Implicit proofs

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Abstract

We describe a general method how to construct from a propositional proof system P a possibly much stronger proof system iP. The system iP operates with exponentially long P-proofs described "implicitly" by polynomial size circuits.

As an example we prove that proof system iEF, *implicit* EF, corresponds to bounded arithmetic theory V_2^1 and hence, in particular, polynomially simulates the quantified propositional calculus G and the Π_1^b-consequences of S_2^1 proved with one use of exponentiation. Furthermore, the soundness of iEF is not provable in S_2^1. An iteration of the construction yields a proof system corresponding to $T_2 + Exp$ and, in principle, to much stronger theories.

Extended Frege system EF is considered to be a strong propositional proof system. The qualification strong means that EF smoothly formalizes many arguments in elementary combinatorics or algebra and it seems very hard to come up with tautologies that would be hard to prove in EF (i.e. that they would require long proofs). Another strong proof system is the quantified propositional calculus G which operates with quantified propositional formulas. We can move up in this hierarchy allowing a proof system to quantify also over boolean functions, functionals, etc. But besides simulating definitions from higher order arithmetic or set theory we do not really have any other way of directly constructing strong proof systems.

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The qualification directly is important here as we do have a general correspondence between proof systems and first-order theories (obeying certain tame technical conditions satisfied by all "usual" theories, including set theory) and, in particular, we can define a strong proof system from a strong theory. This correspondence is very useful and it is the deepest information applying to all proof systems (as oppose to statements about particular ones) that we have. In particular, the statements above that EF and G are strong could be substantiated by identifying theories corresponding to them (S^1_2 and U^1_2, respectively). (The proof system extending G by allowing the quantification over functions, functionals, etc. corresponds to $T_2 + Exp$ or to a bit stronger theory, depending on the exact definition).

However, our aim here is to investigate a possibility of a direct, essentially combinatorial, description of strong proof systems that would, in particular, not refer to first order theories. This appears of interest in connections with several problems (e.g. a combinatorial characterization of hard tautologies and of consistency statements in particular, the existence of an optimal proof system, constructions of models of strong bounded arithmetic theories, etc.).

As it is with all known non-trivial proof complexity upper bounds or polynomial simulations, they are much simpler to prove using bounded arithmetic than using direct proof manipulations. Thus although we want to bypass the reference to theories in definitions of strong proof systems, we shall use the correspondence between proof systems and theories in proofs. However, the concept of implicit EF (and iP in general) is defined without any reference to arithmetic.

Let us now describe a part of this correspondence that we will need (and fix the notation in the process). A $\forall\Pi^b_1$-sentence $\forall x, \psi(x)$, with $\psi(x)$ having the form $\forall y(|y| \leq |x|^{O(1)}), \psi_0(x,y)$ for some p-time predicate ψ_0, determines an infinite sequence of propositional formulas $||\psi(x)||^n$ as follows. The formula has n atoms $p_1, \ldots, p_n$ for bits of an x, some $n^{O(1)}$ atoms $q_1, \ldots, q_m$ for bits of a y in ψ_0, and further it has $n^{O(1)}$ atoms $r_1, \ldots, r_s$ for bits of values of subcircuits of a fixed (canonically constructed) circuit computing from $\overline{p}$, $\overline{q}$ the truth value of $\psi_0(x,y)$. The formula $||\psi(x)||^n$ expresses in a DNF form that if $\overline{r}$ are correctly computed by the circuit from the inputs $\overline{p}$, $\overline{q}$ then the output of the computation is 1. A number b of length n is identified with a binary string $(b_1, \ldots, b_n)$ of length n, and these bits will make $||\psi(x)||^n(p_i/b_i)$ a tautology iff $\psi(b)$ is true.

The correspondence between a theory T and a proof system P implies, in particular, the following:

- If T proves $\forall x; \psi(x)$ then tautologies $||\psi(x)||^n(b)$ have polynomial size P-proofs.

