Finite Paths are Universal

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Abstract

We prove that any countable (finite or infinite) partially ordered set may be represented by finite oriented paths ordered by the existence of homomorphism between them. This (what we believe a surprising result) solves several open problems. Such path-representations were previously known only for finite and infinite partial orders of dimension 2. Path-representation implies the universality of other classes of graphs (such as connected cubic planar graphs). It also implies that finite partially ordered sets are on-line representable by paths and their homomorphisms. This leads to a new on-line dimensions.

1 Introduction

An *oriented path* P is any oriented graph (V, E) where $V = \{v_0, v_1, \ldots, v_n\}$ and for every $i = 1, 2, \ldots, n$ either $(v_{i-1}, v_i) \in E$ or $(v_i, v_{i-1}) \in E$ (but not both), and there are no other edges. Thus an oriented path is any orientation of an undirected path. An example is on Fig.1 (in all our drawings all arcs are oriented upwards).

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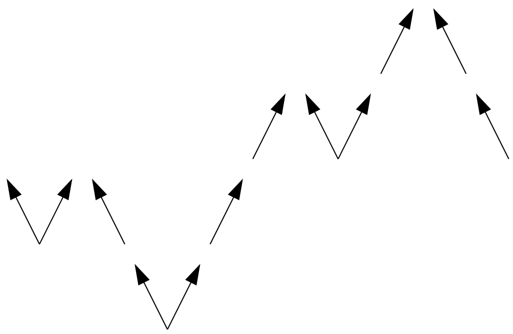

Figure 1: Oriented path P

The *length* $l(P)$ of a path P is the number of edges in P. The *algebraic length* $al(P)$ of a path P is the number of forwarding minus the number of backwarding arcs in the code of P. Thus the algebraic length of a path could be negative. The *level* $l_p(p_i)$ of p_i is the algebraic length of the subpath $(p_0, p_1, \ldots, p_i)$ of P.

One can also express a given path P by a code $c(P) = (a_1 a_2, \ldots, a_n)$, where n is the number of edges in P, and $a_i = 0$ or 1 depending on whether the i-th edge is a forward or a backward edge. For example the path P on Fig.1 has the code $c(P) = 010011101100$. Given a code $c = (a_1, a_2, \ldots, a_n)$ the code c^{-1} is the code $(a'_1, a'_2, \ldots, a'_n)$ given by $a'_i + a_{n+1-i} = 1$. The code $c(P)^{-1}$ corresponds to the flipping of the path P (i.e. exchanging the initial and terminal vertices).

Clearly P is a subgraph of P' iff $c(P)$ is a subsequence of $c(P')$ or $c(P)^{-1}$. Thus finite 0–1 vectors encode finite paths. It follows easily that any finite partial order can be represented by finite paths with the subgraph relation. However this representation (by paths with the subword relation) is no longer possible for infinite partial orders and even for well founded posets. Also finite vectors with the relation defined coordinatewise (like in Dushnik-Miller dimension) do not represent all countably infinite partial orders. In this paper we prove that the ordering of finite oriented paths which is induced by the existence of homomorphism is already rich enough to represent every countable partially ordered set.

More precisely given paths $P = (V, E)$, $P' = (V', E')$ a *homomorphism* is a mapping $f : V \to V'$ which preserves edges:

$$(x, y) \in E \implies (f(x), f(y)) \in E'.$$

Denote by $\mathcal{P}$ the partial order generated by all paths and the existence of a homomorphism (actually, we have to restrict ourselves to *cores* — minimal retracts — to obtain a partial order; otherwise we have a quasiorder which we can factorize by the hom-equivalence, see [22]).

It has been proved in [27] that $\mathcal{P}$ is a dense partial order (with exception of a few gaps which were characterized; these gaps are formed by all core-path of height ≤ 4). [27] also rises (seemingly too ambitious) question whether $\mathcal{P}$ is universal partial order. A path representation is found for finite partial orders and all 2-dimensional infinite partial orders (the representation of countable chains follows from density of paths). Here we give a full solution of this problem:

Theorem 1.1 *$\mathcal{P}$ is universal partial order.*

It is perhaps worth to formulate this result explicitly. Given partially ordered set P and P' an injective mapping $f : P \to P'$ is called *embedding* of P into P' if it satisfies:

$$x \leq_P y \text{ iff } f(x) \leq_{P'} f(y)$$

Theorem 1.1 has the following form:

Theorem 1.1' *For every countable partially ordered set P there exist an embedding $\phi : P \to \mathcal{P}$; in the other words to every $x \in P$ we can associate a finite path $\phi(x)$ such that*

$$x \leq_P y \textit{ iff there exist a homomorphism } \phi(x) \to \phi(y).$$

Homomorphisms of paths were studied from the complexity point of view e.g. in [8] and from the structural point of view e.g. [27]. As opposed to homomorphisms of general graphs (which generalize colorings) it is easy to decide whether there exists a homomorphism from P to P'. This is also reflected by the description of path homomorphisms by means of the path-codes with a simple rewriting scheme.

For two oriented paths P_1 and P_2 with codes $c(P_1)$ and $c(P_2)$ respectively, the concatenation P_1P_2 of P_1 and P_2 is the path with code $c(P_1P_2) = c(P_1)c(P_2)$. In other words, P_1P_2 is the path obtained from the disjoint union of P_1 and P_2 by identifying the terminal vertex of P_1 with the initial vertex of P_2.

One can relate homomorphisms and codes by the following (see [22, 27]):

Theorem 1.2 (Word homomorphism) *For paths P and P' the following two statements are equivalent:*

1. *P is homomorphic to P';*

2. *There exist words c, c' on alphabet $(0,1)$ such that $c(P)$ is a subword segment of c and c' is a subword of $c(P')$ and c' can be obtained from c by means of a sequence of substitutions of the form*

$$ww^{-1}w \to w. \tag{1}$$

(w is any word on alphabet $\{0,1\}$).

The universality of the partial order $\mathcal{P}$ may be interpreted in finite terms by means of an on-line representation. (This is similar to universal homogeneous partial orders which can be captured by the notion of extension.)

By an *on-line representation* of a class $\mathcal{K}$ of partial orders, we mean that one can construct a representation of any partial order R in class $\mathcal{K}$ under the circumstances that the elements of R are revealed one by one. The on-line representation of a class of partial orders can be considered as a game between two players A and B. Player B chooses a partial order P in the class $\mathcal{K}$, and reveals the elements of P one by one to player A (B is a bad guy). Whenever an element of x of P is revealed to A, the relations among x and previously revealed elements are also revealed. Player A is required to construct an oriented path to represent x before the next element is revealed. Player A wins a game if he succeeds in constructing a representation of P. The class $\mathcal{K}$ of partial orders is on-line representable if player A has a winning strategy.

