithmetic progression $i_2 - 2, (i_2 + i_5 - 1)/2, i_5 + 1$. Therefore, $c((i_2 + i_5 - 1)/2) = G$. Similarly, the arithmetic progressions $i_2, (i_2 + i_5 + 1)/2, i_5 + 1$ and $i_2 - 1, (i_2 + i_5 + 1)/2, i_5 + 2$ show that $c((i_2 + i_5 + 1)/2) = R$. But this is in contradiction with the observation that G and R never appear next to each other between i_1 and i_6.

- $i_2 + i_5$ **is even:** Considering the arithmetic progressions $i_2 - 2, (i_2 + i_5 - 2)/2, i_5$ and $i_2 - 1, (i_2 + i_5 - 2)/2, i_5 - 1$ shows that $c((i_2 + i_5 - 2)/2) = G$. Also, $c((i_2 + i_5 + 2)/2) = R$ because of the arithmetic progressions $i_2, (i_2 + i_5 + 2)/2, i_5 + 2$ and $i_2 + 1, (i_2 + i_5 + 2)/2, i_5 + 1$. Since G and R never appear next to each other between i_1 and i_6, $(i_2 + i_5)/2$ cannot be colored with green or red. Therefore, it is colored with blue. Thus, $(i_2 + i_5 - 2)/2), (i_2 + i_5)/2, (i_2 + i_5 + 2)/2$ is a rainbow $AP(3)$, which is a contradiction.

Therefore, the assumption that every two distinct colors appear next to each other leads to a contradiction in both cases. □

Lemma 3 shows that for any rainbow-free 3-coloring, there is a color z, such that for every two consecutive numbers that are colored with different colors, at least one of them is colored with z. We call such a color a *dominant color*. In the rest of this proof, we assume, without loss of generality, that red is the dominant color. In other words, we will assume that B and G do not appear next to each other in c.

Lemma 4 *Let c be a 3-coloring of $\mathbb{N}$ and assume red is the dominant color in c. If there are infinitely many i's such that i and $i + 1$ are both colored blue, and infinitely many j's such that j and $j + 1$ are both colored green, then c contains a rainbow $AP(3)$.*

Proof: Assume, for contradiction, that c is a 3-coloring of $\mathbb{N}$ with no rainbow $AP(3)$ in which BB and GG appear infinitely many times, and R is the dominant color. Therefore, there is $i_1 < i_2 < i_3$, such that BB appears at positions i_1 and i_3, and GG appears at position i_2. Let j_1 be the largest number less than i_2 such that a BB appears at position j_1, and j_2 be the smallest number greater than i_2 such that a BB appears at position j_2. Let $k_1, k_1+1, \ldots, k_2$ be the longest sequence of consecutive numbers between j_1 and j_2 that are colored green (i.e., $j_1 < k_1 < k_2 < j_2$, $c(k) = G$ for every $k_1 \leq k \leq k_2$, and $k_2 - k_1 + 1$ is maximum). By the definition of j_1 and j_2, neither $j_1 + 2$ nor $j_2 - 1$ is colored blue. Therefore, since red is the dominant color, $c(j_1+2) = c(j_2-1) = R$. Consider one of the numbers j_1 or $j_1 + 1$ that has the same parity as $j_2 - 1$. The arithmetic progression consisting of this number, $j_2 - 1$, and their midpoint $\lfloor (j_1 + j_2)/2 \rfloor$ shows that $c(\lfloor (j_1+j_2)/2 \rfloor) \neq G$. Similarly, the red at $j_1 + 2$ and one of the blues at j_2 or $j_2 + 1$ imply that $c(\lceil (j_1 + j_2)/2 \rceil + 1) \neq G$. Therefore, since $k_1 < k_2$, we either have $k_2 < \lfloor (j_1 + j_2)/2 \rfloor$, or $k_1 > \lceil (j_1 + j_2)/2 \rceil + 1$.

Assume $k_2 < \lfloor (j_1 + j_2)/2 \rfloor$. For every i, $k_1 \leq i \leq k_2$, the arithmetic progressions $j_1, i, 2i - j_1$ and $j_1 + 1, i, 2i - j_1 - 1$ show that $2i - j_1 - 1$ and $2i - j_1$ are not colored red. Therefore, none of the numbers between $2k_1 - j_1 - 1$ and $2k_2 - j_1$ is red. This, together with the fact that red is the dominant color, imply that all of the numbers between $2k_1 - j_1 - 1$ and $2k_2 - j_1$ must be colored with the same color, either blue or green. If they are all blue, we get a contradiction with the definition of j_1 and j_2, as these definitions imply that no BB appears after j_1 and before j_2. If they are all green, we have a contradiction with the definition of k_1 and k_2, since by the assumption $k_2 < \lfloor (j_1 + j_2)/2 \rfloor$, the sequence $2k_1 - j_1 - 1, \ldots, 2k_2 - j_1$ is a sequence greens between j_1 and j_2 that is longer than the sequence $k_1, \ldots, k_2$.

Therefore, we get a contradiction in either case. A symmetric argument leads to a similar contradiction for the case $k_1 > \lceil (j_1 + j_2)/2 \rceil + 1$. □

Next we show that the density assumption (2) implies that the dominant color must appear in c with a high frequency. We start with the following simple lemma.

Lemma 5 *Let c be a 3-coloring of $\mathbb{N}$ that satisfies the density assumption (2). Then there is a $k \leq 5$ such that for every i, there exists $j > i$ such that j and $j + k$ are both colored green.*

Proof: Assume not, then there is an i such that every two numbers greater than i that are colored green are at least 6 apart. Therefore, $G_c(n) \leq n/6 + i$, which is a contradiction with (2). □

Lemma 6 *If c is a rainbow-free 3-coloring of $\mathbb{N}$ that satisfies the density assumption (2), and red is a dominant color in c, then there is n_0 such that for every $i > n_0$, either $c(i)$ or $c(i + 1)$ is red.*

Proof: By Lemma 4, the number of appearances of either BB or GG in c is finite. Assume, without loss of generality, that GG appears only a finite number of times in c. That is, there is an n_0 such that no GG appears in c after n_0. If no BB appears after n_0, then we are done. Otherwise, consider a BB at position $i > n_0$.

By Lemma 5, there exists $k \leq 5$ and $j > i$ such that j and $j + k$ are both colored green. The arithmetic progressions $i, j, 2j - i$ and $i + 1, j, 2j - i - 1$ imply that $2j - i - 1$ and $2j - i$ are not red. Therefore, since red is the dominant color, either they are both blue, or both green. The latter case is impossible, since $2j - i - 1 > n_0$. This shows that there is a BB at position $2j - i - 1$. Similarly,

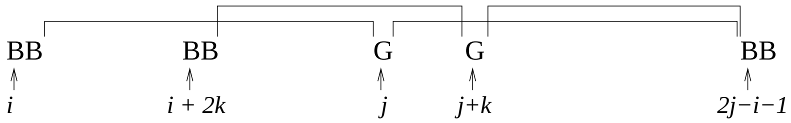

Figure 3: Lemma 6

having a BB at position $2j - i - 1$ and a G at position $j + k$ implies that there is another BB at position $i + 2k$. (See Figure 3).

Repeating the same argument, we conclude that BB appears at positions $i + 2kt$ for every integer $t \geq 0$. Using Lemma 1 it is not difficult to see that if there is a BB at position i_1, and a G at position $i_2 > i_1$, then $i_2 \geq i_1 + 6$. Similarly, if there is a BB at position i_1 and a G at position $i_2 < i_1$, then $i_2 \leq i_1 - 5$. Since $k \leq 5$, these facts imply that for every $t \geq 0$, none of the numbers between $i + 2kt$ and $i + 2k(t + 1)$ is colored green. Therefore, the number of greens is finite, which is a contradiction. □

Lemma 7 *If c be a rainbow-free 3-coloring of $\mathbb{N}$ that satisfies the density assumption (2), and red is a dominant color in c, then there is n_0 such that for every $i > n_0$, either $c(i)$ or $c(i + 2)$ is red.*

Proof: By Lemma 6, there is an n_0 such that for every $i > n_0$, either i or $i + 1$ is colored red. Assume, for contradiction, that there exists $i > n_0$ such that neither i nor $i + 2$ is colored red. By Lemma 6, $c(i + 1) = R$. Therefore, i and $i + 2$ are either both green, or both blue. Assume, without loss of generality, that they are both blue. Consider an arbitrary $l > i$ whose parity is the same as the parity of i. If l is colored green, then the arithmetic progressions $i, (i + l)/2, l$ and $i + 2, (i + l)/2 + 1, l$ show that neither $(i + l)/2$ nor $(i + l)/2 + 1$ is red, which is a contradiction with Lemma 6. Therefore, no $l > i$ with the same parity as i is colored green.

