PRIESTLEY CONFIGURATIONS AND HEYTING VARIETIES

RICHARD N. BALL, ALEŠ PULTR, AND JIŘÍ SICHLER

Abstract. We investigate Heyting varieties determined by prohibition of systems of configuraations in Priestley duals; we characterize the configuration systems yielding such varieties. On the other hand, the question whether a given finitely generated Heyting variety is obtainable by such means is solved for the special case of systems of trees.

Priestley duality provides a correspondence between bounded distributive lattices and certain ordered topological spaces. (See [11] and [12], and for an elementary exposition [4].) In a previous article [2] we showed, among other things, that the class of all Heyting algebras whose Priestley spaces contain no copy of a given single (finite) configuration formed a variety iff the configuration was a tree.

Consequently, prohibiting any system of trees presents a Heyting variety as well. But if a system of prohibited configurations has more than one element, it need not consist of trees alone, or be replaceable by such, to yield a variety. One of our two main results characterizes just such configuration systems.

Our other major result addresses the opposite question: given a variety, when can it be obtained by prohibiting a system of trees? (The general question of when a variety results from the prohibition of any system of configurations is beyond the scope of this article.) We present a complete characterization for the case of finitely generated varieties; special consideration is given to varieties generated by a single object.

There are two topics that we needed to discuss in some detail, in our context for more or less technical purposes, but these subjects may be of some interest in their own right. One of them is the nature of the Priestley equivalents of Heyting homomorphisms, and the other is

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the order structure of coproducts of Priestley spaces, a topic which in general seems to be far from fully understood. To each of these we devote a special section.

1. Preliminaries

1.1. General. No special knowledge of category theory—beyond notions such as functor, product or coproduct—is assumed. More general (co)limits and the comma-construction come in remarks and are not essential for the main text. ([9], [1])

From universal algebra we assume the notion of variety and Birkhoff's Theorem. Subdirect irreducibility and Jónsson's Lemma are also mentioned, again mostly in remarks. ([3])

From topology, the reader is assumed to be acquainted with the standard properties of compact Hausdorff spaces and with the Stone-Čech compactification. ([6])

1.2. Distributive lattices and Heyting algebras. Distributive lattices are assumed to have a bottom element 0 and a top element 1, and the homomorphisms are assumed to preserve them. The resulting category will be denoted by

$$\mathbf{DLat}.$$

In the category of Heyting algebras, posessing a binary operation $\to$ satisfying $a \wedge b \leq c$ iff $a \leq b \to c$, the homomorphisms also preserve the extra operation. Thus, the resulting category

$$\mathbf{Hey}$$

is not a full subcategory of **DLat**. Note, however, that the products in **Hey** are the same as in **DLat**, namely the standard cartesian products.

1.3. Filters and ideals. Filters and ideals in a distributive lattice L will always be non-void and proper ($\neq L$). Recall that each ideal (filter) is contained in a prime one. Moreover, if $J \cap F = \emptyset$ for an ideal J and filter F, then there exist both a prime ideal $J' \supseteq J$ and a prime filter $F' \supseteq F$ such that $J' \cap F' = \emptyset$.

1.4. Posets. Finite posets are sometimes referred to as *configurations* and typically denoted by P, Q, R, while for general posets we use X, Y, Z. The set of all minimal (maximal) elements in X is denoted by $\min X$ ($\max X$). If M is a subset of $(X, \leq)$ we write

$$\downarrow M = \{x \mid \exists y \in M,\ x \leq y\} \quad \text{and} \quad \uparrow M = \{x \mid \exists y \in M,\ x \geq y\},$$

and M is said to be *decreasing*, or a *down-set* (*increasing*, or an *up-set*) if $M = \downarrow M$ ($M = \uparrow M$). We abbreviate $\uparrow\{x\}$ and $\downarrow\{x\}$ to $\uparrow x$ and $\downarrow x$, respectively.

In the finite case we denote by $\mathrm{wd}(P)$ (the *width* of P) the maximum size of an antichain (independent set) in P, and $\mathrm{ht}(P)$ (the *height* of P) the maximum length n of a chain $x_0 < x_1 < \cdots < x_n$ in P. Recall Dilworth's Theorem stating that $|P| \leq \mathrm{ht}(P) \cdot \mathrm{wd}(P)$.

The *immediate successor* relation (the fact that $x < y$ and if $x \leq z \leq y$ then $x = z$ or $z = y$) will be denoted by $x \prec y$.

A poset P is a *forest* if $\uparrow x$ is a chain for each $x \in P$. A (finite) P is said to be *sharp* if it has a top element, denoted by $\top_P$. *Trees* are sharp forests; here they will typically be denoted by T.

1.5. Priestley duality (see e.g. [11],[12]). A Priestley space is an ordered compact Hausdorff space X such that if $x \not\leq y$ then there is a clopen down-set U such that $x \notin U \ni y$. Note that this sort of separation entails the Hausdorff property. Priestley maps are monotone and continuous; the resulting category will be denoted by

$$\mathbf{PSp}.$$

Contravariant functors $\mathcal{P} : \mathbf{DLat} \to \mathbf{PSp}$ and $\mathcal{D} : \mathbf{PSp} \to \mathbf{DLat}$ are defined by

$$\mathcal{P}(L) = (\{x \mid x \text{ is a prime ideal in } L\}, \subseteq, \tau), \quad \mathcal{P}(h)(x) = h^{-1}(x)$$
$$\mathcal{D}(X) = (\{a \mid a \text{ is a clopen down-set in } X\}, \cap, \cup), \quad \mathcal{D}(f)(a) = f^{-1}(a).$$

Note that finite posets correspond as finite Priestley spaces precisely to finite distributive lattices, and hence to finite Heyting algebras; the maps in the Heyting case are more special, see Section 2. Finite sharp posets correspond precisely to subdirectly irreducible finite Heyting algebras.

1.6. Representing and prohibiting configurations. We say that P is represented in X, and write $P \hookrightarrow X$, if there is a map $f : P \to X$ such that $x \leq y$ iff $f(x) \leq f(y)$. If there is no such map we write $P \not\hookrightarrow X$. If $\mathbb{P}$ is a class of configurations we set

$$\mathfrak{F}(\mathbb{P}) = \{X \mid \forall P \in \mathbb{P},\ P \not\hookrightarrow X\} \subseteq \mathbf{PSp}.$$

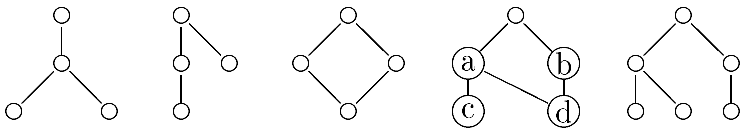
1.7. Some special posets.