It is easy to see that the density of $\mathcal{P}$ implies that the class of countable linear orders sets is on-line representable.

For the benefit of the reader we include the following easy (see e.g. [27]):

Theorem 1.3 *The following three statements are equivalent:*

1. *Every countable partial order is path representable.*

2. *The class of all finite partial orders is on-line path representable.*

3. *The class of all countable partial orders is on-line path representable.*

Proof. It is obvious that 3. $\Longrightarrow$ 1. To see that 2. $\Longrightarrow$ 3., we note that if player A has a winning strategy for the class of all finite partial orders, then this strategy can be applied to construct an on-line representation of any countable partial order P, because at each step the revealed part of P induces a finite partial order.

We now prove that 1. $\implies$ 2.: Let $\mathcal{G}$ be universal homogeneous partial order [10, 29]. Assume that $f : \mathcal{G} \to \mathcal{P}$ is a path representation of $\mathcal{G}$. As $\mathcal{G}$ has the extension property the path representation yields an on-line representation of any finite partial order. $\square$

The on-line formulation of universality clearly indicated why the usual representation of partially ordered sets by vectors ordered coordinatewise cannot succeed. Given two vectors $\vec{v} = (v_1, \ldots, v_t) \leq \vec{v}' = (v'_1, \ldots, v'_t)$ there are only finitely many vectors w with $\vec{v} \leq \vec{w} \leq \vec{v'}$ (and we need infinitely many of them). Even if we allow rational numbers for coordinates then still the interval $\vec{v} \leq \vec{w} \leq \vec{v}'$ would be finitely dimensional (and we need arbitrarily large dimension in this, as in any other interval). Quite symptomatic to our modern times the player A does not know where the next move B will be.

We shall prove Theorem 1.1 in two basic steps. Given a partial order P we first find an embedding

$$\phi^* : P \to \mathcal{P}^*$$

where $\mathcal{P}^*$ is the class of all linear forests, i.e. forests where each component is a path. Such an embedding was found already in [14]. But we need some additional properties of ϕ^* and thus we have to reprove [14] in a stronger form. This will be done in Section 2. In Section 3 we show how to merge components of a linear forest to a single path. This is non-trivial and it will be achieved by means of set-valued trees and universal transition sequences which are respecting the syntactic reductions similar to (1). This is proved in Section 3. In Section 4 we list some corollaries and introduce some naturally defined on-line dimensions. They indicate that our construction, despite of its complexity, is nearly optimal.

2 Path Forest Embedding

The key ingredient of our proof is a particular ordering of finite 0–1 vectors (of variable length) which can be simulated by path homomorphisms. (Of course, chronologically this is exactly opposite how we proceeded: by understanding of homomorphisms between (special) paths we arrived to the vector ordering.)

Definition 2.1 *Let $\vec{v} = (v_1, \ldots, v_t)$, $\vec{v}' = (v'_1, \ldots, v'_{t'})$ be 0–1 vectors. We put:*

$$\vec{v} \leq \vec{v}' \textit{ iff } t \geq t' \textit{ and } v_i \geq v'_i \textit{ for } i = 1, \ldots, t'.$$

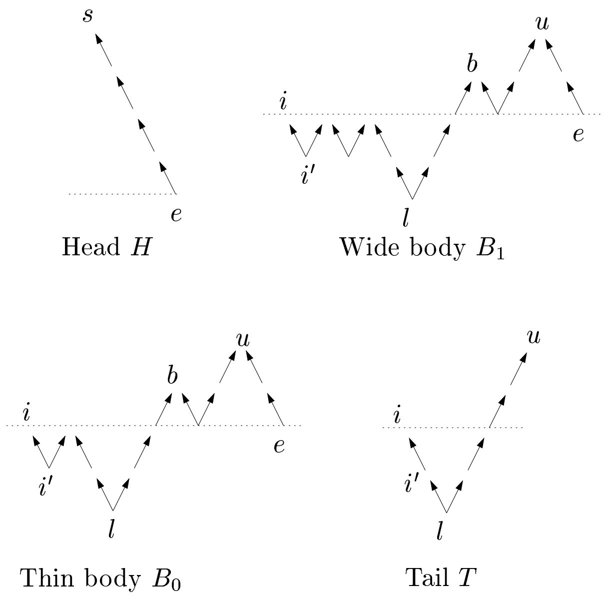

Figure 2: Paths used to construct multipedes

Thus we have e.g. $10111 < 1001$ and $1001 > 1111$. An example of infinite descending chain is e.g.

$$1 > 11 > 111 > \dots$$

Any finite partially ordered set is representable by vectors with this ordering: for vectors of a fixed length we have just reverse ordering used in the (Dushnik-Miller) dimension of partially ordered sets, see e.g. [26].

Now we describe special paths which will represent the above vector ordering. Let H, B_0, B_1 and T be paths defined by Figure 2. Some vertices of the paths are labeled to be easily referred in the text and thus the sets of vertices of individual graphs are partly overlapping.

Definition 2.2 *For a vector vector $\vec{v} = (v_1, \dots, v_t)$ we define the path $M(\vec{v})$ is the concatenation of paths $M_0, M_1, \dots, M_t, M_{t+1}$, where*

$$M_0 = H,$$

$$M_n = B_0 \textit{ iff } 1 \le n \le t \textit{ and } v_n = 0,$$

$$M_n = B_1 \textit{ iff } 1 \le n \le t \textit{ and } v_n = 1,$$

$$M_{t+1} = T.$$

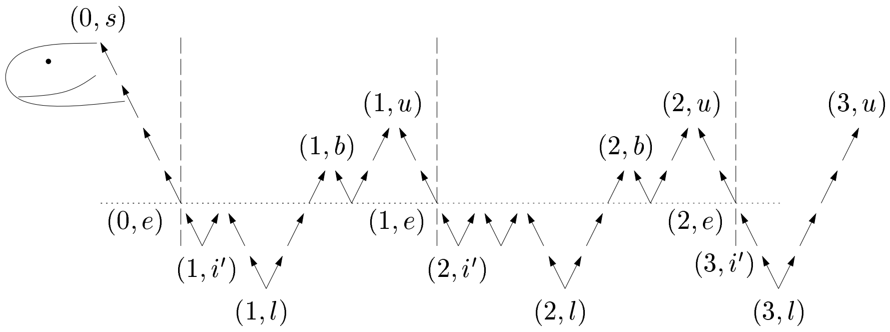

Figure 3: Multipede $M(0,1)$

Here is an explicit definition of $M(\vec{v})$: The vertices of $M(\vec{v})$ form the set

$$\bigcup_{n=0}^{t+1}(\{n\} \times (M_n - \{i\})).$$

There is an edge from (n,u) to (n,u') if and only if there is an edge from u to u' in M_n. There is an edge from $(n+1,i')$ to (n,i) for each $0 \leq n \leq t$.

