Polychromatic Cliques

Pavol Hell[*] and Juan Jose Montellano[†]

Abstract

The sub-Ramsey number $sr(K_n, k)$ is the smallest integer m such that in any edge-colouring of K_m which uses every colour at most k times some subgraph K_n has all edges of different colours. It was known that, for a fixed k, the function $sr(K_n, k)$ is $O(n^3)$ and $\Omega(n)$. We improve these bounds to $O(n^2)$ and $\Omega(n^{3/2})$ (slightly less for small values of k).

1 Background and Notation

An edge-colouring of a graph G (i.e., a mapping of $E(G)$ to a set of 'colours') is *k-bounded* if each colour is used at most k times. If $k = 1$, i.e., if all edges of G have different colours, we say that G is *polychromatic*. The *sub-Ramsey number* $sr(G, k)$ is the smallest integer m such that each k-bounded edge-colouring of K_m contains a polychromatic subgraph isomorphic to G. Recall that the *Ramsey number* $r(G, k)$ is the smallest integer m such that each edge-colouring of K_m with k colours contains a monochromatic (i.e., all edges coloured with the same colour) subgraph isomorphic to G. Thus sub-Ramsey numbers are in this sense dual to Ramsey numbers, and it is easy to see that each $sr(G, k) \leq r(G, k)$, and hence each $sr(G, k)$ is guaranteed to be finite [1].

F. Galvin appears to be the first person to suggest investigating $sr(G, k)$ [8]. In [1] it is shown that $sr(K_n, k)$ is $O(kn^3)$ and $\Omega(kn)$. In [11] the authors show that $sr(P_n, k) = sr(C_n, k) = n$ when n is large enough with respect to k. (P_n and C_n denote respectively the path and the cycle with n vertices.) Results on sub-Ramsey numbers of stars can be found in [9, 10, 4]. Related questions on k-bounded edge-colourings without polychromatic subgraphs which are not necessarily of fixed size (such as polychromatic Hamilton paths or cycles) are investigated in [7, 11, 12, 3].

In this note we improve both the upper bound and the lower bound for $sr(K_n, k)$. In particular we prove that $sr(K_n, k)$ is $O(kn^2)$ (better than $O(kn^3)$) and, for k greater than or equal to 15, $sr(K_n, k)$ is $\Omega(n^{3/2})$ (better than $\Omega(kn)$ for any fixed k). The lower bound for k between 3 and 15 is $\Omega(n^{4/3})$. (We have no improvement of the lower bound when $k = 2$.)

2 The Upper Bound

Theorem 1 *Let $n \geq 3$ and $k \geq 2$ be positive integers. Then*

$$sr(K_n, k) \leq (2n - 3)(n - 2)(k - 1) + 3.$$

Proof. Assume that K_m admits a *good* edge-colouring, i.e., a k-bounded edge-colouring in which no subgraph isomorphic to K_n is polychromatic. We shall show that $m \leq (2n-3)(n-2)(k-1)+2$.

[*]School of Computing Science, Simon Fraser University, Burnaby, B.C., Canada, V5A 1S6; pavol@cs.sfu.ca
[†]Instituto de Matematicas, UNAM, Ciudad Universitaria, Mexico 04510, D.F.; juancho@matem.unam.mx

Fix a good edge-colouring of K_m, with colours $1, 2, \ldots, r$, and denote by d_i the number of edges coloured i. Note that each $d_i \leq k$. Let W denote the number of unordered pairs of edges of K_m coloured by the same colour. We claim that

$$W \leq \binom{m}{2} \frac{k-1}{2}. \tag{1}$$

Indeed, $\binom{m}{2} = |E(K_m)| = \sum_{i=1}^{r} d_i$ and $W = \sum_{i=1}^{r} \binom{d_i}{2}$. Thus

$$W = \sum_{i=1}^{r} d_i \frac{d_i - 1}{2} \leq \sum_{i=1}^{r} d_i \frac{k-1}{2} = \binom{m}{2} \frac{k-1}{2}.$$