Posets $D(n,k)$ and $T(n,k)$ are both defined on the set

$$\{\top\} \cup (\{1,2,\ldots,k\} \times \{1,2,\ldots,n\}),$$

ordered in the first case by $(i,j) < \top$ and $(i,j) \leq (i',j')$ iff either $(i,j) = (i',j')$ or $i > i'$, and in the second case by $(i,j) < \top$ and $(i,j) \leq (i',j')$ iff $i > i'$ and $j = j'$. The "broom" $B(n,k)$ is the subposet of $D(n,k+1)$ consisting of the top element $\top$ together with those pairs (i,j) such that $j = 1$ whenever $i \leq k$.

Finally, the "fork" F, the "hook" H, the "diamond" D, N and M are as in the following picture, in that order.



Of course F is $B(1,2)$. The labeling of N is intentional and will be used in 6.12.

2. H-SPACES AND H-MAPS

2.1. It is a well-known fact that a Priestley space X is isomorphic to a $\mathcal{P}(H)$ for a Heyting algebra H iff

whenever $U \subseteq X$ is clopen then also $\uparrow U$ is clopen.

Priestley spaces with this property will be referred to as *h-spaces*. Furthermore, the Priestley maps $f : X \to Y$ corresponding to Heyting homomorphisms, henceforth called *h-maps*, are known to be characterized by the formula

$$f(\downarrow x) = \downarrow f(x) \text{ for every } x \in X.$$

The resulting subcategory of **PSp** will be denoted by

HPSp.

2.2. The *one-one h-maps* correspond precisely to the Heyting onto homomorphisms. The following is an immediate

Observation. *A Priestley embedding $j : X \subseteq Y$ is an h-map iff X is closed and $X = \downarrow X$.*

This situation will be indicated by

$$X \sqsubset Y.$$

2.3. The onto h-maps, which will be referred to as *h-quotients*, correspond precisely to the one-one Heyting homomorphisms. The existence of an h-quotient $Y \to X$ will be indicated by

$$X < Y.$$

2.3.1. An *h-congruence* on a finite poset P is an equivalence $\sim$ such that

(1) $x \leq y \leq z$ and $x \sim z$ implies $x \sim y \sim z$,
(2) if $x \leq y \sim y'$ then there is an $x' \sim x$ such that $x' \leq y'$.

Proposition. *A relation $\sim$ on P is an h-congruence iff there is an h-quotient $f : P \to Q$ such that*

$$x \sim y \quad \equiv \quad f(x) = f(y).$$

Proof. $\Leftarrow$ is immediate.

$\Rightarrow$: Let $\sim$ be an h-congruence. Denote by $[x]$ the equivalence class containing x and set $[x] \leq [y]$ iff $[y] \subseteq \uparrow [x]$. Then $\leq$ is obviously reflexive and transitive, and if $[x] \leq [y] \leq [x]$ there are $y' \sim y$ and $x' \sim x$ such that $x \geq y' \geq x'$, and $x' \sim y' \sim x$ by (1). Thus $\leq$ is a partial order on $Q = P/\sim$. Consider the mapping $f = (x \mapsto [x]) : P \to Q$. If $x \leq y$ and $y \sim y'$ we have by (2) an $x' \in [x]$ such that $x' \leq y'$; thus $[x] \leq [y]$ and f is monotone. Finally, if $[x] \leq [y] = f(y)$, that is, $[y] \subseteq \uparrow [x]$, we have an $x' \in [x]$ such that $x' \leq y$, and $f(x') = [x'] = [x]$. $\square$

2.4. Lemma. *Let either $Q < P$ or let there be a monotone one-one map $j : Q \to P$. Then* $\mathrm{ht}(Q) \leq \mathrm{ht}(P)$.

Proof. Let $f : P \to Q$ be an h-quotient and consider $x_n \prec x_{n-1} \prec \cdots \prec x_1$ in Q. Then we can inductively choose $y_1, \ldots, y_n$ in P such that $f(y_i) = x_i$ and such that $y_n < y_{n-1} < \cdots < y_1$. Extend this chain to a maximal one. The second statement is trivial. $\square$

2.5. Write

$$Q \sqsubseteq\!\!< P$$

if there is an R such that $Q \sqsubset R < P$.

Lemma. *Let P, Q be sharp with* $\mathrm{ht}(P) = \mathrm{ht}(Q)$. *Then $Q \sqsubseteq\!\!< P$ iff $Q < P$.*

Proof. Let $Q \sqsubset R < P$ with h-quotient $f : P \to R$, and identify Q with a down-set in R. Then by 2.4 $\mathrm{ht}(Q) \leq \mathrm{ht}(R) \leq \mathrm{ht}(P) = \mathrm{ht}(Q)$, hence $\top_R = \top_Q$ and $R = \downarrow \top_R = Q = f(P)$. $\square$

2.6. Let P be a finite poset. The *universal cover* $\widetilde{P}$ of P is the poset

$$(\{x_n x_{n-1} \cdots x_1 \mid x_{k+1} \prec x_k,\ x_1 \text{ maximal}\}, \leq)$$

with

$$\xi \leq \eta \quad \text{iff} \quad \xi \text{ is an extension of } \eta.$$

Define $\gamma_P : \widetilde{P} \to P$ by setting $\gamma_P(x_n \cdots x_1) = x_n$. We have the straightforward

2.6.1. Observations. *1. $\widetilde{P}$ is a forest and it is a tree iff P is sharp.*
2. $\gamma_P : \widetilde{P} \to P$ is an h-quotient.

2.6.2. Lemma. *Let $f : X \to Y$ be an h-quotient and let $g : P \to Y$ be monotone. Then there is a monotone $h : \widetilde{P} \to X$ such that $f \cdot h = g \cdot \gamma_P$.*

Proof. For a maximal $x \in P$ choose $h(x) \in X$ such that $fh(x) = g(x)$. Now let $h(x_n \cdots x_1)$ be already chosen so that $fh(x_n \cdots x_1) = g(x_n)$. If $x_{n+1}x_n \cdots x_1 \in \widetilde{P}$ we have $g(x_{n+1}) \leq g(x_n)$ and hence we can choose $h(x_{n+1}x_n \cdots x_1) = z$ such that $z \leq g(x_n)$ and $f(z) = g(x_{n+1})$. $\square$

2.7. A poset Q is a *cover* of P if there is a monotone $\delta : Q \to P$ and an onto monotone $\varphi : \widetilde{P} \to Q$, such that $\delta \cdot \varphi = \gamma_P$. Note that under these circumstances δ is obviously an h-quotient.

2.7.1. Proposition. *Let $f : X \to Y$ be an h-quotient and let P be a finite subposet of Y. Then there is a cover $\delta : Q \to P$ such that Q is a subposet of X.*

Proof. For the embedding $j : P \subseteq Y$ take a monotone $h : \widetilde{P} \to X$ such that $fh = j\gamma_P$ by 2.6.2. Set $Q = h(\widetilde{P})$, ordered as a subposet of X, let $j' : Q \subseteq X$ be the inclusion map, and define $h' : \widetilde{P} \to Q$ by $h'(\sigma) = h(\sigma)$. The onto map h' is evidently monotone.