Now consider an arbitrary $i' \geq i$ that is colored blue and has the same parity as i. Using Lemma 5, there is $j > i'$ such that j and $j + k$ are both colored green (for a fixed $k \leq 5$). By the above argument, neither j nor $j + k$ has the same parity as i. Therefore, k is either 2 or 4. The arithmetic progression $i', j, 2j - i'$ shows that $2j - i'$ is not red. Also, since it has the same parity as i, it cannot be green. Therefore, $c(2j - i') = B$. Similarly, the arithmetic progression $i' + 2k, j + k, 2j - i'$ and the fact that $i' + 2k$ has the same parity as i show that $c(i' + 2k) = B$. This means that for every $i' > i$ with the same parity as i, if i' is colored blue, then so is $i' + 2k$. Thus, all numbers $i + 2kt$ and $i + 2kt + 2$ for $t \geq 0$ must be colored blue.

- If $k = 2$, this means that every number greater than i that has the same parity as i is colored blue. Therefore, by Lemma 3 no number greater than i is colored green, which is a contradiction.

- If $k = 4$, this means that for every integer $t \geq 0$, $i + 8t$, $i + 8t + 2$ and $i + 8t + 8$ are colored blue. Therefore, by Lemma 3, $i + 8t + 1$, $i + 8t + 3$, and $i + 8t + 7$ are not green. Also, $i + 8t + 4$ and $i + 8t + 6$ have the same parity as i and therefore cannot be colored green. Thus, the only numbers that can be colored green are in the form $i + 8t + 5$. Therefore, $G_c(n) \leq n_0 + \frac{1}{8}n$, which is a contradiction with (2).

$\square$

Now, we are ready to prove Theorem 1.

Proof of Theorem 1: Assume c does not contain any rainbow $AP(3)$. Therefore, by Lemma 3, there is a dominant color. Assume without loss of generality that the dominant color is red. By Lemmas 6 and 7 there is an n_0 such that for every $i > n_0$, at least two of the numbers $i, i+1, i+2$ are colored red. Therefore, for every n, $R_c(n) \geq \frac{2}{3}(n - n_0)$. Thus, $\min(G_c(n), B_c(n)) \leq \frac{1}{2}(n - \frac{2}{3}(n - n_0)) = \frac{1}{6}n + \frac{1}{3}n_0$, contradicting (2). $\square$

A natural question is whether the assumption (2) in Theorem 1 can be weakened. Notice that Conjecture 2 suggests that the conclusion of Theorem 1 is true with the weaker assumption that $\limsup_{n\to\infty}\left(\min(R_c(n), G_c(n), B_c(n)) - \frac{1}{6}n\right) > \frac{4}{6}$. We still haven't been able to prove this fact. However, the following proposition shows that the constant 1/6 in the density assumption cannot be substituted with a smaller constant.

Proposition 1 *There is a rainbow-free 3-coloring c of $\mathbb{N}$ such that for every n,*

$$\min(R_c(n), G_c(n), B_c(n)) = \lfloor (n+2)/6 \rfloor.$$

Proof: Consider the following coloring of $\mathbb{N}$:

$$c(i) := \begin{cases} B & \text{if } i \equiv 1 \pmod 6 \\ G & \text{if } i \equiv 4 \pmod 6 \\ R & \text{otherwise} \end{cases}$$

It is easy to see that c contains no rainbow $AP(3)$ and $\min(R_c(n), G_c(n), B_c(n)) = G_c(n) = \lfloor (n+2)/6 \rfloor$. $\square$

The following proposition shows that Conjecture 2, if true, is the best possible.

Proposition 2 *For every $n \geq 3$, there is a rainbow-free 3-coloring c of $[n]$ in which the size of the smallest color class is $r(n)$, where r is the function defined in (1).*

Proof: For $n \not\equiv 2 \pmod 6$, Proposition 1 gives such a coloring. Assume $n = 6k + 2$ for an integer k. We define a coloring c as follows:

$$c(i) := \begin{cases} B & \text{if } i \leq 2k+1 \text{ and } i \text{ is odd} \\ G & \text{if } i \geq 4k+2 \text{ and } i \text{ is even} \\ R & \text{otherwise} \end{cases}$$

Since every blue number is at most $2k+1$, and every green number is at least $4k+2$, a blue and a green number cannot be the first and the second, or the second and the third terms of an arithmetic progression with all terms in $[n]$. Also, since blue numbers are odd and green numbers are even, a blue and a green cannot be the first and the third terms of an arithmetic progression. Therefore, c does not contain any rainbow $AP(3)$. It is not difficult to see that c contains no rainbow $AP(3)$ and $\min(R_c(n), G_c(n), B_c(n)) = k + 1 = (n+4)/6$. $\square$

3 Rainbow arithmetic progressions in $\mathbb{Z}_n$

A 3-term arithmetic progression ($AP(3)$) in $\mathbb{Z}_n$ is a sequence a_1, a_2, a_3 such that $a_1 + a_3 \equiv 2a_2 \pmod{n}$. For a 3-coloring $c : \mathbb{Z}_n \mapsto \{R, G, B\}$ of $\mathbb{Z}_n$, we define $R_c := \{i : c(i) = R\}$. G_c and B_c are defined similarly. An interesting corollary of Theorem 1 is the following.

Theorem 2 *Every 3-coloring c of $\mathbb{Z}_n$ with $\min(|R_c|, |G_c|, |B_c|) > n/6$ contains a rainbow $AP(3)$.*

Proof: For a 3-coloring c of $\mathbb{Z}_n$, we define a 3-coloring $\bar{c}$ of $\mathbb{N}$ as follows: For every $i \in \mathbb{N}$, $\bar{c}(i) := c(i \bmod n)$. The assumption $\min(|R_c|, |G_c|, |B_c|) > n/6$ implies that

$$\limsup_{n\to\infty} (\min(R_{\bar{c}}(n), G_{\bar{c}}(n), B_{\bar{c}}(n)) - n/6) = +\infty.$$

Therefore, by Theorem 1, there is a rainbow $AP(3)$ in $\bar{c}$. By computing the terms of this arithmetic progression modulo n we obtain a rainbow $AP(3)$ in c. □

A natural question is whether the condition $\min(|R_c|, |G_c|, |B_c|) > n/6$ in Theorem 2 can be weakened. For n divisible by 6, the coloring defined in Proposition 1 shows that this condition is tight. However, for most other values of n it is possible to use number theoretic properties of $\mathbb{Z}_n$ to substitute this condition with a weaker assumption. The following theorem is an example.

Theorem 3 *Let n be an odd number and q be the smallest prime factor of n. Then every 3-coloring c of $\mathbb{Z}_n$ with $\min(|R_c|, |G_c|, |B_c|) > n/q$ contains a rainbow $AP(3)$.*

First, we prove the following lemma.

