# RECURSIVE FAULT-TOLERANCE OF FIBONACCI CUBE IN HYPERCUBE

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Abstract. Fibonacci cube is a subgraph of hypercube induced on nodes without two consecutive 1's. If we remove from Fibonacci cube the nodes with 1 both in the first and the last position, we obtain Lucas cube. We consider the problem of determining the minimum number of nodes in $n$-dimensional hypercube whose removal leaves no subgraph isomorphic to $m$-dimensional Fibonacci cube. The exact values for small $m$ are given and several recursive bounds are established using the symetry property of Lucas cubes and the technique of labeling. The relation to the problem of subcube fault-tolerance in hypercube is also shown.

Keywords: Fibonacci cube; Lucas cube; fault-tolerance; hypercube; direct embedding

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## 1. Introduction

Fibonacci cube was proposed [9] as a new interconnection topology for parallel computers. It has a self-similar recursive structure usefull for a design of parallel algorithms [4,5,10]. Its properties have been studied [6,12] and several modifications have been proposed [1,3,14,15].

Let $Q_n$ denote $n$-dimensional hypercube, where the nodes are all binary strings of length $n$ and two nodes are adjacent if and only if they differ in exactly one bit. We say that an edge $\{u, v\} \in E(Q_n)$ has dimension $i$ if $u$, $v$ differ in $i$-th bit, counted from left.

Fibonacci cube $FC_n$ of dimension $n$ is a subgraph of $Q_n$ induced on nodes

$$V(FC_n) = \{0u; u \in V(FC_{n-1})\} \cup \{10v; v \in V(FC_{n-2})\} \qquad \text{for } n \geq 2, \tag{1}$$

and $V(FC_n) = V(Q_n)$ for $n < 2$. In hypercube $Q_n$ for $n \geq 2$, $FC_n$ is given recursively by connecting $FC_{n-2}$ in $10Q_{n-2}$ subcube with the corresponding nodes of $FC_{n-1}$ in $0Q_{n-1}$ subcube. For our purposes, let us use the notation $FC_n = 0FC_{n-1} \cup 10FC_{n-2}$. The number of nodes of $FC_n$ is $f_n$, the $n$-th Fibonacci number, where $f_0 = 1$, $f_1 = 2$, $f_{n+2} = f_{n+1} + f_n$.

Observe from (1) that the nodes of Fibonacci cube can be characterized as follows:

$$FC_n \text{ is a subgraph of } Q_n \text{ induced on nodes without two consecutive 1's.} \tag{2}$$

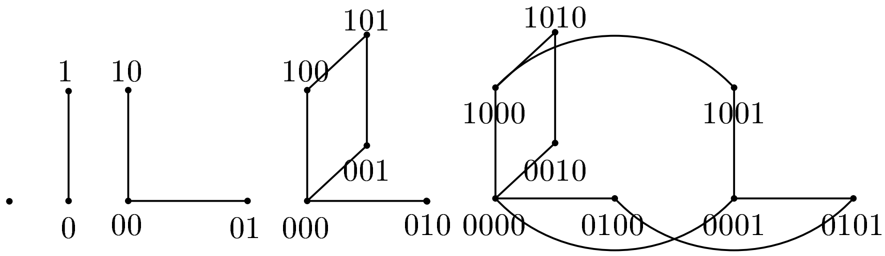



Fig 1. $FC_0$, $FC_1$, $FC_2$, $FC_3$, $FC_4$.

In this paper, we study the problem of determining the sets of nodes whose removal from $n$-dimensional hypercube leaves no subgraph isomorphic to $m$-dimensional Fibonacci cube. Let $\mathcal{S}(n, m)$ be the collection of all such sets and denote $\psi(n, m)$ the minimum size of a set in $\mathcal{S}(n, m)$. Note that $\mathcal{S}(n, m) \subseteq \mathcal{S}(n, m + 1)$ so $\psi(n, m) \geq \psi(n, m + 1)$. For a simplicity, define $\psi(n, m) = 0$ for $m > n$ and $\psi(n, m) = \psi(n, 0) = 2^n$ for $m < 0$. The informal term *faulty set* will always mean a set in $\mathcal{S}(n, m)$.

This question arises in a design of fault-tolerant applications for hypercube parallel architectures. If we consider busy nodes as faulty, how many faults in the worst case can appear in $n$-dimensional hypercube while it is still posssible to run applications designed for $m$-dimensional Fibonacci cube? The maximum number of tolerable faults is $\psi(n, m) - 1$.

Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ via an isomorphism $h : FC_n \to G$. We say that $h$ is a direct embedding of $FC_n$ into $Q_n$. It is known [2] that $h$ can be uniquely extended to a hypercube automorphism $h : Q_n \to Q_n$. Thus, for a set $S$ of nodes of hypercube $Q_n$,

$$S \in \mathcal{S}(n, n) \text{ if and only if } h^{-1}(S) \cap FC_n \neq \emptyset \text{ for every hypercube automorphism } h. \tag{3}$$

Let $h : Q_n \to Q_n$ be a hypercube automorphism. It is a well known fact that there exists exactly one permutation $\pi$ on $\{1, \ldots, n\}$ and exactly one binary string $w = w_1 \ldots w_n$ such that for every node $u = u_1 \ldots u_n$ we have

$$h(u_1 \ldots u_n) = v_1 \ldots v_n, \text{ where } v_i \equiv u_{\pi(i)} + w_{\pi(i)} \pmod 2 \text{ for } 1 \le i \le n. \tag{4}$$

In other words, hypercube automorphisms are composed of a permutation and a negation of bits in binary strings representing nodes.

There are several results on Fibonacci cube fault-tolerance in hypercube. A direct embedding of the $\lfloor \frac{n}{2} \rfloor$-th order Generalized Fibonacci cube of dimension $n + \lfloor \frac{n}{2} \rfloor$ into hypercube of dimension $n$ with no more than three faulty nodes was given [8].

In [2] the authors present a construction of a direct embedding of $n$-dimensional Fibonacci cube into faulty $n$-dimensional hypercube with less or equal $2^{\lceil n/4 \rceil - 1}$ faulty nodes. The idea is to find a permutation $\pi$ and a binary string $w$ such that the automorphism $h$ from (4) maps the given set of faulty nodes $F$ to nodes with two consecutive 1's. By (2) it means that $h(F) \subseteq Q_n \setminus FC_n$ and by (3) it follows that $F \notin \mathcal{S}(n, n)$. Since $F$ is arbitrary we conclude that $\psi(n, n) > 2^{\lceil n/4 \rceil - 1} = \Omega(2^{0.25n})$. Also the exact values of $\psi(n, n)$ for small $n$ and the best known lower and upper bounds on the number of tolerable faulty nodes are given, in terms of $\psi(n, n)$ :

$$\Omega(2^{cn}) = \frac{2^n}{4 f_{n-2}} < \psi(n, n) \le \kappa(n, \lceil n/2 \rceil) = O(2^{dn}), \tag{5}$$

where $c = 2 - log_2(1 + \sqrt{5}) \doteq 0.31$, $d = 2 - \frac{3}{4} log_2 3 \doteq 0.82$ and $\kappa(n, m)$ is defined below.

The problem of a Fibonacci cube fault-tolerance in hypercube is closely related to a subcube fault-tolerance in hypercubes. Let $\kappa(n, m)$ denote the minimum number of nodes whose removal from $n$-dimensional hypercube leaves no $m$-dimensional subcube. There are many results on $\kappa(n, m)$, see [7] for survey. In this paper, we derive similar results for $\psi(n, m)$.

In section 2 we give the exact values for $0 \le m \le 3$. For $\psi(n, 3)$, we modify a proof of Johnson and Entringer [11] who determined the exact value of $\kappa(n, 2)$. In section 3 we introduce Lucas cube, a special subgraph of Fibonacci cube with a symetry property that leads to a recursive bound $\psi(n, n) \le \psi(n-1, n-3)$. Next, the main scope of the paper, the technique of labeling from [7], is introduced. We use it in section 4 to establish recursive upper bounds on $\psi(n, m)$ for the case $m = n$, for example $\psi(n, n) \le 2\psi(n-3, n-5)$ for $n \ge 3$. Several labelings are constructed in section 5. In section 6 we extend this technique also for $m < n$. In section 7 we show that if we multiply with 3 any labeling for $\kappa(n, m)$-problem, we obtain a labeling for $\psi(n, m)$-problem. In section 8 we establish also recursive lower bounds $\psi(n, m) \ge 2\psi(n-4, m-4)$ for $n \ge m \ge 4$ and $\psi(n, m) \ge \frac{2^r}{f_r} \psi(n-r, m-r)$ for $n \ge m \ge r$. In section 9 we combine the results in previous sections to give the exact values or tight bounds on $\psi(n, m)$ for $0 \le m \le n \le 10$ and we conclude with a discussion of related open problems.

