Rainbow Arithmetic Progressions

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Abstract

Consider a coloring of $\{1, 2, \ldots, n\}$ in 3 colors, where $n \equiv 0 \pmod 3$. If all the color classes have the same cardinality, then there is a 3-term arithmetic progression whose elements are colored in distinct colors. This rainbow variant of van der Waerden's theorem proves the conjecture of the second author.

1 Introduction

Given a coloring of a set of natural numbers, we say that a subset is *monochromatic* if all of its elements have the same color and we say that it is rainbow if all of its elements have distinct colors. A classical result in Ramsey theory is van der Waerden's theorem [vW27], which states that for every k and t and sufficiently large n, every k-coloring of $[n] := \{1, 2, \ldots, n\}$ contains a monochromatic arithmetic progression of length t. Jungić et. al. [JLMNR] considered, for the first time in the literature, a rainbow counterpart of van der Waerden's theorem. They proved that every 3-coloring of the set of natural numbers $\mathbb{N}$ with the upper density of each color greater than $1/6$ contains a rainbow AP(3). They also asked whether the "finite" version of their theorem also holds, and, backed by the computer evidence ($n \leq 56$), posed the following conjecture.

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Conjecture 1 *For every $n \geq 3$, every partition of $[n]$ into three color classes R, G, and B with $\min(|R|,|G|,|B|) > r(n)$, where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

Moreover, they constructed a 3-coloring of $[n]$ with $\min(|R|,|G|,|B|) = r(n)$, where r is the function defined in (1), that contains no rainbow $AP(3)$. This shows that Conjecture 1, if true, is the best possible.

A weaker form of this conjecture, due to the second author, was posed at the open problem session of the 2000 MIT Combinatorics Seminar [JLMNR].

Conjecture 2 *Let $n \equiv 0 \pmod 3$. For every equinumerous 3-coloring of $[n]$, that is, a coloring in which all color classes have the same cardinality, there exists a rainbow $AP(3)$.*

In this paper, we prove Conjecture 2.

2 Proof of Conjecture 2

Given a 3-coloring of $[n]$ with colors Red (R), Green (G) and Blue (B), a *B-block* is a string of consecutive elements of $[n]$ that are colored Blue. R-block and G-block are defined similarly. We say that the coloring is *rainbow-free* if it contains no rainbow $AP(3)$.

First, we show that every rainbow-free 3-coloring contains a **dominant color**, that is, a color $z \in \{B,G,R\}$ such that for every two consecutive numbers that are colored with different colors, one of them is colored with z. We will need the following two lemmata.

Lemma 1 *Let $c : [n] \to \{B,G,R\}$ be a 3-coloring of $[n]$ such that every two different colors appear next to each other. Then there exist $p, r \in [n]$, $p < r$, such that*

1. $c(p) = c(r)$,
2. $c(p+1) \neq c(p)$,

3. $c(r-1) \notin \{c(p), c(p+1)\}$, *and*

4. no element in the interval $[p+1, r-1]$ *is colored by the color* $c(p)$.

Proof: Let G be the first color to appear and let the first G-block be followed by an R-block. Since G appears next to B, there exists a G-block that is next to a B-block. If this G-block is preceded by a B-block,

$$\begin{array}{cccccc} G\dots G & R\dots R & \dots & B\dots B & G\dots G & \dots \\ \quad\;\uparrow & & & & \uparrow\quad\; & \\ \quad\;p & & & & r\quad\; & \end{array}$$

then the lemma follows. So, suppose that $\mathcal{G}$, the first G-block that is next to a B-block, is preceded by an R-block and followed by a B- block $\mathcal{B}$.

$$\begin{array}{ccccccc} G\dots G & R\dots R & \dots & R\dots R & G\dots G & B\dots B & \dots \\ & & & & \uparrow & \uparrow & \\ & & & & \mathcal{G} & \mathcal{B} & \end{array}$$

Suppose there is a B-block between the two G-blocks. Consider the last such B-block and denote it by $\mathcal{B}'$. The lemma immediately follows, since we can take p and r to be the last element of $\mathcal{B}'$ and the first element of $\mathcal{B}$.

Now, suppose there is no B-block between any the two G-blocks.