Consider now a subgraph Q of K_m isomorphic to K_q, for some integer $q > n$. We claim that Q must have many pairs of edges of the same colour, otherwise some subgraph of Q isomorphic to K_n would be polychromatic. Specifically, denote by $T(q,n)$ the maximum number of edges of a graph with q vertices which does not have a complete subgraph of n vertices. (The value of $T(q,n)$ is known by Turan's theorem [2].) If Q obtained more than $T(q,n)$ colours then taking one edge of each colour would result in a graph G with q vertices and more than $T(q,n)$ edges, which therefore would have to contain a subgraph isomorphic to K_n. This contradicts the fact that we have a good edge-colouring. Therefore, Q is coloured by at most $T(q,n)$ colours, say, the colours $1, 2, \ldots, t$, where $t \leq T(q,n)$. Denote by $w(Q)$ the number of unordered pairs of edges of Q which obtain the same colour, and by $d_i(Q)$ the number of edges of Q of colour $i, i \leq t$. We claim that

$$w(Q) \geq \binom{q}{2} - T(q,n). \tag{2}$$

Indeed,

$$w(Q) = \sum_{i=1}^{t} \binom{d_i(Q)}{2} = \sum_{i=1}^{t} (d_i(Q) - 1) \frac{d_i(Q)}{2} \geq \sum_{i=1}^{t} (d_i(Q) - 1) =$$

$$= (\sum_{i=1}^{t} d_i(Q)) - t = \binom{q}{2} - t \geq \binom{q}{2} - T(q,n).$$

Let $\mathcal{Q}$ denote the set of all subgraphs of K_m isomorphic to K_q. Since each pair of edges of K_m belongs to $\binom{m-3}{q-3}$ or $\binom{m-4}{q-4}$ graphs in $\mathcal{Q}$, depending on whether the two edges are adjacent or not, we see that $MW \geq \sum_{Q \in \mathcal{Q}} w(Q)$, where $M = \max\{\binom{m-3}{q-3}, \binom{m-4}{q-4}\}$.

Suppose first that $\binom{m-3}{q-3} \leq \binom{m-4}{q-4}$. This inequality implies that $m - 3 \leq q - 3$ and thus $m \leq q$. Therefore our desired conclusion $m \leq (2n-3)(n-2)(k-1) + 2$ will follow as long as we choose q not too big. Let us in fact choose $q = 2n - 2$.

In the remaining case, $\binom{m-3}{q-3} > \binom{m-4}{q-4}$, we have $\binom{m-3}{q-3} W > \sum_{Q \in \mathcal{Q}} w(Q)$ which by (1) and (2) implies that

$$\binom{m-3}{q-3} \binom{m}{2} \frac{k-1}{2} \geq \binom{m-3}{q-3} W > \binom{m}{q} \left(\binom{q}{2} - T(q,n) \right). \tag{3}$$

Using

$$T(q,n) = \frac{n-2}{2n-2} q^2$$

[2], and $q = 2n - 2$, we have

$$\binom{q}{2} - T(q,n) = \binom{q}{2} - \frac{n-2}{2n-2}q^2 = \frac{q}{2}.$$

Therefore, we obtain from (3) that

$$\frac{(m-3)!}{(q-3)!(m-q)!} \cdot \frac{m(m-1)}{2} \cdot \frac{(k-1)}{2} \geq \frac{m!}{(m-q)!q!} \cdot \frac{q}{2}.$$

Simplifying the last inequality we reach the conclusion that $m - 2 \leq (k-1)(q-1)(q-2)/2$ which is equivalent to $m \leq (2n-3)(n-2)(k-1) + 2$. □

3 The Lower Bound

Let m and t be positive integers.

Let $U(t)$ denote the maximum number of ways a set can be written as a union of two t-element subsets. For instance, $U(2) = 3$ because the set $\{1,2,3,4\}$ can be written as $\{1,2\} \cup \{3,4\}$, or $\{1,3\} \cup \{2,4\}$, or $\{1,4\} \cup \{2,3\}$, and no set can be written as the union of two 2-element subsets in four different ways. It is easy to check also that $U(3) = 15$.

Let $F(m,t)$ be the maximum size of a union free family of t-element subsets of $\{1,2,\ldots,m\}$. (A family of sets $S_i, i \in I$ is *union free* if all the $\binom{|I|}{2}$ unions $S_i \cup S_j$, $S_i, S_j \in F$, are distinct.)

