A NOTE ON PROJECTIVE GRAPHS

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ABSTRACT. We show that all graphs with a simple extension property are projective. As a consequence of this result we settle in the affirmative a conjecture of Larose and Tardif and characterize all homogeneous graphs which are projective.

1. Introduction

For graphs G, H, a *homomorphism* $f : G \to H$ is a map $f : V(G) \to V(H)$, for which $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. An *operation* (*polymorphism* in [2, 8, 19]) on H is any homomorphism $f : H^n \to H$, for some $n \geq 1$. Here $H^n = H \times \ldots \times H$ is the direct product defined on $V(H) \times \ldots \times V(H)$ coordinatewise. (Alternatively, H^n can be defined as the maximal graph on the set $V(H) \times \ldots \times V(H)$ with the property that all projections $\pi_i : (x_1, \ldots, x_n) \to x_i$ are homomorphisms.) Larose and Tardif [12] defined (in the analogy with posets) *projective graphs* as graphs for which any idempotent operation (i.e., any homomorphism $f : H^n \to H$ which satisfies $f(x, \ldots, x) = x$) is a projection. They also asked if most graphs on a large set are projective.

In this note we show that all graphs with a natural extension property (called *fair graphs*) are projective. Since almost all graphs on n vertices have good extension properties, this result implies that the fraction of all graphs on n vertices which are projective tends to one as n tends to infinity. Another consequence of our theorem is the characterization of all homogeneous projective graphs.

Projective graphs were studied intensively (see e.g. [19, 14, 5, 15, 12, 13, 17, 18, 2, 1]), see also remarks at the end of this paper.

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2. Fair graphs

We say that a graph H is *fair* if for any two sets A, B, of vertices of H, such that $A \subseteq B$, $|A| \leq 2$, $|B| \leq 5$, and A is independent, there exists a vertex v of H adjacent to all vertices from A and to no vertices from $B \setminus A$. (One can reduce 5 to 3 in the definition of fair graph by elaborating the proof but this is irrelevant to our purpose.) Our main result states that each fair graph, finite or infinite, is projective.

Theorem 1. *Every fair graph is projective.*

Proof. Our argument is based on the following non-trivial result of Larose and Tardif [12].

Theorem 2. *A graph H is projective if and only if it is 2-projective; i.e., if the only homomorphisms $f : H \times H \to H$ satisfying $f(v, v) = v$ are the projection.*

Thus, let H be fair graph and let a homomorphism $f : H \times H \to H$ be such that $f(v, v) = v$ for every $v \in V(H)$. Let us start with the following observation.

Claim 1. *If $f(v, w) = u$ for some $v, w, u \in V(H)$, then $v = u$, or $w = u$.*

Proof. Suppose that $u \notin \{v, w\}$ and v is not adjacent to w. Since H is fair, there exists a vertex $t \in V(H)$ which is adjacent to both v and w and is not adjacent to u. Then vertices (v, w) and (t, t) are adjacent in $H \times H$ but $u = f(v, w)$ is not adjacent to $t = f(t, t)$, which contradicts the fact that f is a homomorphism.

Now let $u \neq v, w$ and $\{v, w\} \in E(H)$. Since H is fair, there are two different non-adjacent vertices $v', w' \in V(H) \setminus \{v, w, u\}$ such that the only edges between sets $S = \{v, w, u\}$ and $T = \{v', w'\}$, are $\{v, v'\}$ and $\{w, w'\}$. Since $\{(v, w), (v', w')\} \in E(H \times H)$, the vertices u and $u' = f(v', w')$ are adjacent in G. But in this case the vertex u' must be different from both v' and w', which, since $\{v', w'\} \notin E(H)$, contradicts the first part of the proof. □

Claim 2. *If for $v, w \in V(H)$, $v \neq w$, we have $f(v, w) = v$, then $f(r, s) = r$ for every $r, s \in V(H)$.*