Lemma 8 *Let c be a 3-coloring of $\mathbb{Z}_n$, a be an integer relatively prime to n, and b be an arbitrary integer. Let $c'(i) := c((ai + b) \bmod n)$ for every $i \in \mathbb{Z}_n$. Then c contains a rainbow $AP(3)$ if and only if c' contains a rainbow $AP(3)$. Furthermore, $|R_{c'}| = |R_c|$, $|G_{c'}| = |G_c|$, and $|B_{c'}| = |B_c|$.*

Proof: It is enough to note that since a is relatively prime to n, the mapping $i \mapsto ai + b \pmod{n}$ is an automorphism of $(\mathbb{Z}_n, +)$. □

Proof of Theorem 3: Assume, for contradiction, that we have a 3-coloring c of $\mathbb{Z}_n$ with no rainbow $AP(3)$ such that $\min(|R_c|, |G_c|, |B_c|) > n/q$. Assume, without loss of generality, that $|G_c| = \min(|R_c|, |G_c|, |B_c|)$. Since $|G_c| > n/q$, there exist $k < q$ and i such that i and $i + k$ are both colored green. Since $k < q$ and q is the smallest prime factor of n, k is relatively prime to n. Therefore, Lemma 8 with $a = k$ and $b = i$ gives a coloring with the same properties as c in which 0 and 1 are both colored green. From now on, we let c denote this coloring. Therefore, c does not contain any rainbow $AP(3)$, and it satisfies $|G_c| = \min(|R_c|, |G_c|, |B_c|) > n/q$ and $c(0) = c(1) = G$.

From c, we construct a coloring $\bar{c}$ of $\mathbb{N}$ as in the proof of Theorem 2. Lemma 3 shows that there is a dominant color in $\bar{c}$. We consider the following two cases:

Case 1: G is the dominant color in $\bar{c}$. Since $\bar{c}$ is periodic, Lemma 4 implies that $\bar{c}$ cannot contain BB and RR at the same time. Assume, without loss of generality, that $\bar{c}$ does not contain any BB. This, together with the fact that G is the dominant color imply that in c every B is followed by a G (i.e., for every $i \in \mathbb{Z}_n$, if $c(i) = B$, then $c(i+1) = G$). Furthermore, since by Lemma 3 no R can be followed by a B in $\bar{c}$, there must be at least one R in c that is followed by a G. Thus, $|G_c| \geq |B_c| + 1$, contradicting the assumption that $|G_c| = \min(|R_c|, |G_c|, |B_c|)$.

Case 2: G is not the dominant color in $\bar{c}$. Without loss of generality, assume R is the dominant color in $\bar{c}$. In $\bar{c}$, GG appears at positions nt for every $t > 0$. Therefore, by Lemma 4 no BB appears in c. On the other hand, the assumption $|B_c| \geq n/q$ implies that there exist $k < q$ and i such that i and $i + k$ are both colored blue. Now, consider the arithmetic progressions $0, i, 2i$ and $1, i, 2i - 1$. These arithmetic progressions show that neither of $2i - 1$ and $2i$ can be red. Therefore, since c doesn't contain BB, they must be both green. Similarly, the arithmetic progressions $2k, i + k, 2i$ and $2k + 1, i + k, 2i - 1$ show that there is a GG at position $2k$. Repeating the same argument implies that there is a GG at position $2kt \pmod{n}$ for every $t \geq 0$. But since k is smaller than the smallest prime factor of n, and n is odd, $2k$ is relatively prime to n. Thus, we have proved that every number in $\{2kt \pmod{n} : t \geq 0\} = \mathbb{Z}_n$ is colored green, which is a contradiction. □

For any integer n, we define $m(n)$ as the largest integer m for which there is a rainbow-free 3-coloring c of $\mathbb{Z}_n$ such that $|R_c|, |G_c|, |B_c| \geq m$. Theorems 2 and 3 show that for every integer n, $m(n) \leq \min(n/6, n/q)$, where q is the smallest prime factor of n. Computing the exact value of $m(n)$ for every n remains a challenge. The following theorem gives a general lower bound for the value of $m(n)$.

Theorem 4 *For every integer n that is not a power of 2, if q denotes the smallest odd prime factor of n, then $m(n) \geq \lfloor \frac{n}{2q} \rfloor$.*

Proof: It suffices to show that there is a rainbow-free 3-coloring c of $\mathbb{Z}_n$ with $\min(|R_c|, |G_c|, |B_c|) \geq \lfloor \frac{n}{2q} \rfloor$. We know that exactly n/q elements of $\mathbb{Z}_n$ are divisible by q. Color $\lfloor \frac{n}{2q} \rfloor$ of these numbers with green and the remaining $\lceil \frac{n}{2q} \rceil$ multiples of q with blue. Color other elements of $\mathbb{Z}_n$ with red. Since q is odd, if two elements of a 3-term arithmetic progression are divisible by q, the third term should also be divisible by q. Therefore, the coloring c constructed above does not contain any rainbow $AP(3)$, and we have $\min(|R_c|, |G_c|, |B_c|) \geq \lfloor \frac{n}{2q} \rfloor$. □

In the following theorem we characterize the set of natural numbers n for which $m(n) = 0$.

Theorem 5 *For every integer n, there is a rainbow-free 3-coloring of $\mathbb{Z}_n$ with non-empty color classes if and only if n does* not *satisfy any of the following conditions:*

(a) n is a power of 2.

(b) n is a prime and $\mathrm{ord}_n(2) = n - 1$ (i.e., 2 is a generator of $\mathbb{Z}_n$).

(c) n is a prime, $\mathrm{ord}_n(2) = (n-1)/2$, and $(n-1)/2$ is an odd number.

Proof: We first prove the **if** part. We need to prove that for every n that does not satisfy any of the above conditions, there is a rainbow-free coloring of $\mathbb{Z}_n$ with no empty color class. We consider the following two cases: n is not prime, and n is prime.

If n is not a prime number, then by conditions above n can be written as $n = pq$ where p is an odd number and $q > 1$. Let c denote the coloring of $\mathbb{Z}_p$ obtained by coloring 0 with red, other multiples of p with green, and other numbers with blue. In this coloring, every rainbow $AP(3)$ must contain 0 and a multiple of p. Since p is odd, the other term in such an arithmetic progression must also be a multiple of p. Therefore, c is rainbow-free.

If n is a prime number, then we define the coloring c as follows: 0 is colored with red, all numbers in $\{2^i \bmod n : i \in \mathbb{Z}\} \cup \{-2^i \bmod n : i \in \mathbb{Z}\}$ are colored with green, and other numbers are colored with blue. By conditions (b) and (c) we know that either $\mathrm{ord}_n(2) < (n-1)/2$, or $\mathrm{ord}_n(2) = (n-1)/2 = 2k$ for an integer k. In the former case, $|G_c| \leq 2\mathrm{ord}_n(2) < n-1$. In the latter case, we have $2^k = -1$ and therefore $|G_c| = \mathrm{ord}_n(2) < n-1$. Thus, B_c is non-empty in either case. Also, every rainbow $AP(3)$ in c must contain 0. Since G_c is closed under multiplication/division by 2 and -1, any 3-term arithmetic progression that contains 0 and an element of G_c, must contain another element of G_c. Thus, c is rainbow-free.

For the **only if** part, we need to argue that if n satisfies any of the conditions (a), (b), or (c), then every coloring of $\mathbb{Z}_n$ with non-empty color classes contains a rainbow $AP(3)$. If n satisfies one of the conditions (b) or (c), then by Theorem 3 any coloring c of $\mathbb{Z}_n$ with $\min(|R_c|, |G_c|, |B_c|) > 1$ contains a rainbow $AP(3)$. If $\min(|R_c|, |G_c|, |B_c|) = 1$, then assume without loss of generality that 0 is the only number colored with red and 1 is colored with green. For every number $i \in \mathbb{Z}_n \setminus \{0\}$ that is colored green, $2i$ must also be green; otherwise $0, i, 2i$ will be a rainbow $AP(3)$. Similarly, if i is green, then $-i$ must also be green. This implies that every number in $\{2^i \bmod n : i \in \mathbb{Z}\} \cup \{-2^i \bmod n : i \in \mathbb{Z}\}$ must be colored green. However, if one of the conditions (b) or (c) hold, then $\{2^i \bmod n : i \in \mathbb{Z}\} \cup \{-2^i \bmod n : i \in \mathbb{Z}\} = \mathbb{Z}_n \setminus \{0\}$. This contradicts with the assumption that B_c is non-empty.