Now if $h'(\sigma) \leq h'(\tau)$ then

$$j\gamma_P(\sigma) = fh(\sigma) = fj'h'(\sigma) \leq fj'h'(\tau) = fh(\tau) = j\gamma_P(\tau),$$

and since j is an embedding, $\gamma_P(\sigma) \leq \gamma_P(\tau)$. Thus the rule

$$\delta(h'(\sigma)) = \gamma_P(\sigma)$$

correctly defines a monotone map $\delta : Q \to P$. $\square$

3. Notes on coproducts

3.1. Proposition. *Let X_i, $i \in J$, be Priestley spaces; consider them disjoint to simplify the notation. In the Stone-Čech compactification βJ of (the discrete) J, identify the $i \in J$ with the corresponding principal ultrafilters. Then the coproduct $X = \coprod_{i \in J} X_i$ can be represented so that*

(1) *$X = \bigcup_{u \in \beta J} X_u$, where the union is disjoint, the X_i with $i \in J$ are the original spaces, and all the X_u with $u \in \beta J$ are order-independent;*

(2) *$\bigcup_{i\in J} X_i$ is dense in X, each X_i is clopen in X, and the coproduct injections ι_i are the embeddings $X_i \subseteq X$.*

Proof. Let D_i be the Priestley duals of X_i. Then $\prod D_i$ corresponds to $\coprod X_i$. Consider the embeddings $\mu_i : \mathbf{2} \to D_i$ and the resulting $\mu : \mathbf{2}^J \to \prod D_i$. The Priestley dual of $\mathbf{2}^J$ is βJ, with the trivial order, and hence for the onto map $\varepsilon : \coprod X_i \to \beta J$ corresponding to μ, the $X_u = \varepsilon^{-1}(u)$ are order-independent. It is easy to check that $\varepsilon^{-1}(i) = \iota_i(X_i)$. Finally, since $\coprod_{j\in J} X_j = X_i \sqcup \coprod_{j\neq i} X_j$ we easily conclude that X_i is clopen in X. The density of $\bigcup_{i\in J} X_i$ is obvious. $\square$

3.2. Proposition. *Let X be a finite poset. Then the co-power ${}^J X$ can be represented as $(\iota_i = (x \mapsto (x,i)) : X \to X\times\beta J)_{i\in J}$, with $X \times \beta J$ ordered by $(x,u) \le (y,v)$ iff $x \le y$ and $u = v$.*

Proof. If $f_i : X \to Y$ are Priestley maps define $g_x : J \to Y$ by $g_x(i) = f_i(x)$ and consider the continuous extensions $\overline{g}_x : \beta J \to Y$. Set $f(x,u) = \overline{g}_x(u)$. Then f is obvously continuous, and $f\iota_i(x) = f_i(x)$. Since the uniqueness of f is a consequence of the density of $X \times J$ in $X \times \beta J$, it remains to be shown that f is monotone. Let $x \le y$ and $f(x,u) = \overline{g}_x(u) \not\le \overline{g}_y(u) = f(y,u)$. Then there is a clopen decreasing U with $\overline{g}_x(u) \notin U \ni \overline{g}_y(u)$, and $u \in \overline{g}_y^{-1}(U) \cap \overline{g}_x^{-1}(X\setminus U) = V$ open in βJ. As J is dense there is an $i \in V \cap J$. Then $\overline{g}_y(i) = f_i(y) \in U$ and since U is decreasing, $f_i(x) \in U$ contradicting $\overline{g}_x(i) \in X \setminus U$. $\square$

3.3. Proposition. *Let T be a tree and let $T \hookrightarrow \coprod_{i\in J} X_i$. Then there is an $i \in J$ such that $T \hookrightarrow X_i$.*

Proof. Recall from [2] that a T-complement of a map $a : T \to L$ is a map $c : T \to L$ such that

- for $t \in \min T$, $a(t) \wedge c(t) \le a'(t)$ where $a'(t) = \bigvee_{t\not\le\tau} a(\tau)$,
- $a(t) \wedge c(t) \le \bigvee_{\tau\prec t} c(\tau) \vee a'(t)$ otherwise,
- and $a(\top_T) = 1$,

and that $T \not\hookrightarrow X$ iff in $L = (X)$ every $a : T \to L$ has a T-complement. It is easy to see that for any system L_i, $i \in J$, if $a : T \to L_i$ have complements, this holds also for $\prod L_i$. $\square$

3.4. Thus, prohibiting a configuration P is preserved in a coproduct if P is a tree and $\coprod X_i$ is general, or if P is a general connected poset and the system X_i, $i \in J$ consists of any number of finite spaces with only finitely many isomorphism types. This situation changes at the first opportunity, namely with the simplest sharp non-tree, the diamond D and a coproduct of a countable increasing system of finite X_i. This will be discussed in this subsection.

Recall from [2] that $D \not\hookrightarrow X$ iff in $L = \mathcal{D}(X)$ we have

$$\text{(noD)} \qquad \begin{aligned} &\forall a_1, a_2\ \exists k\ \exists b^i, c_1^i, c_2^i, i = 1, \dots, k, \quad \text{such that } \bigwedge b_i = 0, \\ &c_1^i \vee c_2^i = 1,\ a_1 \wedge c_1^i \le b^i \vee a_2, \quad \text{and} \quad a_2 \wedge c_2^i \le b^i \vee a_i. \end{aligned}$$

3.4.1. Lemma. *For every n there is a finite X_i containing no diamond such that in the Heyting algebra $L_n = \mathcal{D}(X_n)$ there are a_1, a_2 for which the b^i from (noD) have to be at least n in number.*

Proof. Set $X_n = \{\perp, 1, 2, \top\} \times \{1, 2, \dots, n\}$ with the order defined by (write xy for (x, y))

$$\begin{aligned} \perp i \le 1j \le \top j \quad &\text{iff} \quad i = j, \\ \perp i \le 2j \le \top j \quad &\text{iff} \quad i \ne j. \end{aligned}$$

Obviously, $D \not\hookrightarrow X_n$. In L_n, the down-set lattice of X_n, consider

$$a_i^n = \{i, \perp\} \times \{1, 2, \dots, n\}.$$

Claim. *Let b, c_1, c_2 be such that $a_1^n \cap c_1 \subseteq b \cup a_2^n$, $a_2^n \cap c_2 \subseteq b \cup a_1^n$ and $\perp i, \perp j \notin b$ for two distinct i, j. Then $c_1 \cup c_2 \ne X$.*

Indeed, since b is a down-set it cannot contain any of $1i, 1j, 2i, 2j$. Since $a_1^n \cap c_1 \subseteq b \cup a_2^n$, $1i, 1j \notin c_1$ and hence $\top i, \top j \notin c_1$. Since $a_2^n \cap c_2 \subseteq b \cup a_1^n$, $2i, 2j \notin c_2$ and hence $c_2 \cap (\{\top\} \times \{1, \dots, n\}) = \emptyset$. Thus, $c_1 \cup c_2$ does not contain $\top i, \top j$.