Proof. First let us prove the claim under additional assumptions that all vertices v, w, r, s are different and H contains no edges between the sets $U = \{v, w\}$ and $W = \{r, s\}$. Claim 1 implies that $f(r, s)$ is equal to either r or s. Let us suppose then that $f(r, s) = s$. Since H is fair, there are vertices $\bar{s} \neq v, w$ and $\bar{v} \neq r, s$, such that $\bar{s}$ is adjacent to v and r, but not to s, while $\bar{v}$ is adjacent to s and w, but not to v. Then

$(\bar{s}, \bar{v})$ is adjacent in $H \times H$ to both (v, w) and (r, s), so $f(\bar{s}, \bar{v})$ must be adjacent in H to both v and s. However, neither $\bar{s}$ nor $\bar{v}$ have this property, which contradicts Claim 1.

Now let $v, w, r, s \in V(H)$ be such that $v \neq w$ and $f(v, w) = v$. Since H is fair, there are two different vertices t, u of H which are adjacent to none of vertices v, w, r, s. Thus, applying twice the argument from the first part of the proof we infer that $f(t, u) = t$ and $f(r, s) = r$. □

Now Theorem 1 is an immediate consequence of Claim 2. □

Let us remark that the projectivity of the graph G follows already from Claim 1 as proved in [12]. However under stronger assumption of fairness we have an easy proof of this fact.

It is well known and easy to check that the probability that a graph $\mathbb{G}_n$ chosen at random from the family of all graphs with vertex set $\{1, 2, \ldots, n\}$ is fair tends to 1 as $n \to \infty$ (see, for instance, the proof of Theorem 10.5 in [9]), so Theorem 1 implies that $\mathbb{G}_n$ is asymptotically almost surely projective.

Theorem 3. *The probability that the random graph $\mathbb{G}_n$ is projective tends to 1 as $n \to \infty$.* □

Remark. Since the probability that $\mathbb{G}_n$ is rigid tends to one as $n \to \infty$, the above result implies that with probability $1 - o(1)$ the random graph $\mathbb{G}_n$ is also strongly rigid (for the definitions and discussion of these notions see [16, 19]).

Some particular graphs are of independent interest. For example, the projectivity of complete graphs emerges naturally in the analysis of uniquely colorable graphs. The fact that a finite complete graph is projective was observed by V. Müller [15] (see also [17] for an easy proof which applies to a more general class of finite graphs). Here we give yet another simple argument which shows that a complete graph of any cardinality is projective.

Proposition 4. *Every finite or infinite complete graph with at least* 3 *vertices is projective.*

Proof. Let $f : K_\alpha \times K_\alpha \to K_\alpha$ be a homomorphism such that $f(v, v) = v$ for each $v \in V(K_\alpha)$. We show first that for every $u \in V(K_\alpha)$ the set $f^{-1}(u)$ is contained in one line, i.e., either $f^{-1}(u) \subseteq \{u\} \times V(K_\alpha)$, or $f^{-1}(u) \subseteq V(K_\alpha) \times \{u\}$.

Indeed, suppose that this is not the case and there exist $(v, w), (r, s) \in V(K_\alpha) \times V(K_\alpha)$ and $u \in V(K_\alpha)$ such that $f(v, w) = f(r, s) = u$, and pairs $(v, w), (r, s)$ do not lie on one line. This means that $v \neq r$ and $w \neq s$; but then $\{(v, w), (r, s)\}$ is an edge of $K_n \times K_n$ mapped by f

into a loop at the vertex u contradicting the fact that K_α contains no loops. Hence, for every $u \in V(K_\alpha)$, $f^{-1}(u)$ is contained in a line.