The only case that remains to check is when n satisfies (a), i.e., we need to prove that when $n = 2^k$ for an integer k, there is no rainbow-free coloring of $\mathbb{Z}_n$ with non-empty color classes. We prove this statement by induction on k. The induction basis is easy to verify. Assume this statement holds for $k-1$, and (for contradiction) consider a rainbow-free coloring c of $\mathbb{Z}_{2^k}$ with non-empty color classes.

We can partition $\mathbb{Z}_{2^k}$ into two sets: the set of even numbers $\mathbb{Z}_{2^k}^E = \{2i \bmod 2^k : i \in \mathbb{Z}_{2^k}\}$ and the set of odd numbers $\mathbb{Z}_{2^k}^O = \{2i+1 \bmod 2^k : i \in \mathbb{Z}_{2^k}\}$. It is clear that each of $\mathbb{Z}_{2^k}^E$ and $\mathbb{Z}_{2^k}^O$ is isomorphic to $\mathbb{Z}_{2^{k-1}}$. Therefore, by the induction hypothesis if c restricted to either one of them has non-empty color classes, then c will contain a rainbow $AP(3)$. Thus, we may assume without loss of generality that no element of $\mathbb{Z}_{2^k}^E$ is colored blue and no element of $\mathbb{Z}_{2^k}^O$ is colored green. Also, assume without loss of generality that $|G_c| \geq |B_c|$, and let $G_c = \{a_1, a_2, \ldots, a_{|G_c|}\} \subseteq \mathbb{Z}_{2^k}^E$. Now, consider an arbitrary $x \in \mathbb{Z}_{2^k}^O$ that is colored blue, and some i, $1 \leq i \leq |G_c|$. Since $2a_i - x \bmod 2^k$ belongs to $\mathbb{Z}_{2^k}^O$, it can not be green. Also, it can not be red, since otherwise $2a_i - x, a_i, x$ will be a rainbow $AP(3)$. Thus, for every i and every x that is blue, $2a_i - x$ is also blue. Starting from a fixed blue element x and using the above statement, we obtain that all the elements of $\{2a_i - x \bmod 2^k : 1 \leq i \leq |G_c|\} \cup \{2(a_i - a_1) + x \bmod 2^k : 1 \leq i \leq |G_c|\}$ are colored blue. We know that for distinct i, j, $2a_i - x \not\equiv 2a_j - x \pmod{2^k}$ and $2(a_i - a_1) + x \not\equiv 2(a_j - a_1) + x \pmod{2^k}$. Also, if for some i, j, $2a_i - x = 2(a_j - a_1) + x \pmod{2^k}$, then $2(a_j - a_1 - a_i + x)$ must be divisible by 2^k, which is impossible since $a_j - a_1 - a_i + x$ is an odd number. Thus, there are $2|G_c|$ distinct numbers in $\{2a_i - x \bmod 2^k : 1 \leq i \leq |G_c|\} \cup \{2(a_i - a_1) + x \bmod 2^k : 1 \leq i \leq |G_c|\}$ that are all colored blue. This shows that $|B_c| \geq 2|G_c|$, which is in contradiction with the assumption $|G_c| \geq |B_c| > 0$.
□

4 Additive number theory and rainbows in $\mathbb{Z}_p$

Strong inverse theorems from additive number theory have proved to be useful tools in Ramsey theory. For example, Gowers' proof of Szemerédi theorem relies on the theorem of Freiman [11]. Likewise, we will use a recent theorem of Hamidoune and Rødseth [14], generalizing the classical Vosper's theorem [28], to prove that *almost* every coloring of $\mathbb{Z}_p$ with three colors has rainbow solutions for *almost* all linear equivalence relations in three variables in $\mathbb{Z}_p$. Moreover, we classify all the exceptions.

We write p to denote a prime number and (m, n) to denote the greatest common divisor of m and n. For $a, b \in \mathbb{Z}_p$, we define sequence $\{i\}_a^b$ in $\mathbb{Z}_p$ as $\{i | i \in \mathbb{Z}_p, a \leq i \leq b\}$, if $a \leq b$, and $\{i | i \in \mathbb{Z}_p, a \leq i \leq p-1 \text{ or } 0 \leq i \leq b\}$, otherwise. For $X, Y \subset \mathbb{Z}_p$ and $j \in \mathbb{Z}_p$, let $jX = \{jx \mid x \in X\}$, $X - j = \{x - j \mid x \in X\}$ and $X + Y = \{x + y \mid x \in X, y \in Y\}$. We also define the distance function $D_p(k, l)$ in $\mathbb{Z}_p$ as the smallest nonnegative value of $|k - l + pj|$, over all $j \in \mathbb{Z}$. Hence, $D_p(k) := D_p(k, 0) = \min\{k, p - k\}$. Note that $D_p(k) \leq \frac{p}{2}$.

Theorem 6 *Let $a, b, c, e \in \mathbb{Z}_p$, with $abc \not\equiv 0 \pmod p$. Then every coloring of $\mathbb{Z}_p = R \cup B \cup G$ with $|R|, |B|, |G| \geq 4$, contains a rainbow solution of $ax + by + cz \equiv e \pmod p$ with the only exception being the case when $a = b = c$ and every color class is an arithmetic progression with the same common difference d, so that $d^{-1}R = \{i\}_{i=a_1}^{a_2-1}$, $d^{-1}B = \{i\}_{i=a_2}^{a_3-1}$ and $d^{-1}G = \{i\}_{i=a_3}^{a_1-1}$, where $(a_1 + a_2 + a_3) \equiv e + 1$ or $e + 2 \pmod p$.*

Before proving Theorem 6, we recall the classical theorem of Cauchy and Davenport [18] and the recent result of Hamidoune and Rødseth [14].

Theorem (Cauchy-Davenport) *If $S, T \subset \mathbb{Z}_p$, then $|S + T| \geq \min\{p, |S| + |T| - 1\}$.*

Theorem (Hamidoune-Rødseth) *Let $S, T \subset \mathbb{Z}_p$, $|S| \geq 3$, $|T| \geq 3$, $7 \leq |S + T| \leq p - 4$. Then either $|S + T| \geq |S| + |T| + 1$, or S and T are contained in arithmetic progressions with the same common difference and $|S| + 1$ and $|T| + 1$ elements respectively.*

We also need the following two lemmas.

Lemma 9 *If $S \subset \mathbb{Z}_p$ is contained in an arithmetic progression of length $|S| + 1$ with common difference d, then there are at most two pairs of elements of $\mathbb{Z}_p$ of the form $(x, x + d)$ such that $x \in S$ and $x + d \notin S$.*

Proof: Let $S \subset \{a + di\}_{i=0}^{|S|}$. Define $X = \{a + (i + 1)d \mid 0 \leq i \leq |S|, a + id \in S, a + (i + 1)d \notin S\}$. Then $X \cap S = \emptyset$ and $X \cup S \subset \{a + di\}_{i=0}^{|S|+1}$. Therefore, $|X| + |S| \leq |S| + 2$, and $|X| \leq 2$. Note that X is precisely the set of elements of the form $x + d$ such that $(x + d) \notin S$ and $x \in S$. $\square$

Lemma 10 *Let $p > 7$ and let $S \subset \mathbb{Z}_p$, $3 \leq |S| \leq p - 5$, be contained in an arithmetic progression of length $|S| + 1$ and common difference d, $(d, p) = 1$. Then every arithmetic progression of length $|S| + 1$ containing S has the common difference equal to d or $p - d$.*

Proof: Suppose that S is contained in an arithmetic progression of length $|S|+1$ and common difference $d' \neq d, p-d$. Applying the group isomorphism $\mathbb{Z}_p \to d^{-1}\mathbb{Z}_p$, we can assume that S is contained in the arithmetic progression $A := \{a+i\}_{i=0}^{|S|}$, as well as in the arithmetic progression $\bar{A}$ of length $|S|+1$ and common difference $\bar{d}$ $(= D_p(d^{-1}d'))$. We have the following three cases:

1. $2 \leq \bar{d} \leq 4$.
   View $\mathbb{Z}_p$ as a circle on p elements and consider the process of looping around the circle and removing the terms of $\bar{A}$ with respect to their order in $\bar{A}$. Let j be the smallest integer such that all the terms of $\bar{A}$ have been removed after j loops. Number of elements of A, removed after j loops is at most $\sum_{i=0}^{j-1}\lceil\frac{|S|+1-i}{\bar{d}}\rceil$. Since $S \subset A$, we have $|S| \leq \sum_{i=0}^{j-1}\lceil\frac{|S|+1-i}{\bar{d}}\rceil$. Number of elements x, $x \in \bar{A}$ and $x \notin A$, removed after j loops is at least $\sum_{i=0}^{j-2}\lfloor\frac{p-(|S|+1)+i}{\bar{d}}\rfloor$. Hence, to finish the proof by contradiction, it suffices to show that if j is the smallest integer with $|S| \leq \sum_{i=0}^{j-1}\lceil\frac{|S|+1-i}{\bar{d}}\rceil$, then $\sum_{i=0}^{j-2}\lfloor\frac{p-(|S|+1)+i}{\bar{d}}\rfloor > 1$.

   (a) $\bar{d} = 2$.
   $|S| \leq \lceil\frac{|S|+1}{2}\rceil$ does not hold and, thus, $j \geq 2$. Then $\lfloor\frac{p-(|S|+1)}{2}\rfloor \geq 2$.

   (b) $\bar{d} = 3$.
   $|S| \leq \lceil\frac{|S|+1}{3}\rceil + \lceil\frac{|S|}{3}\rceil$ only if $|S| \in \{3,4\}$. Hence, either $j=3$, or $j=2$ and $|S| \in \{3,4\}$. If $j=3$, then $\lfloor\frac{p-(|S|+1)}{3}\rfloor + \lfloor\frac{p-|S|}{3}\rfloor \geq 2$. If $j=2$, then $\lfloor\frac{p-(|S|+1)}{3}\rfloor \geq 2$, since $|S| \leq 4$ and $p > 7$.

   (c) $\bar{d} = 4$.
   If $j \leq 2$, then $|S| \leq \lceil\frac{|S|+1}{4}\rceil + \lceil\frac{|S|}{4}\rceil$. Hence, $|S| \leq 2$, which contradicts $|S| \geq 3$. Therefore, $j \geq 3$. Then $\lfloor\frac{p-(|S|+1)}{4}\rfloor + \lfloor\frac{p-|S|}{4}\rfloor \geq 2$, since $p - |S| > 4$.

2. $4 \leq \bar{d} \leq |A|$.
   Exactly 1 element of A is not in S. Since $\mathbb{Z}_p \setminus S$ has at least 5 elements, no element of $U := \{a+|S|+1+i\}_{i=0}^{3}$ is in S. Since every element of $U - \bar{d}$ is in A, $U - \bar{d}$ contains at least 3 elements of S. This contradicts Lemma 9, because there are 3 pairs $(x, x+\bar{d})$, $x \in S$, $x + \bar{d} \notin S$.

3. $\bar{d} > |A|$.
   No element of $V := \{a+i+\bar{d}\}_{i=0}^{3}$ is in S because $a+\bar{d} > a+|S|+1$ and $a+\bar{d}+3 \leq a+\frac{p}{2}+3 < a+p$. However, every element of $V - \bar{d}$ is in A. Thus, at least 3 elements of V are in S. This contradicts Lemma 9, because there are 3 pairs of elements $(x, x+\bar{d})$, $x \in S$, $x+\bar{d} \notin S$.

$\square$

Proof of Theorem 6: Assume that that there exist $a, b, c, e \in \mathbb{Z}_p$, with $abc \not\equiv 0 \pmod{p}$, and a coloring $\tilde{c}_p$ of $\mathbb{Z}_p = R \cup B \cup G$ into 3 color classes $(|R|, |B|, |G| \geq 4)$, containing no rainbow solution of $ax + by + cz \equiv e \pmod{p}$. Let R', B', G' be a permutation of R, B, G, and let a', b', c' be a permutation of a, b, c. Since $a'b'c' = abc \not\equiv 0 \pmod{p}$, $|a'R'| = |R'|, |b'B'| = |B'|, |c'G'| = |G'|$. If $|a'R' + b'B'| \geq |a'R'| + |b'B'| + 1$, then by the theorem of Cauchy and Davenport and $|R'| + |B'| + |G'| = p$, $|a'R' + b'B' + c'G'| \geq \min\{p, (|R'|+|B'|+1) + |G'| - 1\} = p$. Hence, there exists a rainbow solution of $ax + by + cz \equiv e \pmod{p}$, which is a contradiction. Therefore,

$$|a'R' + b'B'| < |a'R'| + |b'B'| + 1 = |R'| + |B'| + 1 < p - 3,$$

$$|b'B' + c'G'| < |b'B'| + |c'G'| + 1 = |B'| + |G'| + 1,$$
$$|a'R' + c'G'| < |a'R'| + |c'G'| + 1 = |R'| + |G'| + 1.$$

Moreover, using the condition $|R|, |B|, |G| \geq 4$ and the theorem of Cauchy and Davenport, we obtain

$$|a'R' + b'B'| \geq 7, |b'B' + c'G'| \geq 7, |a'R' + c'G'| \geq 7.$$

Hence, for every $X, Y \in \{R, B, G\}, X \neq Y$, and every $x, y \in \{a, b, c\}, x \neq y$, we can apply the theorem of Hamidoune and Rødseth on sets xX and yY; that is, xX and yY are contained in arithmetic progressions with the same common difference and $|X| + 1$ and $|Y| + 1$ elements respectively.

Set xX is contained in an arithmetic progression of length $|X| + 1$ if and only if X is contained in an arithmetic progression of length $|X| + 1$. Thus, R, B and G are contained in arithmetic progressions of lengths $|R|+1$, $|B|+1$ and $|G|+1$, respectively. Since every arithmetic progression in $\mathbb{Z}_p$ of common difference d is also an arithmetic progression of common difference $p - d$, Lemma 10 implies that there exist unique common differences d_R, d_B and d_G $(\leq \frac{p}{2})$ for all arithmetic progressions of lengths $|R| + 1$, $|B| + 1$ and $|G| + 1$, containing R, B and G, respectively.

Let $X, Y \in \{R, B, G\}, X \neq Y$, and $x, y \in \{a, b, c\}, x \neq y$. Since xX and yY are contained in arithmetic progressions with the same common difference and $|X| + 1$ and $|Y| + 1$ elements respectively, $y^{-1}xX$ and Y are contained in arithmetic progressions with the common difference d_Y and $|X| + 1$ and $|Y| + 1$ elements respectively. Hence, $y^{-1}xd_X = d_Y$.

Similarly, yX and xY are contained in arithmetic progressions with the same common difference and $|X| + 1$ and $|Y| + 1$ elements respectively. Thus, $x^{-1}yX$ and Y are contained in arithmetic progressions with the common difference d_Y and $|X| + 1$ and $|Y| + 1$ elements respectively. Hence, $x^{-1}yd_X = d_Y$.

It follows that $y^{-1}xd_X = x^{-1}yd_X$, that is, $|x| = |y|$. Therefore, $|a| = |b| = |c| =: t$ and $d_R = d_B = d_G =: d$.