## 2. Exact values of $\psi(n, m)$ for small $m$

In this section we focus on what we know about the concrete values of $\psi(n, m)$ for $0 \le m \le 3$. Let us start with some notations. Given $S \subseteq V(Q_n)$ and a binary string

$a_1 \dots a_r$ of length $r$, $0 \le r \le n$, let us denote $S(a_1 \dots a_r) = \{v_1 \dots v_n \in S \mid v_i = a_i$ for all $1 \le i \le r\}$. For a subgraph $G = (V, E)$ of $Q_n$ let $G(a_1 \dots a_r)$ denote its subgraph induced on $V(a_1 \dots a_r)$.

**Lemma 1.** *$\psi(n, m) \le \kappa(n, \lceil m/2 \rceil)$, for $n \ge m \ge 0$.*

*Proof.* Fibonacci cube of dimension $m$ contains a subcube of dimension $\lceil m/2 \rceil$ as a subgraph. To see this, consider the subcube on nodes $*0 \cdots * 0*$ for $m$ odd or $*0 \cdots * 0$ for $m$ even, where $*$ is 0 or 1. These nodes do not contain two consecutive 1's so by (2) this subcube of dimension $\lceil m/2 \rceil$ is a subgraph of Fibonacci Cube of dimension $m$. Thus any faulty set for a direct embedding of $Q_{\lceil m/2 \rceil}$ into $Q_n$ is also a faulty set for a direct embedding of $FC_m$ into $Q_n$. $\square$

**Theorem 2.**

i) *$\psi(n, 0) = 2^n$, for $n \ge 0$,*
ii) *$\psi(n, 1) = 2^{n-1}$, for $n \ge 1$,*
iii) *$\psi(n, 2) = 2^{n-1}$, for $n \ge 2$.*

*Proof.* Part i) and ii) follow immediatelly from $FC_0 = Q_0$, $FC_1 = Q_1$ and $\kappa(n, 0) = 2^n$, $\kappa(n, 1) = 2^{n-1}$, [7]. For iii), by Lemma 1 we have that $\psi(n, 2) \le \kappa(n, 1) = 2^{n-1}$. But also $\psi(n, 2) \ge 2^{n-1}$ since we can divide $Q_n$ into $2^{n-2}$ disjoint $Q_2$ and every $Q_2$ must contain at least 2 nodes from any faulty set for $FC_2$. $\square$

**Theorem 3.** *$\psi(n, 3) = \lfloor 2^n/3 \rfloor$, for $n \ge 3$.*

*Proof.* Johnson and Entringer [11] determined, in terms of $\kappa$, that $\kappa(n, 2) = \lfloor 2^n/3 \rfloor$, so we obtain from Lemma 1 that $\psi(n, 3) \le \lfloor 2^n/3 \rfloor$.

Further, we modify their proof [11] to show that, for $n \ge 3$,

$$\text{if } S \subseteq V(Q_n) \text{ and } |S| < \lfloor 2^n/3 \rfloor \text{ then } Q_n \setminus S \text{ contains } FC_3 \text{ as a subgraph.} \tag{6}$$

We prove (6) by induction but first of all we need the following observation:

$$\lfloor 2^{n+1}/3 \rfloor = \begin{cases} \frac{2^{n+1}-2}{3} = 2\frac{2^n-1}{3} = 2\lfloor 2^n/3 \rfloor, & n \text{ even}, \\ \frac{2^{n+1}-1}{3} = 2\frac{2^n-2}{3} + 1 = 2\lfloor 2^n/3 \rfloor + 1, & n \text{ odd}. \end{cases} \tag{7}$$

For $n = 3$ we have $\lfloor 2^n/3 \rfloor = 2$. If we remove any node from $Q_3$ we still have a subgraph isomorphic to $FC_3$, so (6) holds. Observe that if $\{u, v\} \in \mathcal{S}(3, 3)$ then $u$, $v$ are antipodal nodes.

For $n = 4$ suppose that $|S| < \lfloor 2^n/3 \rfloor = 5$. If $Q_4 \setminus S$ does not contain $FC_3$ as a subgraph we are led to a contradiction as follows.

First, it must be that $|S(0)| = |S(1)| = 2$ and $u'$, $v'$ as well as $u''$, $v''$ are antipodes, where $S(0) = \{0u', 0v'\}$, $S(1) = \{1u'', 1v''\}$, for otherwise we can find a subgraph isomorphic to $FC_3$ in either $Q'_3 \setminus S(0)$ or $Q''_3 \setminus S(1)$, where $Q_4 = 0Q'_3 \cup 1Q''_3$. See Figure 2 for an ilustration. Then there exists 6-cycle $T'$ in $Q'_3 \setminus S(0)$ and let $T''$ denote 6-cycle in $Q''_3$ adjacent to $T'$. There must exist two adjacent nodes $a$, $b \in T''$ disjoint from $u''$ and $v''$. Denote $c$, $d$ their

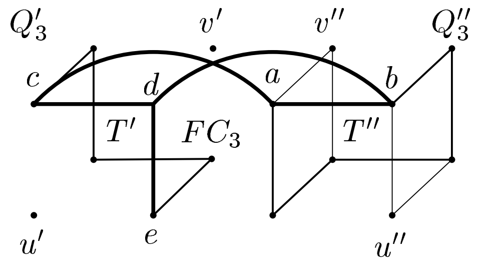



FIG 2. $\psi(4,3) \geq 5$ for otherwise we are led to a contradition.

corresponding nodes in $T'$ and $e$ the other node in $T'$ adjacent to $d$. These nodes form $FC_3$ in $Q_4 \setminus S$, a contradition, so (6) holds.

For $n = 5$ if we have that $|S| = |S(0)| + |S(1)| < \lfloor 2^n/3 \rfloor = 10$ then either $|S(0)| < 5$ or $|S(1)| < 5$, so by the previous case for $n = 4$ we have that (6) holds.

Suppose (6) is true for $3, \ldots, n$ and let $S \subseteq V(Q_{n+1})$ with $|S| < \lfloor 2^{n+1}/3 \rfloor$. If $Q_{n+1} \setminus S$ does not contain $FC_3$ as a subgraph we are led to a contradiction as follows.

First, $n$ must be odd for otherwise, by (7) and the induction hypothesis, we would have

$$|S| = |S(0)| + |S(1)| \geq 2\lfloor 2^n/3 \rfloor = \lfloor 2^{n+1}/3 \rfloor.$$

Then, since $n$ is odd, we have, again from (7) and the induction hypothesis,

$$\lfloor 2^{n+1}/3 \rfloor - 1 \geq |S| = |S(0)| + |S(1)| \geq 2\lfloor 2^n/3 \rfloor = \lfloor 2^{n+1}/3 \rfloor - 1,$$
$$\text{so } |S(0)| = |S(1)| = \lfloor 2^n/3 \rfloor.$$

Similarly,

$$\lfloor 2^n/3 \rfloor = |S(0)| = |S(00)| + |S(01)| \geq 2\lfloor 2^{n-1}/3 \rfloor = \lfloor 2^n/3 \rfloor,$$
$$\lfloor 2^n/3 \rfloor = |S(1)| = |S(10)| + |S(11)| \geq 2\lfloor 2^{n-1}/3 \rfloor = \lfloor 2^n/3 \rfloor,$$
$$\text{so } |S(00)| = |S(01)| = |S(10)| = |S(11)| = \lfloor 2^{n-1}/3 \rfloor.$$

Next, for $n$ odd we have from (7) that $\lfloor 2^{n-1}/3 \rfloor$ is odd so that, for some $i \in \{0,1\}$ we have $|S(00i)| = \lfloor 2^{n-2}/3 \rfloor + 1$ and $|S(001-i)| = \lfloor 2^{n-2}/3 \rfloor$. Since

$$|S(10i)| + |S(00i)| = \lfloor 2^{n-1}/3 \rfloor = |S(001-i)| + |S(101-i)|,$$

we obtain that $|S(10i)| = \lfloor 2^{n-2}/3 \rfloor$ and $|S(101-i)| = \lfloor 2^{n-2}/3 \rfloor + 1$. Similarly

$$|S(00i)| = |S(101-i)| = |S(011-i)| = |S(11i)| = \lfloor 2^{n-2}/3 \rfloor + 1,$$
$$|S(001-i)| = |S(10i)| = |S(01i)| = |S(111-i)| = \lfloor 2^{n-2}/3 \rfloor.$$

From (7) we have that $\lfloor 2^{n-2}/3 \rfloor$ is even, so that

$$|S(001-i1)| = |S(10i1)| = |S(01i1)| = |S(111-i1)| = \lfloor 2^{n-3}/3 \rfloor.$$

Since $|S(11)| = \lfloor 2^{n-1}/3 \rfloor$, observe that by the same argument $|S(1**1)| = \lfloor 2^{n-1}/3 \rfloor$ and

$$|S(10i1)| + |S(11i1)| + |S(111-i1)| + |S(101-i1)| = \lfloor 2^{n-1}/3 \rfloor,$$

and this, combined with the previous result, gives

$$|S(11i1)| + |S(101-i1)| = \lfloor 2^{n-1}/3 \rfloor - 2\lfloor 2^{n-3}/3 \rfloor = 2\lfloor 2^{n-3}/3 \rfloor + 1.$$

Similar argument yields

$$\begin{aligned} |S(011-i1)| + |S(11i1)| &= 2\lfloor 2^{n-3}/3 \rfloor + 1, \\ |S(101-i1)| + |S(011-i1)| &= 2\lfloor 2^{n-3}/3 \rfloor + 1, \end{aligned}$$

so that, finally,

$$2(|S(101-i1)| + |S(011-i1)| + |S(11i1)|) = 6\lfloor 2^{n-3}/3 \rfloor + 3.$$

Since the left side is even but the right side is odd, this is a contradiction and (6) holds. Hence, $\psi(n,3) \geq \lfloor 2^n/3 \rfloor$. $\square$

## 3. Lucas cube

In this section we introduce Lucas cube, a special subgraph of Fibonacci cube with a symetry property that leads to recursive bounds on $\psi(n,n)$. It has been studied in [13].

