Hamilton cycles in prisms over graphs

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Note: In the black-and-white version of this paper, the mapping of colors to gray shades is as follows:

yellow	gray
orange	red
green	blue

1 Introduction

The concept of a Hamilton cycle is central in graph theory. It is the subject of many results and conjectures, and moreover, a number of other concepts derive from this one. For instance, one can view a Hamilton cycle as a special sort of a closed walk — namely one that visits each vertex exactly once. If we define a

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k-*walk* to be a spanning closed walk visiting no vertex more than k times, then a Hamilton cycle is just a 1-walk (and thus a k-walk for any $k \geq 1$). A closely related notion is that of a k-*tree*, which is a spanning tree with all vertices of degree at most k (in particular, a 2-tree is precisely a Hamilton path). It is not hard to show (see [13]) that any graph with a k-tree has a k-walk, and that the existence of a k-walk guarantees the existence of a $(k+1)$-tree, for any k. Thus

$$\text{1-walk (Hamilton cycle)} \Rightarrow \text{2-tree (Hamilton path)} \Rightarrow \text{2-walk} \Rightarrow \text{3-tree} \Rightarrow \dots$$

For information on the existence of k-walks, k-trees and related topics, refer, e.g., to [7].

The central theme of the present paper is another relaxation of hamiltonicity: the property of having a hamiltonian prism. (The *prism* over a graph G is obtained by taking two copies of G and joining the two copies of each vertex by an edge.) A crucial observation (shown below) is that the property of having a hamiltonian prism is 'sandwiched' between the existence of a 2-tree and the existence of a 2-walk. That is:

$$\text{2-tree} \Rightarrow \textbf{hamiltonian prism} \Rightarrow \text{2-walk} \tag{1}$$

and none of the implications can be reversed.

Let us consider the first of the implications. If G has a Hamilton path, then its prism is hamiltonian (just take the Hamilton path in each copy and add the two edges necessary to make a Hamilton cycle in the prism). The converse implication does not hold, as demonstrated by the complete bipartite graph $K_{2,4}$ which has no Hamilton path, but does have a hamiltonian prism (see Fig. 1).

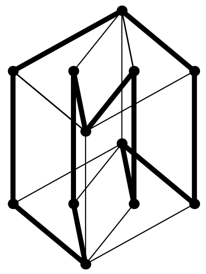


Figure 1: A Hamilton cycle in the prism over $K_{2,4}$.

As for the second implication in (1), it is easy to observe that any graph G with a hamiltonian prism has a 2-walk which follows the edges of G corresponding to the edges of the Hamilton cycle in the prism. There is again no implication in the converse direction, as shown by the graph in Fig. 2 which has a 2-walk, but its prism is clearly non-hamiltonian. Similar examples can be constructed with arbitrarily large connectivity, see Section 6.

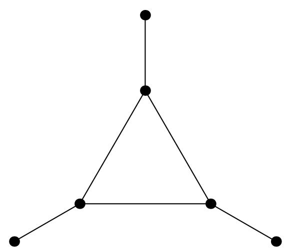


Figure 2: A graph with a 2-walk but with a non-hamiltonian prism.

Summing up, we have shown that the property of having a hamiltonian prism (or *prism-hamiltonicity* as we call it occasionally) is indeed situated between the hamiltonicity and the existence of a 2-walk. In this sense, it is a relaxation of hamiltonicity which is, however, not too far from the latter. Many classical questions that have been asked about the existence of Hamilton cycles make good sense when related to prisms. Indeed, each of Sections 3 to 7 of the present paper is inspired by such a classical problem. Over the time, some of the problems have been solved in the positive, some in the negative, still others remained conjectures, but in each case, we can obtain interesting questions concerning prisms.

Our main areas of interest are: planar graphs (Section 3), line graphs (Section 4), 4-regular graphs (Section 5), tough graphs (Section 6), and squares of graphs (Section 7)

Let us remark that historically, the interest in prisms was partly due to the (still open) conjecture of D. Barnette that the graphs of 4-polytopes are hamiltonian, in which prisms appear as a particular case. Rosenfeld and Barnette [16] showed, in 1973, that cubic planar 3-connected graphs have hamiltonian prisms if the Four Color Conjecture (open at that time) was true. Fleischner [10] found a proof avoiding the use of the Four Color Theorem. Eventually, Paulraja [15] showed that planarity is inessential here.

Theorem 1.1 *Any 3-connected cubic graph has a hamiltonian prism.*

One of the key problems in this paper is whether Rosenfeld and Barnette's result can be generalized in a different direction: retaining the planarity, but dropping the hypothesis that the graph is cubic. This is the first question we consider after we introduce some necessary notation and conventions.

Let us conclude this introduction by an intriguing problem which does not quite fit into any of the subsequent sections: Is there an analogue of the well-known Bondy–Chvátal closure concept for prism-hamiltonicity? In particular, we ask the following:

Problem 1.2 Let G be a graph on n vertices, and let x, y be two non-adjacent vertices such that the sum of their degrees is at least n. Is it true that G has a hamiltonian prism if and only if $G + xy$ does?

2 Notation and definitions

The graphs we shall deal with are without loops and multiple edges. In cases where we do allow multiple edges, we make this clear by referring to a *multigraph*.

The definition of the prism given in Section 1 can be rephrased as follows: The prism over G is the Cartesian product $G\square K_2$ of G with K_2. We identify G with one of its two copies in $G\square K_2$, so the two 'clones' of a vertex $v \in V(G)$ may be denoted by v and v^*. The same notation is used for edges. Edges of the form vv^* are referred to as *vertical*.

There is a convenient way of representing 2-factors in $G\square K_2$ by certain edge colorings in the graph G. (We remark that a similar coloring scheme was defined in [5] in relation to hamiltonian decompositions.) Any 2-factor F in $G\square K_2$ has an associated coloring of a subset of $E(G)$ in three colors (blue, yellow and green), defined as follows. For any edge $e \in E(G)$,

$$e \text{ is colored } \begin{cases} \text{blue} & \text{if } F \text{ contains } e \text{ but not } e^*, \\ \text{yellow} & \text{if } F \text{ contains } e^* \text{ but not } e, \\ \text{green} & \text{if } F \text{ contains both } e \text{ and } e^*. \end{cases}$$

If F uses none of e and e^*, then e gets no color at all.

Which colorings come from 2-factors in $G\square K_2$ in this way? One necessary condition is obvious. Consider G with a subset of $E(G)$ colored in blue (B), yellow (Y) and green (G). We define the *type* of a vertex v to be the multiset of the colors of all edges incident with v. If the coloring was obtained from a 2-factor, then the type of any vertex contains the same number of blue edges and yellow edges. Furthermore, the numbers of green and (say) blue edges must sum up to either 1 or 2, the former corresponding to the edge vv^* being used by the 2-factor. It follows that the only possible types are G, BY, GG, GBY and BBYY. An edge coloring in which each vertex has one of these types will be called *admissible*.