If $\mathcal{B}$ is followed by an R-block, then the lemma clearly follows. So, let $\mathcal{B}$ be followed by a G-block. The same reasoning as above, combined with the assumption that R appears next to B, implies that one of the following two scenarios happens:

$$\begin{array}{cccccc} \dots & G\dots G & R\dots R & G\dots G & B\dots B & R\dots R \\ & & \quad\;\uparrow & & & \uparrow\quad\; \\ & & \quad\;p & & & r\quad\; \end{array}$$

or

$$\begin{array}{cccccc} \dots & G\dots G & B\dots B & G\dots G & R\dots R & B\dots B \\ & & \quad\;\uparrow & & & \uparrow\quad\; \\ & & \quad\;p & & & r\quad\; \end{array}$$

which completes the proof. $\square$

Lemma 2 *Let $c : [n] \to \{B, G, R\}$ be a 3-coloring of $[n]$ such that*

$$\min\{|c^{-1}(B)|, |c^{-1}(G)|, |c^{-1}(R)|\} \geq 5$$

and every two different colors appear next to each other. Then there is a rainbow $AP(3)$.

Proof: By Lemma 1, we can assume that there are p, q, r, such that

$$\begin{array}{cccccccc} \ldots & G & R & \ldots & R & B\ldots B & G & \ldots \\ & \uparrow & & \uparrow & \uparrow & \uparrow & \uparrow & \\ & p & & R/B & q & B & r & \\ & & & \text{only} & & \text{only} & & \end{array}$$

So, we assume that $p + 2 \leq q < r - 2$, $c(p) = G$, $c(p + 1) = R$, $c(q) = R$, $c(r) = G$, $c(i) = B$ for all $q < i < r$ and $c(j) \neq G$ for all $p + 1 < j < q$. Since $|c^{-1}(G)| \geq 5$, then $p \geq 3$ or $r \leq n - 2$. Without loss of generality, assume $p \geq 3$.[1] If $r + q$ is even, then q, $\frac{r+q}{2}$, r is a rainbow $AP(3)$. So, let $r + q$ be odd. It follows that an even number of elements between q and r are colored by B. If $c(q - 1) = R$ then $q - 1$, $\frac{r+q-1}{2}$, r is a rainbow $AP(3)$. Let $c(q - 1) = B$ and let $s = \min\{i \in [p, q] | c(i) = B\}$. Then,

$$\begin{array}{cccccccccc} \ldots & G & R\ldots R & B & \ldots & B & R & B\ldots B & G & \ldots \\ & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & \\ & p & \text{only } R & s & R/B & & q & \text{only } B & r & \\ & & & & & & & \text{even\#} & & \end{array}$$

Notice that s could be equal to $q - 1$. Now, if $p + s$ is even, then p, $\frac{p+s}{2}$, s is a rainbow $AP(3)$. Otherwise, an even number of elements between p and s are colored by R. If $c(p - 1) = G$, then $p - 1$, $\frac{p+s-1}{2}$, s is a rainbow $AP(3)$. So, let $c(p - 1) = R$. If $c(s + 1) = B$ (here, $s \neq q - 1$), then p, $\frac{p+s+1}{2}$, $s + 1$ is a rainbow $AP(3)$. Hence, let $c(s + 1) = R$. Then, the interval $[p - 1, r]$ is colored as follows.

$$\begin{array}{cccccccccccc} \ldots & R & G & R\ldots R & B & R & \ldots & B & R & B\ldots B & G & \ldots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \uparrow & \uparrow & \uparrow & \\ & p-1 & p & \text{only } R & s & s+1 & & & q & \text{only } B & r & \\ & & & \text{even\#} & & & & & & \text{even\#} & & \end{array}$$

[1]Otherwise, consider the coloring $c' : [n] \to \{B, G, R\}$, defined by $c'(i) = c(n - i + 1)$ for every $i \in [n]$.

Suppose that $p+r-1$ is even. If $c(\frac{p+r-1}{2}) = R$, then p, $\frac{p+r-1}{2}$, $r-1$ is a rainbow $AP(3)$. If $c(\frac{p+r-1}{2}) = B$, then $p-1$, $\frac{p+r-1}{2}$, r is a rainbow $AP(3)$.

