On cube-free median graphs

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Abstract

Let G be a cube-free median graph. It is proved that $\frac{k}{2} \geq \sqrt{n} - 1 \geq \frac{m}{2\sqrt{n}} \geq \sqrt{s} \geq r - 1$, where n, m, s, k, and r are the number of vertices, edges, squares, Θ-classes, and the number of edges in a smallest Θ-class of G, respectively. Moreover, the equalities characterize Cartesian products of two trees of the same order. The cube polynomial of cube-free median graphs is also considered and it is shown that planar cube-free median graphs can be recognized in linear time.

Keywords: median graph, cube-free graph, Cartesian product, recognition algorithm

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1 Introduction

Cube-free median graphs are, by definition, median graph without an induced 3-cube Q_3. This class of graphs naturally appears in different contexts. For instance, cube-free median graphs are precisely the bipartite absolute retracts without induced $K_{2,3}$ [3], and they play an important role in the location theory [2, 11]. Cube-free median graphs are also precisely those median graphs for which the equality is attained in an Euler-type formula for median graphs [9].

Edges xy and uv of a graph G are in the Djoković-Winkler relation Θ [5, 14] if

$$d_G(x,u) + d_G(y,v) \neq d_G(x,v) + d_G(y,u)\,.$$

Relation Θ is reflexive and symmetric in general and transitive on median graphs. Hence it partitions the edge set of a median graph into equivalence classes, called Θ-*classes*.

Let G be a cube-free median graph. Then the following invariants of G are important to us:

n ... the number of its vertices,
m ... the number of its edges,
s ... the number of its (induced) squares,
k ... the number of its Θ-classes, and
r ... the number of the edges in its smallest Θ-class.

The main result of this note asserts that

$$\frac{k}{2} \geq \sqrt{n} - 1 \geq \frac{m}{2\sqrt{n}} \geq \sqrt{s} \geq r - 1.$$

Moreover, if G is not a tree, then in any of the above inequalities, the equality holds if and only if G is the Cartesian product of two trees of the same order.

In the next section we recall concepts and results needed later. Then we follow with a section in which the main result is proved. In the concluding section we give some more properties of cube-free median graphs. We give few remarks on the cube polynomial of cube-free median graphs—we show that they always have two real zeros, and we give a combinatorial interpretation to their extreme points. Finally, we show that planar cube-free median graphs can be recognized in linear time.

2 Preliminaries

The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever either $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The *r-cube* Q_r is the Cartesian product of r copies of the complete graph on two vertices K_2.

The *interval* $I(u, v)$ between two vertices u and v in G is the set of all vertices on shortest paths between u and v. A subgraph H of G is *convex* if we have $I(u, v) \subseteq V(H)$ for every u, $v \in V(H)$. A graph G is a *median graph* if $|I(u, v) \cap I(u, w) \cap I(v, w)| = 1$ holds for every triple of vertices u, v, and w. The vertex of this intersection is called the *median* of the triple u, v, w. It is easy to see that median graphs are bipartite and that the Cartesian product operation preserves median graphs.

Let $G = (V, E)$ be a graph, V_1 and V_2 subsets of V with nonempty intersection, and $V = V_1 \cup V_2$. Suppose that V_1 and V_2 induce isometric subgraphs of G and that no vertex of $V_1 \setminus V_2$ is adjacent to a vertex of $V_2 \setminus V_1$. In addition, let $V_1 \cap V_2$ be a convex set in G. Then the *convex expansion* of a graph G with respect to V_1 and V_2 is the graph obtained from G by the following procedure:

(i) replace each vertex $v \in V_1 \cap V_2$ by vertices v_1, v_2 and insert the edge $v_1 v_2$.

(ii) insert edges between v_1 and the neighbors of v in $V_1 \setminus V_2$ as well as between v_2 and the neighbors of v in $V_2 \setminus V_1$.

(iii) insert the edges $v_1 u_1$ and $v_2 u_2$ whenever $v, u \in V_1 \cap V_2$ are adjacent in G.

Mulder [12, 13] proved that a graph is a median graph if and only if it can be obtained from K_1 by a sequence of convex expansions.