Conversely, every admissible coloring determines a 2-factor F in an obvious way, as illustrated in Fig. 3: all the edges of F in the two copies of G are given by the above rule (e.g. F will use e^* but not e if the color of $e \in E(G)$ is yellow). The *vertical* edges vv^* are added wherever necessary to make F into a 2-factor. This is a bijective correspondence between admissible colorings and 2-factors in $G\square K_2$. However, we shall use admissible colorings mostly to describe Hamilton cycles in the prism.

Let us pause to present a useful sufficient condition for prism-hamiltonicity from [5]. A *spanning cactus* in a graph G is a spanning connected subgraph H such that every vertex of degree ≥ 3 in H is contained in precisely one cycle of H, and all other vertices are contained in at most one cycle of H. The cactus is said to be *even* if all of its cycles are of even length (i.e. if it is bipartite). Assume now that G has a spanning even cactus H, color the edges of the cycles blue and yellow (in an alternating way), and color all the other edges of H green.

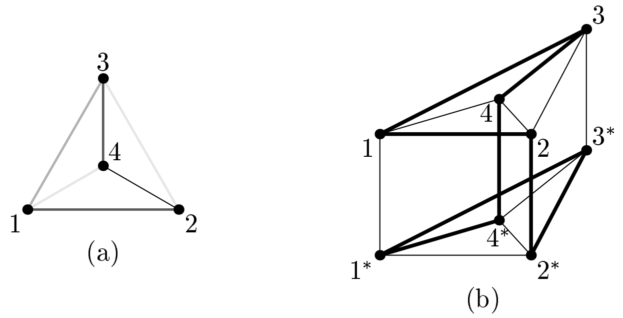


Figure 3: (a) A coloring of the complete graph K_4 (an uncolored edge is shown black). (b) The corresponding Hamilton cycle (bold).

(The edges in $E(G) - E(H)$ are left uncolored.) We refer the reader to [5] for an (easy) proof of the following proposition and for a (less easy) application of it, providing a new proof of Theorem 1.1.

Proposition 2.1 ([5]) *If G contains a spanning even cactus, then its prism is hamiltonian.*

3 Planar graphs

A classical theorem of Tutte [20] states that all 4-connected planar graphs are hamiltonian. There are well-known examples of non-hamiltonian 3-connected planar graphs. On the other hand, Gao and Richter [11] showed that 3-connected planar graphs have 2-walks, extending the result of Barnette [2] that they have 3-trees.

It is natural to ask whether prisms over 3-connected planar graphs are hamiltonian. Surprisingly, this is an open question.

Conjecture 3.1 ([16]) *Any 3-connected planar graph is prism-hamiltonian.*

Note that *cubic* 3-connected graphs (planar or not) are prism-hamiltonian by Theorem 1.1. In fact, cubic 3-connected planar graphs are conjectured in [1] to have a much stronger property, namely that their prism has a decomposition into two Hamilton cycles. For cubic 3-connected planar *bipartite* graphs, the existence of such a decomposition was proved in [5].

We remark that Conjecture 3.1 cannot be extended to 2-connected planar graphs: for instance, the complete bipartite graph $K_{2,5}$ is 2-connected and planar, but its prism is not hamiltonian.

Conjecture 3.1 is open even for planar triangulations. In the following subsections, we prove the conjecture for two classes of 3-connected planar graphs, namely for chordal 3-connected planar graphs and for Halin graphs.

3.1 Kleetopes

A *kleetope* is a plane graph obtained from a drawing of the complete graph K_4 by successive subdivisions of internal faces. (A face F is *internal* if it differs from the infinite face; the subdivision of F consists in adding a new vertex v_F inside F, and joining it to all the three vertices of F.) See Fig. 4a for an example of a kleetope.

Recall that a graph is *chordal* if it contains no chordless cycle of length at least 4. It is known that kleetopes coincide with (drawings of) 3-connected chordal planar graphs. (This observation is implicit in [19, Section 2].) As a partial result in the direction of Conjecture 3.1, we shall show that 3-connected chordal planar graphs (i.e., kleetopes) are prism-hamiltonian.

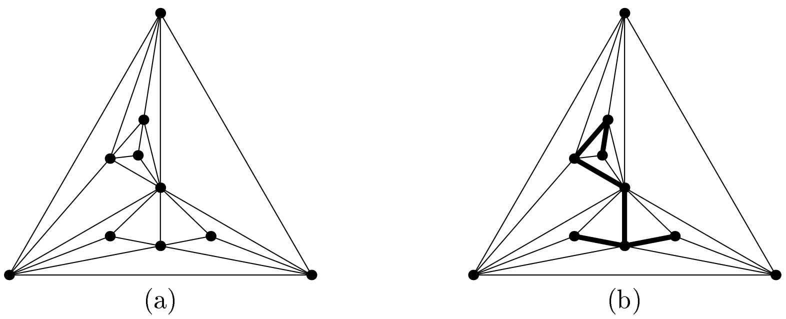


Figure 4: (a) A kleetope G. (b) The structure tree of G.

An *internal vertex* of a kleetope G is any vertex not incident with the infinite face of G. The *depth* $\lambda(v)$ of a vertex v is defined as follows. In any drawing of K_4, the depth of the internal vertex is 0 and the depths of the other vertices are -1. If G arises from G' by subdividing an internal face $F = x_1x_2x_3$, then the depth of the new vertex v_F is $\lambda(v_F) = \max \lambda(x_i) + 1$ and all the other depths are as in G'.

Observe that in any G, there is no edge joining internal vertices of the same depth. It follows that every vertex v with $\lambda(v) > 0$ has a *unique* neighbor $p(v)$ (the *parent* of v) such that $\lambda(p(v)) = \lambda(v) - 1$.

We may define the *structure tree* $T(G)$ of G as the tree on all internal vertices of G such that every edge of T joins an internal vertex v (of positive depth) to its parent. (See Fig. 4b.) In particular, note that if G is any drawing of K_4, then $T(K_4)$ consists of a single vertex. In all other cases, the leaves of $T(G)$ are precisely all internal vertices of degree 3, plus possibly the unique vertex of zero depth.

Lemma 3.2 *Every kleetope G can be obtained from K_4 by a sequence of steps, each of which is the simultaneous subdivision of one, two or three faces containing a common internal vertex of degree 3.*

Proof. Consider the structure tree $T(G)$ of a given kleetope G. We may assume $G \neq K_4$, so let m be a leaf of $T(G)$ of the largest depth, and let u be the parent of m. Remove all *children* of u (the leaves whose parent is u) to obtain a graph G' which, by the induction hypothesis, can be constructed as stated in the lemma. The vertex u, being a leaf of $T(G')$, is of degree 3 in G'. Subdividing all the faces of G' incident with u which correspond to the removed leaves of $T(G)$, we obtain the desired construction of G. $\square$

Consider a coloring of the graph G associated to a Hamilton cycle C of the prism over G. Recall that an edge e is green in this coloring if and only if both e and e^* are in C. We shall say that the green edge e is *balanced* (in the coloring) if C traverses e and e^* in different directions.