- T proves the soundness of P and for any another proof system Q, if T proves also the soundness of Q then P polynomially simulates Q.

We shall not repeat other definitions and basic facts from proof complexity or bounded arithmetic. The reader can find those in [5] (or in the other original references listed in the bibliography).

1 Implicit EF

Let EF be a fixed Extended Frege system in the DeMorgan language. The set of all DeMorgan tautologies is denoted $TAUT$. We shall assume that EF proofs are written in an *enhanced* form where each step caries an information about the rule and the previous steps that were used in its derivation. This is an inessential change that does not affect the proof complexity of EF (more than by a logarithmic factor).

The symbol $\leq_{\text{lex}}$ denotes the lexicographic ordering on any fixed $\{0,1\}^k$. If we identify $i = (i_1, \ldots, i_k) \in \{0,1\}^k$ with the number $\sum_{j:i_j \neq 0} 2^j$ then $\leq_{\text{lex}}$ corresponds to the usual ordering on $\{0, \ldots, 2^k - 1\}$.

Definition 1.1 *Let $\tau \in TAUT$. An* implicit *EF proof of τ is a pair (α, β) such that:*

1. *β is a many-output boolean circuit in variables $i_1, \ldots, i_k$.*
2. *The sequence $\beta(\overline{0}), \ldots, \beta(i), \ldots, \beta(\overline{1})$ is an EF-proof of τ (the i's are ordered by $\leq_{lex}$).*

   The EF-proof described by β is denoted β^*.
3. *α is an EF-proof of a (canonical) tautology $Correct_\beta(x_1, \ldots, x_k)$ expressing that*

   "the formula in the step $\beta(x_1, \ldots, x_k)$ has been derived

   in β^* according to the EF-rules specified in $\beta(x_1, \ldots, x_k)$"

The proof system so defined is denoted iEF.

Note that we do not need to require that α also contains an EF-proof of the fact that the last step of β^* is τ (plus the auxiliary information); that is expressed by a true boolean sentence written using a circuit and so it always has a polynomial size proof in EF. Further note that as we consider enhanced EF-proofs the formula $Correct_\beta(\overline{x})$ is indeed expressible without existential quantification over steps in β^*, and hence if β^* is a correct EF-proof the formula is a tautology (when considering only polynomial size proofs such a quantification posses no problem as the quantifiers range only over a polynomial size set).

Let us start with the obvious.

Lemma 1.2 *iEF is a proof system in the sense of Cook-Reckhow [3], and it polynomially simulates EF.*

Proof :

It is clear that iEF is sound and complete. The third condition in the Cook-Reckhow's definition is that the relation "(α, β) is an iEF -proof of τ" is decidable in polynomial time. That follows as it is sufficient to check that the formula in the last step of β^* is τ, and that "α is an EF-proof of $Correct_\beta$" which is a polynomial time relation obviously.

A p-simulation of EF by iEF proceeds as follows. Let π be an EF-proof of τ of size m. Let β be a circuit in $\log(m)$ inputs that simply copies π into β^*, i.e. $\beta^* = \pi$. Clearly such β exists of size $O(|\pi|)$.

For α we take an EF-proof of $||Prf(u, v)||^m(\pi, \tau)$, where $Prf(u, v)$ is the polynomial time relation "u is an EF-proof of v". This has an EF-proof of size $O(|\pi|^2)$ that is constructed by a polynomial time algorithm from π and τ. This completes the p-simulation.

q.e.d.

Another p-simulation of EF by iEF follows from Lemma 3.2.

2 The strength of iEF

Now we calibrate the strength of iEF.