We call the path $M(\vec{v})$ *multipede* assigned to $\vec{v}$.

Example 2.1 *For the vector* $(0,1)$ *the resulting multipede is shown at* Fig.3.

The homomorphisms of multipedes encode the vector ordering (Definition 2.1):

Proposition 2.1 *For any two* 0–1 *vectors* $\vec{u}, \vec{v}$ *holds:*

$$\vec{u} \leq \vec{v} \text{ iff } M(\vec{u}) \to M(\vec{v}).$$

Moreover, if $M(\vec{u}) \to M(\vec{v})$ *then there exists a homomorphism* $f : M(\vec{u}) \to M(\vec{v})$ *which maps the terminal vertex of* $M(\vec{u})$ *to the terminal vertex* $M(\vec{v})$.

Remark. We shall see that any homomorphism $M(\vec{u}) \to M(\vec{v})$ maps the initial vertex of $M(\vec{u})$ to the initial vertex of $M(\vec{v})$ so we do not have to demand this.

Proof. Assume that $f : M(\vec{u}) \to M(\vec{v})$ is a path homomorphism. We shall prove that the multipedes $M(\vec{u})$, $M(\vec{v})$ are chosen so that f corresponds to $\vec{u} \leq \vec{v}$. First, we make several observations:

1. $f(0,s) = (0,s)$.

   This follows directly from the fact that the only monotone subpath of length 5 of $M(\vec{u})$ is between vertices $(0,s)$ and $(1,i')$.

2. $f(n,l) = (n',l)$.

   This follows from the fact that each homomorphism preserves algebraic distances. It follows from 1. that the algebraic distance of any vertex x in $M(\vec{u})$ to $(0,s)$ must be equivalent to the algebraic distance of $f(x)$ to $(0,s)$ in $M(\vec{v})$. Thus f preserves the levels of vertices. There are no other vertices at level -6.

3. $f(n,l) = (n',l) \implies f(n,b) = (n',b)$ for each $n \leq |\vec{u}|$, $n' \leq |\vec{v}|$.

   Similarly to 2., there no other vertices at level -3 in $M(\vec{v})$ whose distance from (n',l) is at most 3.

4. $f(n,l) = (n',l) \implies f(n,u) = (n',u)$ for each $n \leq |\vec{u}|$, $n' \leq |\vec{v}|$.

   $f(n,b) = (n',b)$ follows from 3. There no other vertices in $M(\vec{v})$ at level -2 whose distance from (n',l) is at most 3.

5. $f(n,l) = (n',l) \implies f(n+1,i') = (n'+1,i')$ for each $n \leq |\vec{u}|-1$, $n' \leq |\vec{v}|-1$.

   From 4. we get that $f(n,u) = (n',u)$. The distance of (n,u) to $(n+1,i')$ in $M(\vec{u})$ is 3. $(n'+1,l)$ is the only vertex having level -5 and whose distance from (n',u) in $M(\vec{v})$ is at most 3.

6. $f(n,l) = (n',l) \implies f(n+1,l) = (n'+1,l)$ for each $n \leq |\vec{u}|-1$, $n' \leq |\vec{v}|-1$.

   It follows from 5. that $f(n,i') = (n',i')$. The distance of $(n+1,i')$ to $(n+1,l)$ in $M(\vec{u})$ is at most r. $(n'+1,l)$ is the only vertex having level -6 and whose distance from $(n'+1,i')$ in $M(\vec{v})$ is at most 5.

It follows from 1. that $f(1,l) = (1,l)$. From 5. we get that $f(n,l) = (n,l) \implies f(n+1,l) = (n+1,l)$ for each $n \leq |\vec{v}|-1$. By induction $f(n,l) = (n,l)$ for $n \leq |\vec{v}|$. We also have that $|\vec{u}| \geq |\vec{v}|$. It is easy to see that $u_n \geq v_n$ for each $n \leq |\vec{v}|$ (the distances of (n,l) and $(n+1,l)$ in $M(\vec{v})$ must be shorter or equal to the distances of the corresponding lowest vertices in $M(\vec{u})$). It follows that the existence of a homomorphism $f : M(\vec{u}) \to M(\vec{v})$ implies that $\vec{u} \leq \vec{v}$.

Now assume $\vec{u} \leq \vec{v}$. Put $\vec{u} = (u_1, \ldots, u_r)$, Put $\vec{v} = (v_1, \ldots, v_s)$. It is $r \geq s$ and $u_i \geq v_i$ for $i = 1, \ldots, s$. Let $M(\vec{u})$ be constructed by concatenation of paths M_0, M_1,…,M_{s+1} and $M(\vec{v})$ be constructed by concatenation of paths M'_0, M'_1,…,M'_{t+1}. We can construct homomorphism $f : M(\vec{u}) \to M(\vec{v})$ as follows:

Put $f(n, x) = (n, x)$ for each n such that $M_n = M'_n$. In the case $M_n = B_1$ and $M'_n = B_0$ we put $f(n, x) = (n, h(x))$, where h is a homomorphism of B_1 to B_0 such that $h(i) = i$ and $h(e) = e$. In the case either $M_n = B_0$ or $M_n = B_1$ and $n > r$ we put $f(n, x) = (s + 1, h'(x))$ where h' is a homomorphism of $B_0 \to T$ (or $B_1 \to T$) such that $h'(i) = i$. Finally we put $f(r + 1, x) = (s + 1, x)$.

It is easy to verify that f is a homomorphism $M(\vec{u}) \to M(\vec{v})$. $\square$

Now it seems that all it suffices to do is to represent any partially ordered set by finite vectors with the above ordering. However this is not possible. For example we cannot represent even an infinite ascending chain by vectors encoding (and the universality cannot be saved by adding more general, say integral, coordinates). Fortunately finite set of vectors are universal as we shall show now.