Theorem 3.3 *The prism over any kleetope is hamiltonian.*

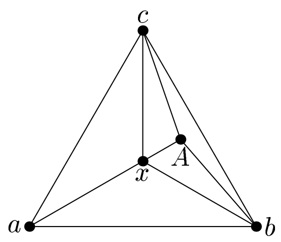
Proof. We prove (by induction) the stronger statement that the prism over any kleetope G contains a Hamilton cycle H such that in the associated coloring of G,

> (*) every degree 3 internal vertex v is incident with at most two colored edges, and if these are two green edges, then they are balanced.

For $G = K_4$, such a Hamilton cycle is shown in Fig. 3b. Thus let G arise from G' by subdividing some faces sharing an internal vertex x of degree 3, as in Lemma 3.2. The neighbors of x in G' are denoted by a, b, c. We let the new vertices of G be denoted by (some of) the letters A, B, C, where A subdivides the face *not* containing a, and analogously for B and C. By symmetry, we may distinguish only 3 cases: the set N of the new vertices is $\{A\}$, $\{A, B\}$ or $\{A, B, C\}$. Each of the cases splits up into several subcases depending on which edges are used by the Hamilton cycle H' of G'. We give tables indicating how the coloring is to be extended to G in each subcase (symmetric subcases omitted). The first column of the tables lists all the possible combinations of colors of the edges adjacent to x (subject to (*) and up to symmetry). To perform the modification, first uncolor all edges listed in the first column, and then apply the coloring in the second column. A path with all edges green is referred to as a green path. Similarly, a blue-yellow path has edges colored alternatingly blue and yellow, starting with blue.

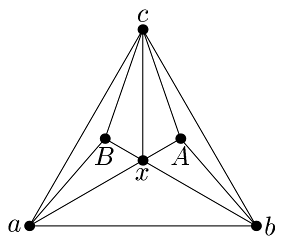
It is straightforward to check that the extended colorings do correspond to Hamilton cycles satisfying (*).

Case 1. There is one new vertex A.



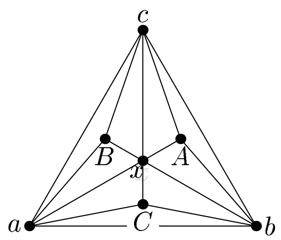
replace:	with:
green ax	green axA
green cx	green cxA
blue-yellow axb	blue-yellow axb + green xA
blue-yellow cxb	blue-yellow cxb + green xA
green axb	green $axAb$
green cxb	green $cxAb$

Case 2. There are two new vertices: A and B.



replace:	with:
green ax	green $aBxA$
green cx	green $cBxA$
blue-yellow axb	blue-yellow $aBxAb$
blue-yellow cxb	blue-yellow $cBxAb$
green axb	green $aBxAb$
green cxb	green $cBxAb$

Case 3. The new vertices are A, B and C. In this case, there is more symmetry and only 3 subcases to distinguish.



replace:	with:
green ax	blue-yellow $aBxCa$ + green xA
blue-yellow axb	blue-yellow $aBxAb$ + green xC
green axb	blue-yellow $aBxAbCa$

The last subcase of Case 3 deserves a comment. This is where the provision on balanced green edges is used. Indeed, if the edges in the green path axb were not balanced, we would obtain a disconnected 2-factor after the modification. □

3.2 A generalization of Halin graphs

A *Halin graph* is a plane graph such that removing all edges of its outer face F, we obtain a tree T whose leaves are precisely the vertices on F, and T has no vertices of degree 2.

We consider Halin graphs in connection to Conjecture 3.1 as an interesting class of 3-connected planar graphs. It turned out that our proof that Halin graphs have hamiltonian prisms can be applied to a much larger class of graphs, defined as follows. A *generalized Halin graph (over C)* is any union of a cycle C and a tree T such that C and T are edge-disjoint, and $V(C)$ is the set of all leaves of T. Thus, compared to the definition of Halin graphs, we do not require the planarity and allow degree 2 vertices in the tree. (See Fig. 5 for a drawing of the Petersen graph as a generalized Halin graph.)

Although these generalized Halin graphs may be non-planar, we decided to include this discussion in the section on planar graphs, mainly because of the original motivation.

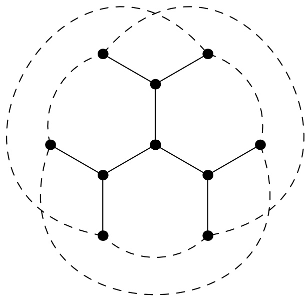


Figure 5: The Petersen graph as a generalized Halin graph over the dashed cycle.

Lemma 3.4 *Let T be a tree and let r be a vertex of T of degree at least 2. Then T contains a spanning system $\mathcal{P}$ of (possibly trivial) paths, such that*

(i) the paths in $\mathcal{P}$ are vertex-disjoint, and

(ii) each $P \in \mathcal{P}$ contains exactly one leaf v of T, and v is an endvertex of P,

(iii) r is an endvertex of some path in $\mathcal{P}$.

Proof. Orient all edges of T away from r to obtain an oriented tree $\vec{T}$. Let $\mathcal{P}$ be a system of directed paths spanning all leaves of T, satisfying (i) and (ii), and spanning as much of T as possible. (To see that at least one system with the required properties exists, consider the system of trivial paths $\{v\}$ for all leaves v of T.) Assume $\mathcal{P}$ does not span T and choose $v \notin \bigcup_{P \in \mathcal{P}} V(P)$. Since v is not a leaf of T, there is a vertex v^+ such that $vv^+ \in E(\vec{T})$. By a suitable choice of v, we may assume that v^+ is contained in some path $P \in \mathcal{P}$. Since all vertices of $\vec{T}$ have in-degree ≤ 1, the path P must begin at v^+. But then we can augment it by the edge vv^+, a contradiction with the choice of $\mathcal{P}$. $\square$

Theorem 3.5 *Generalized Halin graphs have hamiltonian prisms.*

Proof. Let G be a generalized Halin graph over a cycle C. If C is even, then color it alternatingly blue and yellow, find a system $\mathcal{P}$ of paths in $T = G - E(C)$ as in Lemma 3.4 (for an arbitrary $r \notin V(C)$), and color every path from $\mathcal{P}$ green. This coloring clearly corresponds to a Hamilton cycle of $G \square K_2$.