Now if b^i, c_1^i, c_2^i satisfy (noD) then each of the b^i omits at most one $\perp i$ and hence they are at least n in number. $\square$

3.4.2. Proposition. *There exist finite X_i of constant height such that $D \hookrightarrow \coprod_{n=1}^{\infty} X_n$ while $D \not\hookrightarrow X_n$ for all n.*

Proof. In the example in 3.4.1, (noD) cannot be satisfied in $\prod L_n = \mathcal{D}(\coprod X_n)$ since if $a_i = (a_i^n)_n$ had the b^i, c_1^i, c_2^i, $i = 1, \dots, k$ as required, the k would suffice for all the coordinates. $\square$

3.4.3. In [2] it was shown that prohibiting a tree (or forest, co-tree or co-forest) P in a Priestley space X could be expressed by a first order lattice condition on $\mathcal{D}(X)$. The problem of whether this was true of other configurations remained open. Now we see from the example of the diamond D that the answer is negative. More generally, we have

Proposition. *Let P be a connected configuration such that the condition $P \not\hookrightarrow X$ can be expressed by a first order formula in the language of lattices holding in $\mathcal{D}(X)$. Then the condition $P \not\hookrightarrow X$ is preserved by coproducts.*

Proof. Let $P \not\hookrightarrow X_i$, $i \in J$. By 3.1, $\coprod_{i\in J} X_i$ can be represented as a disjoint union $\bigcup_{u\in\beta J} X_u$ with X_u order independent. Thus, if $P \hookrightarrow \coprod_{i\in J} X_i$, we have $P \hookrightarrow X_u$ for some of the ultrafilters u.

Now by [7], X_u is the Priestley dual of the ultraproduct $\prod_u \mathcal{D}(X_i)$. Suppose there is a first order formula F on $\mathcal{D}(X)$ characterizing $P \not\hookrightarrow X$. Then F holds for all the $\mathcal{D}(X_i)$ and hence, by Łoś's Theorem ([8]), it holds for $\prod_u \mathcal{D}(X_i) \cong \mathcal{D}(X_u)$, contradicting $P \hookrightarrow X_u$. $\square$

3.4.4. If $D \hookrightarrow \coprod_{i\in J} X_i$ then the order structure of X_i is not quite arbitrary, though. We have (recall 1.6)

Proposition. *If $D \hookrightarrow \coprod_{i\in J} X_i$ then for some $i \in J$ either $D \hookrightarrow X_i$ or $T(2,2) \hookrightarrow X_i$.*

Proof. The condition that $D \not\hookrightarrow X$ and $T(2,2) \not\hookrightarrow X$ can be expressed as

(1) no two incomparable non-minimal x_1, x_2 have a common upper bound.

We will show that this is equivalent to

(2) for every $a_1, a_2 \in L = \mathcal{D}(X)$ there exist b_1, b_2, c_1, c_2 such that $a_i \wedge c_i = 0$, $b_i \wedge a_i \le a_j \vee c_j$ and $b_1 \vee b_2 = 1$.

Since (2) is obviously preserved by products, the statement will follow. We will work with $X = \mathcal{P}(L)$.

(2)$\Rightarrow$(1): Let x_1, x_2 be incomparable non-minimal elements, and assume for the sake of argument that they are both contained in some x. Then there are $a_i \in x_i \setminus x_j$ such that if $c \wedge a_i = 0$ then $c \in x_i$. (Indeed, let $y \subsetneq x_i$. Choose $a'_i \in x_i \setminus y$, $a''_i \in x_i \setminus x_j$ and set $a_i = a'_i \vee a''_i$. Then $a_i \notin x_j$ and if $c \wedge a_i = 0$ we have $c \wedge a'_i = 0$ and hence $c \in y \subseteq x_i$.) Now for the b_i, c_i from (2), $c_i \in x_i$, $a_j \vee c_j \in x_j$ and hence $b_i \wedge a_i \in x_j$ and consequently $b_i \in x_j \subseteq x$ and $1 = b_1 \vee b_2 \in x$, a contradiction.

(1)$\Rightarrow$(2): Let (2) not hold. Then there are a_1, a_2 such that if $b_i \wedge a_i \le a_j \vee c_j$ and $a_j \wedge c_j = 0$ then $b_1 \vee b_2 \ne 1$. Then

$$I = \{b_1 \vee b_2 \mid \exists c_1, c_2,\ a_i \wedge c_i = 0 \text{ and } b_i \wedge a_i \le a_j \vee c_j\}$$

is a proper ideal and we can choose a proper prime ideal $x \supseteq I$. Set $F = L \setminus x$ and let

- I_i be the ideal generated by a_i and all the c with $c \wedge a_i = 0$,
- and F_i be the filter generated by F and a_i.

Then $I_j \cap F_i = \emptyset$. Indeed, if $b \wedge a_i \le a_j \vee c$ and $a_j \wedge c = 0$ then $b = 0 \vee b \in I \subseteq x$ and hence $b \notin F$. Choose prime ideals $x_i \supseteq I_i$ such that $x_i \cap F_j = \emptyset$. Now, since $x_i \cap (L \setminus x) = \emptyset$, $x_i \subseteq x$, and as $a_j \notin x_i \ni a_i$ they are incomparable. As x_i contains all the c with $c \wedge a_i = 0$, it is not minimal. (Let K be the proper filter $\{a' \wedge d \mid a' \ge a_i,\ d \notin x_i\}$. Then by 1.3 there is a prime ideal y disjoint from K and therefore contained in x_i and omitting a_i). $\square$

3.5. Since $T(2,2)$ is the cover $\widetilde{D}$ of the diamond D, the Proposition below is an extension of the statement 3.4.4 for coproducts of h-spaces. It can be conjectured that it holds in the general context; for our purposes the restricted statement suffices.

Proposition. *Let X_i, $i \in J$, be h-spaces and let $P \hookrightarrow \coprod X_i$ for a connected P. Then there is a cover Q of P and $i \in J$ such that $Q \hookrightarrow X_i$.*

Proof. For $p \in P$ choose a clopen down-set U_p such that $U_p \cap P = {\downarrow}p \cap P$, and such that $U_p \subseteq U_q$ whenever $p \leq q$. Set

$$A_p = \bigcap_{q \geq p} U_q \cap \bigcap_{q \not\geq p} (X \setminus U_q)$$

Note that for all p and q,

$$(*) \qquad \exists x \in A_q \ \exists y \in A_p, \ x \geq y \quad \text{iff} \quad q \geq p$$

For if $y \leq x \in A_q \subseteq U_q$ then $y \in U_q$ since U_q is a down-set, hence $A_p \cap U_q \neq \emptyset$ and $q \geq p$.