Definition 2.3 *We denote by $\mathcal{V}^*$ the class of all finite vector-sets. Let $\vec{V}$ and $\vec{V}'$ be two finite set of 0–1 vectors. We put $\vec{V} \leq \vec{V}'$ iff for every $v \in \vec{V}$ there exists $v' \in \vec{V}'$ such that $\vec{v} \leq \vec{v}'$.*

It is easy to see that $\mathcal{V}^*$ is a partial order. Similarly we define $\mathcal{P}^*$:

Definition 2.4 *We denote by $\mathcal{P}^*$ the class of all finite path-forests. For finite path-sets $\{P_i; i \in I\}, \{P'_i; i \in I'\}$ we put*

$$\sum_{i \in I} P_i \leq_{\mathcal{P}^*} \sum_{i \in I'} P'_i$$

iff there exists a homomorphism

$$\sum_{i \in I} P_i \to \sum_{i \in I'} P'_i.$$

(By $\sum_{i \in I} P_i$ we denote the disjoint union of paths $P_i, i \in I$)

We have the following

Proposition 2.2

1. *$\mathcal{V}^*$ is universal partial order;*

2. *$\mathcal{P}^*$ is universal partial order.*

We shall need a more specific (and more technical) formulation of Proposition 2.2 which is proved in [14]. Towards this end denote by $\mathcal{G}$ the generic partially ordered set (i.e. universal and homogeneous partially ordered set; cf. [4, 10, 29]). It is well known that $\mathcal{G}$ exists. It can be constructed by an iterated amalgamation as *Fraïssé limit* of the class of all finite partial orders [29, 12, 4]. $\mathcal{G}$ has as well a nice finite presentation (see our companion paper [13]). $\mathcal{G}$ contains any finite partial order and in fact any finite partial suborder P of $\mathcal{G}$ can be in $\mathcal{G}$ extended to any partial order P' containing it. For the sake of simplicity let $\mathcal{G}$ be partially ordered set $(\mathbb{N}, \leq_{\mathcal{G}})$. An embedding of $\mathcal{G}$ into $\mathcal{V}^*$ and of of $\mathcal{G}$ into $\mathcal{P}^*$ will be constructed on-line in the (standard) ordering of natural numbers.

We shall prove

Proposition 2.3 *There exists a function h assigning each natural number n path $h = P_n$ with the following properties:*

1. *(end-points): If $P_m \to P_n$ then there exists a homomorphism mapping initial vertex of P_m to the initial vertex of P_n and the terminal vertex of P_m to the terminal vertex of P_n.*

2. *(ordering): $P_m \to P_n$ iff $m \geq n$ and $m \leq_{\mathcal{G}} n$.*

3. *(rigidity): Let P be the concatenation of paths $P_{i(1)}, \ldots, P_{i(t)}$. Then for any homomorphism $f : P_n \to P$ there exists j, $1 \leq j \leq t$ such that $f(V(P_n)) \subset (\{j\} \times V(P_{i(j)}))$. (There may be several such j, it is possible $i(j) = i(j')$)*

4. *(representation): The function $\phi^* : \mathbb{N} \to \mathcal{P}^*$ defined by*

$$\phi^*(n) = \sum_{\substack{m \leq n \\ m \leq_{\mathcal{G}} m}} P_m$$

   is an embedding of $\mathcal{G}$ into $\mathcal{P}^*$. $P_m \to P_n$ iff $m \geq n$ and $m \leq_{\mathcal{G}} n$.

The proof of Proposition 2.3 is a combination of an embedding $\phi^* : \mathcal{G} \to \mathcal{V}^*$ and the multipede encoding provided by Proposition 2.1.

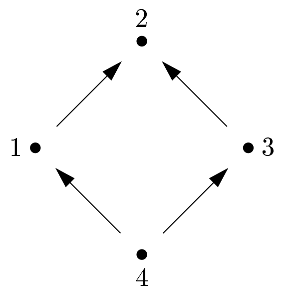

Figure 4: Partially ordered set P

Proof of proposition 2.2 (1). To every $n \in \mathbb{N}$ we assign the vector $\vec{v}(n) = (v_1, v_2, \ldots, v_n)$ where $v_m = 1$ iff $n \leq_{\mathcal{G}} m$, $m \leq n$. Further we define the set of vectors $\vec{V}(n) \in \mathcal{V}^*$ as follows

$$\vec{V}(n) = \{\vec{v}(m); m \in \mathbb{N}, m \leq n, m \leq_{\mathcal{G}} n\}.$$

1. Assume $m \leq_{\mathcal{G}} n$. We prove $\vec{V}(m) \leq_{\mathcal{V}^*} \vec{V}(n)$:

Consider arbitrary $m' \in \mathbb{N}$ such that $\vec{v}(m') \in \vec{V}(m)$. From the definition of $\vec{V}$ it follows easily that $m' \leq_{\mathcal{G}} m$. To prove the inequality $\vec{V}(m) \leq_{\mathcal{V}^*} \vec{V}(n)$ it suffices to find a vector $\vec{u} \in \vec{V}(n)$ such that $\vec{v}(m') \leq \vec{u}$. We examine two possibilities:

If $m' \leq n$, then $\vec{v}(m') \in \vec{V}(n)$ follows directly from the definition of $\vec{V}$ and inequality $m' \leq_{\mathcal{G}} n$.

In the case $m' > n$ we consider any $r \leq m'$ such that $\vec{v}_r(n) = 1$. We have inequalities $m' \leq_{\mathcal{G}} m \leq_{\mathcal{G}} n \leq_{\mathcal{G}} r$. Further $r \leq n < m'$ and thus $\vec{v}_r(m) = 1$. Consequently $\vec{v}(m') \leq \vec{v}(n)$.

2. Assume $\vec{V}(m) \leq_{\mathcal{V}^*} \vec{V}(n)$. We prove $m \leq_{\mathcal{G}} n$:

There exist r, $\vec{v}(r) \in \vec{V}(n)$ such that $\vec{v}(m) \leq \vec{v}(r)$. We have $|\vec{v}(r)| \leq |\vec{v}(m)|$ and thus $r \leq m$. From the definition of $\vec{v}(r)$ we have $\vec{v}(r)_r = 1$. It follows that $\vec{v}(m)_r = 1$. From the definition of $\vec{v}(m)$ we have that $m \leq_{\mathcal{G}} r$. From the definition of $\vec{V}(n)$ we get $r \leq_{\mathcal{G}} n$ and thus $m \leq_{\mathcal{G}} n$. $\square$

Proposition 2.2(2) will follow from Proposition 2.3.

Proof of Proposition 2.3. The end-point property and the rigidity property 2.3(1),(3) follows from the form of multipedes $M(\vec{v})$.

The ordering property 2.3(2) follows as we have $\vec{v}(m) \not\leq \vec{v}(n)$ for $m < n$ (as $|\vec{v}(m)| < |\vec{v}(n)|$ for $m < n$).