In the rest of the proof, we assume that C is odd. For each $e \in E(C)$, let C_e be the unique cycle in $T \cup \{e\}$. Writing $|C_e|$ for the length of C_e, we claim that

$$\sum_{e \in E(C)} |C_e| \text{ is odd.} \tag{2}$$

To begin with,

$$\begin{aligned}
\sum_{e \in E(C)} |C_e| &= |C| + \sum_{e \in E(T)} |\{ f \in E(C) : e \in C_f \}| \\
&= |C| + \sum_{e \in E(T)} |\{ f \in E(C) : f \text{ joins the components of } T - e \}|.
\end{aligned}$$

For $e \in E(T)$, let R_e denote the set of all edges of C joining the two components of $T - e$. Since $|C|$ is odd, it is enough to prove that $|R_e|$ is even for any $e \in E(T)$. Clearly, R_e is an edge cut in $G - e$. It is a standard fact that any cycle intersects any edge cut in an even number of edges. Applying this fact to the cycle C, we infer that R_e (which is a subset of C) contains an even number of edges, and (2) is established.

From (2) it follows that there is an edge $h \in E(C)$ with odd $|C_h|$. Let U be the set of vertices of C_h not incident with h. Apply Lemma 3.4 to the graph obtained from T by the contraction of U to a single vertex u (discarding loops), setting $r = u$. In the resulting system of paths, remove u from the path which contains it (as an endvertex). The outcome is a system $\mathcal{P}$ of vertex-disjoint paths in G spanning $V(G) - U$, disjoint from U, and such that each path has precisely one endvertex on C.

Make the even cycle $C' = C \cup C_h - h$ an alternating blue-yellow cycle. Furthermore, color each path in $\mathcal{P}$ green. As before, this coloring determines a Hamilton cycle in the prism over G. The proof is finished. $\square$

4 Line graphs

Recall that if $G = (V, E)$ is a graph, then its *line graph* $L(G)$ has vertex set E, and $e_1, e_2 \in E$ are joined by an edge in $L(G)$ if e_1 is adjacent to e_2 in G. We simply say that H is a line graph if there exists a graph G such that $H = L(G)$.

A prominent conjecture concerning the hamiltonicity of line graphs was stated by Thomassen [18].

Conjecture 4.1 (Thomassen's conjecture) *Every 4-connected line graph is hamiltonian.*

The conjecture is open even if we replace '4-connected' by '6-connected'. Zhan [22] and Jackson [12] independently proved that 7-connected line graphs

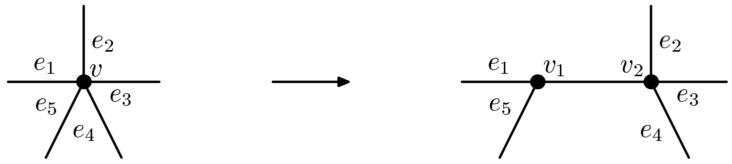


Figure 6: Replacing a vertex v with two new vertices in the proof of Lemma 4.2.

are hamiltonian. On the other hand, there are examples of 3-connected non-hamiltonian line graphs: for instance, let P' be obtained by subdividing each edge of the Petersen graph P by one vertex. The line graph of P' is P with each vertex 'inflated' to a triangle. Thus it is 3-connected, and any Hamilton cycle in $L(P')$ would clearly yield a Hamilton cycle in P, which does not exist.

In contrast, we show that for prism-hamiltonicity, it is enough if the line graph is 2-connected.

4.1 2-connected line graphs are prism-hamiltonian

In the rest of this section, multiple edges are allowed — so we deal with multigraphs. Most graph definitions carry over naturally to this setting. Multiplicities are counted in vertex degrees (which is important when we speak of *cubic* multigraphs) and in the size of an edge cut (which affects the notion of a *bridgeless* multigraph). In the line graph of a multigraph G, vertices corresponding to a pair of parallel edges are joined by parallel edges. Thus, $L(G)$ is, properly speaking, a multigraph too.

In the following lemma, the *contraction* of a subtree $T \subset G$ consists in contracting every edge of T, discarding the loops but preserving any multiple edges.

Lemma 4.2 *Let G be a bridgeless multigraph with minimum degree at least 3. Then, there exists a cubic bridgeless multigraph G_3 such that G can be obtained by the contraction of some pairwise disjoint induced subtrees of G_3.*

Proof. We define the *excess* of G to be the following sum:

$$\mathrm{exc}(G) = \sum_{v \in V(G)} (\deg_G(v) - 3)$$

The proof proceeds by induction on the excess $\mathrm{exc}(G)$ of the graph. If $\mathrm{exc}(G) = 0$, then G is cubic and the statement is trivial.

Assume $\mathrm{exc}(G) > 0$ and consider a vertex v with $d := \deg_G(v) > 3$. Let e_1 be any edge incident with v. Since G is bridgeless, there is a cycle of G which contains the edge e_1. Let e_2 be the other edge of this cycle which is incident with v. Let $e_3, \ldots, e_d$ be the remaining edges incident with the vertex v. Replace the vertex v by two new vertices v_1 and v_2, making the edges e_1 and e_d incident with v_1, the remaining edges e_2, e_3, ..., e_{d-1} with v_2, and adding a new edge v_1v_2 (see Fig. 6). Let G' be the resulting graph. Note that $\mathrm{exc}(G') = \mathrm{exc}(G) - 1$ and G may be obtained from G' by contracting the edge v_1v_2. Furthermore, G' is bridgeless: the only possible bridge may be v_1v_2, but this edge is contained in the above cycle through e_1 and e_2, so G' is bridgeless indeed. We apply induction to G', obtaining a graph G_3. To get the subtree T_v of G_3 corresponding to the vertex v of G, take the subtrees of G_3 corresponding to $v_1, v_2 \in V(G')$ and join them by an edge of G_3 incident with both these subtrees. Note that such an edge exists since $v_1v_2 \in E(G')$. Clearly, T_v is an induced subtree of G_3, because there are no multiple edges between v_1 and v_2 in G'. $\Box$

We define an *eulerian factor* of a multigraph G to be a (not necessarily connected) spanning subgraph of G with all degrees even. Note that the terminology is not quite unified here as different authors might use the term 'even factor' or 'spanning cycle', reserving 'eulerian factor' for a connected spanning subgraph with even degrees.

