On High-Degree Crossing-Critical Graphs

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Abstract

We construct an infinite sequence of graphs that are 2-crossing-critical in the projective plane, and that contain vertices of arbitrarily high degrees. In particular, such a sequence has an unbounded bandwidth. This example brings significant doubts concerning the conjecture of Richter and Salazar that k-crossing-critical graphs in the plane have bandwidth bounded in k.

Keywords: crossing number, graph drawing, crossing-critical graph, path-width, bandwidth.
2000 Math Subjects Classification: **05C10, 05C62**

1 Background on Crossing Number

We assume that the reader is familiar with basic terms of graph theory. In a *drawing* of a graph G in the plane the vertices of G are points and the edges are simple curves joining their endvertices. Moreover, it is required that no edge passes through a vertex (except at its ends), and no three edges cross in a common point. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of crossings of edges in a drawing of G in the plane (thus, a graph is planar if and only if its crossing number is 0). The concept of crossing number naturally generalizes to other compact surfaces, namely to the projective plane.

For $k \geq 1$, we say that a graph G is *k-crossing-critical* if $\mathrm{cr}(G) \geq k$ but $\mathrm{cr}(G - e) < k$ for each edge $e \in E(G)$. It is important to study crossing-critical graphs in order to understand structural properties of the crossing number. The only 1-crossing-critical graphs are, by the Kuratowski theorem, the subdivisions of K_5 and $K_{3,3}$. The first construction of an infinite family of 2-crossing-critical graphs of minimal degree 3 was published by Kochol [4]. Many more crossing-critical constructions have appeared since. Observing their common structure, Salazar and Thomas conjectured in [1] that all k-crossing-critical graphs have path-width bounded in k.

That conjecture has been proved by the author in [3]. As a natural generalization of this result the following extended conjecture has appeared in [6]. Briefly, a graph G has *bandwidth* at most k if there is a bijection $\beta : V(G) \to \{1, \ldots, |V(G)|\}$ such that $|\beta(u) - \beta(v)| \leq k$ for each edge $e = uv$ in G.

Conjecture 1 (Richter, Salazar) *There is an integer function b such that each k-crossing critical graph in the plane has bandwidth at most $b(k)$.*

We do not know whether the conjecture holds true or not. However, the conjecture would immediately imply that k-crossing-critical graphs have maximal degree bounded in k, and so our further result, Theorem 2, brings up significant doubts about validity of Conjecture 1.

2 Projective Critical Construction

¿From now on we consider drawings of graphs in the projective plane. We present the following new construction:

Theorem 2 *There exists an infinite sequence of simple 3-connected graphs H_k, $k \geq 3$, such that each H_k is 2-crossing-critical in the projective plane, and that H_k has a vertex of degree $6k$.*

We remark that our construction has been partly inspired by the concept of "planar tiles" [5] in building crossing-critical graphs. We call a *tile* T the graph on 11 vertices and 15 edges depicted in Figure 1 in solid lines, and we distinguish its "boundary" vertices a, b and c, d. Let T' be a disjoint copy of T, and denote by T^2 the graph obtained from $T \cup T'$ by identifying the vertices c with a' of T', and d with b' of T'.

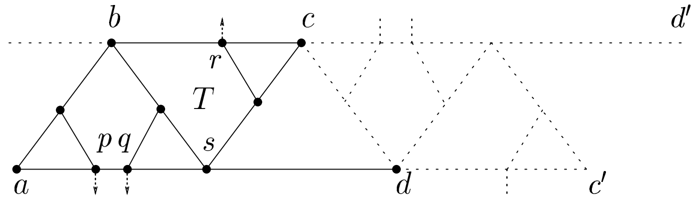

Figure 1: One "tile" T used in our construction (in solid lines); and another attached copy of T (a scheme in dashed lines), together forming T^2.