Now take two vertices $v', v'' \in V(K_\alpha)$, $v' \neq v''$. Then, since $f(v', v') = v'$ and $f(v'', v'') = v''$, we have $f(v', v'') \in \{v', v''\}$; let us assume that $f(v', v'') = v'$. Take any $v''' \in V(K_\alpha)$, $v''' \neq v'$, and suppose that $f(v', v''') = v'''$. Then any choice of $f(v''', v')$ would contradict the assumption that each of the sets $f^{-1}(v')$ and $f^{-1}(v''')$ is contained in one line. Hence, for all $u \in V(K_\alpha)$, we have $f(v', u) = v'$, i.e., $f^{-1}(v') = \{v'\} \times V(K_\alpha)$. Thus, for every $v \in V(K_\alpha)$, the set $f^{-1}(v)$ is a full line, which clearly implies that f is a projection. □

As another consequence of Theorem 1 we characterize all projective homogeneous graphs. A countable graph G is called *homogeneous* (sometime *ultrahomogeneous*) if any partial isomorphism between two finite subgraphs of G extends to an isomorphism of G. All homogeneous undirected graphs were classified by Lachlan and Woodrow [11] as follows. The finite homogeneous graphs are: complete graphs K_n; their complements – empty graphs D_m; balanced n-partite graphs on nm vertices $K_n[D_m]$; the complements of $K_n[D_m]$; and two 'special' graphs: the cycle C_5 and the Window graph W which is the line graph of $K_{3,3}$. The infinite homogeneous graphs are: infinite graphs of type $K_n[D_m]$ and their complements; the Rado graph which is the (unique) example of random infinite countable graph; graphs U_k, $k \geq 3$, which are universal and homogeneous K_k-free graphs, uniquely determined by the amalgamation property of the class $\mathrm{Forb}(K_k)$ of all finite K_k-free graphs (U_k can be constructed as *Fraissé limit* of this class; see, for instance, [7, 11]); and the complements of U_k.

From the above result of Lachan and Woodrow and Theorem 1 we obtain the following characterization of homogeneous projective graphs.

Theorem 5. *The family of all projective homogeneous undirected graphs consists of the following elements:*

(i) C_5;
(ii) *complete graphs;*
(iii) *graphs* U_k, $k \geq 2$, *and their complements;*
(iv) *the Rado graph.*

Proof. Note first that each projective graph is connected and its vertices can be distinguished by their neighbourhood. It rules out all finite homogeneous graphs except of two special graphs and complete graphs. C_5 is projective by [12], the fact that the *window graph* W fails to be projective can be verified by a direct inspection (in fact no non-trivial product graph can be projective). The complete graphs are projective by Proposition 4. Finally all graphs U_k, $k \geq 3$, and Rado graph, are easily seen to be fair. The complementary graphs are handled similarly as in Theorem 1. Thus the assertion follows. □

We conclude this note with two remarks. First we observe that a similar argument can be used to other countable universal graphs of the type Forb($\mathcal{F}$) where $\mathcal{F}$ is a set of 'forbidden' subgraphs, as well as to some other structures (such as oriented graphs, posets and relational structures). Moreover, according to Lachlan [10] (see also [4]) the homogeneous structures are of two types: the stable and unstable ones. In the undirected graph case the unstable ones are formed by U_k, their complements, and the Rado graph; all other homogeneous graphs are stable. Thus our result seems to suggest that all unstable structures tend to be projective while the stable projective ones perhaps could be characterized (it is possible at least for some particular cases like oriented graphs). We shall return to this problem elsewhere.

Despite beeing defined only recently, projectivity seems to be an important property. The fact that complete graphs are projective have been discovered in various context (and proved by various techniques) by Müller, Sauer, Greenwell, Lovász, Nešetřil, Zhu, Larose, Tardif, Alon et al. and possibly others. The proof given above is particularly simple. Projectivity is related to the unique extension of colorings (see [14, 15] and in fact (as shown in [18]) it is equivalent with it. A bit surprisingly, projectivity is related to the complexity of coloring. The fact that the relations invariant to all polymorphisms are exactly the definable sets of relations ([3], see also [8]) implies that the H-coloring problem is NP-complete for any projective graph H. Particularly, this implies that the H-coloring problem is NP-complete for almost all graphs H. Thus (as observed in [2]) for almost all graphs we do not have to use the combinatorially complicated proof of [6]. Finally, note that very recently the projectivity has been related to Furier analysis [1].

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