Theorem 2.1 *iEF corresponds to bounded arithmetic theory V_2^1. In particular,*

1. *V_2^1 proves the soundness of iEF.*
2. *Whenever a $\forall\Pi_1^b$-sentence $\forall x \psi(x)$ is provable in V_2^1 then the sequence of tautologies $||\psi(x)||^n$ has polynomial size iEF-proofs.*
3. *If V_2^1 proves the soundness of a proof system Q then iEF polynomially simulates Q.*

Moreover, an iEF-proof of $||\psi(x)||^n$ *can be constructed by a polynomial-time algorithm (from a string of length n) and the construction can be formalized in* S^1_2*, and the polynomial simulation in item 3. can be also defined in* S^1_2*.*

Proof :

We start by proving the soundness of iEF in V^1_2. Work in a model of V^1_2 where we have an iEF-proof (α, β) (coded by a number, say b) of formula τ. Let a be a number coding a truth assignment to atoms of τ.

By induction on $\overline{i} \in \{0,1\}^k$ (ordered by $\leq_{\text{lex}}$) construct a set $A_{\overline{i}}$ coding a truth assignment to extension atoms in β^* introduced in steps $\leq_{\text{lex}} \overline{i}$ such that all their extension axioms are true when atoms of τ are evaluated by a. The induction step is trivial and the statement that such a set exists is $\Sigma^{1,b}_1$, hence the $\Sigma^{1,b}_1$-induction implies that there is such a set $A := A_{\overline{1}}$ for $\overline{i} = \overline{1}$.

Using A, a and b as parameters prove by Π^b_1-induction on $\overline{i}$ that all formulas in β^* are true under the assignment given by a and A. The induction step uses the proof α: EF is sound in any model of V^1_2 and hence each step of β^* is indeed derived correctly via EF-rules, which are all sound. Hence τ is satisfied by (any) assignment a. This completes the proof of the first part.

Assume that V^1_2 proves a $\forall\Pi^b_1$-sentence $\forall x, \psi(x)$ which is of the form $\forall y \psi_0(x, y)$ with y implicitly bounded in ψ_0. We shall describe polynomial size iEF-proofs of tautologies $||\psi(x)||^n$, $n \geq 1$. In fact, the proof π_n of $||\psi(x)||^n$ will be constructed by a polynomial time algorithm from a string of length n, and the construction itself could be formalized in S^1_2.

By [4] the hypothesis implies (is equivalent to, in fact) that there is a term $t(x)$ of the language of S^1_2 such that S^1_2 proves:

$$(*) \qquad\qquad t(x, y) \leq |z| \longrightarrow \psi_0(x, y)\ .$$

Furthermore, we may assume that $(*)$ has an S^1_2 proof in which all formulas are strict Σ^b_1; let Ω be one such proof. The algorithm that will construct π_n will use Ω as an advice (but it is common for all n and so the algorithm is uniform).

A general sequent in Ω looks like

$$\exists u A(x, y, z, u), \ldots \ \longrightarrow\ \exists v B(x, y, z, v), \ldots$$

To simplify the notation we show just one formula per cedent and we do not show explicit bounds in the existential quantifiers.

The proof β^* will contain n atoms p for bits of x, $n^{O(1)}$ atoms q for bits of y and $t(2^n) \leq 2^{n^{O(1)}}$ atoms r for bits of z. Proof Ω is translated into β^* step by step. If we were constructing a simulation in EF, a sequent of the form as above would be translated into a sequent of the form

$$||A(x,y,z,u)||(p,q,r,u),\ldots \quad\longrightarrow\quad ||B(x,y,z,v)||(p,q,r,v),\ldots$$

where we denote new atoms assigned to bits of u and v ($\leq 2^{n^{O(1)}}$ of them) also u and v for simplicity of the notation. Here u are new atoms that are not extension atoms while v are extension atoms depending possibly on all p,q,r,u. But as there are exponentially many atoms r already, such a sequent would be exponentially long and could not be produced by a polynomial size circuit.