Define inductively $B_p \ni p$ as follows. For p minimal set $B_p = A_p$. If B_q have been already determined for $q < p$ set

$$B_p = A_p \cap \bigcap_{q<p} {\uparrow}B_q.$$

Then by 2.1 all the B_p are clopen. Obviously $p \in B_p$, and by the definition and $(*)$,

$$q \leq p \text{ iff } B_p \subseteq {\uparrow}B_q.$$

Since $Y = \bigcup X_i$ is dense in $\coprod X_i$ we have non-empty $C_p = Y \cap B_p$ and since $Y = {\uparrow}Y = {\downarrow}Y$ we see that

$$(**) \qquad q \leq p \text{ iff } C_p \subseteq {\uparrow}C_q.$$

Now choose for $p_n p_{n-1} \dots p_1 \in \widetilde{P}$ values $\varphi(p_n p_{n-1} \dots p_1) \in C_{p_n}$ as follows: $\varphi(p_1)$ is chosen arbitrarily, and if we already have $\varphi(p_{n-1} \dots p_1) \in C_{p_{n-1}}$ choose, by $(**)$, $\varphi(p_n \dots p_1) \in C_{p_n}$ such that

$$\varphi(p_n \dots p_1) \leq \varphi(p_{n-1} \dots p_1).$$

Set $Q = \{\varphi(\sigma) \mid \sigma \in \widetilde{P}\}$ and define

$$\delta : Q \to P$$

by setting $\delta(\varphi(p_n \dots p_1)) = p_n$. This is correct since if $\varphi(p_n \dots p_1) = \varphi(p'_m \dots p_1)$ then $C_{p_n} \cap C_{p'_m} \neq \emptyset$ and hence $p_n = p'_m$. Since φ is monotone by construction, it remains only to note that δ is monotone by $(**)$. $\square$

4. VARIETIES DETERMINED BY PROHIBITING CONFIGURATIONS

4.1. The class $\mathfrak{F}(\mathbb{P})$, consisting of those $X \in \mathbf{PSp}$ such that $P \not\hookrightarrow X$ for all $P \in \mathbb{P}$, is obviously closed under subspaces and finite coproducts. Sometimes it is closed under general coproducts (see 3.3) and sometimes not (see 3.4.2). With trivial exceptions it cannot be closed under finite products, for if X contains $x < y$ then every configuration is representable in a sufficiently large power X^n - see, e.g., [5], [10], or [13]. Nor is $\mathfrak{F}(\mathbb{P})$ closed under quotients, for such closure, together with finite coproducts, yields colimits, and any finite poset can be obtained from points and one-chains as a finite colimit – see the comma-construction, e.g. in [9].

4.2. In [2] we have shown, however, that prohibiting trees yield varieties of Heyting algebras. This is possible since the nature of quotients in **HPSp** differs from that in **PSp**, so that the above note on colimits does not apply. In this section we will deal with the question of when $\mathfrak{F}_H(\mathbb{P})$, the subcategory of **HPSp** defined analogously to $\mathfrak{F}(\mathbb{P})$ in **PSp**, constitutes in the duality a variety. By Birkhoff's Theorem, 2.2 and 2.3, this happens iff it is closed under

- all coproducts, and
- all h-quotients.

4.3. Proposition. *Let $\mathbb{P}$ be a class of finite connected posets. Then the following statements are equivalent.*

(1) *The dual image of $\mathfrak{F}_H(\mathbb{P})$ is a variety.*
(2) *The dual image of $\mathfrak{F}_H(\mathbb{P})$ is a quasivariety.*
(3) *For every $P \in \mathbb{P}$ and for every cover Q of P there is a $P' \in \mathbb{P}$ such that $P' \hookrightarrow Q$*

Proof. Since the subobjects come free, (1)≡(2).

(2)⇒(3): Let P be in $\mathbb{P}$ and let Q be a cover of P. Then we have an h-quotient $f : Q \to P$ and since $P \notin \mathfrak{F}_H(\mathbb{P})$ also $Q \notin \mathfrak{F}_H(\mathbb{P})$ and there has to be a $P' \in \mathbb{P}$ with $P' \hookrightarrow Q$.

(3)⇒(2): I. $\mathfrak{F}_H(\mathbb{P})$ is closed under coproducts.

Let $\coprod_{i\in J} X_i \notin \mathfrak{F}_H(\mathbb{P})$. Then $P \hookrightarrow \coprod X_i$ for some $P \in \mathbb{P}$. By 3.5 there is a cover Q of P, and an $i \in J$ such that $Q \hookrightarrow X_i$. If $P' \in \mathbb{P}$ is such that $P' \hookrightarrow Q$ we have $P' \hookrightarrow X_i$ and $X_i \notin \mathfrak{F}_H(\mathbb{P})$.

II. $\mathfrak{F}_H(\mathbb{P})$ is closed under h-quotients.

Let $f : X \to Y$ be an h-quotient and let $Y \notin \mathfrak{F}_H(\mathbb{P})$. Then there is a $P \in \mathbb{P}$ such that $P \hookrightarrow Y$ and by 2.7.1 there is a cover Q of P such that $Q \hookrightarrow X$. If $P' \in \mathbb{P}$ is such that $P' \hookrightarrow Q$ we have $P' \hookrightarrow X$ and $X \notin \mathfrak{F}_H(\mathbb{P})$. $\square$

4.4. Example. Thus we have for instance the following two distinct varieties based on the prohibition of the diamond: $\mathfrak{F}_H(D, T(2,2))$ and $\mathfrak{F}_H(D, T')$, where T' is obtained from $T(2,2)$ by omitting one of the two minimal points. Note that neither of the two systems can be replaced by a system of trees to obtain the same variety. For if either system, denoted $\mathfrak{V}$, were of the form $\mathfrak{F}_H(\mathbb{T})$ for a system $\mathbb{T}$ of trees then we would have a $T \in \mathbb{T}$ such that $T \hookrightarrow D$. But every $T \hookrightarrow D$ lies in $\mathfrak{V}$.