Lucas cube $LC_n$ of dimension $n$ is a subgraph of $Q_n$ induced on nodes

$$V(LC_n) = \{0u; u \in V(FC_{n-1})\} \cup \{10v0; v \in V(FC_{n-3})\} \qquad \text{for } n \geq 3, \tag{8}$$

and $V(LC_n) = V(FC_n)$ for $n < 3$. In our notation, $LC_n = 0FC_{n-1} \cup 10FC_{n-3}0$. For $n \geq 2$, the number of nodes is $l_n$, the $n$-th Lucas number, where $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$.

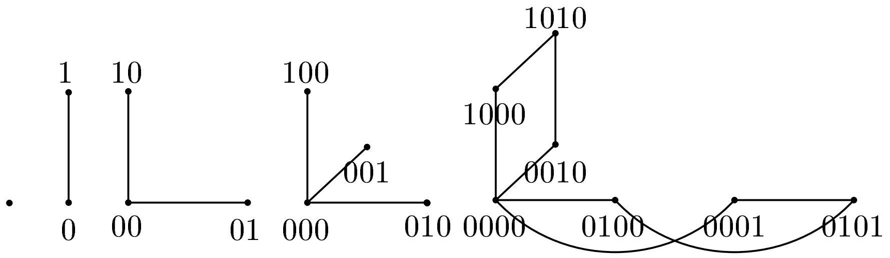



Fig 3. $LC_0$, $LC_1$, $LC_2$, $LC_3$, $LC_4$.

Observe from (1) and (8) that Lucas cube is a subgraph of Fibonacci cube: if we remove from $FC_n$ all nodes with both the first and the last bit set to 1, we obtain $LC_n$. We have

$$FC_n = \left\{ \begin{array}{l} LC_n = \left\{ \begin{array}{l} 0FC_{n-1} \\ 10FC_{n-3}0 \end{array} \right. \\ \phantom{LC_n =}\ \left. \begin{array}{l} \\ 10FC_{n-4}01 \end{array} \right\} = 10FC_{n-2} \end{array} \right\} = FC_n \qquad \text{for } n \geq 4,$$

and for $n = 3$ we have 101 instead of $10FC_{n-4}01$.

From (8) we obtain that the nodes of Lucas cube can be characterized as follows:

$$\begin{array}{c} LC_n \text{ is a subgraph of } Q_n \text{ induced on nodes without two consecutive 1's} \\ \text{and the first and last bits are considered to be consecutive.} \end{array} \tag{9}$$

For $0 \leq i \leq n$ define a hypercube automorphism $\rho_i(v_1 \dots v_n) = v_{n-i+1} \dots v_n v_1 \dots v_{n-i}$ for $v \in V(Q_n)$, i.e. $\rho_i$ rotates the binary strings representing nodes $i$-times to the right. We have from (9) that this maps $LC_n$ to $LC_n$, so restricted on $LC_n$ it is an automorphism of Lucas cube. This leads to a symetry property expressed in the following lemma.

**Lemma 4.** *For any $1 \leq i \leq n$, $n \geq 3$, if we remove all edges of dimension $i$ from $LC_n$ then it splits into two subgraphs isomorphic to $FC_{n-1}$ and $FC_{n-3}$.*

*Proof.* Split $LC_n$ along the first dimension into two subgraphs isomorphic to $FC_{n-1}$ and $FC_{n-3}$ by definition (8). Rotate it $(i-1)$-times to the right, so the removed edges of the first dimension are mapped to the edges of $i$-th dimension. Since the rotation is an automorphism of $LC_n$, we are done. □

**Theorem 5.** *$\psi(n, n) \leq \psi(n-1, n-3)$, for $n \geq 3$.*

*Proof.* Split $Q_n$ along an arbitrary dimension $i$ into two disjoint $Q'_{n-1}$ and $Q''_{n-1}$. Let $S \subseteq V(Q'_{n-1})$ be a set of size $\psi(n-1, n-3)$ in $\mathcal{S}(n-1, n-3)$. We will show that $S$ is in $\mathcal{S}(n, n)$. For a contradition, suppose that $G$ is a subgraph of $Q_n \setminus S$ isomorphic to $FC_n$ via hypercube automorphism $h$. $G$ contains a subgraph isomorphic to $LC_n$ since $FC_n = LC_n \cup 10FC_{n-4}01$. Let $j$ be the dimension which automorphism $h$ maps to dimension $i$: $j = \pi^{-1}(i)$ where $\pi$ is given by (4). By Lemma 4, if we split $LC_n$ along the $j$-th dimension, it must be $h(LC_n) \cap Q'_{n-1} = h(FC_{n-1})$ or $h(LC_n) \cap Q'_{n-1} = h(FC_{n-3})$. In any case, this contradicts the choice of $S$. See Figure 4 for an illustration. □

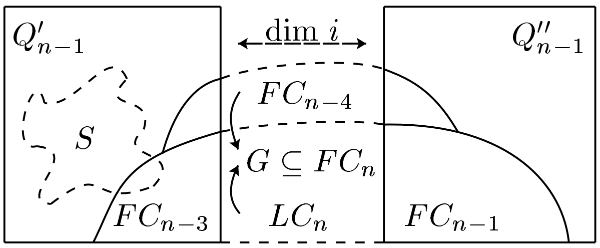



FIG 4. Intersection of $G$ with $S$.

Theorem 5 is based on the observation that any subgraph in hypercube of dimension $n$ isomorphic to Fibonacci cube of dimension $n$ intersects any subcube of dimension $n-1$ with a graph containing a subgraph isomorphic to Fibonacci cube of dimension $n-3$.

## 4. Labeling technique

In this section we introduce the concept of labeling [7], an useful tool for studying the recursive fault-tolerance in hypercubes and other self-similar networks.

Consider $(n-r)$-dimensional subcubes in $Q_n$ induced on nodes with first $r$ bits fixed. These $2^r$ subcubes form $r$-dimensional hypercube, denote it $C_r$, where each node $u$ represents one subcube $Q_n(u)$ and each edge represents a collection of all edges between two adjacent subcubes.

The following theorem [7] shows the use labelings for the problem of subcube fault-tolerance in hypercube.

**Theorem 6.** *Label the nodes of $C_r$ with non-negative integers such that for every $0 \leq i \leq r$,*

$$\text{every } i\text{-dimensional subcube in } C_r \text{ has a node with label at least } i. \tag{10}$$

*Then for every $n \geq r$ and $m \leq n$ we have that*

$$\kappa(n,m) \leq \sum_{u \in V(C_r)} \kappa(n-r, m-l(u)),$$

*where $l(u)$ is the label of $u$ and for $m < l(u)$ we set $\kappa(n-r, m-l(u)) = \kappa(n-r, 0) = 2^{n-r}$.*

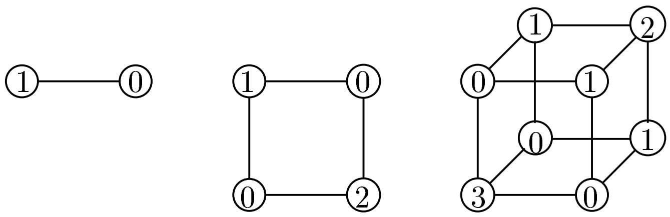


Fig 5. Examples of a labeling for the $\kappa(n,m)$-problem, $n = 1, 2, 3$.

To avoid distraction, let us say that the labeling (10) is a labeling for the $\kappa(n,m)$-problem. See Figure 5 for examples. We modify this concept for the $\psi(n,m)$-problem, in this section for $m = n$.

For a subgraph $G$ of $Q_n$ let us define its index of an inclusion of Fibonacci cube as

$$in(G) = \min\{i \mid G \text{ contains a copy of } FC_{n-i} \text{ as a subgraph}\}, \tag{11}$$

and leave $in(G)$ undefined if $G$ is an empty graph. Recall that $FC_0$ is a single node.