In the next proposition we recall two properties of cube-free median graphs that will be needed later. The first one was given in [10, Corollary 3] and the second follows from the main result of [9]. However, to be self-contained as much as possible we give here their short (unified) proofs.

Proposition 2.1 *Let G be a cube-free median graph with n vertices, m edges, s squares, and k classes of the relation Θ. Then*

$$s = m - n + 1 \qquad \textit{and} \qquad k = -m + 2n - 2\,.$$

Proof. We prove the claim by induction on the number of expansion steps. The statement is true for K_1. So let G be obtained by an expansion from a cube-free median graph G' over G_0. Then G_0 is a tree. Let n', m', k', and s' be the corresponding invariants of G' and let $|V(G_0)| = x$. Then $n = n' + x$, $m = m' + x + (x-1)$, $s = s' + (x-1)$, and $k = k' + 1$. Hence, using the induction hypothesis, we compute

$$s = s' + (x-1) = (m' - n' + 1) + x - 1 = m - n + 1\,,$$

and

$$k = k' + 1 = (-m' + 2n' - 2) + 1 = -m + 2n - 2\,.$$

$\square$

Let uv be an edge of a median graph G. Let G_1 be the subgraph of G induced by the vertices closer to u than to v and G_2 be the subgraph induced by the vertices closer to v than to u. As G is bipartite, $G_1 \cap G_2 = \emptyset$, and the pair G_1, G_2 is called a *split* of G. Two splits G_1, G_2 and H_1, H_2 are called *crossing splits* if $G_i \cap H_j \neq \emptyset$ for $i, j = 1, 2$. We will apply the following result:

Theorem 2.2 [11, Theorem 11] *A median graph G contains r pairwise crossing splits if and only if G contains a Q_r as an induced subgraph.*

3 Inequalities for cube-free median graphs

In this section we present a sequence of inequalities for cube-free median graphs that involve the number of vertices, edges, Θ-classes, squares, and the number of edges in a smallest Θ-class. All these inequalities are characteristic for cube-free median graphs (in the class of median graphs) and turn into equalities precisely for the Cartesian product of two trees of the same order.

Before the main result we state two lemmas.

Lemma 3.1 *Let G be a cube-free median graph with k Θ-classes. Let $r \geq 2$ be the number of edges in its smallest Θ-class. Then $k \geq 2r - 2$. Moreover, the equality holds if and only if G is the Cartesian product of two trees of the same order.*

Proof. The proof is by induction of r. Let $r = 2$. Then G contains at least one square, so that $k \geq 2$. Moreover, $k = 2$ if and only if $G = P_2 \Box P_2$.

Let now $r \geq 3$ and let $uvwz$ be a square of G. Let G' be the median graph obtained by first contracting the Θ-class E_1 containing the edge uv and then the Θ-class E_2 containing uz. Then G' has $k' = k - 2$ Θ-classes. Let F be a

Θ-class different from E_1 and E_2. Since G is cube-free, Theorem 2.2 implies that at most two edges of F are identified while contracting E_1 and E_2. It follows that $r' \geq r-1$. By the induction assumption, $k' \geq 2r'-2$, and hence $k-2 = k' \geq 2r'-2 \geq 2(r-1)-2$, so that $k \geq 2r-2$. Moreover, the equality will hold if and only if $r' = r-1$ and $k' = 2r'-2$. Thus by the induction assumption, $G' = T_1'\Box T_2'$ where T_1', T_2' are trees on $r-1$ vertices. Since any Θ-class of G has at least r elements, every Θ-class of G is intersected by E_1 or E_2. This is possible only if in both expansion steps the intersection consists of layers isomorphic to T_1', and T_2', respectively. Thus $G = T_1 \Box T_2$, where T_i can be obtained from T_i' by one expansion step. $\Box$

Lemma 3.2 *Let G be a cube-free median graph with k Θ-classes and s squares. Then, $k^2 \geq 4s$. Moreover, the equality holds if and only if G is the Cartesian product of two trees of the same order.*