5. Finitely generated varieties determined by systems of trees

5.1. Let $\mathbb{G}$ be a finite system of finite posets. Denote by

$$\mathfrak{V}(\mathbb{G})$$

the dual image of the variety of Heyting algebras generated by all the $\mathcal{D}(P)$, $P \in \mathbb{G}$. By abuse of language we will speak of a *variety of h-spaces.* It is easy to see that

$$\mathfrak{V}(\mathbb{G}) = \{X \mid X \precsim \coprod_{i\in J} P_i,\ P_I \in \mathbb{G}\}. \tag{5.1.1}$$

In this section we will present a procedure to decide whether $\mathfrak{V}(\mathbb{G})$ can be described as a $\mathfrak{F}_H(\mathbb{T})$ where $\mathbb{T}$ is a system of trees. (The symbol $\mathbb{T}$ will always be used in this sense). Recall that by [2], but also by 4.3, $\mathfrak{F}_H(\mathbb{T})$ is always a variety.

5.2. Set $\tau(\mathbb{G}) = \{T \mid T \text{ is a tree, } \forall P \in \mathbb{G},\ T \not\hookrightarrow P\}$. We have

Proposition. *There is a* $\mathbb{T}$ *such that* $\mathfrak{V}(\mathbb{G}) = \mathfrak{F}_H(\mathbb{T})$ *iff* $\mathfrak{V}(\mathbb{G}) = \mathfrak{F}_H(\tau(\mathbb{G}))$.

Proof. Since obviously $\mathbb{G} \subseteq \mathfrak{F}_H(\tau(\mathbb{G}))$ and $\mathfrak{V}(\mathbb{G})$ is the smallest variety containing $\mathbb{G}$ we have

$$\mathfrak{V}(\mathbb{G}) \subseteq \mathfrak{F}_H(\tau(\mathbb{G})). \tag{5.2.1}$$

Now obviously $\mathbb{T}_1 \subseteq \mathbb{T}_2$ implies $\mathfrak{F}_H(\mathbb{T}_1) \supseteq \mathfrak{F}_H(\mathbb{T}_2)$. Thus, if $\mathfrak{V}(\mathbb{G}) = \mathfrak{F}_H(\mathbb{T})$ then we see, first, that $\mathbb{T} \subseteq \tau(\mathbb{G})$ and then $\mathfrak{F}_H(\mathbb{T}) = \mathfrak{V}(\mathbb{G}) \subseteq \mathfrak{F}_H(\tau(\mathbb{G})) \subseteq \mathfrak{F}_H(\mathbb{T})$. $\square$

5.3. For a poset X set

$$\mathcal{T}(X) = \{T \text{ a tree} \mid T \not\hookrightarrow X\}.$$

In view of 5.2, $\mathfrak{V}(\mathbb{G})$ can be obtained as $\mathfrak{F}_H(\mathbb{T})$ for a system of trees $\mathbb{T}$ iff we have the implication

if X is such that $T \not\hookrightarrow X$ for every $T \in \tau(\mathbb{G})$ then $X \in \mathfrak{V}(\mathbb{G})$,

in other words,

$$((\forall P \in \mathbb{G},\ T \not\hookrightarrow P \ \Rightarrow\ T \not\hookrightarrow X) \ \Rightarrow\ X \in \mathfrak{V}(\mathbb{G})$$

which can be rewritten as

$$\mathcal{T}(X) \subseteq \bigcup_{P\in\mathbb{G}} \mathcal{T}(P) \quad\Rightarrow\quad X \in \mathfrak{V}(\mathbb{G}). \tag{5.3.1}$$

5.4. Interpreting for our context Jónsson's Lemma on subdirectly irreducibles, we obtain

Lemma. *Let X be sharp and finite, $X \in \mathfrak{V}(\mathbb{G})$. Then $X \sqsubseteq\!\!\!< P$ for some $P \in \mathbb{G}$.*

This statement can also be easily deduced from 3.4.

5.5. For a system $\mathbb{G}$ set

$$\mathbf{n}(\mathbb{G}) = \max\{|P| \mid P \in \mathbb{G}\}$$

Proposition. *A finitely generated variety $\mathfrak{V}(\mathbb{G})$ is $\mathfrak{F}_H(\mathbb{T})$ for a system of trees $\mathbb{T}$ iff for every sharp X with $|X| \le \mathbf{n}(\mathbb{G})^2$,*

$$\mathcal{T}(X) \subseteq \bigcup_{P\in\mathbb{G}} \mathcal{T}(P) \quad\Rightarrow\quad \exists P \in \mathbb{G},\ X \sqsubseteq\!\!\!< P.$$

Thus, the question can be decided in a finite number of steps.

Proof. I. First, we will prove that it suffices to check the condition (5.3.1) just for the sharp X.

Let Y be a general h-space. For $m \in \max(Y)$ consider $Y(m) =\downarrow m$. Then $Y(m)$ is an h-space and the embeddings $j_m : Y(m) \subseteq Y$ are h-maps. Thus we have an h-map $f : \coprod Y(m) \to Y$ defined by $f \cdot \iota_m = j_m$, obviously an h-quotient. Now let (5.3.1) hold for sharp X and let $\mathcal{T}(Y) \subseteq \bigcup_{P\in\mathbb{G}} \mathcal{T}(P)$. We have $\mathcal{T}(Y(m)) \subseteq \mathcal{T}(Y)$ and hence $\mathcal{T}(Y(m)) \subseteq \bigcup_{P\in\mathbb{G}} \mathcal{T}(P)$, and $Y(m)$ is in $\mathfrak{V}(\mathbb{G})$. As $Y < \coprod Y(m)$, also $Y \in \mathfrak{V}(\mathbb{G})$.

II. Now about the size. If a sharp X is bigger than $\mathbf{n}(\mathbb{G})^2$ then it contains, by Dilworth's Theorem, a tree bigger than $\mathbf{n}(\mathbb{G})$ (this also includes the case of infinite X) and hence not contained in any of the $P \in \mathbb{G}$.

III. Finally, by 5.4 we can replace $X \in \mathfrak{V}(\mathbb{G})$ by the existence of a $P \in \mathbb{G}$ such that $X \sqsubseteq\!\!\!< P$. $\square$

6. Well-characterized configurations

6.1. A sharp finite poset P is said to be *well-characterized* if

$$\mathcal{T}(X) \subseteq \mathcal{T}(P) \quad \Rightarrow \quad X \sqsubseteq\!\!\!< P.$$

By (5.3.1) this is the same as saying that $\mathfrak{V}(P)$ is $\mathfrak{F}_H(\mathbb{T})$ for a class of trees. The terminology is borrowed from computer science, where "good characteristics" means, loosely speaking, that both a property V and its negation $\neg V$ may be recognized by specific finite means. If P has the property, obviously weaker, that

$$\text{for every tree } T, \quad T \hookrightarrow P \;\Rightarrow\; T \sqsubseteq\!\!\!< P$$

we will say that P is *tree-varietal.*

6.2. Notation. For $p \in P$ set

$$\mathfrak{m}(p) = \min P \cap \downarrow\! p, \quad \text{and} \quad P_{\langle k \rangle} = \{p \in P \mid |\mathfrak{m}(P)| = k\}.$$