We overcome this difficulty by systematically introducing new extension atoms for all (sub)formulas that appear in the translation. Hence the sequent gets translated into a sequent of the form

$$w_A,\ldots \quad\longrightarrow\quad w_B,\ldots$$

where w_A and w_B are extension atoms depending on p,q,r,u and p,q,r,v (and hence u too) respectively. Having the sequent from Ω this introduction of the extension atoms is exponential in size but very canonical and can be constructed by a polynomial size circuit with an access to Ω. By this phrase we mean that the circuit has size $n^{O(1)}$ and produces the extension atoms and axioms bit by bit.

The whole proof β^* is parcelled into peaces parametrized by sequents in Ω. Each peace has its own canonical assignment of extension atoms and is constructed by a suitable polynomial size circuit. It remains to show how these peaces are put together to form an EF-proof. That is, how are the inferences in Ω simulated.

We shall consider only the most complicated case, the simulation of a Σ^b_1-LIND inference

$$\frac{\exists u A(t,u) \ \rightarrow\ \exists v A(t+1,v)}{\exists u' A(0,u') \ \rightarrow\ \exists v' A(|w|,v')}$$

(we leave out the free parameters and the quantifier bounds). Assume that the proof β^* contains a derivation of a sequent of the from $w_A \longrightarrow w_B$ representing

$$||A||(t,u) \quad\longrightarrow\quad ||A||(s,v)$$

where t are new atoms (not extension atoms), s are extension atoms introduced so that they define the number represented by t plus 1, and v are extension atoms depending on (p,q,r and) t, u.

Take $|w| = 2^{n^{O(1)}}$ copies of this derivation (canonically listed) in disjoint atoms t,s,u,v, say t^i,s^i,u^i,v^i for $0 \leq i < 2^{n^{O(1)}}$. Piece them together by postulating that $t^0 = \overline{0}$ (represents 0), $s^i = t^{i+1}$, and $v^i = u^{i+1}$. This is correct as atoms t^i and u^i were not extension atoms and so we can add conditions on them. This concatenation of the $|w|$ subproofs is again quite canonical and it constitutes a proof of a sequent of the form $w_A \longrightarrow w_B$ corresponding to:

$$||A||(0, u^0) \ \longrightarrow \ ||A||(|w|, v^{|w|}) \ .$$

To finish the description of β^* we need only to derive the (translation of the) antecedent $t(x, y) \leq |z|$ of $(*)$. This is done by stipulating that all atoms r are equal to 1 and using a canonical EF-proof of the valid inequality saying that the term $t(x, y)$ produces from x and y of the lengths n and $n^{O(1)}$ respectively at most $2^{n^{O(1)}}$ bits.

The EF-proof α of the correctness of the description of β^* by β is easy and uses the parcellization of β^* given by the steps in Ω; it is essentially an EF-proof of the fact that Ω is indeed a proof in $S^1_2 + 1 - Exp$ of $(*)$. This concludes the proof of the second part of the theorem.

The third property of the correspondence between iE and V^1_2 stated in the theorem is actually a consequence of the first two (this is a standard argument, cf. [5]). The formalization of the constructions in items 2. and 3. is routine (note that the formalization starts with Ω, not with any V^1_2-proof). This concludes the proof of Theorem 2.1.

q.e.d.

Now we note some corollaries of the theorem. The first one just restates explicitly what has been used in the proof of the theorem (the last sentence in the corollary follows by a general well-known argument using the correspondence between a theory and a proof system).

Corollary 2.2 *Let $\forall x \psi(x)$ be a $\forall\Pi^b_1$-sentence that is provable in $S^1_2 + 1 - Exp$, i.e so that S^1_2 proves*

$$|y| \geq t(x) \rightarrow \psi(x) \ .$$

Then the sequence of tautologies $||\psi(x)||^n$, $n \geq 1$, admits polynomial size iEF-proofs.

Moreover, the set of all $\forall\Pi^b_1$-sentences provable in $S^1_2 + 1 - Exp$ is axiomatized over S^1_2 by the canonical (see [5]) $\forall\Pi^b_1$-sentence expressing the soundness of iEF.