Lemma 4.3 *Let G be a bridgeless multigraph with minimum degree at least 3. Then, G contains an eulerian factor G' such that the degree of each vertex is non-zero in G'.*

Proof. By Lemma 4.2, let G_3 be a cubic bridgeless multigraph such that G can be obtained from G_3 by the contraction of pairwise disjoint induced subtrees $\{ T_v : v \in V(G) \}$. Since G_3 is bridgeless and cubic, it contains a 1-factor by the well-known Petersen theorem (which does apply to multigraphs). The complement of the 1-factor is a 2-factor. Let E be the set of the edges of the 2-factor which correspond to the edges of the original graph G (i.e., they are not included in any T_v). Consider a vertex v. Since each cycle which enters T_v has to leave it, the number of edges of the 2-factor incident with T_v is even. This number is non-zero because T_v is acyclic. Since the vertices of G can be obtained from G_3 by the contraction of the subtrees T_v, each vertex of G is incident with an even number of edges of E. Hence E forms the desired eulerian factor of G. $\Box$

Theorem 4.4 *Let G be a multigraph. If $L(G)$ is 2-connected, then the prism over $L(G)$ contains a Hamilton cycle.*

Proof. We first modify G to get another multigraph G_0. Let E_1 be the set of edges of G such that their one of their endvertices has degree one. If $E_1 = E(G)$, G has to be a star and the statement of the theorem is trivial. Assume henceforth that $E_1 \neq E(G)$. Remove the edges in E_1, along with all the isolated vertices this creates. The resulting graph G^- is bridgeless, since any bridge would yield a cut-vertex in $L(G)$, contradicting the assumption that $L(G)$ is 2-connected.

If all the vertices of G^- have even degrees, then G^- contains an Euler tour. Hence $L(G^-)$ is hamiltonian. In fact, it is easy to see that $L(G)$ is hamiltonian as well, so in particular, its prism has a Hamilton cycle.

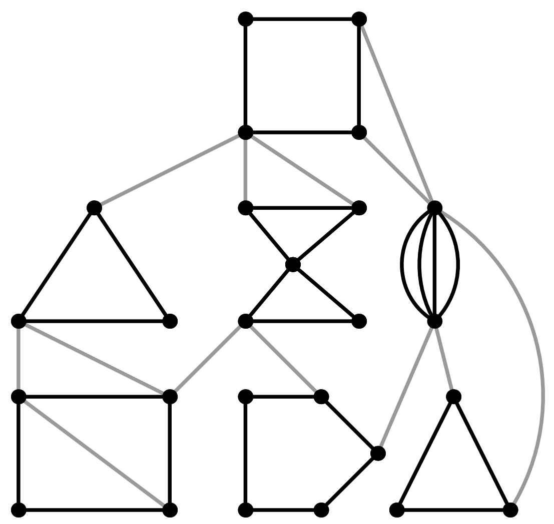


Figure 7: The graph $G^{=}$ with the eulerian factor F. For clarity, some vertices of $G^{=}$ are shown with degree 2.

In the following, we assume that G^{-} contains a vertex of odd degree.

Suppress all the vertices of degree two of G^{-}. The resulting multigraph $G^{=}$ is bridgeless and its minimum degree is at least 3. Fix an eulerian factor F of $G^{=}$ which exists by Lemma 4.3. Let k be the number of the components of F. We shall assign colors to edges of $G^{=}$ and later also to those of G. All the edges included in F will be colored black (see Fig. 7).

Let us introduce two operations we shall use in the proof. A *splitting* of a vertex v amounts to replacing it with two new vertices v' and v'' in such a way that each edge incident with v is made incident with exactly one of v' and v''. No edge $v'v''$ is added. Note that if a multigraph H is obtained by splitting some vertices of G, then $L(H)$ is a spanning subgraph of $L(G)$. In particular, to prove that $L(G)$ is (prism-)hamiltonian, it suffices to prove the same for $L(H)$.

The *detachment* of an edge uv from the vertex v consists in splitting v into two vertices such that one of the new vertices is incident only with the edge uv.

Assume first that the eulerian factor F is 2-regular, i.e., each of its components is a cycle. (See Fig. 8.) The general case will be addressed at the end of the proof. Choose a set of $k-1$ edges such that F together with these edges forms a connected subgraph of $G^{=}$. Color these $k-1$ edges red. We claim that $G^{=}$ must contain an edge which is neither black nor red. If it does not, then any red edge is a bridge in $G^{=}$, which is assumed not to exist. Thus all edges are black, which implies $k = 1$. Hence $G^{=}$ is a cycle, which contradicts the assumption that G^{-} contains a vertex of odd degree. We have shown that there is some edge o which is neither red nor black as claimed.

Color the edge o in orange and detach it from one of its endvertices. The edges that have no color so far are now colored gray, and each gray edge is detached

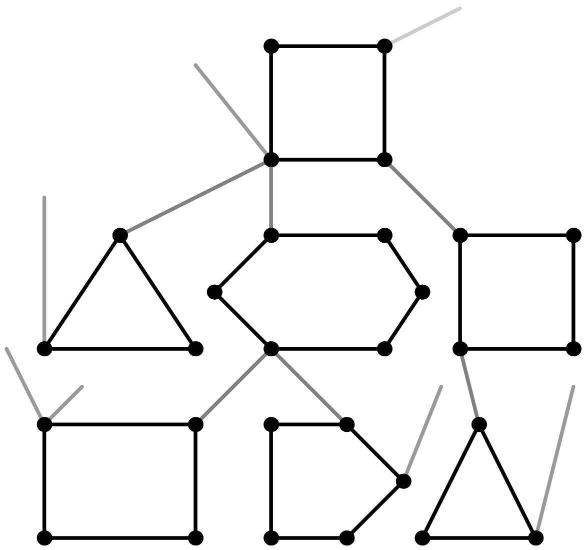


Figure 8: Black, red, orange and gray edges of $G^=$. Again, some degrees are 2 for the sake of clarity.

from an arbitrary endvertex of it. (Cf. Fig. 8.)

We now carry the coloring over to the original graph G. All the edges of each path comprised of suppressed vertices of degrees 2 get the color of the corresponding edge of $G^=$ (Fig. 9). The removed pendant edges of $E - E_1$ are colored gray and each of them is detached from one of its endvertices. Gray edges incident solely with orange (red) edges are recolored to orange (red), respectively. Let G_0 be the resulting colored graph (Fig. 10).

We use the coloring scheme introduced in Section 2 to prove that the prism over $L(G_0)$ (and thus also the prism over $L(G)$) contains a Hamilton cycle. The black cycles together with red edges form a tree-like structure. Root this tree at the (unique) cycle incident with the orange edge o. We form yellow-blue cycles first (stressing that the cycles exist in the line graph, not in G_0 itself). Apply the following to each black cycle of G_0. Let $E_{\mathcal{C}}$ be the set of all black edges forming the cycle together with all the gray, red and orange edges incident with it. Let e_r be the red edge joining the black cycle to the parent cycle, i.e., the cycle closer to the root. If the black cycle is the root cycle, then e_r is the orange edge incident with it. The yellow-blue cycle is obtained as follows (cf. Fig. 11):

- If $|E|_{\mathcal{C}}$ is even, the cycle is created on the vertices of $L(G)$ corresponding to the edges in $E_{\mathcal{C}} \subseteq E(G_0)$.