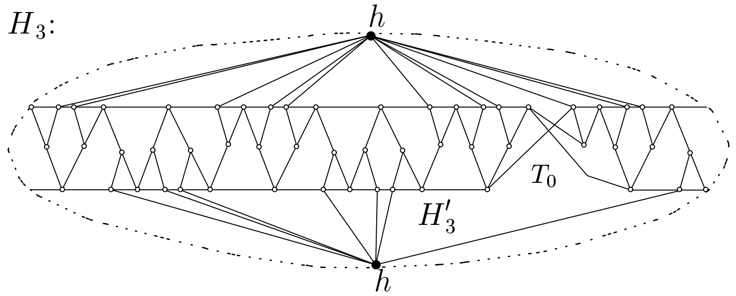

Figure 2: The graph H_3 drawn in the projective plane (i.e., the pairs of opposite points on the dashed ellipse are indentified).

Let us take k disjoint copies $T_1^2, \dots, T_k^2$ of the graph T^2 and identify the vertex pair a_{i+1}, c'_i and the pair b_{i+1}, d'_i. Denote the resulting graph by H'_k. (Roughly speaking k copies of T^2 are sticked together in a circular fashion to make H'_k. Notice that, unlike in classical crossing-critical constructions, the resulting graph is "not twisted" – H'_k is a planar graph!) Now realize that H'_k is made of $2k$ copies of the tile T. Denote by p_i, q_i, r_i and p'_i, q'_i, r'_i for $i = 1, \dots, k$ the copies of the vertices p, q, r (cf. Figure 1) in the T-copies. The graph H_k results by adding a new vertex h to H'_k, and joining h to all p_i, q_i, r_i and p'_i, q'_i, r'_i for $i = 1, \dots, k$. An example, the graph H_3, is presented in Figure 2.

Validity of Theorem 2 for the above constructed sequence H_k is established in two Lemmas 3 and 5.

Lemma 3 *For every $k \geq 2$ the graph H_k can be drawn in the projective plane with at most two edge crossings, and, moreover; for each edge $e \in E(H_k)$ the graph $H_k - e$ can be drawn in the projective plane with at most one edge crossing.*

Proof. The first part is easy to see. A projective drawing of the graph H_3 with two edge crossings is given in Figure 2. Generally, the idea is to place the vertex h arbitrarily, and draw the subgraph $H'_k = H_k - h$ around the Möbius band which results if we delete a topological neighbourhood of h from the projective plane. For that to work, the natural planar drawing of H'_k has to be "twisted" at some place. We formalize the idea as follows.

Let T^+ denote the graph obtained from the tile T by adding a new vertex b^+ adjacent only to b. (Then, if T_0 is a copy of T in H'_k, the vertex b_0^+ of T_0^+ coincides with the s-vertex of the adjacent T-copy in H'_k.) We say that a drawing of a spanning subgraph $G \subseteq T^+$ is *twisted* if the vertices a, b^+, d, c appear on the outer face in this cyclic order. A suitable twisted drawing of T_0^+ with two crossings is sketched as a part in Figure 2, and the rest of the graph H_k is filled in with no additional crossings for all $k \geq 2$.

Let us move to the second part of the lemma. Assume that $e \in E(H'_k)$ and, moreover, that $e \in E(T_0)$ where T_0 is an appropriate T-copy in H'_k. We say that a vertex $x \in \{p_0, q_0, r_0\}$ of T_0 is *accessible* in a twisted drawing of $T_0^+ - e$ in the plane if x lies on the outer face between a_0, c_0 or between b_0^+, d_0. (Practical meaning of this notion is that such a twisted drawing can be extended to a drawing of the whole H'_k around a Möbius band, in which the accessible vertices are on the boundary. A small arrow is used in our pictures to indicate accessibility.) In particular, a twisted drawing of $T_0^+ - e$ with just one crossing and accessible p_0, q_0, r_0 easily extends to a drawing of the whole $H_k - e$ with one crossing in the projective plane.

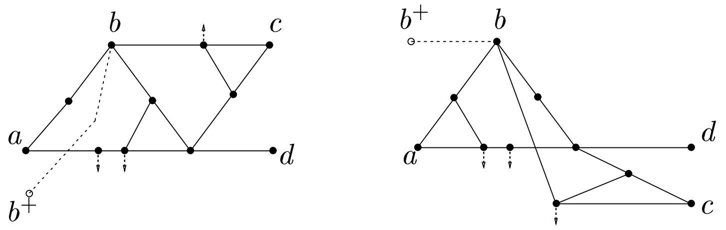

Figure 3: How to twist a drawing of $T^+ - e$; examples.