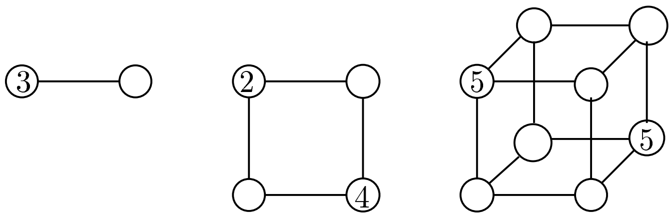
Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $r \leq n$. For every node $u \in V(C_r)$, consider subgraph $G(u)$ in $Q_n(u)$. We say that $in(G(u))$ is an indexing of $C_r$ with respect to $G$. We are considering partial indexings since $in(G(u))$ may be undefined. For two subgraphs $G_1$, $G_2$ of $Q_n$ isomorphic to $FC_n$, we say that their respective indexings are equivalent if there exists an automorphism $g : C_r \to C_r$ such that $in(G_1(u)) = in(G_2(g(u)))$ for all $u \in V(C_r)$.

Further, we define a (partial) labeling for the $\psi(n, n)$-problem. Label some nodes $u \in V(C_r)$ with non-negative integers $l(u)$ such that for $n \geq r$

$$\text{for every } G \simeq FC_n \text{ subgraph of } Q_n \text{ there exists } v \in V(C_r) \text{ with } in(G(v)) \leq l(v), \tag{12}$$

i.e. $G(v)$ contains Fibonacci cube of dimension at least $n - l(v)$ as a subgraph.

Observe from the definition that a single node with label 0 is the only such labeling of $C_0$. For another examples see Figure 6.




a) b) c)

FIG 6. Examples of a labeling for the $\psi(n, n)$-problem, $n = 1, 2, 3$.

**Theorem 7.** *Let $l$ be a (partial) labeling satisfying (12). Then for every $n \geq r$ we have*

$$\psi(n, n) \leq \sum_{\substack{u \in V(C_r) \\ l(u) \text{ defined}}} \psi(n - r, n - l(u)).$$

*Proof.* For a node $u \in V(C_r)$ with $l(u)$ defined, let $S(u) \subseteq V(Q_n(u))$ be a faulty set of size $\psi(n - r, n - l(u))$ in $\mathcal{S}(n - r, n - l(u))$. We show that $S = \bigcup_{\substack{u \in V(C_r) \\ l(u) \text{ defined}}} S(u)$ is in $\mathcal{S}(n, n)$.

For a contradition, let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and disjoint from $S$. By (12) we have a node $v$ with $in(G(v)) \leq l(v)$, hence $Q_n(v)$ contains $FC_{n-in(G(v))}$ as a subgraph which contains $FC_{n-l(v)}$ as a subgraph and disjoint from $S$. This contradicts the choice of $S$. $\square$

In the next section we show that labelings on Figure 6 satisfy (12) so we obtain

**Corollary 8.**

i) $\psi(n, n) \leq \psi(n - 1, n - 3)$ *for* $n \geq 1$,
ii) $\psi(n, n) \leq \psi(n - 2, n - 2) + \psi(n - 2, n - 4)$ *for* $n \geq 2$,
iii) $\psi(n, n) \leq 2\psi(n - 3, n - 5)$ *for* $n \geq 3$.

Note that i) is the same as in Theorem 5. We will see in the next section that labelings for the $\psi(n, n)$-problem can be constructed with a repeated use of a symetry property of Lucas cubes from Lemma 4.

## 5. Construction of labelings

First of all, recall that for every subgraph $G$ of $Q_n$ isomorphic to $FC_n$ via $h : FC_n \to G$ there exists exactly one permutation $\pi$ on $\{1, \dots, n\}$ and exactly one binary strings $w$ of length $n$ such that $h$ is given by (4). The following lemma gives us a sufficient condition (13) for an equivalence of indexings with respect to two subgraphs isomorphic to $FC_n$.

**Lemma 9.** *For $r \leq n$, let $w_1$, $w_2$ be binary strings of length $n$ and $\pi_1$, $\pi_2$ be permutations on $\{1, \dots, n\}$ such that*

$$\pi_1^{-1}(i) \leq r \text{ if and only if } \pi_2^{-1}(i) \leq r, \quad \text{for all } 1 \leq i \leq n. \tag{13}$$

*Denote $h_1$, $h_2$ automorphisms of $Q_n$, $h_1$ given by $\pi_1$, $w_1$ and $h_2$ given by $\pi_2$, $w_2$. Then the indexings of $C_r$ with respect to $G_1 = h_1(FC_n)$ and $G_2 = h_2(FC_n)$ are equivalent.*

*Proof.* Define hypercube automorphism $h = h_2 \circ h_1^{-1}$ and let a permutation $\pi$ and a binary string $w$ be given by (4). By (13) we have that

$$\pi(i) = \pi_1^{-1}(\pi_2(i)) \leq r \text{ if and only if } i \leq r, \quad \text{for all } 1 \leq i \leq n,$$

so we can define a permutation $\tau$ on $\{1, \dots, r\}$ as a restriction of $\pi$ and a binary string $v$ of length $r$ as a prefix of $w$. Let $g$ denote the automorphism of $C_r$ from (4) given by $\tau$, $v$. Since $h(h_1(z)) = h_2(z)$ for all $z \in V(Q_n)$ it follows that

$$\text{if } z \in G_1(u) \text{ then } h(z) \in G_2(g(u)), \quad \text{for all } u \in V(C_r),$$

and we conclude $in(G_1(u)) = in(G_2(g(u)))$ for all $u \in V(C_r)$. $\square$

**Lemma 10.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 1$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 7.*

*Proof.* By inspection of all cases, see Appendix.

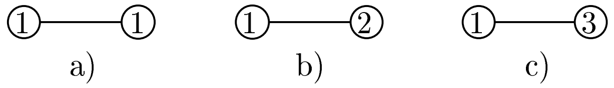



Fig 7. All indexings of $C_1$ up to an equivalence.

**Lemma 11.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 2$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 8.*

*Proof.* By inspection of all cases, see Appendix.

**Corollary 12.** *Labelings a) and b) on Figure 6 satisfy (12).*

*Proof.* Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r$. Let $I$ denote the indexing on Figure 7 (resp. Figure 8) equivalent to the indexing with respect to $G$, i.e. $in(G(u)) = I(g(u))$ for all $u \in V(C_r)$ for some automophism $g : C_r \to C_r$. On Figure 7

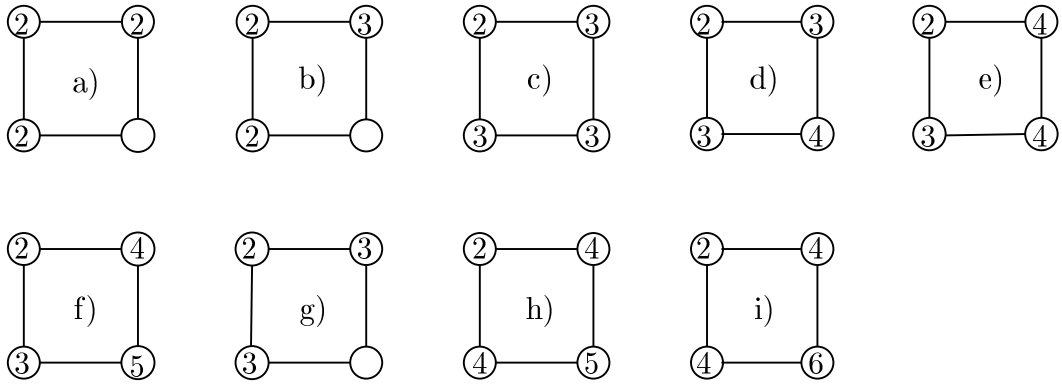



FIG 8. All indexings of $C_2$ up to an equivalence.

none index is higher than 3 and on Figure 8 if a node has index $> 4$ then its antipodal node has index 2. Thus there exists a node $v \in V(C_r)$ with $in(G(v)) = I(g(v)) \leq l(v)$ where $l$ is labeling on Figure 6a) (resp. Figure 6b). $\square$

For $r > 2$ we can construct labelings for the $\psi(n,n)$-problem without the need to list indexings with respect to all subgraphs $G$ of $Q_n$ isomorphic to $FC_n$. Instead of this we find a (partial) function that majorizes them. Nodes of $C_r$ without two consecutive 1's form $FC_r$ in $C_r$. For $u \in V(C_r)$ define

$$m(u) = \begin{cases} 2w(u) + r & u \in V(FC_r), \\ \text{not def.} & u \notin V(FC_r), \end{cases} \tag{14}$$

where $w(u)$ denotes the weight of $u$, i.e. the number of 1's in $u$. See Figure 9 for examples.

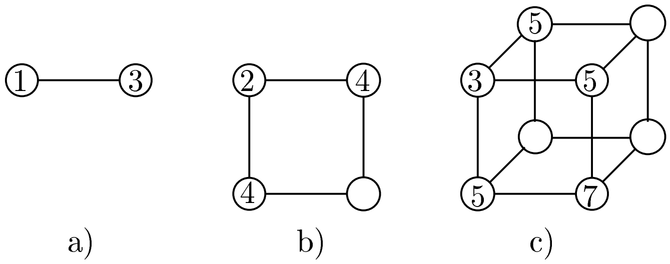



FIG 9. Majorant (14) of all indexings of $C_r$ with respect to some $G$.