- If $|E|_{\mathcal{C}}$ is odd, the cycle is created on the vertices of $L(G)$ corresponding to the edges of G_0 in $E_{\mathcal{C}} - \{e_r\}$, and the vertex corresponding to e_r is joined by a green edge to an arbitrary vertex corresponding to a black edge incident with e_r.

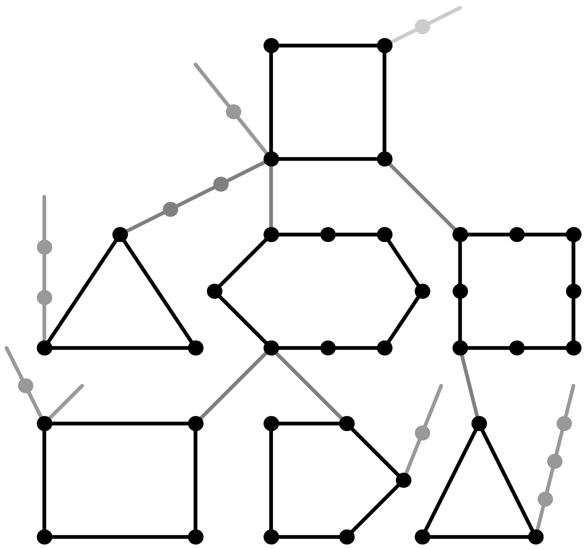


Figure 9: Colors assigned to edges of G before restoring any pendant edges.

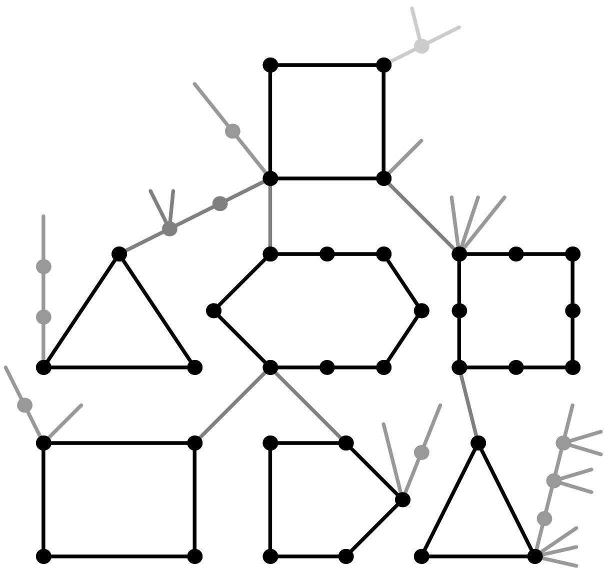


Figure 10: The graph G_0 with all its edges.

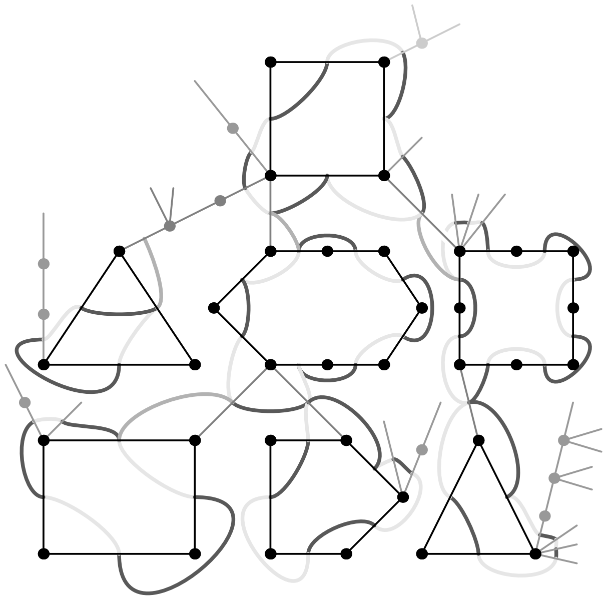


Figure 11: The graph G_0 with the created yellow-blue cycles corresponding to the black cycles and green edges joining red edges not included into the yellow-blue cycles.

Next, we add green edges (cf. Fig. 12):

- Add green paths corresponding to red paths with suppressed vertices of degree two. These paths include incident red pendant edges if there are any.
- Add green paths corresponding to gray subtrees ending in a gray edge incident with a black cycle. Do the same for the orange subtree ending in the orange edge incident with the root black cycle. These green paths end at the vertex corresponding to their green/orange edges incident with the black cycles.

It is straightforward to check that the yellow-green-blue edges represent a Hamilton cycle in the prism over G_0 through the correspondence explained in Section 2.

It remains to explain how to deal with the case where F (the eulerian factor of $G^=$) has vertices of degree larger than 2. We make each component of F into a cycle by splitting all vertices w of degree ≥ 4 in F; the edges incident with w can be made incident with any of the vertices obtained by splitting w (compare Figs. 7 and 8). As remarked above, it suffices to prove the hamiltonicity of the prism over the multigraph obtained from the splitting. We have reduced the general situation to the former case. $\square$

An immediate corollary of Theorem 4.4 is the following:

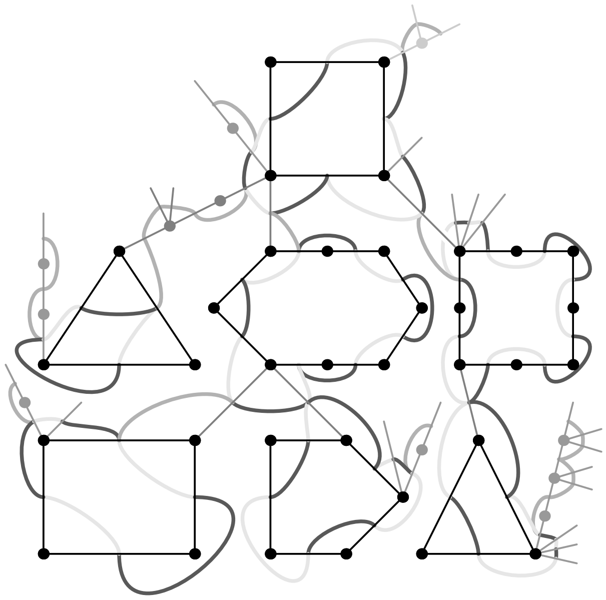


Figure 12: The graph G_0 with all the yellow, green and blue connections.