Multiplying by t^{-1} and using the symmetry in the variables x, y, z, the equation $ax + by + cz \equiv e \pmod{p}$ reduces to one of the following four equations: $x+y-z \equiv -e$, $x+y-z \equiv e$, $x+y+z \equiv -e$, $x + y + z \equiv e$ in $\mathbb{Z}_p$. The first two cases further reduce to the equation $x + y \equiv z \pmod{p}$, after shifting $\mathbb{Z}_p$ by $-e$ and e, respectively. Next, we use the following result of Schönheim [23]:

Theorem (Schönheim) *Let $\mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$ be a partition of $\mathbb{N}$, with no rainbow solutions of $x+y = z$. Let $\mathcal{G}$ be the class containing the largest smallest element, denoted by m. Let E, F, be subsets of $\mathcal{E}, \mathcal{F}$ consisting of the elements smaller than m. Then for $i \in \mathbb{N}$,*

1. $e \in E \to e + im \in \mathcal{E}$
2. $f \in F \to f + im \in \mathcal{F}$,

with exceptions occurring only for one of the classes $\mathcal{E}$, $\mathcal{F}$, and only at the multiples of some fixed nontrivial divisor of m.

Consider the coloring $\tilde{c} : \mathbb{N} \to \{R, G, B\}$ defined by $\tilde{c}(x) = \tilde{c}_p(x \pmod{p})$, for all $x \in \mathbb{N}$. Assume that G is the color class containing the largest smallest element denoted by m. Schönheim's theorem (with $R \equiv E$, $B \equiv F$, $G \equiv \mathcal{G}$) implies that there exists a divisor s of $m \leq p$, such that every element x with $c(x) = G$ is a multiple of s. If $(p, s) = 1$, then $c(m + p) = G$. However, s does not divide $m + p$, so $c(m + p) \neq G$, and we have a contradiction. If $s = p$, then $c(x) = G$ for $x \in \mathbb{N}$ if and only if $x \in \{p, 2p, 3p...\}$. Then the coloring c_p of Z_p has only 1 element with color G, namely 0. This contradicts the condition $|G| \geq 4$.

Therefore, we can assume that the equation $ax + by + cz \equiv e \pmod p$ is of the form $x + y + z \equiv e \pmod p$. Since $d = d_R = d_B = d_G$, after applying the group isomorphism $\mathbb{Z}_p \to d^{-1}\mathbb{Z}_p$, we can assume that R, B and G are contained in strings of $|R|+1$, $|B|+1$ and $|G|+1$ consecutive elements, respectively. One of the following two cases occurs:

1. There exist at least two color classes, say R and B, that are not contained in strings of $|R|$ and $|B|$ consecutive elements, respectively.

   Then $R = \{a_1 + i\}_{i=0}^{|R|-2} \cup \{a_1 + |R|\}$ and $B = \{a_1 + |R| - 1\} \cup \{a_1 + |R| + i\}_{i=1}^{|B|-1}$. Then $R + B = \{2a_1 + |R| + i\}_{i=-1}^{|R|+|B|-1}$, so that $|R + B| = |R| + |B| + 1$. By the theorem of Cauchy and Davenport, $|R + B + G| = p$, which implies that the equation $x + y + z \equiv e \pmod p$ has a rainbow solution. Contradiction.

2. R, B and G are contained in the strings of $|R|$, $|B|$ and $|G|$ consecutive elements, respectively. Then $R = \{i\}_{i=a_1}^{a_2-1}$, $B = \{i\}_{i=a_2}^{a_3-1}$, $G = \{i\}_{i=a_3}^{a_1-1}$, in which case $R + B + G = \{i\}_{i=a_1+a_2+a_3}^{a_1+a_2+a_3-3}$. Clearly, if there is no rainbow solution to the equation $x+y+z \equiv e \pmod p$, then $a_1+a_2+a_3 \equiv e + 1$ or $e + 2 \pmod p$.

Therefore, if the equation $ax + by + cz \equiv e \pmod p$ has no rainbow solutions, then $a = b = c$ and every color class is an arithmetic progression with the same common difference d, so that $d^{-1}R = \{i\}_{i=a_1}^{a_2-1}$, $d^{-1}B = \{i\}_{i=a_2}^{a_3-1}$ and $d^{-1}G = \{i\}_{i=a_3}^{a_1-1}$, where $(a_1 + a_2 + a_3) \equiv e + 1$ or $e + 2 \pmod p$. □

5 Future directions

The problems and conjectures stated in the previous sections deal with the existence of rainbow structures in the sets of integers. There are many more directions and generalization one might consider.

One natural direction is generalizing the problems above for rainbow solutions of any homogeneous equation, imitating Rado's theorem about the monochromatic analogue. We have already showed an example of this in Theorem 6.

Search for a rainbow counterpart of the Hales-Jewett theorem, though an exciting possibility, led us to some negative results. First, recall some notation from [12]. Define C_t^n, the n-cube over t elements by $C_t^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, t-1\}\}$. A *geometric line* in C_t^n is a set of (suitably ordered) points $\mathbf{x_0}, \ldots, \mathbf{x_{t-1}}$, $\mathbf{x_i} = (x_{i,1}, \ldots, x_{i,n})$ so that in each coordinate j, $1 \leq j \leq n$, either $x_{0,j} = x_{1,j} = \ldots = x_{t-1,j}$, or $x_{s,j} = s$ for every $0 \leq s < t$, or $x_{s,j} = n - s$ for every $0 \leq s < t$. The Hales-Jewett theorem states that for every t and k, if n is sufficiently large, every k-coloring of C_t^n contains a monochromatic geometric line. This motivates the following question: Is it true that for every equinumerous t-coloring of C_t^n there exists a rainbow geometric line? The following coloring show that the answer is negative even for small values of t and n. A 3-coloring of C_3^3 defined by $C_1 = \{000, 002, 020, 200, 220, 022, 202, 222, 001\}$, $C_2 = \{011, 021, 101, 201, 111, 221, 010, 210, 012\}$, and $C_3 = \{100, 110, 120, 121, 211, 102, 112, 122, 212\}$ (parentheses and commas being removed for clarity), has no rainbow geometric lines.

Another generic direction we considered is increasing the number of colors and the length of a rainbow AP.

Proposition 3 *For every n and $k > 3$, there exists a k-coloring of $[n]$ with no rainbow $AP(k)$ and with each color of size at least $\lfloor \frac{n+2}{3\lfloor (k+4)/3 \rfloor} \rfloor$.*

Proof: First, we partition the set of k colors into three sets C_1, C_2, and C_3 of sizes l_1, l_2, and l_3, respectively, where (l_1, l_2, l_3) is defined as follows:

$$(l_1, l_2, l_3) = \begin{cases} (l+1, l, l-1) & \text{if } k = 3l \\ (l+1, l+1, l-1) & \text{if } k = 3l+1 \\ (l+2, l, l) & \text{if } k = 3l+2 \end{cases}$$

Notice that by the above definition, $\max(l_1, l_2, l_3) = \lfloor (k+4)/3 \rfloor$, and there are always i, j such that $|l_i - l_j| = 2$. Now, for $i = 1, 2, 3$, we color the numbers in $N_i := \{x \in [n] : x \equiv i \pmod 3\}$ with colors in C_i, so that for each two colors in C_i, the number of times they are used differ by at most 1 (one can achieve this by coloring N_i with the colors in C_i cyclically). Thus, it is easy to verify that each color is used at least $\lfloor \frac{n+2}{3\max(l_1, l_2, l_3)} \rfloor = \lfloor \frac{n+2}{3\lfloor (k+4)/3 \rfloor} \rfloor$ number of times. Also, every arithmetic progression $\mathcal{A}$ is either completely contained in one of N_i's, or satisfies $||\mathcal{A} \cap N_i| - |\mathcal{A} \cap N_j|| \le 1$ for every $i, j \in \{1, 2, 3\}$. Thus, the existence of i, j with $|l_i - l_j| = 2$ shows that there is no rainbow $AP(k)$ in this coloring. □

The above proposition can be thought of as a generalization of Proposition 1 for $k > 3$. One is tempted to also generalize Theorem 1 and conjecture that any partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_k$ into k color classes, with every color class having density greater than $\frac{1}{3\lfloor (k+4)/3 \rfloor}$, contains a rainbow $AP(k)$. However, it is easy to verify that the following equinumerous colorings of $\mathbb{N}$ do not contain any rainbow $AP(5)$, and hence the generalization of Radoičić's conjecture is not true for $k = 5, 6$.

$$c_5(i) := \begin{cases} 1 & \text{if } i \equiv 1,3 & \pmod{10} \\ 2 & \text{if } i \equiv 2,5 & \pmod{10} \\ 3 & \text{if } i \equiv 4,8 & \pmod{10} \\ 4 & \text{if } i \equiv 6,7 & \pmod{10} \\ 5 & \text{if } i \equiv 9,0 & \pmod{10} \end{cases} \qquad c_6(i) := \begin{cases} 1 & \text{if } i \equiv 1,3 & \pmod{12} \\ 2 & \text{if } i \equiv 2,4 & \pmod{12} \\ 3 & \text{if } i \equiv 5,7 & \pmod{12} \\ 4 & \text{if } i \equiv 6,8 & \pmod{12} \\ 5 & \text{if } i \equiv 9,11 & \pmod{12} \\ 6 & \text{if } i \equiv 10,0 & \pmod{12} \end{cases}$$

We still do not know whether there is a similar example when the number of colors is $k = 4$ or $k > 6$. If the number of colors is infinite, the following proposition shows that one cannot guarantee even the existence of a rainbow $AP(3)$ with the assumption that each color has a positive density.