Proof. Suppose that G is a smallest cube-free median graph with $4s > k^2$. Let r be the number of edges in its smallest Θ-class. If $r = 1$ then let G' be the cube-free median graph obtained by contraction an equivalence class with one edge. Let s' and k' be the number of squares and Θ-classes of G', respectively. Clearly, $s' = s$ and $k' = k-1$. By the minimality, $4s' \leq k'^2$ and so $k^2 < 4s = 4s' \leq k'^2 = (k-1)^2$, which implies $k \leq 0$, a contradiction. So let $r \geq 2$. We now contract a Θ-class of G containing r edges to obtain the cube-free median graph G' with $s' = s-(r-1)$ squares and $k' = k-1$ Θ-classes. By the minimality, $4(s-(r-1)) \leq (k-1)^2$, so that

$$k^2 < 4s \leq k^2 - 2k + 1 + 4r - 4\,,$$

which implies $2k \leq 4r-3$ and so $k \leq 2r-2$. Now Lemma 3.1 implies that $G = T_1 \Box T_2$, where T_1 and T_2 are trees of same order. But in this case it is straightforward to verify that $k^2 = 4s$, a contradiction. $\Box$

Here is our main result.

Theorem 3.3 *Let G be a cube-free median graph with n vertices, m edges, s squares, k Θ-classes, and r edges in its smallest Θ-class. Then*

$$\frac{k}{2} \geq \sqrt{n} - 1 \geq \frac{m}{2\sqrt{n}} \geq \sqrt{s} \geq r - 1\,.$$

Moreover, if G is not a tree then in any of the above inequalities the equality holds if and only if G is the Cartesian product of two trees of the same order.

Proof. If G is a tree then $k = m = n - 1$, $s = 0$, $r = 1$. In this case, it is easy to verify the theorem. So, assume that $s > 0$. We first prove the following claim:

Claim 1. *$\sqrt{n} - 1 \geq \sqrt{s}$ with equality if and only if G is the Cartesian product of two trees of the same order.*

From Proposition 2.1 we infer that $n = k + s + 1$, hence by Lemma 3.2 we have

$$\sqrt{n} - 1 = \sqrt{k + s + 1} - 1 \geq \sqrt{s + 2\sqrt{s} + 1} - 1 = \sqrt{s}. \tag{1}$$

Note that the equality is preserved if and only if $k^2 = 4s$, and this happens precisely when G is the Cartesian product of two trees on same number of vertices due to Lemma 3.2. So Claim 1 is established.

Using Claim 1, we obtain the first inequality as follows:

$$\frac{k}{2} = \frac{n - 1 - s}{2} \geq \frac{n - 1 - (\sqrt{n} - 1)^2}{2} = \sqrt{n} - 1\,.$$

For the second inequality we argue as

$$\frac{m}{2\sqrt{n}} = \frac{s + n - 1}{2\sqrt{n}} \leq \frac{(\sqrt{n} - 1)^2 + n - 1}{2\sqrt{n}} = \sqrt{n} - 1\,,$$

and for the third one, using (1), as

$$\frac{m}{2\sqrt{n}} = \frac{s + n - 1}{2\sqrt{n}} = \frac{(\sqrt{n} - \sqrt{s})^2 + 2\sqrt{sn} - 1}{2\sqrt{n}} \geq \frac{1 + 2\sqrt{sn} - 1}{2\sqrt{n}} = \sqrt{s}\,.$$

Note that in each of the above three arguments, we preserve equality if and only if it is preserved in Claim 1.

It remains to prove that $r - 1 \leq \sqrt{s}$ where equality holds in the claimed case. Note first that, since G is not a tree, the inequality is trivially true for $r = 1, 2$. Moreover, for $r = 1$ we have strict inequality, while for $r = 2$ the equality holds if and only if $G = P_2 \square P_2$. So let $r \geq 3$ and let S be a square of G. Let E_1, E_2 be the two Θ-classes that contain the edges of S: we may assume that $|E_1| = r$. Let G' be the median graph constructed from G by contracting first the Θ-class E_1 and afterwards contracting the Θ-class E_2. Then for G' we have $k' = k - 2$ and $s \geq s' + 2r - 3$. Moreover, invoking Theorem 2.2 again, we also have $r' \geq r - 1$. Hence we can compute

$$(r - 1)^2 \leq r'^2 = (r' - 1)^2 + 2(r' - 1) + 1 \leq s' + 2(r - 1) - 1 \leq s' + 2r - 3 \leq s\,.$$