By [4] $S^1_2 + 1 - Exp$ is not $\forall\Pi^b_1$-conservative over S^1_2. Hence Corollary 2.2 immediately yields

Corollary 2.3 *The soundness of iEF is not provable in S^1_2.*

Note that it is not known if S^1_2 proves the soundness of the quantified propositional calculus G.

Theorem 2.1 yields an information about the relative strength of G and iEF.

Corollary 2.4 *iEF p-simulates G.*

Proof :

By [8] the proof system G corresponds to theory U^1_2 and, in particular, the two properties of the correspondence singled out in the introduction are valid for U^1_2 and G. This implies (as U^1_2 is weaker than V^1_2) that V^1_2 proves the soundness of G, and hence iEF polynomially simulates G by the third property stated in Theorem 2.1.

q.e.d.

Proving Corollary 2.4 directly would be rather challenging to a formalization. It is not very difficult to prove directly (via a witnessing style argument) that iEF polynomially simulates G_1. But the simulation of full G, say via Herbrand theorem, would lead to very convoluted formulas (similarly as formulas in Herbrand theorem get complex with the growth of the quantifier complexity).

3 Iteration of the construction

In defining iP from a general proof system P (say a non-deterministic acceptor of $TAUT$ or a polynomial time function whose range is $TAUT$) the problem is not what α should be (a P-proof) but what should β be. In particular, I have not found a natural definition of parcelling a P-proof into steps that could be then computed by β (a minimal requirement for such a definition is that it specializes to the usual notion of a step if applied to ordinary proof calculi). However, there is a way out. Any proof system P can be p-simulated by a proof system Q that extends EF by a polynomial time subset of $TAUT$ as new axioms (cf.[6], [5]). We shall denote this form of Q by $Q \supseteq EF$. In fact, many usual proof systems p-simulating EF are actually p-equivalent to some $Q \supseteq EF$. Hence without loosing to much in generality we can restrict to such proof systems in the next definition.

For $Q \supseteq EF$ it is straightforward to say what β should be: $\beta(\bar{\imath})$ is either a step correctly derived from earlier steps (whose indices are encoded in $\beta(\bar{\imath})$ too) by an

EF-rule, or it is a Q-axiom (which is a p-time property). Let $Correct^Q_\beta$ be the tautology expressing this, constructed analogously to $Correct_\beta$ from Definition 1.1 (in particular, $Correct_\beta = Correct^{EF}_\beta$).

Definition 3.1 *Let $P, Q \supseteq EF$ be proof systems. Define new proof system $[P, Q]$ as follows. A $[P, Q]$-proof of $\tau \in TAUT$ is a pair (α, β) such that:*

1. *β is a many-output boolean circuit in variables $i_1, \ldots, i_k$.*
2. *The sequence $\beta(\overline{0}), \ldots, \beta(i), \ldots, \beta(\overline{1})$ is a Q-proof of τ. The Q-proof described by β is denoted β^*.*
3. *α is a P-proof of the tautology $Correct^Q_\beta(x_1, \ldots, x_k)$.*

Further put $iP := [P, P]$.

We note few simple properties of the bracket operation. The symbols $\leq_p$ and $\equiv_p$ denote the p-simulation and the p-equivalence, respectively.

Lemma 3.2 *For $P \supseteq EF$, $P \leq_p [P, EF]$.*

Proof :
Let $\tau(x_1, \ldots, x_n)$ be a tautology. Circuit β will describe the following trivial, exponential derivation of τ. For each $a \in \{0,1\}^n$, β^* has a segment where it computes the truth value of $\tau(a)$: this is simply the derivation of subformulas which are true, respectively of the negations of subformulas which are false.

Then it contains 2^{n-1} segments, one for each $(a_2, \ldots, a_n) \in \{0,1\}^{n-1}$ where it derives $\tau(x_1, a_2, \ldots, a_n)$ from $\tau(0, a_2, \ldots, a_n)$ and $\tau(1, a_2, \ldots, a_n)$ (using $x_1 \equiv 0 \vee x_1 \equiv 1$).