6.3. Lemma. *Let P be a tree. Then for $p, q \in P$ either $\mathfrak{m}(p) \cap \mathfrak{m}(q) = \emptyset$ or p, q are comparable.*

Proof. In a tree we cannot have $x < p, q$ with incomparable p, q. $\square$

6.4. Lemma. *Let $f : P \to Q$ be an h-quotient. Then $f(\min P) = \min Q$.*

Proof. If $x \in \min P$ then $\downarrow\! f(x) = f(\{x\})$ and hence $f(x) \in \min Q$. If $y \in \min Q$ choose $x' \in P$ with $f(x') = y$, and an $x \in \mathfrak{m}(x')$. Then $f(x) \in \downarrow\! f(x') = \{y\}$. $\square$

6.5. Lemma. *Let $R \sqsubset Q$, let $f : P \to Q$ be an h-quotient and let $|\min R| \geq |\min P|$. Then f restricts to a bijection from $\min P$ onto $\min Q = \min R$.*

Proof. Since $R = \downarrow\! R$ in Q we have $\min R \subseteq \min Q$. By 6.4, $|\min R| \leq |\min Q| \leq |\min P| \leq |\min R|$. Thus, $\min Q = \min R$ and f restricted to $\min P$ is one-one. $\square$

6.6. Lemma. *Let $f : P \to Q$ be an h-quotient and let $|\min Q| = |\min P|$. Then $f(P_{\langle 1\rangle}) = Q_{\langle 1\rangle}$ and $f(P \setminus P_{\langle 1\rangle}) = Q \setminus Q_{\langle 1\rangle}$.*

Proof. Let $x \in P_{\langle 1\rangle}$ and suppose for the sake of contradiction that $y_1, y_2 \in \mathfrak{m}(f(x))$ are distinct. Choose $x_1', x_2' \le x$ with $y_i = f(x_i')$, and choose $x_i \in \mathfrak{m}(x_i')$. Then $f(x_i) = y_i$ and hence x_1, x_2 are distinct elements of $\mathfrak{m}(x)$, a contradiction. Now let $x \notin P_{\langle 1\rangle}$, say $x_1, x_2 \in \mathfrak{m}(x)$ are distinct. Then by 6.5 $f(x_1), f(x_2)$ are distinct in $\mathfrak{m}(f(x))$. □

6.7. Lemma. *Let $R \sqsubset Q$, let $f : P \to Q$ be an h-quotient, let $|\min R| = |\min Q|$ and $|R_{\langle 1\rangle}| = |P_{\langle 1\rangle}|$. Then $f = f_1 \cup f_c$ where $f_1 : P_{\langle 1\rangle} \to R_{\langle 1\rangle} = Q_{\langle 1\rangle}$ is a bijection onto and $f_c : P \setminus P_{\langle 1\rangle} \to Q \setminus Q_{\langle 1\rangle}$ is onto. Consequently,*

$$\text{(6.7.1)} \qquad \textit{for } x \in P_{\langle 1\rangle} \textit{ and } y \in P, \quad x \le y \textit{ iff } f(x) \le f(y).$$

Proof. Take the decomposition from 6.6. Since $R = \downarrow R$ in Q, $R_{\langle 1\rangle} \subseteq Q_{\langle 1\rangle}$ and hence $|R_{\langle 1\rangle}| \le |Q_{\langle 1\rangle}| \le |P_{\langle 1\rangle}| = |R_{\langle 1\rangle}|$ and the statement immediately follows. □

6.8. Lemma. *Let P be tree-varietal and let $\mathfrak{m}(x') \subseteq \mathfrak{m}(y) = \min P$. Then there is an $x \le y$ such that $\mathfrak{m}(x) = \mathfrak{m}(x')$.*

Proof. Take a chain C of length ht(P) and consider the tree $T = C \cup \min P \hookrightarrow P$. By 2.5 there is an h-quotient $f : P \to T$. On T define an equivalence $\sim$ by setting

$$s \sim t \quad \text{iff} \quad \mathfrak{m}(s) = \mathfrak{m}(t).$$

If $s \le r \le t$ and $s \sim t$ then obviously $s \sim r$; if $s \le t \sim t'$ we have $\mathfrak{m}(s) \subseteq \mathfrak{m}(t) = \mathfrak{m}(t')$ and by 6.3, s is comparable with t'. If $\mathfrak{m}(s) = \mathfrak{m}(t')$ then $s \sim t' \le t$, and if $\mathfrak{m}(s) \ne \mathfrak{m}(t')$ then $s < t'$ and we have $s \sim s \le t'$. Thus $\sim$ is an h-congruence (see 2.3.1), and consequently also the equivalence $\approx$ defined by

$$x \approx y \quad \text{iff} \quad f(x) \sim f(y)$$

is an h-congruence on P. Since by 6.5 f is a bijection on $\min P$ we have

$$x \approx y \quad \text{iff} \quad \mathfrak{m}(x) = \mathfrak{m}(y).$$

Now we have $x' \le \top \approx y$ and hence there is an $x \le y$ such that $x' \approx x$. □

6.9. Proposition. *Let P be tree-varietal with $\mathrm{wd}(P) = n > 1$. Then $P = P_{\langle 1\rangle} \cup P_{\langle n\rangle}$, $|\min P| = n$, and $P_{\langle 1\rangle}$ is a disjoint union of independent chains.*

Proof. Let T consist of top t_0 and n independent elements $y_1, \dots, y_n < y_0$. Since $T \hookrightarrow P$ we have an h-quotient $f : P \to Q$, and $T = \downarrow T$ in

Q. As $|\min P| \le n = |\min T|$, by 6.5 f maps $\min P$ bijectively onto $\min Q = \min T$. Let $x_i \in \min P$ be sent to y_i. For $x \in P$ we have

$$x \ge x_i \quad \text{iff} \quad f(x) \ge y_i. \tag{6.9.1}$$

(For if $y_i = f(x')$, $x' \le x$ and $x'' \in \mathfrak{m}(x')$ we have $f(x'') = y_i$ and hence $x'' = x_i$). Take an x_0 such that $f(x_0) = y_0$. Let $x \in P$ be arbitrary. Since $\mathfrak{m}(x) \subseteq \mathfrak{m}(x_0) = \min P$ there is, by 6.8, an $x' \le x_0$ such that $\mathfrak{m}(x') = \mathfrak{m}(x)$. If $f(x') = y_0 = f(x_0)$, x is in $P_{\langle n\rangle}$. If $f(x') = y_i$ for some i, $\mathfrak{m}(x) = \mathfrak{m}(x') = \{x_i\}$ and $x \in P_{\langle 1\rangle}$.