So the task is to find a twisted drawing of the graph $T_0^+ - e$ with at most one crossing such that all three vertices p_0, q_0, r_0 are accessible. That is a matter of an elementary case-check, where two of the cases of e are illustrated in Figure 3. The rest can be done analogously.

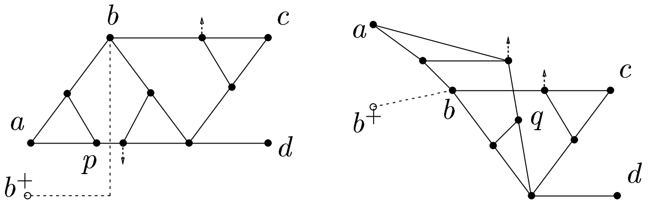

Figure 4: Twisted drawings of T^+ with inaccessible p or q.

Lastly, assume that $e \in E(H_k)$ is incident with the vertex h, and denote by T_0 the copy of T in H'_k incident with the other end of e. Up to symmetry, there are now three cases to consider $e = hp_0$, hq_0, or hr_0, and the last case of hr_0 is analogous to hp_0. We show that there always is a twisted drawing of T_0^+ with at most one edge crossing such that those two vertices of p_0, q_0, r_0 which are not incident with e are accessible. The cases are both drawn in Figure 4. Then such a drawing easily extends to a drawing of $H_k - e$ with at most one crossing in the projective plane. ∎

3 Proving the Lower Bound

The main task of this section is to finish the proof of Theorem 2 in Lemma 5. We are going to use the following claim from [2]. An *embedding* is a drawing with no edge crossing.

Lemma 4 [2] *The graphs R_1 and R_2 from Figure 5 have no embeddings in the projective plane.*

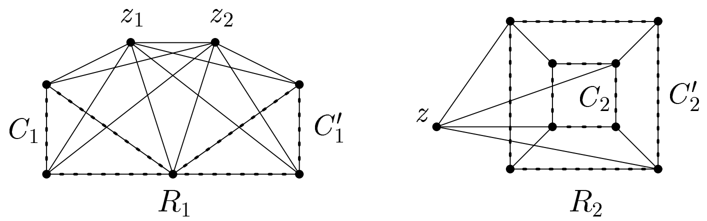

Figure 5: The graphs R_1 ($\simeq K_7 - C_4$) and R_2 ($\simeq K_{4,5} - M_4$); two of the "excluded minors" for the projective plane.

Lemma 5 *The graph H_k (defined above) for $k \geq 3$ cannot be drawn in the projective plane with at most one edge crossing.*

Proof. Roughly speaking, we are going to show, that even if we "remove" one crossing from an arbitrary drawing of H_k, we still find a minor isomoprhic to one of R_1 or R_2 (Figure 5) in the rest of the graph. This will be stated precisely soon. Since the property of a graph having an embedding in a surface is preserved under taking minors, Lemma 4 together with the previous idea implies that any drawing of H_k in the projective plane has at least two edge crossings.

First, as a worm-up, we show that the graph H_k for $k \geq 2$ has a minor isomorphic to R_2. Let C_b denote the cycle in H'_k which results as the union of copies of the path $brcp'q's'd' \subset T^2$. Let C_a similarly denote the union of copies of the path $apqsdr'c' \subset T^2$ in H'_k. Then C_a, C_b are two disjoint cycles in H_k, and our goal is to contract them to form the cycles C_2, C'_2 of R_2 (emphasized by thick dots in Figure 5). The vertex h of H_k then becomes z in R_2, and the rest follows easily (see also in Figure 2).

Let $e_p = pq$, $e_r = br$, and $e_s = sd$ denote edges of T (Figure 1). Consider an edge $e \in E(H_k)$ which is not a copy of e_p, e_r or e_s in H'_k. Then a more careful consideration of the arguments in the last paragraph reveals that even the graph $H_k - e$ has a minor isomorphic to R_2: That is easy if $e \notin E(C_a) \cup E(C_b)$. Otherwise, say for $e \in E(C_a)$, the edge e could be "bridged" by another two edges in H'_k not incident with C_b, unless e is a copy of e_p, e_r or e_s above. Therefore, if an edge crossing in the graph H_k involves such an edge e, then we delete e and argue that there is another crossing left in $H_k - e$ by Lemma 4.