**Lemma 13.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$. Then there exists an automorphism $g : C_r \to C_r$ such that for all $u \in V(C_r)$,*

*if $m(g(u))$ is defined then $G(u)$ is nonempty and $in(G(u)) \leq m(g(u))$.*

*Proof.* We can modify the mapping $h_1 : FC_n \to G$ to a mapping $h_2 : FC_n \to H$ such that the permutation $\pi_2$ and the binary string $w_2$ given by $h_2$ from (4) satisfy the condition that $\pi_2(i) \leq \pi_2(j)$ and $w_{2,\pi(i)} = 0$ for all $1 \leq i \leq j \leq r$. Moreover, we do it in a such way that also (13) is true, where $\pi_1$ and $w_1$ are given by $h_1$, so we can construct by Lemma 9 the automorphism $g : C_r \to C_r$ with $in(G(u)) = in(H(g(u)))$ for all $u \in V(C_r)$. We will show that for all $u \in V(C_r)$, if $m(u)$ is defined then $H(u)$ is nonempty and $in(H(u)) \leq m(u)$. So $g$ is the desired automorphism.

First of all, observe that $H(u)$ is nonempty for all $u = u_1 \ldots u_r \in V(FC_r)$. Define $v = v_1 \ldots v_n \in V(Q_n)$ with $v_{\pi_2(i)} = u_i$ for $1 \leq i \leq r$ and $v_{\pi_2(i)} = 0$ for $r < i \leq n$, so $h_2(v) \in Q_n(u)$. Since $u$ is without two consecutive 1's it follows that also $v$ is without two consecutive 1's, i.e. $v \in V(FC_n)$, and $h_2(v) \in H(u)$.

Secondly, we prove by induction on $r$ that $in(H(u)) \leq m(u)$ for all $u \in V(FC_r)$. By Lemma 10 and Lemma 11 it is true for $r \leq 2$. Suppose it is true for $r - 1$ and let $v \in V(FC_{r-1})$ be the prefix of $u$, i.e. $u = v0$ or $u = v1$. Since $in(H(v))$ is defined we have $H(v)$ contains $h_2(FC_{n-in(H(v))})$ which contains $h_2(LC_{n-in(H(v))})$ as a subgraph. For $n - in(H(v)) \geq 3$, we have by a symetry property in Lemma 4 that $h_2(LC_{n-in(H(v))})$ splits into $h_2(FC_{n-in(H(v))-1}) \subseteq H(v0)$ and $h_2(FC_{n-in(H(v))-3}) \subseteq H(v1)$. For $n - in(H(v)) < 3$, $LC_2$ splits into $FC_1$ and $FC_0$ and $LC_1$ splits into $FC_0$ and $FC_0$. Thus we conclude that

$$\begin{aligned} in(H(v0)) &\leq in(H(v)) + 1 \leq m(v) + 1 = m(v0), \\ in(H(v1)) &\leq in(H(v)) + 3 \leq m(v) + 3 = m(v1). \end{aligned}$$

□

**Corollary 14.** *Labeling c) on Figure 6 satisfies (12).*

*Proof.* Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 3$. By Lemma 13 we have an automorphism $g : C_3 \to C_3$ with $in(G(u)) \leq m(g(u))$ for all $u \in V(C_3)$ with $m(g(u))$ defined. Denote $Z$ the set of nodes $u$ on Figure 9c) with $m(u) \leq 5$ and denote $v_1$, $v_2$ the nodes of Figure 6c) with label 5. Observe that for any automorphism $g' : C_3 \to C_3$ we have that $g'(Z)$ and $\{v_1, v_2\}$ intersect. Hence we conclude that there exist a node $v \in \{v_1, v_2\} \subset V(C_3)$ with $in(G(v)) \leq m(g(v)) \leq l(v) = 5$, where $l$ is the labeling on Figure 6c). □

To construct good labelings for a general $r$ is a challenging combinatorial problem in itself. However, we can generalize Corollary 14 to construct labelings of $C_r$ for a special $r$ in the form $r = 2^k - 1$, where $k$ is some integer. This result is based on well-known Hamming code, a perfect one-error correcting binary code, which can be interpreted as a partition of hypercube of dimension $r = 2^k - 1$ into $2^r/(r + 1)$ disjoint 1-spheres. For an integer $z$, $z$-sphere with centre node $u$ is a subgraph of $Q_r$ induced on all nodes in a distance less or equal $z$ from $u$.

**Corollary 15.** *For every integer $k$ and $n \geq r = 2^k - 1$*

$$\psi(n, n) \leq \frac{2^r}{r + 1} \psi(n - r, n - r - 2).$$

*Proof.* Let $T$ denote the set of centre nodes of disjoint 1-spheres given by Hamming codes. Label them with $r + 2$ and other nodes leave unlabeled. Denote $Z$ the subgraph induced on nodes $u \in V(C_r)$ with $m(u) \leq r + 2$. Since $Z$ is also 1-sphere, observe that for any automorphism $g' : C_r \to C_r$ we have that $g'(Z)$ intersects $T$. We can conclude by Lemma 13 that the constructed labeling satisfy (12). The rest follows from Theorem 7. □

## 6. Recursive fault-tolerance for $m < n$

In this section we start with a generalization of Theorem 5 and we extend the labeling technique for $m < n$ in a way analogous to (10). First of all, note that a subgraph $G$ of $Q_n$ isomorphic to $FC_m$ is contained within a unique subcube given by $\{0, 1, *\}$-string $s = s_1 \dots s_n$ defined as follows. Let $h$ denote the isomorphism $h : FC_m \to G$, $u = h(0 \dots 0)$ and $I$ be the set of dimensions of edges on which edges of $FC_m$ from $0 \dots 0$ are mapped, i.e. $I = \{i; \exists v \in V(FC_m), \{0 \dots 0, v\} \in E(FC_m)$ and edge $\{\mathrm{h}(0 \dots 0), \mathrm{h(v)}\}$ has dimension $i\}$. Define $s_i = *$ for $i \in I$ and $s_i = u_i$ else.

Similarly like for $m = n$ the mapping $h$ can be uniquely extended to the automorphism of the whole subcube.

**Theorem 16.** *For $n \geq m \geq 0$*

i) $\psi(n, m) \leq \min\{\psi(n-1, m) + \psi(n-1, m-3),\ 2\psi(n-1, m-1)\}$,
ii) $\psi(n, m) \geq \max\{2\psi(n-1, m),\ \psi(n-1, m-1)\}$.

*Proof.* For i), split $Q_n$ along an arbitrary dimesion $i$ into two disjoint $Q'_{n-1}$ and $Q''_{n-1}$. Let $S_1 \subseteq V(Q'_{n-1})$ and $S_2 \subseteq V(Q''_{n-1})$ be sets of size $\psi(n-1, m)$ and $\psi(n-1, m-3)$ (resp. both $\psi(n-1, m-1)$) in $\mathcal{S}(n-1, m)$ and $\mathcal{S}(n-1, m-3)$ (resp. both in $\mathcal{S}(n-1, m-1)$). We show that $S_1 \cup S_2$ is in $\mathcal{S}(n, m)$. Clearly, any $FC_m$ in $Q_n$ disjoint from $S_1 \cup S_2$ cannot be all in either $Q'_{n-1}$ or $Q''_{n-1}$. But it also cannot be in both subcubes, because, in that case, we have by Lemma 4 that $Q'_{n-1}$ and $Q''_{n-1}$ contain $FC_{m-1}$ and $FC_{m-3}$ as a subgraphs which contradicts the choice of $S_1 \cup S_2$.

For ii), note that at least $\psi(n-1, m)$ nodes must be removed from each $Q'_{n-1}$ and $Q''_{n-1}$ so that no $FC_m$ remains in either $Q'_{n-1}$ or $Q''_{n-1}$. Thus $\psi(n, m) \geq 2\psi(n-1, m)$. To prove the second inequality, let $S_1 \cup S_2$ be set of size $\psi(n, m)$ in $\mathcal{S}(n, m)$ and $S_1 \subseteq V(Q'_{n-1})$, $S_2 \subseteq V(Q''_{n-1})$. Denote by $T$ the set of nodes of $Q'_{n-1}$ that are adjacent to $S_2$. If $Q'_{n-1}$ contains $FC_{m-1}$ disjoint from $(S_1 \cup T)$ then we connect this $FC_{m-1}$ with the corresponding $FC_{m-2}$ in $Q''_{n-1}$ and we obtain $FC_n$ in $Q_n$ disjoint from $S_1 \cup S_2$. This contradicts the choice of $S$ so $S_1 \cup T$ must contain at least $\psi(n-1, m-1)$ nodes and, therefore, $\psi(n, m) \geq \psi(n-1, m-1)$. $\square$

Now we can make a step from a labeling for the $\psi(n, n)$-problem towards a labeling for the $\psi(n, m)$-problem for general $m \leq n$. Label nodes $u \in V(C_r)$ with non-negative integers $l(u)$ such that for $m \leq n \geq r$

$$\text{for every } G \simeq FC_m \text{ subgraph of } Q_n \text{ there exists } v \in V(C_r) \text{ with } in(G(v)) - n + m \leq l(v), \tag{15}$$

i.e. $G(v)$ contains Fibonacci cube of dimension at least $m - l(v)$ as a subgraph.