The following result of Frankl and Füredi gives the asymptotic behaviour of $F(m,t)$:

Theorem 2 *[5] Let t be a positive integer.*
There exist positive constants c_t, c_t' so that

$$c_t' m^{\lceil \frac{4t}{3} \rceil /2} \leq F(m,t) \leq c_t m^{\lceil \frac{4t}{3} \rceil /2}.$$

The relevance of the functions $U(t)$ and $F(m,t)$ to our problem arises from the following fact:

Theorem 3 *If $n > F(m,t)$ and $k \geq U(t)$, then*

$$sr(K_n, k) \geq \binom{m}{t}.$$

Proof. Consider the complete graph $K(m,t)$ whose vertices are all the t-element subsets of $\{1,2,\ldots,m\}$, in which the edge SS' has colour $S \cup S'$. Then this colouring of $K(m,t)$ is k-bounded, since $k \geq U(t)$. At the same time, there is no polychromatic K_n, since $n > F(m,t)$. □

Corollary 1 *If $k \geq 15$, then (for a positive constant c)*

$$sr(K_n, k) \geq cn^{3/2},$$

and, if $3 \leq k < 15$, then

$$sr(K_n, k) \geq cn^{4/3}.$$

Proof. If $k \geq 15 = U(3)$, then we can choose $t = 3$ in Theorem 3. For $t = 3$, any choice of m with $n > c_t m^{\lceil \frac{4t}{3} \rceil /2} = c_3 m^2$ assures that we have $n > F(m, 3)$. We may choose $m = \lceil (c_3^{-1/2} - \epsilon) n^{1/2} \rceil$, for a small positive ϵ. Such a value of m yields

$\binom{m}{3} \geq cn^{3/2}$, thus $sr(K_n, k) \geq cn^{3/2}$ (for a positive constant c). For $k \geq 3 = U(2)$, we proceed analogously, taking $t = 2$ and m satisfying $n > c_2 m^{3/2}$, yielding $sr(K_n, k) \geq \binom{m}{3} \geq cn^{4/3}$. $\square$

We remark that using [6], in which $F(m, 3)$ is evaluated exactly as $F(m, 3) = \lfloor m(m-1)/6 \rfloor$, we find, for $k \geq 15$, that

$$sr(K_n, k) \geq (\sqrt{6} - \epsilon) n^{3/2}.$$

References

[1] B. Alspach, M. Gerson, G. Hahn, and P. Hell, On sub-Ramsey numbers, *Ars Combinatoria* 22 (1986) 199-206.

[2] P. Dirac, Extension of Turan's Theorem on Graphs, *Acta Math. Sci. Hungar.* 14 (1963) 417 – 422.

[3] P. Erdos, J. Nesetril, and V. Rodl, Some problems related to partitions of edges of a graph, in **Graphs and other combinatorial topics**, Teubner, Leipzig 1983, pp. 54-63.

[4] P. Fraisse, G. Hahn, and D. Sotteau, Star sub-Ramsey numbers, *Annals of Discrete Mathematics* 34 (1987) 153-163.

[5] P. Frakl, Z. Füredi, Union-free families of sets and equations over fields, *J. Number Theory* 23 (1986) 210-218.

[6] P. Frakl, Z. Füredi, A new extremal property of Steiner triple-systems, *Discrete Math.* 48 (1984) 205-212.

[7] A. Frieze and B. Reed, Polychromatic Hamilton cycles, *Discrete Math.* 118 (1993) 69-74.

[8] F. Galvin, Advanced Problem number 6034, *Amer. Math. Monthly* 82 (1975), 529.

[9] G. Hahn, Some star anti-Ramsey numbers, *Congressus Num.* 19 (1977) 303-310.

[10] G. Hahn, More star sub-Ramsey numbers, Discrete Math. 43 (1981) 131-139.

[11] G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and an edge-colouring conjecture, *Discrete Math.* 62 (1986) 29-33.

[12] M. Maamoun and H. Meyniel, On a problem of G. Hahn about coloured Hamilton paths in K_2n, *Discrete Math.* 51 (1984) 213-214.