So, let $p+r-1$ be odd. If $c(p-2) = B$ then $p-2, p-1, p$ is a rainbow $AP(3)$. Suppose $c(p-2) = R$. If $c(\frac{p+r-2}{2}) = R$, then p, $\frac{p+r-2}{2}$, $r-2$ is a rainbow $AP(3)$. If $c(\frac{p+r-2}{2}) = B$, then $p-2$, $\frac{p+r-2}{2}$, r is a rainbow $AP(3)$. Hence, the only remaining case is when $c(p-2) = G$.

$$\begin{array}{ccccccccccccc}
\dots & G & R & G & R\dots R & B & R & \dots & B & R & B\dots B & G & \dots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \uparrow & \uparrow & \uparrow & \\
 & p-2 & p-1 & p & \text{only}R & s & s+1 & & & q & \text{only}B & r & \\
 & & & & \text{even}\# & & & & & & \text{even}\# & &
\end{array}$$

Since an even number of elements between q and r are colored by B, there exists k, $2 \leq k < \frac{r-p}{2}$, such that $c(p+2k) = B$. Suppose that k is the smallest number with this property. Now, consider elements $\frac{p+p+2k}{2} = p+k$ and $\frac{p-2+p+2k}{2} = p+k-1$. By the property of k and the fact that one of the elements $k, k-1$ is even, we conclude that either $c(p+k) = R$ or $c(p+k-1) = R$. Then, either $p-2$, $p+k-1$, $p+2k$ or p, $p+k$, $p+2k$ is a rainbow $AP(3)$. $\square$

Lemma 2 immediately implies

Corollary 1 *Let $c : [n] \to \{B, G, R\}$ be a rainbow-free 3-coloring of $[n]$ such that*

$$\min\{|c^{-1}(B)|, |c^{-1}(G)|, |c^{-1}(R)|\} \geq 5.$$

Then there exists a dominant color.

Clearly, there can be only one dominant color.

The following lemma will be instrumental in showing that in every rainbow-free 3-coloring of $[n]$ there exists a *recessive color*, that is, a color $w \in \{B, G, R\}$ such that no two consecutive numbers are colored by w.

Lemma 3 *Let $c : [n] \to \{B, G, R\}$ be a 3-coloring of $[n]$ such that R is the dominant color. Suppose there exist i and j, so that $i < j$, $c(i) = c(i+1) = B$, $c(i+2) = c(j-1) = R$, $c(j) = c(j+1) = G$. Suppose also that the interval $[i+2, j-1]$ does not contain two consecutive elements that are both colored by B or by G. Then, there is a rainbow $AP(3)$.*

Proof: Suppose there exists a rainbow-free coloring c with the properties above. Consider the interval $[i, j+1]$.

$$\begin{array}{ccccccc}
\ldots & BB & R & \ldots & R & GG & \ldots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
 & i & i+2 & \text{no } BB, \text{ no } GG & j-1 & j &
\end{array}$$

Suppose no element of $[i+3, j-2]$ is colored by R. Since R is the dominant color, $[i+3, j-2]$ is either a B-block or a G-block. Then, either $i+1$, $i+2$, $i+3$ or $j-2$, $j-1$, j is a rainbow $AP(3)$, contradicting the assumption that c is rainbow-free.

Suppose no element of $[i+3, j-2]$ is colored by G. Since $c(j) = G$ and $c(j-1) = R$, then $c(j-2) = R$. Since $c(j+1) = G$ and $c(j-1) = R$, then $c(j-3) = R$. Iterating this reasoning from right to left, we conclude that $[i+2, j-1]$ is an R-block. Then, clearly, there exists a rainbow $AP(3)$, and, thus, we arrive at a contradiction. A symmetric argument shows that at least one of the elements of $[i+3, j-2]$ is colored by B.

Therefore, there is at least one element of each color in $[i+3, j-2]$.