Note that condition (15) can be rewriten more informaly and analogously to (10) as follows. Label nodes $u \in V(C_r)$ with non-negative integers $l(u)$ such that for every $0 \leq i \leq r$,

$$l \text{ restricted on every } i\text{-dimensional subcube } D_i \text{ in } C_r \text{ satisfies (12).}$$

For example, observe on Figure 10 that every node, i.e. subcube $D_0$, has label at least 0, every edge, i.e. subcube $D_1$, has a node with label at least 3, every 4-cycle, i.e subcube $D_2$,

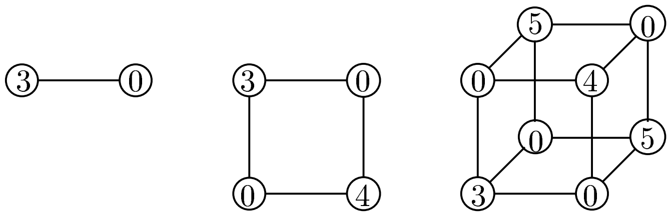


FIG 10. Examples of a labeling for the $\psi(n,m)$-problem.

has two antipodal nodes with labels at least 2 and 4, and labeling c) of $C_3$ satisfies (12) by an argument similar to Corollary 14. Hence all labelings on Figure 10 satisfy (15).

**Theorem 17.** *Let $l$ be a labeling of $C_r$ satisfying (15). For every $n \geq r$, $m \leq n$ we have*

$$\psi(n,m) \leq \sum_{u \in V(C_r)} \psi(n-r, m-l(u)).$$

*Proof.* For a node $u \in V(C_r)$ let $S(u) \subseteq V(Q_n(u))$ be a faulty set of size $\psi(n-r, m-l(u))$ in $\mathcal{S}(n-r, m-l(u))$. We show that $S = \bigcup_{u \in V(C_r)} S(u)$ is in $\mathcal{S}(n,m)$.

For a contradition, let $G$ be a subgraph of $Q_n$ isomorphic to $FC_m$ and disjoint from $S$. By (15) we have a node $v \in V(C_r)$ with $in(G(v)) - n + m \leq l(v)$, hence $Q_n(v)$ contains $FC_{n-in(G(v))} = FC_{m-(in(G(v))-n+m)}$ as a subgraph which contains $FC_{m-l(v)}$ as a subgraph and disjoint from $S$. This contradicts the choice of $S$. □

From Figure 10 we immediatelly obtain

**Corollary 18.** *For $m \leq n$*

i) $\psi(n,m) \leq \psi(n-1,m) + \psi(n-1,m-3)$, *for* $n \geq 1$,
ii) $\psi(n,m) \leq 2\psi(n-2,m) + \psi(n-2,m-3) + \psi(n-2,m-4)$, *for* $n \geq 2$,
iii) $\psi(n,m) \leq 4\psi(n-3,m) + \psi(n-3,m-3) + \psi(n-3,m-4) + 2\psi(n-3,m-5)$, *for* $n \geq 3$.

Note that i) is included also in Theorem 16.

## 7. Relation to labelings for subcube fault-tolerance

In this section we show that labelings for Fibonacci cube fault-tolerance can easily be derived from labelings for subcube fault-tolerance. Although it is an interesting relation, it should be noted that labelings obtained in this way are less effective than labelings constructed directly in previous sections, compare Figure 5 and Figure 10.

**Theorem 19.** *Let $l$ be a labeling of $C_r$ satisfying (10). Then the labeling $3l$ satisfies (15).*

*Proof.* Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_m$ via $h : FC_m \to G$. Denote $A$ the subgraph of $C_r$ induced on nodes $u$ with $G \cap Q_n(u) \neq \emptyset$, i.e. $in(G(u))$ is defined. We will show that for some $s$, $0 \leq s \leq m$, $A$ contains a subcube $C_s$ of dimension $s$ with

$in(G(u)) - n + m \leq 3s$ for all $u \in V(C_s)$. From (10) it follows that there is a node $v \in V(C_s)$ with $3s \leq 3l(v)$, thus $in(G(v)) - n + m \leq 3l(v)$ and (15) holds.

First of all, observe that $A$ is a product of several Fibonacci cubes. Namely $A = \bigotimes_{i=1}^{z} FC_{p_i}$ where $p_i$ is the length of $i$-th block of consecutive bits mapped to $\{1, \dots, r\}$ by isomorphism $h$ and $z$ is the number of such blocks. See Figure 11 for example.

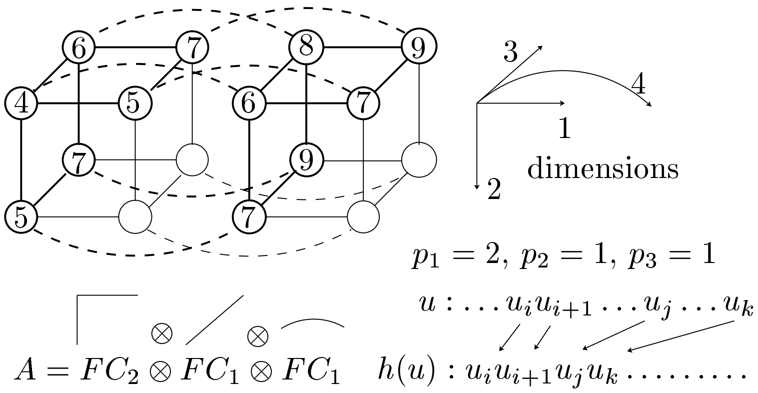



FIG 11. $A$ is a product of Fibonacci cubes.

As we have seen in Lemma 1, every $FC_{p_i}$ contains a subcube of dimension $\lceil p_i/2 \rceil$ so the product $A$ contains a subcube $C_s$ of dimension $s = \sum_{i=1}^{z} \lceil p_i/2 \rceil$. Denote $a$ the number of bits mapped to $\{1, \dots, r\}$ by isomorphism $h$, i.e. $a = \sum_{i=1}^{z} p_i$. Clearly, $in(G(u)) - n + m \geq a$ for all $u \in V(C_s)$, and the equality holds for node $u$ with $h(0 \dots 0) \in Q_n(u)$.

For every edge $\{u, v\}$ of $C_s$, observe that $in(G(u))$ and $in(G(v))$ differ by at most 1 if it is an edge of $FC_{p_i > 1}$, and by at most 2 if it is an edge of $FC_{p_i = 1}$. The first statement follows from the fact that if we denote $j$ the dimension mapped by $h$ to dimension of $\{u, v\}$, then $j + 1$ (or similarly for $j - 1$) is also mapped to some dimension in $\{1, \dots, r\}$, otherwise $p_i = 1$. Thus for some $w \in V(FC_m)$ if $h(w_1 \dots w_{j-1} 00 w_{j+2} \dots w_m) \in Q_n(u)$ then $h(w_1 \dots w_{j-2} 010 w_{j+2} \dots w_m) \in Q_n(v)$. The second statement follows from the similar argument and the symetry property of Lucas cubes.

From above we obtain $in(G(u)) - n + m \leq a + b + 2c$ for all $u \in V(C_s)$, where $b = \sum_{p_i > 1} \lceil p_i/2 \rceil$ and $c = \sum_{p_i = 1} \lceil p_i/2 \rceil$, i.e. $b + c = s$. Since

$$2 \sum_{i=1}^{z} \lceil p_i/2 \rceil \geq \sum_{i=1}^{z} p_i + \sum_{\substack{i=1 \\ p_i = 1}}^{z} p_i, \quad \text{for every } p_1, \dots, p_z,$$

we have $2s \geq a + c$ by definition so $3s \geq a + b + 2c \geq in(G(u)) - n + m$ for all nodes $u \in V(C_s)$ and we are done. $\square$

## 8. Recursive lower bounds

The methods for lower bounds in this section are modifications for a general $m$ of the methods that were essentialy used in [2].

**Lemma 20.** *Let $G$ be a subgraph of $C_r$ isomorphic to $FC_r$ and $S \in \mathcal{S}(n, m)$, $n \geq m \geq r$. Then*

$$\sum_{u \in V(G)} |S(u)| \geq \psi(n-r, m-r).$$

*Proof.* Set $T = \{v \in V(Q_{n-r}); \exists u \in V(G) \text{ and } uv \in S\}$, i.e. $T$ is a subset of $V(Q_{n-r})$ obtained by a projection of $S \cap G$ over last $n-r$ dimensions, so $\sum_{u \in V(G)} |S(u)| \geq |T|$. Suppose that $|T| < \psi(n-r, m-r)$. Then there is a copy of $FC_{m-r}$ in $Q_{n-r}$ disjoint from $T$. By connecting such copies in every $Q_n(u)$ for $u \in V(G)$ we obtain a copy of $FC_r \otimes FC_{m-r}$ in $Q_n$ disjoint from $S$. Since $FC_r \otimes FC_{m-r}$ contains $FC_m$ as a subgraph (more precisely, in our notation $FC_r FC_{m-r} = FC_m \cup FC_{r-2}0110FC_{m-r-2}$ for $n \geq 4$, $r \geq 2$ and similarly for $r < 2$ or $n < 4$), this contradicts the choice of $S$ and we are done. $\square$

Selecting the minimal $S$ yields $\psi(n, m) = |S| \geq \sum_{u \in V(G)} |S(u)| \geq \psi(n-r, m-r)$. Note that this generalizes the second statement in Theorem 16, the argument there is the same. The following results are based on better estimates than $|S| \geq \sum_{u \in V(G)} |S(u)|$.