Then there are 2^{n-2} segments where all $\tau(x_1, x_2, a_3, \ldots, a_n)$ are derived from $\tau(x_1, 0, a_3, \ldots, a_n)$ and $\tau(x_1, 1, a_3, \ldots, a_n)$, etc. The proof ends with a derivation of τ from $\tau(x_1, \ldots, x_{n-1}, 0)$ and $\tau(x_1, \ldots, x_{n-1}, 1)$.

The correctness of the steps in β^* is trivial to prove assuming one knows that all $\tau(a)$'s have been derived, i.e. are true. But if P proves τ, it proves that "all $\tau(a)$ are true", and hence can prove the formula $Correct_\beta$. So $[P, EF]$ p-simulates P.

q.e.d.

Note that the lemma holds (by the same proof) even with $[P, F]$ or even $[P, R]$ in place of $[P, EF]$ (F a Frege system and R resolution).

The next lemma shows that it makes no sense to iterate the construction in the place of α.

Lemma 3.3 *For all $P \supseteq EF$, $iP \equiv_p [iP, P]$.*

Proof :

The p-simulation of iP by $[iP, P]$ follows from Lemma 3.2 (as $P \supseteq EF$). For the opposite p-simulation consider the case $P = EF$.

In the proof of the soundness of iEF in V_2^1 we only used the fact that α is an EF-proof in order to know that what α proves is actually true in the model. That is, we only used that the soundness of EF is provable in V_2^1. Hence α could have been an iEF-proof as well. This shows (by part 3 of Theorem 2.1) that $iEF \geq_p [iEF, EF]$.

The case of general $P \supseteq EF$ is proved analogously, using a theory corresponding to iP in place of V_2^1.

q.e.d.

So if we want to iterate the i-construction we should apply it to the second argument in the bracket operation. For the rest of the section we restrict ourselves to $P = EF$.

Definition 3.4 *Put $i_1 EF := iEF$, and for $k \geq 1$ define*

$$i_{k+1}EF \;:=\; [EF, i_k EF]$$

One can show analogously to Theorem 2.1 (or by applying Theorem 2.1 to its own formalization in S_2^1) that $i_k EF$ corresponds to $S_2^1 + k - Exp$ of [4] (or see [5]) and hence to the Σ_1-induction in a k-th order bounded arithmetic. Analogously to Corollary 2.3, $S_2^1 + k - Exp$ does not prove the soundness of $i_{k+1}EF$. We shall not get into details as we are unable to say anything else sensible about the proof systems besides the next theorem.

Theorem 3.5 *The soundness of each $i_k EF$, $k \geq 1$, is provable in $T_2 + Exp$.*

On the other hand, if a $\forall\Pi_1^b$-sentence $\forall x \psi(x)$ is provable in $T_2 + Exp$ then there is a $k \geq 1$ such that all tautologies $||\psi(x)||^n$, $n \geq 1$, have polynomial size $i_k EF$-proofs.

In the correspondence between $T_2 + Exp$ and $i_k EF$'s the constant k is fixed in proofs of any particular sequence $||\psi(x)||^n$, $n \geq 1$. But we can also allow k unbounded (besides the implicit bound given by the size of the whole proof). In this way we get a proof system that is (presumably) stronger. This is analogous to the situation for G: proofs in T_2 translate into G_k-proofs, fixed $k \geq 1$, while G (unbounded quantifier complexity) corresponds to a stronger theory U_2^1. A formal definition of this very strong proof system might be as follows.

Definition 3.6 *Proof system $i_\infty EF$ is defined as follows. An $i_\infty EF$-proof of $\tau \in TAUT$ is a triple (α, β, w) such that (α, β) is an $i_{|w|}EF$-proof of τ.*

It can be shown that $T_2 + Exp$ does not prove the soundness of $i_\infty EF$. This is an evidence that $i_\infty EF$ may be indeed stronger than any $i_k EF$.