The representation property 2.3(4) is just another formulation of the embedding $\vec{V}$: We have

$$\begin{aligned}
\vec{V}(m) \leq \vec{V}(n) \quad &\text{iff} \quad \{\vec{v}(i); i \leq m, i \leq_{\mathcal{G}} m\} \leq_{\mathcal{V}^*} \{\vec{v}(j); j \leq n, j \leq_{\mathcal{G}} n\} \\
&\text{iff} \quad \sum_{\substack{i \leq m \\ i \leq_{\mathcal{G}} m}} M(\vec{v}(i)) \to \sum_{\substack{j \leq n \\ j \leq_{\mathcal{G}} n}} M(\vec{v}(j))
\end{aligned}$$

which means that the mapping defined by

$$\phi^*(n) = \sum_{\substack{m \leq n \\ m \leq_{\mathcal{G}} m}} P_m$$

is an embedding. $\square$

Example 2.2 *Assume that the first 4 vertices of $\mathcal{G}$ induce partial order P shown in* Fig.4. *The representation will be as follows:*

$$\begin{array}{rclrcl}
\vec{v}(1) & = & (1), & \vec{V}(1) & = & \{(1)\} \\
\vec{v}(2) & = & (1,1), & \vec{V}(2) & = & \{(1,1)\} \\
\vec{v}(3) & = & (0,0,1), & \vec{V}(3) & = & \{(1),(1,1),(0,0,1)\} \\
\vec{v}(4) & = & (0,0,1,1), & \vec{V}(4) & = & \{(1,1),(0,0,1,1)\}
\end{array}$$

3 Tree transitions

We will concatenate the disjoint paths of $\phi^*(x)$ to single path $\phi(x)$. By rigidity condition 2.3(3) and by the definition of ordering $\mathcal{P}^*$ any concatenation does not introduce new homomorphisms. What is more difficult is to guarantee that $\phi(x) \to \phi(y)$ whenever $\phi^*(x) \to \phi^*(y)$.

It is not enough to simply concatenate the paths forming the set $\phi^*(x)$ in some specified order. Consider, for instance the partial order shown at the Figure 5. The linear forest $\phi^*(5)$ consists of the paths P_1, P_2, P_3, P_4, P_5 and for $i = 2, 3, 4$, the linear forest $\phi^*(i)$ consists of paths P_1, P_i.

Assume, for instance that we put $\phi(2) = P_1P_2$. It follows that $\phi(5)$ contains P_1P_2 as a subpath: By combination of the rigidity and the ordering properties 2.3(2),(3) we know that the subpath P_1 of $\phi(2)$ must map to the subpath P_1 of $\phi(5)$. P_2 does not map to P_1, so the subpath P_2 of $\phi(2)$ must map to the subpath neighboring to the subpath P_1 in $\phi(5)$. The only choice of such path is P_2. By similar argument one gets that $\phi(3) = P_3P_1$ because P_3 does not map to P_2 (otherwise we would have $\phi^*(3) \to \phi^*(2)$). Thus $\phi(5)$ contain subpath

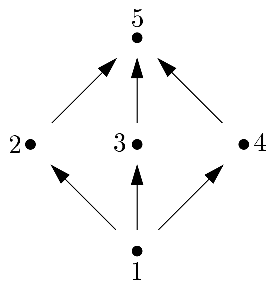

Figure 5: Partially ordered set P

$P_3P_1P_2$. Finally there is no choice how to map $\phi(4)$ to $\phi(5)$ as P_4 does not map to P_3 nor P_2.

Our solution will produce very long sequences consisting of many copies of individual paths. The above example is still simple. We can put:

$$\begin{aligned}
\phi(1) &= P_1,\\
\phi(2) &= P_1P_2,\\
\phi(3) &= P_3P_1,\\
\phi(4) &= P_1P_4,\\
\phi(5) &= P_3P_1P_2P_1P_4P_5.
\end{aligned}$$

Our constructed sequences will be longer and more complicated and we will guarantee that each subset of the paths will be present as a subsequence. We have to deal with even more complex situations for $m < n$ and $n \leq m$ where the mappings in between paths are not inclusions and the only tool at our disposal is the rewriting scheme from Theorem 1.2(1).

Let us return to the description of embedding $\mathcal{G} \to \mathcal{P}$. First we give an informal description of our proof.

For every n we put

$$I(n) = \{m; m \in \mathcal{G}, m \leq n, m \leq_{\mathcal{G}} n\}.$$

From the representation property 2.3(4) we have for every n

$$\phi^*(n) = \sum_{m \in I(n)} P_m.$$

When $m \leq_{\mathcal{G}} n$ we know that either $m \leq n$ and $I(m) \subseteq I(n)$ or $m > n$ and $I(m) = \{i; i \in I(n), i \leq n\}$ (by applying transitivity of $\leq_{\mathcal{G}}$). In the second case we get from the ordering property 2.3(3) that each $P_{m'}$, $m' \in I(m), m' > n$ maps to P_n.

Now we shall describe merging of paths $\{P_i; i \in I\}$ in 3 steps. Assume that we are given the set $\{P_i; i \in I\}$ which form $\phi(n)$.

I. We define subset tree $T(I)$ over I and

II. For a tree $T(I)$ we define (transition) universal word $W(I)$ in the following sense:

Definition 3.1 *Function W from finite sets of natural numbers to the words consisting of the natural numbers is said to* construct universal sequence *if the following conditions holds:*

1. *For each finite set A of natural numbers $W(A)$ is word over the alphabet A containing every $a \in A$.*
2. *For each A finite and $B \subset A$, $W(B)$ is subword of $W(A)$.*
3. *Given A and $n \in A$, $B = \{x; x \in A, x \leq n\}$, There exist a sequence transformation of $W(A)$ to $W(B)$ where each transformation has one of the following forms:*

   (i) replace x by n for any integer $x > n$;
   (ii) $\overrightarrow{w}\overleftarrow{w}\overrightarrow{w} \to \overrightarrow{w}$ for any subword w.

The transformation (ii) is formalized perhaps imprecisely. What we mean is that an sequence of form $a_1 \dots a_t a_t \dots a_1 a_1 \dots a_t$ is replaced by $a_1 \dots a_t$ (for any choice of letters $a_1, \dots, a_t$).

III. Finally, we replace every symbol of $W(I)$ by the symetrization $\widehat{P_i} = P_i P_i^{-1}$ of the path P_i. This replacement will be denoted $\widehat{P_I}$. We then prove that for any I, J holds

$$\widehat{P_I} \to \widehat{P_J} \text{ iff } \sum_{i \in I} Pi \to \sum_{j \in J} Pj.$$

Combining with Proposition 2.3 this will finish the proof of Theorem 1.1.

Now we shall describe steps **I.**, **II.**, **III.** in a greater detail.

I. Let $\{P_i; i \in I\}$ be a fixed set of paths which corresponds to $\phi^*(n) = \sum_{i \in I} P_i$.

Definition 3.2 *The nodes of the* subset tree $T(I)$ over I *are all nonempty chains containing I of non-empty subsets of I. The chains are ordered by*

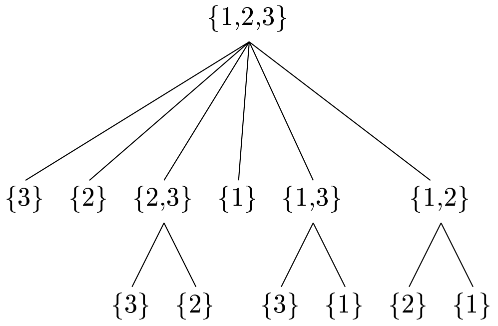

Figure 6: Subset tree of the set {1,2,3}

inclusion. Chain A is a son of chain B iff $A \supset B$ and there is no C such that $B \subset C \subset A$. The label of the vertex *A of $T(I)$ is the inclusion minimal element of A.*

We consider subset tree to be a planted tree by ordering sons via order $\leq_T$ of their labels. Assume that $A \neq B$ are brothers (have the same father), a is label of A and b is label of B. Let n be the minimal element of the symmetric difference of sets a and b. We put $A <_T B$ if and only if $n \in b$.