Corollary 4.5 *The prism over the line graph of any bridgeless graph is hamiltonian.*

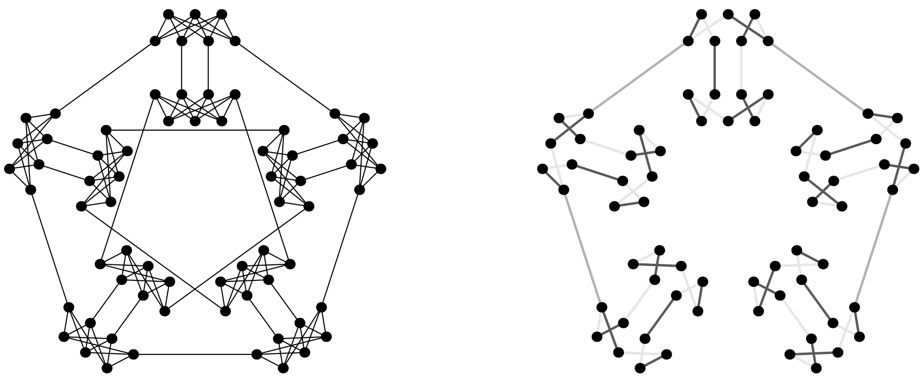
4.2 Claw-free graphs

Many results on line graphs carry over to the wider class of *claw-free graphs*, i.e., graphs containing no induced subgraph isomorphic to $K_{1,3}$. For instance, it is known [17] that 7-connected claw-free graphs are hamiltonian, and that Thomassen's conjecture implies that 4-connected claw-free graphs are hamiltonian. We conjecture the following:

Conjecture 4.6 *Any 2-connected claw-free graph is prism-hamiltonian.*

5 4-regular graphs

Nash-Williams [14] posed the question whether every 4-connected 4-regular graph is hamiltonian. The first example showing that this is not the case was the graph M in Fig. 13a, constructed by Meredith [4]. The construction is as follows. Let P' be the (4-regular) graph obtained by doubling a 1-factor in the Petersen graph. For each vertex v of P', take a copy H_v of the complete bipartite graph $K_{3,4}$. The *Meredith graph* M arises from the disjoint union of these copies by adding edges in such a way that M is 4-regular, and that there is an edge between H_v and H_w whenever vw is an edge of P'. Clearly, M is defined uniquely up to isomorphism. To see that it is non-hamiltonian, observe that any Hamilton cycle would have to enter and leave each $K_{3,4}$ just once, and so it would determine a Hamilton cycle in the Petersen graph, which does not exist.



(a) (b)

Figure 13: (a) The Meredith graph M. (b) A spanning even cactus in M.

Proposition 5.1 *The Meredith graph M is prism-hamiltonian.*

Proof. By Proposition 2.1, it is sufficient to exhibit a spanning even cactus in M. One such cactus is shown in Fig. 13b. $\square$

Proposition 5.1, together with the fact that other similarly constructed examples are also prism-hamiltonian, lead us to the following problem.

Problem 5.2 Are all 4-connected 4-regular graphs prism-hamiltonian?

An interesting class of 4-regular graphs consists of the line graphs of cubic graphs. By Theorem 4.4, if such a graph is 2-connected, then it has a hamiltonian prism. It follows that if Problem 5.2 is restricted to line graphs, the answer is affirmative. In connection to this question, it is perhaps worth noting that Conjecture 4.1 is easily seen to be equivalent to the statement that all 4-connected 4-regular *line* graphs are hamiltonian.

6 Toughness and connectivity

As mentioned in Section 1, there are examples of graphs of arbitrarily high connectivity which have 2-walks, but whose prisms are non-hamiltonian. We shall now construct one such family of graphs. For a positive integer k, let H_k be the graph consisting of three copies H_k^1, H_k^2, H_k^3 of the complete bipartite graph $K_{2k,4k-1}$, and a matching M connecting one half of the smaller color class of H_k^1 to one half of the smaller color class of H_k^2, and similarly for the other pairs of indices. (See Fig. 14 for a picture of H_2.) The graph H_k is $2k$-connected, has a 2-walk, and the following argument shows that its prism is not hamiltonian.

Assume that the prism over H_k has a Hamilton cycle C. The cycle intersects the prism over H_k^1 in a union $C^1 = P_1 \cup \ldots \cup P_s$ of disjoint paths. We aim to show that $s = 1$. Let A and B denote the smaller and the larger color class of H_k^1, respectively. Consider the path P_i, where $1 \leq i \leq s$. Assume P_i contains m vertices of $A \square K_2$. These vertices split P_i up into $m-1$ paths, each of which contains at most 2 vertices from $B \square K_2$. Thus the number of vertices of $B \square K_2$ on P_i is at most $2m-2$. Summing over all i, we get

$$2|B| \leq 4|A| - 2s,$$

since $P_1 \cup \ldots \cup P_s$ spans the prism over H_k^1. Thus

$$s \leq 2|A| - |B| = 1$$

as claimed. By symmetry, C must intersect the prisms over H_k^2 and H_k^3 in contiguous paths (C^2 and C^3, respectively) as well.

Since each H_k^j has an odd number of vertices, the path C^j enters $H_k^j \square K_2$ and leaves it in different copies of H_k. This, however, cannot be true for all three H_k^j's simultaneously. Therefore, the prism over H_k contains no Hamilton cycle C.

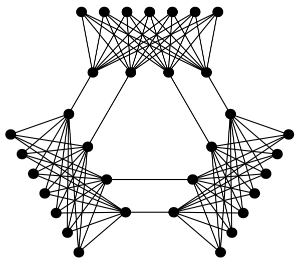


Figure 14: The graph H_2.

The above considerations are closely related to the notion of toughness. A graph G is *k-tough* if the removal of any m vertices yields a graph with at most m/k components. The *toughness* of G is the maximum k such that G is k-tough (or ∞ if G is complete). In 1973, Chvátal [6] stated the beautiful conjecture that every 2-tough graph is hamiltonian. Only in 2000, the conjecture was disproved by Bauer, Broersma and Veldman [3] by constructing non-hamiltonian graphs of toughness $9/4 - \varepsilon$ (for small ε). A weaker form of the conjecture, that there is some k such that toughness k implies hamiltonicity, is still open.

The construction from [3] was modified by Ellingham and Zha [8] who obtained $(17/24 - \varepsilon)$-tough graphs with no 2-walk. An upper bound for toughness that guarantees the existence of a 2-walk was also obtained in [8]: every

4-tough graph has a 2-walk. These are the best bounds available, but a conjecture from [13] states that the truth is much closer to the lower bound; namely that a toughness of 1 is sufficient for the existence of a 2-walk. This would improve a result of Win [21] that all 1-tough graphs have 3-trees.

We present another modification of the above construction which gives $(9/8-\varepsilon)$-tough graphs whose prisms are not hamiltonian. Consider the graph A as in Fig. 15 and observe that its prism has no Hamilton path from a to a^*.