Proposition 4 *There is an infinite coloring of $\mathbb{N}$ with each color having positive density such that there is no rainbow $AP(3)$.*

Proof: For each $x \in \mathbb{N}$, let $c(x)$ be the largest integer k such that x is divisible by 3^k. It is easy to see that the color k has density $2 \times 3^{-k-1} > 0$ in this coloring. Also, if $c(x) \neq c(y)$, it is not difficult to see that $c(2x - y) = c((x + y)/2) = \max(c(x), c(y))$. Therefore, if two elements of an arithmetic progression are colored with two different colors, the third term must be colored with one of those two colors. Thus, there is no rainbow $AP(3)$ in c. □

Yet another direction is limiting our attention to equinumerous colorings and letting the number of colors be different from the desired length of a rainbow AP. Let T_k denote the minimal number $t \in \mathbb{N}$ such that there is a rainbow $AP(k)$ in every equinumerous t-coloring of $[tn]$ for every $n \in \mathbb{N}$. We have the following lower and upper bounds on T_k.

Proposition 5 *For every $k \geq 3$, $\lfloor \frac{k^2}{4} \rfloor < T_k \leq \frac{k(k-1)^2}{2}$.*

Proof: First, we prove the upper bound. Let $m = a(k-1)+b$, with $k \geq 3$, $a \geq 1$, and $0 \leq b \leq k-1$. We note that there is bijective correspondence between the set of all $AP(k)$'s and the set of all 2-element sets $\{\alpha, \beta\} \subseteq [m]$, $\alpha < \beta$, with $\alpha \equiv \beta \pmod{k-1}$. It follows that the number of all $AP(k)$'s in $[m]$ is $b\binom{a+1}{2} + (k-b-1)\binom{a+1}{2}$. Thus,

$$\#\text{ of } AP(k)\text{'s in } [tn] > \frac{tn(tn-2(k-1))}{2(k-1)}.$$

Note that for a t-regular coloring of $[tn]$, in each of the t colors there are $\binom{n}{2}$ pairs that could be the terms of at most $\binom{k}{2}$ different $AP(k)$'s. Therefore, for any t-regular coloring of $[tn]$ there are at most $t\binom{k}{2}\binom{n}{2}$ $AP(k)$'s that are *not* rainbow. Therefore, T_k is bounded by the smallest t that satisfies

$$\frac{tn(tn-2(k-1))}{2(k-1)} \geq t\binom{k}{2}\binom{n}{2} \text{ for all } n,$$

which implies the upper bound.

As for the lower bound, we exhibit colorings c_1 and c_2, showing that $T_{2k+1} > k^2 + k$ and $T_{2k} > k^2$. Let a *j-block* B_j ($j \in \mathbb{N}$) be the sequence $12\ldots j12\ldots j$, where the *left half* and the *right half* of the block are naturally defined.

The coloring c_1 gives the following color assignment to the elements of $[2k^2+2k]$ (bars denoting endpoints of the blocks):

$$\Big|B_k^-\Big|\ldots\Big|B_j^-\Big|\ldots\Big|B_2^-\Big|B_1^-\Big|B_1^+\Big|B_2^+\Big|\ldots\Big|B_i^+\Big|\ldots\Big|B_k^+\Big|,$$

Here, $B_j^- = B_j - \binom{j+1}{2}$ and $B_i^+ = B_i + \binom{i}{2}$, where $X + a$ denotes the set $\{x + a | x \in X\}$, for $a \in \mathbb{Z}$, $X \subset \mathbb{Z}$. Note that c_1 uses each of the $k^2 + k$ colors exactly twice.

The coloring c_2 of $[2k^2]$ is defined similarly:

$$\Big|B_{k-1}^-\Big|\ldots\Big|B_j^-\Big|\ldots\Big|B_2^-\Big|B_1^-\Big|B_1^+\Big|B_2^+\Big|\ldots\Big|B_i^+\Big|\ldots\Big|B_k^+\Big|,$$

thus using each of the k^2 colors exactly twice.

Next, we show that $[2k^2+2k]$, colored by c_1, does not contain a rainbow $AP(2k+1)$. The key observation is that a rainbow AP with common difference d cannot contain elements from opposite halves of any block B_j, where d divides j. Fix a longest rainbow AP $\mathcal{A}$ and let d denote its common difference. If $d > k$, then the length of $\mathcal{A}$ is $\leq 2k$. If $d \leq k$, then $\mathcal{A}$ is one of the following three types:

1. $\mathcal{A}$ is contained in $\Big|B_d^-\Big|B_{d-1}^-\Big|\ldots\Big|B_2^-\Big|B_1^-\Big|B_1^+\Big|B_2^+\Big|\ldots\Big|B_{d-1}^+\Big|B_d^+\Big|$.

   Then $\mathcal{A}$ does not intersect the left half of B_d^- nor the right half of B_d^+. Hence, the length of $\mathcal{A}$ is at most $2d \leq 2k$.

2. $\mathcal{A}$ is contained in $\left|B^-_{(j+1)d}\middle|B^-_{(j+1)d-1}\right|\ldots\left|B^-_{jd}\right|$ or in $\left|B^+_{jd}\middle|B^+_{jd+1}\right|\ldots\left|B^+_{(j+1)d}\right|$, where $(j+1)d \leq k$.

   Assume that the first case occurs. Then $\mathcal{A}$ does not intersect the left half of $B^-_{(j+1)d}$ nor the right half of B^-_{jd}. Hence, the length of $\mathcal{A}$ is at most $\frac{1}{d}(jd+2(jd+1)+2(jd+2)+\ldots+2(jd+d-1)+(jd+d)) \leq 2(j+1)d \leq 2k$.

3. $\mathcal{A}$ is contained in $\left|B^-_{jd+x}\middle|B^-_{jd+x-1}\right|\ldots\left|B^-_{jd}\right|$ or in $\left|B^+_{jd}\middle|B^+_{jd+1}\right|\ldots\left|B^+_{jd+x}\right|$, where $jd+x<k$. Assume that the first case occurs. Then $\mathcal{A}$ does not intersect the right half of B^-_{jd}. Hence, the length of $\mathcal{A}$ is at most $\frac{1}{d}(jd+2(jd+1)+2(jd+2)+\ldots+2(jd+x-1)+2(jd+x)) \leq \frac{1}{d}(jd+2jd(d-1)+d(x-1)) < 2(jd+x) < 2k$.

Similarly, one shows that $[2k^2]$, colored by c_2, does not contain a rainbow $AP(2k)$. □

Note that Proposition 5 gives $3 \leq T_3 \leq 6$, while Conjecture 1 claims that $T_3 = 3$.