It is easy to see that $i(i_\infty EF) \equiv_p i_\infty EF$ and hence the i-operation does not necessarily always produce a stronger proof system. But we can now start iterating the i_∞-operation and proceed forward. We could have also defined the i_∞-operation not as $|w|$-iteration of the i-operation but as w-iteration (or even 2^w-iteration, etc.) enumerated by a polynomial size circuit (or by a circuit produced by a polynomial size circuit, etc.).

In fact, there does now seem to be *the* canonical way how to iterate the basic i-operation. This appears analogous to a situation in proof theory of higher order arithmetic and set theory where there is also no *the* canonical way how to iterate consistency statements or even how to represent ordinals.

We conclude by two remarks about the bracket operation for systems below EF. For example, U^1_2 can be described (its bounded first-order consequences, precisely) as $R^1_2 + 1 - Exp$, where R^1_2 is a subtheory of S^1_2 corresponding to Frege systems F. But we cannot conclude analogously to Theorem 2.1 that iF or $[EF, F]$ correspond to U^1_2. This is because F has no extension atoms and cannot abbreviate a priori exponentially long formulas translating formulas in the starting arithmetical proof, no matter that it is equally canonical as in the case of V^1_2.

The absence of extension atoms in F has another corollary: For any $P \geq_p G_1$ it holds that $[P, F] \equiv_p P$. This can be seen as follows. As $P \geq_p G_1$ we can take for a theory T_P corresponding to P (it is unique only up to $\forall\Pi^b_1$-consequences) a theory containing T^1_2. Now assume that (α, β) is an $[P, F]$-proof of τ in a model of T_P. The P-proof α is sound in the model and hence β^* is indeed an F-proof of τ. As there are no other atoms in β^* than the atoms of τ, a truth assignment falsifying τ would transfer β^* into a sequence of 0's and 1's which has no first occurrence of 0. That contradicts the minimization principle for Δ^b_1-formulas valid in the model (by T^1_2). Hence T_P proves the soundness of $[P, F]$ and we are done (the opposite simulation $[P, F] \geq_p P$ follows by the remark after the proof of Lemma 3.2. In fact, $P \equiv_p [P, F] \equiv_p [P, R]$.

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References

[1] S. R. Buss, Bounded Arithmetic. Naples, (1986), Bibliopolis.

[2] S. A. Cook, Feasibly constructive proofs and the propositional calculus, in: *Proc. 7^{th} Annual ACM Symp.on Theory of Computing*, (1975), pp. 83-97. ACM Press.

[3] S. A. Cook and A. R. Reckhow, The relative efficiency of propositional proof systems, *J. Symbolic Logic*, **44(1)**, (1979), pp.36-50.

[4] J. Krajíček, Exponentiation and Second Order Bounded Arithmetic, *Annals of Pure and Applied Logic*, **48(3)**, (1990), pp. 261-276.

[5] J. Krajíček, *Bounded arithmetic, propositional logic, and complexity theory*, Encyclopedia of Mathematics and Its Applications, Vol. **60**, Cambridge University Press, (1995).

[6] J. Krajíček, P. Pudlák, Propositional Proof Systems, the Consistency of First Order Theories and the Complexity of Computations, *J. Symbolic Logic*, **54(3)**, (1989), pp. 1063-1079.

[7] J. Krajíček, P. Pudlák, Quantified propositional calculi and fragments of bounded arithmetic, *Zeitschrift f. Mathematische Logik u. Grundlagen d. Mathematik*, **36**, (1990), pp. 29-46.

[8] J. Krajíček, G. Takeuti, On Bounded $\sum_1^1$-Polynomial Induction, in: *Feasible Mathematics*, eds. S.R. Buss and P.J. Scott, (1990), Birkhauser, pp. 259-280.

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