Suppose $i+j+1$ is odd. Since R is the dominant color and there are no consecutive elements in $[i+3, j-2]$ both colored by B or by G, at least one of the elements $\left\lfloor \frac{i+j+1}{2} \right\rfloor$, $\left\lfloor \frac{i+j+1}{2} \right\rfloor + 1$ is colored by R. This implies that one of the arithmetic progressions

$$\begin{array}{ccc}
i, & \frac{i+j}{2} = \left\lfloor \frac{i+j+1}{2} \right\rfloor, & j \\
i+1, & \frac{i+j+2}{2} = \left\lfloor \frac{i+j+1}{2} \right\rfloor + 1, & j+1
\end{array}$$

is a rainbow $AP(3)$.

Therefore, if c is rainbow-free then $i+j+1$ is even. It follows that $c\left(\frac{i+j+1}{2}\right) \neq R$, otherwise, i, $\frac{i+j+1}{2}$, j is a rainbow $AP(3)$.

If there exists $i+2 < k < \frac{i+j+1}{2}$, with $c(k) = G$, then $c(2k-i) \neq R$ and $c(2k-i-1) \neq R$, otherwise i, k, $2k-i$ or $i+1$, k, $2k-i-1$ is a rainbow $AP(3)$. Since R is the dominant color, it follows that $2k-i$ and $2k-i-1$ are both colored by B or by G, which contradicts the assumed property of i and j. Therefore, if $c(k) = G$, where $i+2 < k < j-1$, then $k \geq \frac{i+j+1}{2}$.

A symmetric argument implies that if $c(k) = B$, $i+2 < k < j-1$, then $k \leq \frac{i+j+1}{2}$.

Without loss of generality, assume that $c\left(\frac{i+j+1}{2}\right) = G$, the other case being symmetric. Then, interval $[i, j+1]$ is colored as follows.

$$\begin{array}{ccccccccc} \dots & BB & R & \dots & RGR & \dots & R & GG & \dots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ & i & i+2 & \text{no } BB & \frac{i+j+1}{2} & \text{no } GG & j-1 & j & \\ & & & \text{no } G & & \text{no } B & & & \end{array}$$

Hence, there are no elements colored by G in $[i, \frac{i+j-1}{2}]$ and there are no elements colored by B in $[\frac{i+j+1}{2}, j+1]$. Since an element of each color appears in $[i+2, j-1]$, there exists $p \geq 2$ such that $c\left(\frac{i+j+1}{2} - l\right) = R$ for all $l \in [1, p]$ and $c\left(\frac{i+j+1}{2} - p - 1\right) = B$. Moreover, since c is rainbow-free, p must be even. It follows that $c\left(\frac{i+j+1}{2} + p + 1\right) = G$.

Let $x := \frac{i+j+1}{2} - p - 1$ and $y := \frac{i+j+1}{2} + p + 1$. Define intervals $\mathcal{I}_s = [a_s, b_s]$, $\mathcal{J}_s = [c_s, d_s]$, $s \in [0, r-1]$, where $a_s = i+2(sp+s+1)$, $b_s = i+2((s+1)p+s)+1$, $c_s = j - 2((s+1)p+s)$, $d_{s+1} = j - 2(sp+s+1) + 1$, and r is the smallest index such that $1 \leq c_{r-1} - y = x - b_{r-1} \leq 2p$. Notice that the number of elements in each of these intervals is $2p$.

For each $l \in [p]$, the following two arithmetic progressions

$$\begin{array}{ccl} i, & \frac{i+j+1}{2} - l, & i+j+1-2l-i = j-2l+1 \\ i+1, & \frac{i+j+1}{2} - l, & i+j+1-2l-i-1 = j-2l \end{array}$$

are not rainbow only if both $j - 2l$ and $j - 2l + 1$ are red. Hence, $\mathcal{J}_0$ is an R-block. This implies that $\mathcal{I}_0$ is an R-block, since $c(\frac{i+j+1}{2}) = G$ and c is rainbow-free. Since $\mathcal{I}_0$ is an R-block and $c(x) = B$, we conclude that $\mathcal{J}_1$ is an R-block. This, in turn, implies that $\mathcal{I}_1$ is an R-block, since $c(\frac{i+j+1}{2}) = G$. Iterating this argument, we conclude that all the intervals $\mathcal{I}_s$, $\mathcal{J}_s$, $s \in [0, r-1]$, are R-blocks. We have the following two cases.