Finally, the equality will hold if and only if $r' = r-1$, $k' = 2r'-2$, and $s'+2r-3 = s$. So applying Lemma 3.1 we can conclude that G is the Cartesian product of two trees of the same order. (Note that such a graph satisfies $r' = r-1$ and $s'+2r-3 = s$.) $\square$

Consider now the hypercubes Q_d, $d \geq 3$. Clearly,

$$n = 2^d, \quad m = d\,2^{d-1}, \quad k = d, \quad s = \binom{d}{2} 2^{d-2}, \quad \text{and} \quad r = 2^{d-1}\,.$$

For $d \geq 3$ we have

$$\frac{k}{2} < \sqrt{n} - 1 < \frac{m}{2\sqrt{n}} < \sqrt{s} < r - 1\,.$$

that is,

$$\frac{d}{2} < 2^{d/2} - 1 < d\,2^{d/2-2} < \sqrt{d(d-1)}\,2^{d/2-3/2} < 2^{d-1} - 1\,,$$

which can be easily checked for $d = 3$, so by asymptotic reasons the inequalities follows for all $d \geq 3$. Hence the inequalities of Theorem 3.3 are characteristic for cube-free median graphs in the following sense: A median graph is cube-free if and only if for any convex subgraph of G the inequalities of Theorem 3.3 hold.

We conclude this section with an open problem, considering a generalization of Theorem 3.3 to arbitrary median graphs.

Problem 3.4 *Let G be a Q_{d+1}-free median graph with k Θ-classes and n vertices. Let r be the number of edges in a smallest Θ-class of G and denote by α_i the number of Q_i-cubes in G. Do some (or maybe all) of the following inequalities hold:*

$$\frac{k}{d} \geq \sqrt[d]{n} - 1 \geq \sqrt[d]{\alpha_d} \geq \sqrt[d-1]{r} - 1 \quad ?$$

Regarding the above problem, let G be the Cartesian product of d trees of the same order, say p. Then, $n = p^d$, $k = (p-1)d$, $\alpha_d = (p-1)^d$, $r = p^{d-1}$. Note that G is Q_{d+1}-free median graph. So, we obtain that

$$\frac{k}{d} = \sqrt[d]{n} - 1 = \sqrt[d]{\alpha_d} = \sqrt[d-1]{r} - 1 = p - 1.$$

Thus, if the inequalities of the above problem hold, possibly they become equalities if and only if the graph is a product of d trees all of same order.

4 Additional properties

In this section some more properties of cube-free median graphs are obtained. In the first part, we consider the cube polynomial of these graphs, in particular its zeros and extreme points. In the second part we show that cube-free median graphs can be recognized in linear time.

The cube polynomial $c(G,x)$ of a graph G was introduced in [4] and is defined as follows. Let $\alpha_i(G)$ denotes the number of induced i-cubes of G, so that in particular $\alpha_0(G) = |V(G)|$ and $\alpha_1(G) = |E(G)|$. Then

$$c(G,x) = \sum_{i \geq 0} \alpha_i(G) x^i \,.$$

Using the results of previous sections we can prove:

Proposition 4.1 *The cube polynomial of a cube-free median graph G has real zeros. Moreover, it has a unique zero if and only if G is a tree or the Cartesian product of two trees of the same order.*

Proof. Let G be a cube-free median graph with n vertices, m edges, s squares, and k classes of the relation Θ. Then, as G is cube-free, $c(G,x) = sx^2 + mx + n$. By Proposition 2.1 we have

$$s = m - n + 1 \qquad \text{and} \qquad k = -m + 2n - 2\,,$$

and so

$$c(G,x) = sx^2 + (2s+k)x + (k+s+1). \tag{2}$$

If $s = 0$ then G is a tree and the claim follows trivially. So assume that $s > 0$. By Lemma 3.2 we infer that

$$(2s+k)^2 - 4s(k+s+1) = k^2 - 4s \geq 0\,,$$

which establishes that $c(G,x)$ has real zeros. Note that the cube polynomial has a unique zero if and only if $k^2 = 4s$. By Lemma 3.2, this holds if and only if G is the Cartesian product of two trees of the same order. □

Let $\mathcal{F}(G)$ denotes some set of edges consisting of representatives of the Θ-classes of a (cube-free) median graph G. Thus, $|\mathcal{F}(G)| = k$. For an edge $e = uv$ of a median graph G let U_e be the subgraph of G induced by the vertices x of G such that there is an edge $f = xy$ with $e\Theta f$ and $d(u,x) < d(u,y)$.