The subset tree $T(1,2,3)$ is shown at Figure 6.

We will define the n-collapse of a subset tree:

Definition 3.3 *Let $n \in I$ be a natural number. An n-*collapse of the subset tree *$T(I)$ is function φ_n mapping $T(I)$ onto the subset tree $T(I')$ of the set $I' = I \cap \{1, \ldots, n\}$ as follows:*

Vertex A of T is mapped to the vertex A' of T' constructed from A by replacing of $x > n$ in any $y \in A$ by n and deleting the multiplicities of n (clearly φ_n image of a chain is a chain).

A collapse of $T(I)$ has the following local properties:

Lemma 3.1 *For each $T(I)$ and any choice of $n \in I$, $A <_T B$ sons of C in $T(I)$ one of the following conditions hold:*

1. *$\varphi_n(A)$ and $\varphi_n(B)$ are sons of $\varphi_n(C)$ and $\varphi_n(A) <_T \varphi_n(B)$.*
2. *$\varphi_n(C)$ is father of $\varphi_n(A)$ and $\varphi_n(C) = \varphi_n(B)$.*
3. *$\varphi_n(C)$ is father of $\varphi_n(A) = \varphi_n(B)$.*

4. $\varphi_n(A) = \varphi_n(B) = \varphi_n(C)$.

Proof. The vertices A and B are chains and they differ only by their smallest element (i.e. their label). In the case that labels of A and B are identical after replacing all elements greater than n by n, then we get $\varphi_n(A) = \varphi_n(B)$ and thus we get either 3. or 4.

Assume that labels of $\varphi_n(A)$ and $\varphi_n(B)$ are different.

If $|\varphi_n(A)| = |\varphi_n(B)|$, $\varphi_n(A)$ and $\varphi_n(B)$ differ only in their label and thus they are brothers in the tree T_n. From the definition of order $\leq_T$ it is easy to see that the order must be preserved when the largest numbers are cut away. Thus we get 1.

In the remaining case $|\varphi_n(A)| \neq |\varphi_n(B)|$. Because A and B are brothers, we have that $|A| = |B|$ and the chains differ only by the smallest elements. To get $|\varphi_n(A)| \neq |\varphi_n(B)|$, label of exactly one of A or B became equivalent to the second smallest element of the chain A and B and thus the number of elements in the chain has changed differently than the number of elements in the other chain.

It is easy to see that if this happens for A then the label B (which is greater in the $\leq_T$ order) can differ only by the elements greater than n implying $|\varphi_n(A)| = |\varphi_n(B)|$ (a contradiction). We conclude that the label of B got identified with the second smallest element of the chain A and we have 2. $\square$

II. We proceed by construction of the universal word $W(I)$ for the subset tree $T(I)$. We will construct a sequence (word) $W(A)$ for each vertex A of $T(I)$ recursively. Since I is the root vertex of $T(I)$, we compute the $W(I)$ as well.

For any leaf vertex A of $T(I)$ the label consist of single integer x and we put $W(A) = x$.

Assume that A is not a leaf, S its rightmost son. In the case A has no left brother, we put:

$$W(A) = W(S).$$

In the case A has left brother L, we put:

$$W(A) = W(L)\overleftarrow{W(L)}W(S).$$

Lemma 3.2 *For given set M and T subset tree of it, $W(T)$ construct universal sequence (in the sense of Definition 3.1).*

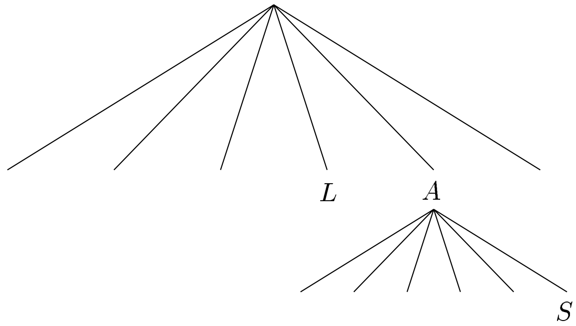

Figure 7: Construction of W

Proof. The conditions 1. and 2. from the definition follows trivially from the construction of $W(I)$. $W(I)$ contain as a subword each $W(Q)$ where Q is an ancestor of I (or $Q \subseteq I$).

To prove 3. from the Definition 3.1, we prove stronger

Claim 3.1 *For any vertex A of $T(I)$ we can transform $W(A)$ to $W(\varphi_n(A))$ using the transformations 3.1$(i)(ii)$.*

We proceed by induction over the depth of the construction of $W(A)$.

For the leaf-vertices of $T(I)$ the Claim 3.1 holds trivially: The word assigned to the leaf vertex L with label l is l and $\varphi_n(L)$ is leaf vertex either with label l or n. In the first case we need no transformation and in the second case we use transformation (i).

Assume that A is a non-leaf vertex. We put $a = W(A)$, $a' = W(\varphi_n(A))$, $s = W(s)$, $s' = W(\varphi_n(S))$, and $l = W(L)$, $l' = W(\varphi_n(L))$ or empty word in case L does not exist.

We consider four possible cases (allowed by Lemma 3.1):

1. $\varphi_n(A) \neq \varphi_n(S)$ and $\varphi_n(A) \neq \varphi_n(L)$ (if L exist).

   From the definition of φ_n it is easy to see that the left brother of $\varphi_n(A)$ (if L exists) is $\varphi_n(L)$ and the rightmost son of $\varphi_n(A)$ is $\varphi_n(S)$. By the induction hypothesis we transform l to l', s to s' and thus also $a = l\overleftarrow{l}s$ to $a' = l'\overleftarrow{l'}s'$.

2. $\varphi_n(A) = \varphi_n(S)$ and $\varphi_n(L) \neq \varphi_n(A)$ (if L exists).

   As above, the left brother of $\varphi_n(A) = \varphi_n(S)$ (if L exists) is $\varphi_n(L)$. s' is $l'\overleftarrow{l'}s''$ where s'' is the word assigned to the rightmost son of $\varphi_n(A)$ or label of $\varphi\!\!/_n(A)$ in the case $\varphi\!\!/_n(A)$ is leaf. By induction we have a transformation of a to $l' \overleftarrow{l'} s'$ and thus:

$$l' \overleftarrow{l'} l' \overleftarrow{l'} s''.$$

We apply $l' \overleftarrow{l'} l' \to l'$ and get a transformation to:

$$l' \overleftarrow{l'} s'' = a'.$$

3. $\varphi\!\!/_n(A) = \varphi\!\!/_n(S) = \varphi\!\!/_n(L)$.