Conjecture 3 *For all $k \geq 3$, $T_k = \Theta(k^2)$.*

The proof of Proposition 5 above is inspired by the proof of the following "canonical version" of van der Waerden's theorem on arithmetic progressions, due to Erdős and Graham [7]. We include this for completeness:

Theorem 7 *For every positive integer $k \geq 3$, there exists a positive integer $n(k)$ such that every coloring of the first $n \geq n(k)$ positive integers contains either a monochromatic $AP(k)$ or a rainbow $AP(k)$.*

Proof: By Szemerédi's Theorem [27], for every $\delta > 0$ there exists a positive integer $s(k,\delta)$ such that for all $n \geq s(k,\delta)$ every subset $C \subset [n]$ with $|C| > \delta n$ contains an $AP(k)$. Fix $\delta = \frac{2}{k(k-1)^2}$ and let $n \geq s(k,\delta)$. Suppose there exists a coloring of $[n] = C_1 \cup C_2 \cup \ldots \cup C_r$ containing no monochromatic or rainbow $AP(k)$. Since a color class C_i does not contain a monochromatic $AP(k)$, then $|C_i| \leq \delta n$. In the proof of Proposition 5, it was shown that the number of $AP(k)$'s is at least $\frac{n(n-2(k-1))}{2(k-1)}$. Since every non-rainbow $AP(k)$ contains a pair of terms of the same color, there are at most $\binom{k}{2}\sum_{i=1}^{r}\binom{|C_i|}{2}$ non-rainbow $AP(k)$'s. Since $0 \leq |C_i| \leq \delta n$ and $\sum_{i=1}|C_i| = n$, the inequality $\binom{k}{2}\sum_{i=1}^{r}\binom{|C_i|}{2} \leq \binom{k}{2}\sum_{i=1}^{1/\delta}\binom{\delta n}{2} = \frac{\binom{k}{2}\binom{\delta n}{2}}{\delta}$. However, the inequality $\frac{n(n-2(k-1))}{2(k-1)} \leq \frac{\binom{k}{2}\binom{\delta n}{2}}{\delta}$ does not hold for our choice of δ. □

It is easy to show that the maximal number of rainbow $AP(3)$'s over all equinumerous 3-colorings of $[3n]$ is $\lfloor 3n^2/2 \rfloor$, this being achieved for the unique 3-coloring with color classes $R = \{n | n \equiv 0 \pmod 3\}$, $B = \{n | n \equiv 1 \pmod 3\}$ and $G = \{n | n \equiv 2 \pmod 3\}$. It seems very difficult to characterize those equinumerous 3-colorings (in general, k-colorings) that minimize the number of rainbow $AP(3)$'s. Letting $f_k(n)$ denote the minimal number of rainbow $AP(k)$'s, over all equinumerous k-colorings of $[kn]$, we pose the following conjecture.

Conjecture 4 *$f_3(n) = \Omega(n)$.*

If we define $g_k(n)$ as the minimal number of rainbow $AP(k)$'s, over all equinumerous k-colorings of $\mathbb{Z}_{kn}$, then a straightforward counting argument shows that $g_3(n) \geq n$, when n is odd.

Finally, the further generalization of Vosper's theorem, due to Serra and Zémor [25], may lead to a generalization of Theorem 6 for more than 3 color classes.

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References

[1] V.E. Alekseev and S. Savchev. Problem M. 1040. *Kvant*, 4:23, 1987.

[2] N. Alon. On a conjecture of Erdős, Simonovits and Sós concerning anti-Ramsey numbers. *Journal of Graph Theory*, 7:91–94, 1983.

[3] N. Alon, H. Lefmann, and V. Rödl. On an anti-Ramsey type result. *Coll. Math. Soc. J. Bolyai 60, Sets, Graphs and Numbers, Budapest (Hungary)*, pages 9–22, 1991.

[4] M. Axenovich, Z. Füredi, and D. Mubayi. On generalized Ramsey theory: the bipartite case. *Journal of Combinatorial Theory, Ser. B*, 79:66–86, 2000.

[5] F.R.K. Chung and R.L. Graham. On multicolor Ramsey numbers for bipartite graphs. *Journal of Combinatorial Theory, Ser. B*, 18:164–169, 1975.

[6] W. Deuber, R.L. Graham, H.J. Prömel, and B. Voigt. A canonical partition theorem for equivalence relations in $\mathbb{Z}^t$. *Journal of Combinatorial Theory, Series A*, 34:331–339, 1983.

[7] P. Erdős and R.L. Graham. Old and new problems and results in combinatorial number theory. *L'Enseignement Mathematique, Univ. Geneve*, 1980.

[8] P. Erdős and R. Rado. A combinatorial theorem. *J. London Math. Soc.*, 25:249–255, 1950.

[9] P. Erdős, M. Simonovits, and V.T. Sós. Anti-Ramsey theorems. In *Coll. Math. Soc. J. Bolyai*, volume 10 of *Infinite and finite sets*, pages 633–642, Keszthely (Hungary), 1973.

[10] P. Erdős and Z. Tuza. Rainbow subgraphs in edge-colorings of complete graphs. In Gimbel, J. and Kennedy, J.W. and Quintas, L.V., editor, *Quo Vadis, Graph Theory?*, volume 55 of *Annals of Discrete Mathematics*, pages 81–88. 1993.

[11] T. Gowers. A new proof of Szemerédi's theorem. *Geom. Funct. Anal.*, 11:465–588, 2001.

[12] R.L. Graham, B.L. Rothschild, and J.H. Spencer. *Ramsey Theory.* John Wiley and Sons, 1990.

[13] R.K. Guy. *Unsolved Problems in Number Theory.* Springer-Verlag, 1994. E11–12.

[14] Y.O. Hamidoune and Ø.J. Rødseth. An inverse theorem mod p. *Acta Arithmetica*, 92:251–262, 2000.

[15] H. Lefmann. A canonical version for partition regular systems of linear equations. *Journal of Combinatorial Theory, Ser. A*, 41:95–104, 1986.

[16] H. Lefmann and V. Rödl. On canonical Ramsey numbers for complete graphs versus paths. *Journal of Combinatorial Theory, Ser. B*, 58:1–13, 1993.

[17] H. Lefmann and V. Rödl. On Erdős-Rado numbers. *Combinatorica*, 15:85–104, 1995.

[18] M.B. Nathanson. *Additive number theory: Inverse problems and the Geometry of sumsets*, volume GTM 165. Springer-Verlag, 1996.

[19] J. Nešetřil, H.J. Prömel, V. Rödl, and B. Voigt. Canonizing ordering theorems for Hales-Jewett structures. *Journal of Combinatorial Theory, Ser. A*, 40:394–408, 1985.

[20] H.J. Prömel and B. Voigt. Canonical partition theorems for parameter sets. *Journal of Combinatorial Theory, Series A*, 35:309–327, 1983.

[21] D. Richer. Unordered canonical Ramsey numbers. *Journal of Combinatorial Theory, Ser. B*, 80:172–177, 2000.

[22] V. Rödl and Z. Tuza. Rainbow subgraphs in properly edge-colored graphs. *Random Structures Algorithms*, 3:175–182, 1992.

[23] J. Schönheim. On partitions of the positive integers with no x, y, z belonging to distinct classes satisfying $x+y=z$. In *R. A. Mollin (ed.) Number theory (Proceedings of the First Conference of the Canadian Number Theory Association, Banff 1988)*, pages 515–528, de Gruyter, 1990.

[24] I. Schur. Über die Kongruenz $x^m + y^m \equiv z^m \bmod p$. *Jahresb. Deutsche Math. Verein*, 25:114–117, 1916.

[25] O. Serra and G. Zémor. On a generalization of a theorem by Vosper. *Integers. Electronic Journal of Combinatorial Number Theory*, 0:A10 , 2000. (electronic, `http://www.integers-ejcnt.org/`).

[26] G. Szekely, editor. *Contests in Higher Mathematics, Miklós Schweitzer Competition 1962-1991.* Problem Books in Mathematics. Springer, 1995. C.22.

[27] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arithmetica*, 27:199–245, 1975.

[28] G. Vosper. The critical pairs of subsets of a group of prime order. *J. London Math. Soc.*, 31:200–205, 1956.