Proposition 4.2 *Let G be a cube-free median graph that is not a tree, and let x_{min} be the minimum point of $c(G,x)$. Then*

$$x_{min} = -1 - \frac{k}{\sum_{e\in\mathcal{F}(G)} |E(U_e)|}\,.$$

Moreover, if G is 2-edge-connected, then $x_{min} \geq -2$, and $x_{min} = -2$ if and only if $G = C_4$.

Proof. Note first that since G is not a tree, $c(G,x)$ has an extreme, more precisely a minimum. Let G has n vertices, m edges, s squares, and k classes of the relation Θ. From (2) we deduce $x_{min} = -1 - \frac{k}{2s}$.

As G is cube-free, every graph U_e is a tree (possibly K_1). Every edge f of an U_e corresponds to a square S of G, and there exists a unique graph $U_{e'}$ (distinct from U_e) such that the edge f also corresponds to S. Therefore, $2s = \sum_{e\in\mathcal{F}(G)}(|V(U_e)|-1) = \sum_{e\in\mathcal{F}(G)}|E(U_e)|$ which proves the first assertion.

Let now G be 2-edge-connected. Then each Θ-class consists of at least two edges, and so each tree U_e has at least one edge. Hence $\frac{k}{\sum_{e\in\mathcal{F}(G)}|E(U_e)|} \geq 1$, and thus by the above, $x_{min} \geq -2$. Finally, the equality is achieved if and only every tree U_e has exactly one edge which is possible only if G is a square. $\square$

In [8] a reduction is given asserting that recognizing median graphs is roughly equivalent to finding triangles in graphs, for the latter problem cf. [1]. In fact, the corresponding median graphs in the reduction are cube-free, so a fast recognition algorithm (say linear or "almost" linear) for cube-free median graphs would imply such an algorithm for recognition of triangle-free graphs.

Hence there is not much hope for a linear recognition algorithm for general cube-free median graphs. However, we are going to show that planar cube-free median graphs can be recognized in linear time. For this sake we need to recall that a subgraph H of a graph G is called *gated* in G if for every $x\in V(G)$ there exists a vertex u in H such that $u\in I(x,v)$ for all $v\in V(H)$. Note that if for some x such a vertex u in $V(H)$ exists, it must be unique.

Proposition 4.3 *Let G be a median graph. Then one can decide in linear time whether G is cube-free.*

Proof. Let u be an arbitrary vertex of G and let T be a BFS-tree with respect to u. We claim that G is cube-free if and only if every vertex of T has down-degree at most two. (By the down-degree of a vertex w we mean the number of neighbors of w that are closer to u than w.)

If G is cube-free then by [7, Lemma 3.35] the down-degrees are bounded by two, otherwise we would have hypercubes of higher dimensions.

Conversely, suppose that $H = Q_k$, $k \geq 3$, is a subgraph of G. It suffices to consider the case $k = 3$, for otherwise just consider any induced 3-cube of H. If $u \in H$, then the vertex w of H with $d_H(u, w) = 3$ lies in the third distance level from u and is of down-degree at least three. So suppose $u \notin H$. As H is convex in G it is also gated (cf. [7, Lemma 2.23]), so there is a unique vertex v of H closest to u. Let $w \in H$ be the vertex of H with $d(v, w) = 3$. By the gatedness, $d(u, w) = d(u, v) + 3$. But then w is of down-degree at least three.

To conclude the argument note that it can be easily checked in linear time whether all the down-degrees of G in T are bounded by two. □

Since planar median graphs can be recognized in linear time by a result of [8], Proposition 4.3 implies:

Corollary 4.4 *Planar cube-free median graphs can be recognized in linear time.*

We conclude by noting that the recognition problem for planar graphs that contain no subgraph homeomorphic to the 3-cube is efficiently solved in [6].

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