**Theorem 21.** $\psi(n, m) \geq 2\psi(n-4, m-4)$, *for $n \geq m \geq 4$.*

*Proof.* Observe from Figure 12 that $C_4$ can be partitioned into two disjoint copies $G$, $G'$ of $FC_4$. So for the minimal set $S \in \mathcal{S}(n, m)$ we have by Lemma 20 that $\psi(n, m) = |S| = \sum_{u \in V(G)} |S(u)| + \sum_{u \in V(G')} |S(u)| \geq 2\psi(n-4, m-4)$. $\square$

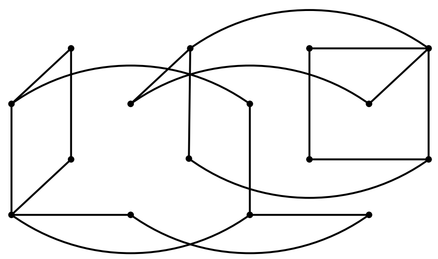


FIG 12. Partition of $C_4$ into two disjoint copies of $FC_4$.

**Theorem 22.** $\psi(n, m) \geq \frac{2^r}{f_r}\psi(n-r, m-r)$, *for $n \geq m \geq r$.*

*Proof.* For $u$, $v \in V(Q_n)$ the number of automorphisms $g : Q_n \to Q_n$ with $g(u) = v$ is $n!$. Thus for $A$, $B \subseteq V(Q_n)$ we obtain

$$\sum_{g: Q_n \to Q_n} |g(A) \cap B| = \sum_{u \in A, v \in B} n! = |A|.|B|.n!.$$

Since there is exactly $n!2^n$ automorphisms of $Q_n$ we deduce that there exists an automorphism $g : Q_n \to Q_n$ with $|g(A) \cap B| \le \frac{|A|.|B|}{2^n}$.

Let $B$ be the minimal set in $\mathcal{S}(n,m)$ and $A$ be the product $FC_r \otimes Q_{n-r}$, i.e. $A$ consists of nodes without two consecutive ones on first $r$ bits. By the argument above we have an automorphism $g : Q_n \to Q_n$ with $|g(A) \cap B| \le \frac{f_r\psi(n,m)}{2^r}$. We can suppose without lost of generality that $g$ maps dimensions $\{1,\dots,r\}$ to themselves, so $g$ restricted on $C_r$ is an automorphism $g'$ of $C_r$. Let $G$ denote the subgraph of $C_r$ isomorphic to $FC_r$ via $g'$, i.e. $g(A) = \bigcup_{u \in V(G)} Q_n(u)$. By Lemma 20 we obtain that

$$\frac{f_r\psi(n,m)}{2^r} \ge \sum_{u \in V(G)} |B(u)| \ge \psi(n-r, m-r),$$

and we are done. $\square$

## 9. Conclusions

The combined results from this paper and a few previous results for small values of $n$ (namely, $\psi(n,n)$ for $n \le 6$, [2]) are summarized in Figure 13. There is still some improvement possible.

| $n \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | |
| 2 | 4 | 2 | 2 | | | | | | | | |
| 3 | 8 | 4 | 4 | 2 | | | | | | | |
| 4 | 16 | 8 | 8 | 5 | 4 | | | | | | |
| 5 | 32 | 16 | 16 | 10 | 8-10 | 6 | | | | | |
| 6 | 64 | 32 | 32 | 21 | 16-20 | 12 | 8 | | | | |
| 7 | 128 | 64 | 64 | 42 | 32-42 | 24 | 16-24 | 8-12 | | | |
| 8 | 256 | 128 | 128 | 85 | 64-84 | 48-56 | 32-48 | 16-24 | 8-20 | | |
| 9 | 512 | 256 | 256 | 170 | 128-170 | 96-120 | 64-112 | 32-64 | 22-48 | 12-24 | |
| 10 | 1024 | 512 | 512 | 341 | 256-340 | 192-240 | 128-240 | 64-165 | 44-128 | 26-68 | 16-48 |

Fig 13. Table of known values or bounds on $\psi(n,m)$ for $0 \le m \le n \le 10$.

All recursive lower bounds were constructive, in a sence that a given faulty sets for smaller $n$ and $m$, we recursively construct the faulty set in $\mathcal{S}(n,m)$ by adding a prefix corresponding to a respective subcube. The faulty sets for $m \le 3$ in section 2 are taken as a faulty sets for $\kappa(n, \lceil m/2 \rceil)$-problem. They can be constructed as a union of several level sets, i.e. nodes $u$ with weight $w(u) \equiv k \mod m$ for a properly chosen $k$.

The upper bound in Theorem 21 is also constructive, in a sence that given a set $S$ of nodes with $|S| < 2\psi(n-4, m-4)$ we recursively find a copy of $FC_m$ in $Q_n$ disjoint from $S$, and each such step can be done in time $O(|S|)$. Thus the Theorem 21 can be used for construction of fault-tolerant embedding algorithm running in polynomial time $O(n.|S|)$. On the other hand, in Theorem 22 the situation is different, we have no better way than to search all $n!2^n$ automorphisms in every step.

The labeling method can be applied in the problem of fault-tolerance also for other networks with recursive structure. What is necessary is some kind of a symetry property, like in Lemma 4 for Lucas cubes.

The construction of labelings satisfying (12) (resp. (15)) for higher dimension $r$ remains a challenging combinatorial problem. In Collorary 15 we have shown that some results from coding theory might be useful for such tasks.

For practical purposes, it might be interesting to consider faulty edges instead of nodes, or faulty subgraphs in general. We believe that similar results could be formulated for this problem as well.

We have shown the close relationchip to the problem of subcube fault-tolerance in hypercubes. This problem is well known also as a problem of $(n, k)$-universal sets, and there are connections to $k$-independence problem, partitions and linear codes.

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## Appendix

All graphs in this section are induced subgraphs of hypercube. For a simplicity, we work with sets of binary strings of the same length representing nodes. Let us write $FC_n$ instead of $V(FC_n)$, and $FC_0$ is a nonempty set containing just the empty string denoted $\lambda$. For $G_1$ subgraph of $Q_i$ and $G_2$ subgraph of $Q_j$ define a concatenation $G_1G_2 = \{uv \mid u \in G_1, v \in G_2\}$ subgraph of $Q_{i+j}$. For $i \geq -1$ define

$$FC_i{}^0 = \begin{cases} \emptyset & i = -1 \\ FC_i0 & i \geq 0 \end{cases} \quad \text{and} \quad {}^0FC_i = \begin{cases} \emptyset & i = -1 \\ 0FC_i & i \geq 0 \end{cases}.$$

Observe from characterization of $FC_n$ in (2) that

i) $FC_iFC_j = FC_{i+j}$ if $i = 0$ or $j = 0$,
ii) $FC_iFC_j = FC_{i+j} \cup FC_{i-2}{}^0 11 {}^0FC_{j-2}$ for all $i$, $j \geq 1$,
iii) $FC_n = FC_{i-1}0FC_{n-i} \cup FC_{i-2}{}^0 1 {}^0FC_{n-i-1}$ for all $1 \leq i \leq n$.

**Lemma 23.** *Let $w$ be a binary string of length $i$, $G$ be a nonempty subgraph of $Q_{n-i}$, $1 \leq i \leq n$.*

1) $in(FC_i) = 0$,
2) $in(w) = i$,
3) $in(Gw) = in(G) + i$ *for* $i < n$,
4) $in(GFC_i) = in(G)$ *for* $i < n$,
5) $in(w_{i_0}FC_{j_1}w_{i_1} \ldots FC_{j_k}w_{i_k}) = \sum_{a=0}^{k} i_a$, *for any binary strings* $w_{i_a}$ *of length* $i_a$, $0 \leq a \leq k$, *and any integers* $j_b$, $1 \leq b \leq k$, *such that* $\sum_{a=0}^{k} i_a + \sum_{b=1}^{k} j_b = n$.

*Proof.* Part 1) and 2) follow directly from the definition. By (11) $G$ contains $FC_{(n-i)-in(G)}$. Subgraph $Gw$ of $Q_n$ is isomorphic to $G$ so the highest dimension of Fibonacci cube in $Gw$ is the same: $FC_{n-(in(G)+i)}$ and 3) holds. Subgraph $GFC_i$ of $Q_n$ contains $FC_{(n-i)-in(G)}FC_i$ which by ii) contains $FC_{n-in(G)}$ and it is Fibonacci cube of the highest dimension in $GFC_i$ so 4) holds. Part 5) follows from 3) and 4).