   By induction we have transformations of l and s to $a' = l' = s'$ and in turn a transformation of a to $a' \overleftarrow{a'} a'$. We apply $a' \overleftarrow{a'} a' \to a'$ to get a transformation to a'.

4. $\varphi\!\!/_n(A) = \varphi\!\!/_n(L)$ and $\varphi\!\!/_n(S) \neq \varphi\!\!/_n(A)$.

   By induction we have transformations of l to l' and s to s' thus a transformation of a to

   $$l' \overleftarrow{l'} s'.$$

   In this case $\varphi\!\!/_n(S)$ is the rightmost son of $\varphi\!\!/_n(A) = \varphi\!\!/_n(L)$. Let k' be the code of the left brother of $\varphi\!\!/_n(L)$ or the empty word if there is no such a vertex. We have $a' = l' = k' \overleftarrow{k'} r'$ so we get:

   $$l' \overleftarrow{l'} s' = k' \overleftarrow{k'} s' \overleftarrow{s'} \overleftarrow{k'} k' s'.$$

   We apply the transformation $\overleftarrow{k'} s' \overleftarrow{s'} k' \overleftarrow{k'} s' \to \overleftarrow{k'} s'$ and we get

   $$k' \overleftarrow{k'} s' = a'.$$

$\square$

Now we are in position to conclude the proof of our main result:

Proof of Theorem 1.1. We find an embedding $\phi : \mathcal{G} \to \mathcal{P}$. By Proposition 2.2 we have a mapping $h : \mathbb{N} \to \mathcal{P}$ and an embedding $\phi^* : \mathcal{G} \to \mathcal{P}^*$ which assigns to every $n \in \mathcal{G}$ the set

$$\phi^*(n) = \sum_{i \in I(n)} h(i) = \sum_{i \in I(n)} P_i.$$

For $I(n)$ consider subset tree $T(I(n))$ and its transition $W(I(n))$. For every n consider the symetrization $\widehat{P_n}$ defined by code

$$c(\widehat{P_n}) = c(P_n)c(P_n)^{-1}.$$

Denote by P_I the concatenation of paths $\widehat{P_i}$ which correspond to letters in $W(I)$. Explicitly: if $W(I) = i(1)i(2)\dots i(t)$ is a word over I then we put

$$P_I = \widehat{P_{i(1)}}\widehat{P_{i(2)}}\dots\widehat{P_{i(t)}}.$$

We put $\phi(n) = P_{I(n)}$ and we prove that ϕ is an embedding $\mathcal{G} \to \mathcal{P}$.

However this is by now easy. We have $m \leq_{\mathcal{G}} n$ iff $\phi^*(m) \leq_{\mathcal{P}^*} \phi^*(n)$. If $m \not\leq_{\mathcal{G}} n$ then $\phi(m) \not\leq \phi(n)$ by rigidity of path encoding.

If $m \leq_{\mathcal{G}} n$ then $\phi^*(m) \leq_{\mathcal{P}^*} \phi^*(n)$.

In the case $m \leq n$, the ordering property gives $I(m) \subseteq I(n)$: The word $W(I(n))$ contains $W(I(m))$ as an subword by Definition 3.1 and thus $\phi^*(m)$ is subpath of $\phi^*(n)$.

Assume that $m > n$. By Lemma 3.2 there exists sequence of words $w_1, w_2, \dots w_t$, where w_i is constructed from w_{i-1} by one of the transformations allowed by 3.1$(i)(ii)$.

We will construct sequence of paths $\phi(m) = P'_1,\ P'_2,\ \dots,\ P'_t = \phi(n)$ and show that $P'_i \to P'_{i+1}$ for each $i = 1, \dots, t-1$.

(i) Let w_{i+1} be constructed from w_i by transformation $x \to n$ for an $x > n$. Then, by ordering property 2.3(3) we know that $P_x \to P_n$ and thus we can create P'_{i+1} from P'_i by replacing $\widehat{P_x}$ by $\widehat{P_n}$.

(ii) Assume that a subword $a(1)\dots a(t)a(t)\dots a(1)a(1)\dots a(t)$ of w_i was replaced by $a(1)\dots a(t)$ to get w_{i+1}. We put

$$P' = \widehat{P_{a(1)}}\dots\widehat{P_{a(t)}}\widehat{P_{a(t)}}\dots\widehat{P_{a(1)}}\widehat{P_{a(1)}}\dots\widehat{P_{a(t)}}$$

$$P'' = \widehat{P_{a(1)}}\dots\widehat{P_{a(t)}}.$$

From the symmetry of $\widehat{P_{a(i)}}$ (i.e. be $\widehat{P_{a(i)}}^{-1} = \widehat{P_{a(i)}}$) it is trivial to see that

$$c(P') = c(P'')c(P'')^{-1}c(P'').$$

implying $P' \to P''$ and thus we can construct P'_{i+1} from P'_i by replacing P' by P''.

Thus we have $\phi(m) \to \phi(n)$. This finishes proof of Theorem 1.1. $\square$

4 Concluding Remarks

4.1 On-line Dimensions

Let P be a finite partially ordered set. We showed that P has an on-line embedding $\phi : P \to \mathcal{V}^*$ (explicitly ϕ is an embedding which can be extended to the embedding of generic partially ordered set $\mathcal{G}$)

Consider the following On-Line Dimensions $\mathrm{OLD}(P)$ as follows:

$$\mathrm{OLD}(P) = \min_{\phi} \max_{x \in P} |\phi(x)|.$$

Thus $\mathrm{OLD}(P)$ measures sizes of vector sets assigned to the vertices of P. The minimum is taken over all on-line embedding $\vec{V} : P \to \mathcal{V}^*$. We can explicitly define this quantity as follows (without reference to mystical beings of A–B players): Let $\mathcal{G}$ be the generic (i.e. homogeneous and universal) partially ordered set.

We can then define the on-line dimensions as follows.

Definition 4.1 *Let P be a finite partially ordered set. We put*

$$\mathrm{OLD}(P) = \min_{\phi} \max_{x \in P} |\phi(x)|$$

where minimum is taken over all embedding $\phi : P \to \mathcal{V}^*$ which can be extended to an embedding $\phi' : \mathcal{G} \to \mathcal{V}^*$.

By Proposition 2.2 this dimension is well defined. From the above proof of Proposition 2.2 we have the following bound:

Corollary 4.1 *Let P be a partial order with n vertices. Then* $\mathrm{OLD}(P) \leq n$.