- **Case 1.** $2 < c_{r-1} - y = x - b_{r-1} \leq 2p$.

  In this case, $y + 2p + 2 \in \mathcal{J}_{r-1}$ and x, y, $y + 2p + 2$ is a rainbow $AP(3)$, contradicting our assumption that c is rainbow-free.

- **Case 2.** $c_{r-1} - y = x - b_{r-1} = 1$ or $c_{r-1} - y = x - b_{r-1} = 2$.

  In this case, $x - p - 1 \in \mathcal{I}_{r-1}$ and $x - p - 1$, x, $\frac{i+j+1}{2}$ is a rainbow $AP(3)$, thus, arriving at a contradiction.

Therefore, c cannot be rainbow-free. $\square$

Corollary 2 *Let $c : [n] \to \{B, G, R\}$ be a rainbow-free 3-coloring of $[n]$ such that R is the dominant color. Then, either B or G is a recessive color.*

Proof: Suppose that neither B nor G is a recessive color. Then, among all pairs of elements (i, j), such that $c(i) = c(i+1) = B$ and $c(j) = c(j+1) = G$, choose the one where $|j - i|$ is minimal. Without loss of generality, assume that $i < j$. Then, by the choice of i and j, $c(i + 2) = R$, $c(j - 1) = R$ and interval $[i + 2, j - 1]$ does not contain two consecutive elements both colored by B or by G. Lemma 3 implies that c contains a rainbow $AP(3)$, which is a contradiction. $\Box$

Finally, we are in a position to prove Conjecture 2.

Theorem 1 *Let $n \equiv 0 \pmod 3$. For every equinumerous 3-coloring of $[n]$ there exists a rainbow $AP(3)$.*

Proof: The claim is true for $n \leq 15$ [JLMNR]. Let $n \geq 15$, $n \equiv 0 \pmod 3$, and let $c : [1, n] \to \{B, G, R\}$ be an equinumerous 3-coloring. Suppose that c is rainbow-free. By Corollary 1, there is a dominant color, say R. By Corollary 2, one of the remaining colors, say G, is recessive. It follows that every element colored by G is followed[2] by an element colored by R. Since there are elements of $[n]$ colored by B, there exists at least one pair $i, j \in [n]$, such that $c(i) = B$, $c(j) = G$, and all the elements between i and j are colored with R. Since the number of elements between i and j must be greater than or equal to two, at least one of these elements is such that both of its neighbors are not colored by G. It follows that $|c^{-1}(G)| < |c^{-1}(R)|$, which contradicts our assumption that c is equinumerous. Therefore, c is not rainbow-free. $\Box$

3 Concluding remarks

This note settles the question of the existence of a rainbow arithmetic progression in equinumerous 3-colorings of $[n]$. However, Conjecture 1 is still open. We hope that our lemmas in Section 2 with some additional ideas could prove that conjecture as well.

[2] or preceded, if $c(n) = G$

There are many directions and generalizations one might consider. For a discussion on this topic, as well as similar results for $\mathbb{Z}_n$, consult [JLMNR]. One natural direction is imitating the well known Rado's theorem for the monochromatic analogue [GRS90] and generalizing the problems above for rainbow solutions of other linear equations, under appropriate conditions on the cardinality of the color classes. The equation $x + y = z$ has already been studied. Alekseev and Savchev [AS87, G94] proved that every equinumerous 3-coloring of $[3n]$ contains a rainbow solution of this equation. Schönheim [Sch90, S95], answering the question of E. and G. Szekeres, proved that for every 3-coloring of $[n]$, such that every color class has cardinality greater than $n/4$, the equation $x+y=z$ has rainbow solutions. Here, $n/4$ is optimal.

Finally, it would be very interesting to prove similar rainbow-type results for longer arithmetic progressions and larger numbers of colors. For example, it is not known whether every equinumerous 4-coloring of $[4n]$ contains a rainbow $AP(4)$. However, note that for every n and $k > 3$, there exists a k-coloring of $[n]$ with no rainbow $AP(k)$ and with each color class of size at least $\lfloor \frac{n+2}{3\lfloor (k+4)/3 \rfloor} \rfloor$ [JLMNR].

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