**Lemma 10.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 1$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 7.*

*Proof.* Let $\pi$ be the permutation and $w$ be the binary string such that $h : FC_n \to G$ where $h$ is given by (4), $i = \pi(1)$. By Lemma 9 we can suppose without a lost of the generality that $w_{\pi(i)} = 0$.

a) For $n = 1$ we have $FC_n = 0 \cup 1$ so

$$G(0) = h(0) = h(0FC_{n-1}) \Rightarrow in(G(0)) = 1 \text{ by Lemma 23}$$
$$G(1) = h(1) = h(0FC_{n-1}) \Rightarrow in(G(1)) = 1 \text{ by Lemma 23.}$$

b1) For $i = 1$ and $n > 1$ we have $FC_n = 0FC_{n-1} \cup 10FC_{n-2}$ so

$$G(0) = h(0FC_{n-1}) \quad \Rightarrow in(G(0)) = 1 \text{ by Lemma 23,}$$
$$G(1) = h(10FC_{n-2}) \Rightarrow in(G(1)) = 2 \text{ by Lemma 23.}$$

b2) For $i = n$ and $n > 1$ we have $FC_n = FC_{n-1}0 \cup FC_{n-2}01$ so

$$\begin{aligned} G(0) = &\ h(FC_{n-1}0) \Rightarrow in(G(0)) = 1 \text{ by Lemma 23,} \\ G(1) = &\ h(FC_{n-2}01) \Rightarrow in(G(1)) = 2 \text{ by Lemma 23.} \end{aligned}$$

c) Else we have from iii) that $FC_n = FC_{i-1}0FC_{n-i} \cup FC_{i-2}010FC_{n-i-1}$ so

$$\begin{aligned} G(0) = &\ h(FC_{i-1}0FC_{n-i}) &&\Rightarrow in(G(0)) = 1 \text{ by Lemma 23,} \\ G(1) = &\ h(FC_{i-2}010FC_{n-i-1}) &&\Rightarrow in(G(1)) = 3 \text{ by Lemma 23.} \end{aligned}$$

□

**Lemma 11.** *Let $G$ be a subgraph of $Q_n$ isomorphic to $FC_n$ and $n \geq r = 2$. Then the indexing of $C_r$ with respect to $G$ is equivalent to one of indexings on Figure 8.*

*Proof.* Let $\pi$ be the permutation and $w$ be the binary string such that $h : FC_n \to G$ where $h$ is given by (4), $i = \pi(1)$, $j = \pi(2)$. By Lemma 9 we can suppose without a lost of the generality that $i < j$ and $w_{\pi(i)} = w_{\pi(j)} = 0$.

a) For $n = 2$ we have

$$h(FC_n) = \begin{cases} h(00) = h(00FC_{n-2}) = G(00) \Rightarrow in(G(00)) = 2, \\ h(01) = h(01FC_{n-2}) = G(01) \Rightarrow in(G(01)) = 2, \\ h(10) = h(10FC_{n-2}) = G(10) \Rightarrow in(G(10)) = 2, \end{cases}$$
$$\emptyset = G(11) \Rightarrow in(G(11)) \text{ not def.}$$

b) For $n > 2$, $i = 1$ and $j = 2$ we have

$$h(FC_n) = \begin{cases} h(0FC_{n-1}) = \begin{cases} h(00FC_{n-2}) &= G(00) \Rightarrow in(G(00)) = 2, \\ h(010FC_{n-3}) &= G(01) \Rightarrow in(G(01)) = 3, \end{cases} \\ \qquad\qquad\qquad h(10FC_{n-2}) = G(10) \Rightarrow in(G(10)) = 2, \end{cases}$$
$$\emptyset = G(11) \Rightarrow in(G(11)) \text{ not def.}$$

c) For $n = 3$, $i = 1$ and $j = 3$ we have

$$h(FC_n) = \begin{cases} h(0FC_{n-1}) = \begin{cases} h(0FC_{n-2}\ 0) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0FC_{n-3}01) = G(01) \Rightarrow in(G(01)) = 3, \end{cases} \\ h(10FC_{n-2}) = \begin{cases} h(10FC_{n-3}0) = G(10) \Rightarrow in(G(10)) = 3, \\ h(10FC_{n-3}1) = G(11) \Rightarrow in(G(11)) = 3. \end{cases} \end{cases}$$

d) For $n > 3$, $i = 1$ and $j = n$ we have

$$h(FC_n) = \begin{cases} h(0FC_{n-1}) = \begin{cases} h(0\ \ FC_{n-2}\ \ 0) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0\ \ FC_{n-3}01) = G(01) \Rightarrow in(G(01)) = 3, \end{cases} \\ h(10FC_{n-2}) = \begin{cases} h(10FC_{n-3}\ \ 0) = G(10) \Rightarrow in(G(10)) = 3, \\ h(10FC_{n-4}01) = G(11) \Rightarrow in(G(11)) = 4. \end{cases} \end{cases}$$

e) For $n > 3$, $i = 1$ and $j = 3$ we have

$$h(FC_n) = \begin{cases} h(0FC_{n-1}) &= \begin{cases} h(0FC_1 0FC_{n-3}) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0010FC_{n-4}) = G(01) \Rightarrow in(G(01)) = 4, \end{cases} \\ h(10FC_{n-2}) &= \begin{cases} h(100\ \ FC_{n-3}) = G(10) \Rightarrow in(G(10)) = 3, \\ h(1010FC_{n-4}) = G(11) \Rightarrow in(G(11)) = 4. \end{cases} \end{cases}$$

f) For $n > 4$, $i = 1$ and $3 < j < n$ we have

$$h(FC_n) = \begin{cases} h(0FC_{n-1}) &= \begin{cases} h(0\ \ FC_{j-2}\ \ 0\ \ FC_{n-j}\quad) = G(00) \Rightarrow in(G(00)) = 2, \\ h(0\ \ FC_{j-3}010FC_{n-j-1}) = G(01) \Rightarrow in(G(01)) = 4, \end{cases} \\ h(10FC_{n-2}) &= \begin{cases} h(10FC_{j-3}\ \ 0\ \ FC_{n-j}\quad) = G(10) \Rightarrow in(G(10)) = 3, \\ h(10FC_{j-4}010FC_{n-j-1}) = G(11) \Rightarrow in(G(11)) = 5. \end{cases} \end{cases}$$

g) For $n > 3$, $i > 1$ and $j = i + 1 < n$ we have

$$h(FC_n) = \begin{cases} h(FC_{i-1}0FC_{n-i}) = \begin{cases} h(FC_{i-1}\ \ 00\ \ FC_{n-i-1}) \Rightarrow in(G(00)) = 2, \\ h(FC_{i-1}\ \ 010FC_{n-i-2}) \Rightarrow in(G(01)) = 3, \end{cases} \\ \qquad\qquad h(FC_{i-2}010\ \ FC_{n-i-1}) \Rightarrow in(G(10)) = 3, \\ \qquad\qquad\qquad \emptyset = G(11) \Rightarrow in(G(11)) \text{ not def.} \end{cases}$$

h) For $n > 4$, $i > 1$ and $j = i + 2 < n$ we have

$$h(FC_n) = \begin{cases} h(FC_{i-1}\ \ 0\ \ FC_{n-i}\quad) &= \begin{cases} h(FC_{i-1}\ \ 0FC_1 0FC_{n-i-2}) \Rightarrow in(G(00)) = 2, \\ h(FC_{i-1}\ \ 0010FC_{n-i-3}) \Rightarrow in(G(01)) = 4, \end{cases} \\ h(FC_{i-2}010FC_{n-i-1}) &= \begin{cases} h(FC_{i-2}0100\ \ FC_{n-i-2}) \Rightarrow in(G(10)) = 4, \\ h(FC_{i-2}01010FC_{n-i-3}) \Rightarrow in(G(11)) = 5. \end{cases} \end{cases}$$

i) For $n > 5$, $i > 1$ and $i + 2 < j < n$ we have

$$h(FC_n) = \begin{cases} h(FC_{i-1}\ \ 0\ \ FC_{n-i}\quad) &= \begin{cases} h(FC_{i-1}\ \ 0\ \ FC_{j-i-1}\ \ 0\ \ FC_{n-j}\quad) = G(00), \\ h(FC_{i-1}\ \ 0\ \ FC_{j-i-2}010FC_{n-j-1}) = G(01), \end{cases} \\ h(FC_{i-2}010FC_{n-i-1}) &= \begin{cases} h(FC_{i-2}010FC_{j-i-2}\ \ 0\ \ FC_{n-j}\quad) = G(10), \\ h(FC_{i-2}010FC_{j-i-3}010FC_{n-j-1}) = G(11). \end{cases} \end{cases}$$

By Lemma 23 it follows $in(G(00)) = 2$, $in(G(01)) = in(G(10)) = 4$, $in(G(11)) = 6$.

j) For $n > 2$, $i > 1$ and $j = n$ observe from (2) that the automorphism $h'$ of $Q_n$ that reverses nodes, i.e. $h'(u_1 \dots u_n) = u_n \dots u_1$, restricted on $FC_n$ is an automorphism of $FC_n$. Thus the indexing with respect to $G = h(FC_n) = h(h'(FC_n))$ is equivalent to the case $i = 1$ and $j < n$. $\square$