Proof. We recall the notation from the proof of Proposition 2.2. The embedding $\vec{V}(x) = \{\vec{v}(y), t(y) \leq t(x), y \leq_P x\}$ (where $t(x)$ is the time of the creation of x) and thus $\vec{V}(x) \leq t(x)$ (as we assume integral time).

$\square$

These bounds can be improved by choosing a convenient ordering; for example for Hiraguchi's partially ordered set S_n [26] we have $\mathrm{OLD}(S_n) \leq \frac{n}{2}$. However we should stop estimating OLD as we can determine it exactly:

Proposition 4.1 $\mathrm{OLD}(P) = 1$ *for every P.*

Proof. This is not a triviality (as the statement may suggest). Assume that $\mathcal{G}$ is the generic partial order defined on $\mathbb{N}$ by relation $\leq_{\mathcal{G}}$ and assume without loss of generality that the partial order on $\{1, 2, \ldots |P|\}$ is isomorphic to P. We modify the definition of $\vec{v}(n)$ and $\vec{V}(n)$ used in the proof of Proposition 2.2 such that for all $n = 1, 2, \ldots |P|$, we shall have $|\vec{V}(n)| = 1$.

For $n \leq |P|$ we put $\vec{v}(n) = (v_1, v_2, \ldots v_{|P|})$ where $v_m = 1$ iff $n \leq_{\mathcal{G}} m$. We put $\vec{V}(n) = \{\vec{v}(n)\}$.

The definition of functions $\vec{v}(n)$ and $\vec{V}(n)$ for $n > |P|$ remains the same as in the proof of Proposition 2.2.

It is easy to see that $\phi(v) = \vec{V}(v)$ is an embedding of P into $\mathcal{V}^*$:

1. Assume that $m \leq_{\mathcal{G}} n$. We prove that $\{\vec{v}(m)\} \leq_{\mathcal{V}^*} \{\vec{v}(n)\}$.

   We need to show that for each r $\vec{v}(n)_r = 1 \implies \vec{v}(m)_r = 1$. From the definition of $\vec{v}(n)$ we have $n \leq_{\mathcal{G}} r$ and from the transitivity we have $m \leq_{\mathcal{G}} r$ that implies $\vec{v}(m)_r = 1$.

2. Assume that $m \not\leq_{\mathcal{G}} n$. We prove that $\{\vec{v}(m)\} \not\leq_{\mathcal{V}^*} \{\vec{v}(n)\}$.

   Assume, for contradiction, that $\vec{v}(m) \leq \vec{v}(n)$. We have that $\vec{v}(m)_i \geq \vec{v}(n)_i$ for each $i \leq |P|$. $\vec{v}(n)_n = 1$ follows from the definition of $\vec{v}(n)$ and $\vec{v}(m)_n = 0$ follows from the definition of $\vec{v}(m)$. This is a contradiction.

It remains to be shown that the $\vec{V}$ embeds $\mathcal{G}$. However the proof for remaining vertices $n \leq |P|$ is analogical to the proof of Proposition 2.2 $\square$

Proposition 4.2 $\mathrm{OLD}(\mathcal{G}) = \omega$.

Proof (sketch). Assume for contradiction $\mathrm{OLD}(\mathcal{G}) \leq k$. Let $\phi : \mathcal{G} \to \mathcal{V}^*$ be any embedding ϕ which assigns to every vertex $x \in P$ a set of at most k vectors. ($\mathcal{G}$ is again the generic partial order).

Fix $x \in \mathcal{G}$ and let K be the maximal length of the vector from $\phi(x)$. The upper set $\{y; y \leq_{\mathcal{G}} x\}$ induces universal poset P' (as $\mathcal{G}$ embeds into it) and by the definition every $\phi(y), y \in P'$ contains vectors of length $\leq |K|$. However as is impossible that applying Ramsey Theorem for partially ordered sets [25] we get easily that the Boolean dimension of P' would be bound by $\leq K$ which is impossible as Boolean dimension of finite partially ordered sets is unbounded. See [24, 11] for the Boolean dimension of a partially ordered set. $\square$

These two results are only seemingly contradictory. What they really mean is that the dimension OLD is not a finite number for generic and every universal partial order (as one would expect) and that the on-line embedding of $\mathcal{G}$ to $\mathcal{V}^*$ can be modified such that at the (arbitrarily long) finite beginning the embedding uses only singleton vector sets.

4.2 On an Overgrown Path

OLD is not the only on-line dimension which we may consider (for example we can optimize the maximal dimension of vectors in $\phi(x)$) and we shall return to this at another occasion. Similar definitions and results can be also introduced for $\mathcal{P}^*$ and $\mathcal{P}$ representations. However note that for path representations (i.e. on-line embedding into $\mathcal{P}$) our bounds drastically change.

Let us define just the following path On Line Dimension:

$$\mathrm{OLD}'(P) = \min_{\phi} \max_{x \in P} |V(\phi(x))|$$

where $\phi : P \to \mathcal{P}$ is an on-line embedding of P into $\mathcal{P}$. $\mathrm{OLD}'(P)$ is well defined by virtue of Theorem 1.1. We have the following

Corollary 4.2 *Let P be a partially ordered set with n vertices. The* $\mathrm{OLD}'(P) \leq 3^{2^{2^n}} n$.

Proof. It suffices to check proof of Theorem 1.1. This large bound is due to the construction underlying the Transition Lemma and it is probably too large. However our lower bound is only logarithmic. $\square$

In view of Proposition 4.1 one should perhaps investigate the growth of embeddings of the generic partial order to ($\mathcal{V}^*$, $\mathcal{P}^*$ or $\mathcal{P}$).

Let $\mathcal{G}$ be the generic partial order on $\mathbb{N}$ and let ϕ by an embedding $\phi : P \to \mathcal{P}$ ($\phi : P \to \mathcal{P}^*$, $\phi : P \to \mathcal{V}^*$). Define as the function $\mathrm{Growth}_\phi : \mathbb{N} \to \mathbb{N}$ by $\mathrm{Growth}_\phi = |\phi(n)|$.

What are the possible growth functions [16]? It follows from our construction that $\mathrm{Growth}_\phi(n) = O(n^2)$ for both $\phi : P \to \mathcal{V}^*$ and $\phi : P \to \mathcal{P}^*$. However for $\phi : P \to \mathcal{P}$ we have $\mathrm{Growth}(n)$ of the order of the function $3^{2^{2^n}} n$.

4.3 Universality of Graphs

The universality of P has some consequences for universality of very special classes of graphs. For example we get the following

Corollary 4.3 *The class of all cubic planar connected graphs is universal.*

This can be proved using rigid gadgets which are used as replacement of edges, see [22, 14] for details of this construction. Let us remark that neither the planar graphs represent all groups [1] nor bounded degree graphs represent all monoids [2]; see also [14].

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