Hence we are left in the proof with a possibility that an edge crossing in a drawing of H_k involves two edges e_1, e_2 which are copies of e_p, e_r or e_s. In that situation we subdivide the edges e_1, e_2 with new vertices x, x', respectively, denoting the new graph by H_k°; and then we indentify x with x', making a graph $H_k^\bullet$. (Graphically, we "place" a new vertex x into the crossing point of e_1, e_2 to get $H_k^\bullet$.) We show that the resulting graph $H_k^\bullet$ has a minor isomorphic to the graph R_1, and so the derived drawing of $H_k^\bullet$ contains another crossing by Lemma 4.

If, say, $e_1, e_2 \in E(C_a)$, then $H_k^\bullet$ contains a (shorter) circuit $C'_a \subset C_a$, which can be used together with C_b to argue that $H_k^\bullet$ has a minor isomorphic to R_2 for $k \geq 3$, as above. So we may assume that $e_1 \in E(C_a)$ and $e_2 \in E(C_b)$, or vice versa. We divide our analysis into three cases, up to symmetry:

(a) e_1 is a copy of e_p, or
(b) e_1 is a copy of e_r and e_2 is a copy of e_r or e_s, or
(c) both e_1, e_2 are copies of e_s.

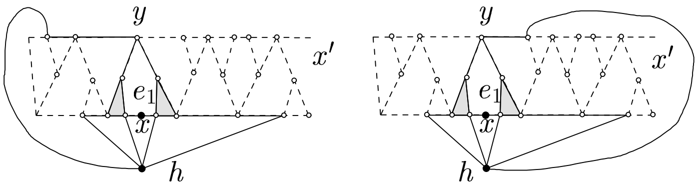

Figure 6: An illustration to the case (a); an R_1-minor.

To solve the case (a), we focus on a fragment of the graph H_k° close to the vertex x, which is drawn in solid lines in Figure 6. Actually, this case has two similar subcases in the picture, and in at least one of them none of the solid edges in the picture is incident with the vertex x'.

Now, in the graph $H_k^\bullet$, we contract the three solid edges incident with the vertex y, and denote by G_0 the subgraph consisting of the two shaded triangles after the contraction (see in Figure 6). Notice that G_0 together with h and the remaining solid edges form a subgraph isomorphic to a subdivision of the graph $R_1 - z_2$, where h corresponds to z_1 and the shaded triangles to the cycles C_1, C_1' in R_1. Moreover, there is a collection of paths (in dashed edges of $H_k^\bullet$) joining the vertex $x = x'$ to all vertices of G_0, and so $x = x'$ is good for z_2 in the graph R_1. Hence the graph $H_k^\bullet$ has a minor isomorphic to R_1.

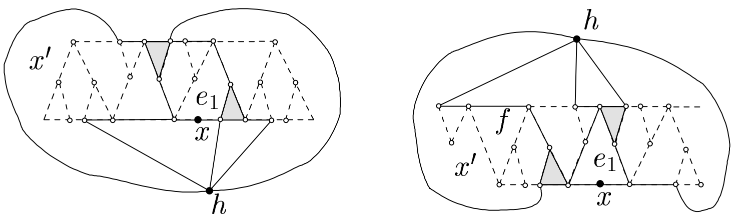

Figure 7: How to get an R_1-minor in the cases (b) and (c).

We similarly show that the graph $H_k^\bullet$ has a minor isomorphic to R_1 in the other two cases (b),(c): Again, e_1, e_2 are chosen without loss of generality such that the subgraphs drawn in solid lines as fragments of H_k° in Figure 7 are not incident with the vertex x'. We would in the case (c) interchange

e_1, e_2 if it happened that $e_2 = f$ where f is as in the picture. Those "solid-line" subgraphs in both cases evidently have minors isomorphic to the graph $R_1 - z_2$, such that the shaded triangles correspond to C_1, C_1' in R_1. Moreover, the vertex $x = x'$ in $H_k^\bullet$ is connected to vertices of the shaded triangles by five paths, and hence the required R_1-minor follows in both cases, as above. ∎

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