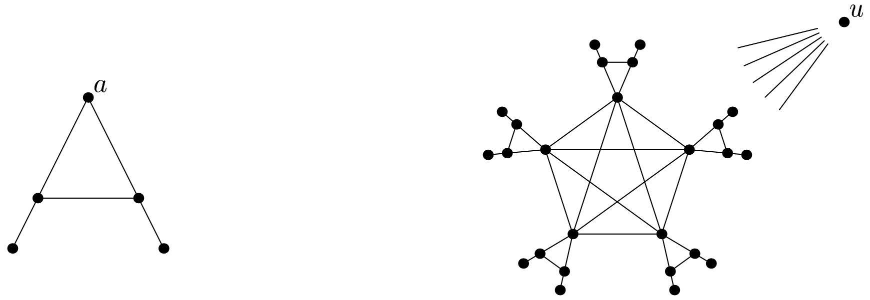


Figure 15: Left: The graph A. Right: The graph G_1 (the vertex u is adjacent to all vertices).

Take $4n+1$ disjoint copies $A_1, \ldots, A_{4n+1}$ of A and add all edges between copies of the vertex a. Form a graph G_n by adding an independent set U of n vertices which are adjacent to every vertex outside U. (See the graph G_1 in Fig. 15.)

Proposition 6.1 *The prism over the graph G_n is non-hamiltonian. The toughness of G_n approaches $9/8$ as $n \to \infty$.*

Proof. This is a straightforward modification of the argument from [3]. Any Hamilton cycle C in $G_n \square K_2$ contains $2n$ vertices equal to u or u^* for some $u \in U$. Removing these $2n$ vertices, C breaks up into at most $2n$ paths, which have a total of $\leq 4n$ endvertices. Since G_n contains $4n+1$ copies of A, some copy A_i contains no endvertex of any of the paths. This means that C covers all of A_i by a single path, entering at (the copy of) a and leaving at (the copy of) a^*. As noted above, this is impossible as $A \square K_2$ has no Hamilton path from a to a^*. Hence C cannot exist.

We compute the toughness of G_n. A *toughness set* is any nonempty proper subset T of $V(G_n)$ with the smallest possible ratio between $|T|$ and the number of components of $G_n - T$. Clearly, if T is a toughness set, then $U \subset T$ and T contains no vertex whose degree in A_i is 1. Thus each A_i has only three candidates for the membership in T. It is not hard to see that one possible toughness set contains (besides U) from each A_i the two vertices of degree 3 in A_i. The toughness of G_n is therefore $(n + 2(4n+1))/(1 + 2(4n+1)) = (9n+2)/(8n+3)$ which tends to $9/8$ as claimed. $\square$

One can consider an analogue of the weaker conjecture of Chvátal.

Conjecture 6.2 *There is a constant k such that the prism over any k-tough graph is hamiltonian.*

7 The square of a graph

The *k-th power* G^k of a graph G is the graph on the same vertices as G, with two distinct vertices x, y joined by an edge whenever their distance in G is at most k. A famous result of Fleischner [9] states that the square G^2 of any 2-connected graph is hamiltonian. For prism-hamiltonicity, we can even relax the assumption of 2-connectivity.

Theorem 7.1 *Let T be a tree with more than one vertex. Then the prism over its square T^2 is hamiltonian.*

Proof. We shall show that the prism over T^2 has a particular type of a Hamilton cycle as described below. Let us first introduce a piece of notation. For a 2-factor C in $T^2 \square K_2$ and the associated coloring of T^2 as in Section 2, we let $S(C)$ denote the spanning subgraph of T^2 consisting of all edges which are assigned some color. We claim that the prism over T^2 contains a Hamilton cycle with the following property (*):

(i) in the associated coloring of T^2, green edges are a subset of $E(T)$, and each green edge is a cut edge in $S(C)$,

(ii) adjacent green edges share a vertex whose degree in T is 2,

(iii) no leaf of T has type GG or BBYY.

The proof is by induction on the number of vertices. The statement is trivial if T is a tree on ≤ 3 vertices. Assume next that T is a star on $n \geq 4$ vertices with central vertex v. The square T^2 is then the complete graph K_n. If n is even, take any Hamilton cycle in T^2 and color it blue-yellow to obtain the coloring defining C. If n is odd, take an even cycle in T^2 through all the leaves of T, color it blue-yellow and add a green edge from any of the leaves to v. Clearly, this coloring has the required properties.

If T is not a star, then let v be a vertex all of whose neighbors are leaves of T, except for exactly one vertex w. Let L be the set of leaves adjacent to v. By induction, there is a Hamilton cycle C' in the prism over $(T-L)^2$ satisfying (*). To extend the associated coloring c' to T^2, we distinguish two cases based on the type of v in $S(C')$.

Case 1. The type of v is BY (see Fig. 16). Noting that the subtree $T_1 \subset T$ on the vertex set $L \cup \{v\}$ is a star, find a coloring c_1 of T_1^2 as described above. The extended coloring is obtained simply as the union (superposition) of c_1 and

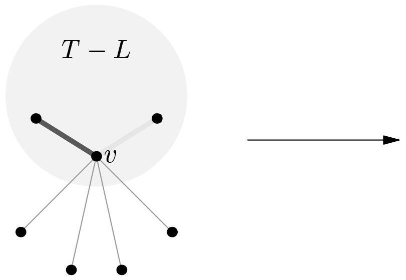


Figure 16: Extending the coloring when v is of type BY (Case 1).

c'. It is straightforward to check that the extension determines a Hamilton cycle in the prism over T^2 and preserves property (*). We omit the details.

Case 2. The type of v is G or BYG (see Fig. 17). This time, consider the star T_2 on $L \cup \{v, w\}$. Find a coloring c_2 of T_2^2 as above, choosing the green edge (if there is one) to be different from vw. By (i), the green edge of $S(C')$ adjacent to v is vw and its removal disconnects $S(C')$. The desired coloring c is obtained from c' by first uncoloring vw and then adding the coloring c_2.

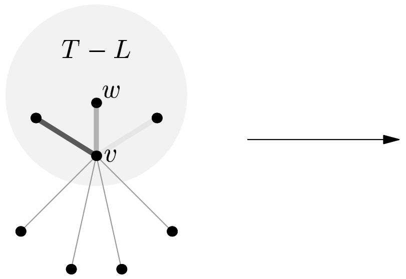


Figure 17: Extending the coloring when v is of type BYG (Case 2).

We need to show that the associated 2-factor C_2 (in the prism) is a Hamilton cycle. Let P be the path in C_2 between v and w, and let Q be the path in C_2 between v^* and w^*. Note that by the construction, P and Q are disjoint, and all their internal vertices are in $L \square K_2$. Furthermore, it is not hard to see that replacing each of P and Q by an edge, we obtain C'. The claim that C_2 is a Hamilton cycle follows. Again, it is easy to check that (*) is preserved. $\square$

Theorem 7.1 directly implies the following corollary.

Corollary 7.2 *The square of any connected graph is prism-hamiltonian.* $\square$

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