As for the second statement, its falsity would imply the existence of some $x \in \min P$ and distinct $z_1, z_2 \in P_{\langle 1\rangle}$ such that $z_1, z_2 > x$, in which case $\{z_1, z_2\} \cup (\min P \setminus \{x\})$ would be an antichain of size $n+1$. □

6.10. Proposition. *Let P be tree-varietal,* $\mathrm{wd}(P) = n > 1$. *Then for every $x \in P_{\langle 1\rangle}$ and $y \in P_{\langle n\rangle}$, $x \le y$.*

Proof. Set $U = \{y \in P_{\langle n\rangle} \mid \forall x \in P_{\langle 1\rangle},\ y \ge x\}$. Obviously $\top_P \in U$. Let

$$C:\quad t_k < t_{k-1} < \cdots < t_0 \tag{6.10.1}$$

be a longest chain in U. Consider the tree

$$T = P_{\langle 1\rangle} \cup C.$$

Let $T \sqsubset Q$ and let $f : P \to Q$ be an h-quotient. Since obviously $T_{\langle 1\rangle} = P_{\langle 1\rangle}$ we can apply Lemma 6.7. Let $f(x) \in C$. If $y \in P_{\langle 1\rangle}$ then $f(y) \in T_{\langle 1\rangle} = Q_{\langle 1\rangle} = P_{\langle 1\rangle}$ and we have, by (6.7.1), $y \le x$ so that $x \in U$. Thus

$$f^{-1}(C) \subseteq U. \tag{6.10.2}$$

Consequently if we find, using the property of h-maps, a chain $x_k < x_{k-1} < \cdots < x_0$ in P such that $f(x_i) = t_i$, this chain is in U, and by the maximality of k, $x_0 = \top_P$. Hence $f(x_0) = \top_Q = \top_T$ and we have $T = {\downarrow}\top_T = {\downarrow}\top_Q = Q$. Now if $x \in P_{\langle n\rangle}$ then $f(x) \in Q_{\langle n\rangle} = T_{\langle n\rangle}$ and hence $f(x) \in C$, and $x \in U$ by (6.10.2) again. □

6.11. Recall the notation from 1.6. We have

Observations. *1. For a tree T, $T \hookrightarrow D(n,k)$ iff T is a subtree of $B(n,k)$.*

2. If P is well characterized and if both $H \hookrightarrow P$ and $F \hookrightarrow P$ then either $N \hookrightarrow P$ or $M \hookrightarrow P$.

Proof. 1. If a tree $T \hookrightarrow D(n,k)$ descends from $\top_T$, the first moment it bifurcates it has to stop.

2. Since the only trees in N are subtrees of F and H, we have $N \sqsubset\!\!\!\sim P$. Consider an h-quotient $f : P \to Q$ with $N \sqsubset Q$ and choose

a', b' such that $f(a') = a$ and $f(b') = b$. Then there are $c', d' < a'$ and $d'' < b'$ such that $f(c') = c$ and $f(d') = f(d'') = d$. Then either we could have chosen $d' = d''$ and $N \hookrightarrow P$ or this was not possible and then $M \hookrightarrow P$. $\square$

6.12. Theorem. *A sharp finite poset with more than one element is well characterized iff it is either $D(n, k)$ or $T(n, k)$.*

Proof. I. Let P be well characterized, $\mathrm{wd}(P) = n$. Then it is tree-varietal and hence can be decomposed into $P_{\langle n\rangle}$ and $P_{\langle 1\rangle}$. Let k be the length of a longest chain in $P_{\langle n\rangle}$. Then $B(n, k) \hookrightarrow P$ and hence, by 6.11.1, $D(n, k) \sqsubseteq P$. Let $D(n, k) \sqsubset Q$ and let $f : P \to Q$ be an h-quotient. Then necessarily $f(P_{\langle 1\rangle}) = \{k\} \times \{1, 2, \ldots, n\}$ so that $f(P_{\langle n\rangle}) \supseteq \{\top\} \cup \{1, 2, \ldots, k-1\} \times \{1, \ldots, n\} = D'$. Now $|P_{\langle n\rangle} \setminus \{\top_P\}| \leq (k-1) \cdot n$ by Dilworth's Theorem, which implies that f is one-one on $P_{\langle n\rangle}$, and being an h-map it yields an isomorphism $P_{\langle n\rangle} \cong D'$. Then by 6.7 and 6.10 we see that P is D' with n independent chains attached below. If $n \geq 2$, $k \geq 2$ and if some of the chains have at least two elements then P contains both F and H and hence by 6.11.2 it contains either N or M, which it does not. Thus either $P = D(n, k)$ or $P = \{\top\} \cup P_{\langle 1\rangle}$. In the latter case, if some chain $c_k < c_{k-1} < \cdots < c_1$ in $P_{\langle 1\rangle}$ were longer than another, say $d_l < d_{l-1} < \cdots < d_1$, then adding $c_k < d_l$ into the order makes a poset Q containing no trees but those representable in P, while $Q \sqsubseteq P$ is impossible. Thus $P = T(n, k)$.

II. Proving that $D(n, k)$ is well characterized is easy. By 6.11.1 we have to prove that if a sharp P contains only subtrees of $B(n, k)$ then $P \sqsubseteq D(n, k)$. But such a P consists of a top $\top$ and at most k layers $L_1, \ldots, L_l$ such that $x \leq y$ iff $y = \top$ or $x = y$ or $x \in L_i$, $y \in L_j$ with $i > j$. Then, producing an h-quotient $f : D(n, k) \to P$ is straightforward.

Now about $T(n, k)$. Let $\mathcal{T}(P) \subseteq \mathcal{T}(T(n, k))$. If P is a chain, $P \sqsubset T(n, k)$. If not, we can have $x \geq x_1, x_2$ with $x_1 \neq x_2$ only if $x = \top$, since $T(n, k)$ does not contain F. Let $x_1, x_2, \ldots, x_l$ be all the elements of P immediately below $\top_P$. Since, for $l \leq k$, $T(n, l) < T(n, k)$ we can assume that $l = k$. Define a mapping $f : T(n, k) \to P$ by setting

$$f(\top) = \top, \quad f(1, i) = x_i,$$
$$f(i+1, j) \prec f(i, j) \text{ if there is such an } f(i+1, j) \in P \text{ and}$$
$$f(i+1, j) = f(i, j) \text{ if there is none.}$$

As the chains in P cannot be longer than those in $T(n, k)$ we readily see that f is an h-quotient. $\square$

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Department of Mathematics, University of Denver, Denver, CO 80208

E-mail address: rball@math.du.edu

Department of Applied Mathematics and ITI, MFF, Charles University, CZ 11800 Praha 1, Malostranské nám. 25

E-mail address: pultr@kam.ms.mff.cuni.cz

Department of Mathematics, University of Manitoba, Winnipeg, Canada R3T 2N2

E-mail address: sichler@cc.umanitoba.ca