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EUROCOMB'03 — Abstracts

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EUROCOMB'03 — Abstracts
J. Fiala (editor)

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The picture on the cover is one of the G-modules by J. Načeradský and J. Nešetřil.

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Preface

This volume contains extended abstracts of all talks presented at EUROCOMB'03 to be held in Prague, Septemebr 8–12. These contributions were selected by the Programme Commitee of EUROCOMB'03:

Martin Aigner (Berlin)
Noga Alon (Tel Aviv)
Peter Cameron (London)
Bruno Codenotti (Pisa)
Pavol Hell (Vancouver)
Gyula O. H. Katona (Budapest)
Jiří Matoušek (Prague)
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André Raspaud (Bordeaux),
Vojtěch Rödl (Atlanta)
Oriol Serra (Barcelona),
Carsten Thomassen (Copenhagen)
Günter M. Ziegler (Berlin)

In addition this volume contains 2 contributions accepted additionally (by Nebeský, Sapozhenko) which replaced contributions which had to be withdrawn. The final versions of most of these papers will appear elsewhere (and we hope that some of them in the special issues of Discrete Mathematics and European Journal of Combinatorics which will be related to the EUROCOMB meeting). In addition this volume also contains abstracts of the invited talks which were known at the moment when this (sizable) volume had to go to the print.

I thank to the all members of the PC commitee and to all our reviewers for a very good work. I also thank to Jiří Fiala who technically administered this volume. The publication of this volume is made possible by the support of European network Combstru and by the Institute for Theoretical Computer Science (ITI) supported by the Ministry of Education of the Czech Republic as project LN00A056.

Jaroslav Nešetřil

List of abstracts

Invited talks

Contributed talks

Invited talks

[1]Besides the abstracts included in this booklet the following ivited lectures are going to be presented at EuroComb'03: Béla Bollobás: TBA, Gil Kalai: Topological Helly Type Theorems, Vera T. Sós: TBA.

Planar Graphs, Parallel Computing, and the Traveling Salesman Problem

William Cook
Georgia Institute of Technology

Abstract

The traveling salesman problem, or TSP for short, asks for the minimum-weight Hamiltonian cycle in a edge-weighted complete graph. The most successful approach known for solving instances of the TSP is the linear-programming based method of Dantzig, Fulkerson, and Johnson from the early 1950s. In this talk we discuss some recent work on extending this approach to larger TSP instances. Following the work of Fleischer and Tardos (1999) and Letchford (2000), we discuss computational techniques for exploiting planarity in improving the linear-programming relaxations. We also discuss how parallel decomposition techniques can be used to extend the Dantzig et al. algorithm to obtain lower bounds for very large instances, including computational tests on a TSP instance having over 500 million vertices. Finally, we present an application to genome sequencing and discuss the role of exact TSP solution methods in this context.

Algebraic connectivity — 30 years after

Miroslav Fiedler
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Abstract

In 1973, the author introduced and studied (Czechoslovak Math. J. 23 (98), 298–305) the notion of algebraic connectivity of a finite undirected weighted graph as the second smallest eigenvalue of the corresponding weighted Laplacian matrix of the graph. Basic properties will be recalled, some applications presented and older as well as newer developments discussed.

On the Structure of Matrices and Matroids

Bert Gerards
CWI Amsterdam and Eindhoven University of Technology

Abstract

Does the class of matrices over a fixed finite field have structure? The class of graphs does; this is the Robertson-Seymour graph structure theorem, which says that for each graph there exists a surface such that all graphs that do not have this graph as a "minor" embed more or less ("more or less" in well specified way) in that surface. Now the notion of minor is geometric rather than graphic and actually the conjecture is that a similar structure theorem holds for matrices over finite fields. It is anticipated that a consequence of such a result is that a minor closed property for binary, ternary, etc. matrices is characterized by a finite list of forbidden minors. Also Rota's conjecture - that representability of a matroid as a matrix over a particular finite field has finitely many forbidden minors - could well follow from such theory.

This talk is based on joint work with Jim Geelen and Geoff Whittle.

Combinatorics and Telecommunication

Martin Grötschel
Konrad-Zuse-Zentrum für Informationstechnik
and Technische Universität Berlin, Germany

Abstract

This talk will begin with a survey of mathematical challenges that arise in telecommunication. Mathematics, and here in particular combinatorial optimization, is involved, e. g., in the design and manufacturing of chips, devices, and network components, the choice of locations, the planning of network topologies, and the dimensioning of the equipment involved. Adequate cryptography, the need of fast data processing, demand routing and failure handling require efficient and reliable mathematical algorithms on the operational side.

The presentation will address, in more detail, the problem of designing low-cost telecommunication networks that provide sufficient capacity to serve a given demand, are based on a chosen technology mix, satisfy various technical side constraints, and survive certain failure situations. This problem is difficult in theory and practice. It will be indicated how algorithms integrating polyhedral combinatorics, linear and integer programming, and various heuristic ideas can help solve real-world instances within reasonable quality guarantees in acceptable running times.

I will also offer a brief overview of new challenges that arise in UMTS, the third generation of mobile telephony.

This talk is based on work of the telecommunications research group at ZIB, the examples discussed and the computational results reported are from joint projects with several telecommunication companies.

Phase transition phenomena in random discrete structures

Tomasz Łuczak
Adam Mickiewicz University, Poznañ

Abstract

It has been known for a long time that in a "critical region" properties of a system can be very sensitive for a miniscule change of its parameters. Numerous phenomena of this kind have been described and thoroughly studied for various probabilistic models of discrete random structures. In the talk we survey some recent results on the behaviour of random structures in the critical region.

LIAR!

Joel Spencer
Courant Institute

Abstract

Paul is trying to ascertain an unknown x, out of n possibilities, from an adversary Carole by asking a series of q queries. In the standard "Twenty Questions" Paul wins if and only if $n \leq 2^q$. In Liar Games, Carole is allowed, under certain restrictions, to give an incorrect response, a lie. She may do this at most k times. In this talk we examine asymptotics for k fixed and $q \to \infty$.

The basic liar game, in which Paul can make arbitrary binary queries, has been well studied. The basic bound is that if $n \sum_{i \leq k} \binom{q}{i} > 2^q$ then Carole will win. The converse almost (but not quite!) holds.

There is a natural connection between Liar Games and Coding Theory. The protocol must allow Paul to determine x despite (at most) k "errors" in the "transmission." The major distinction is that Paul's queries are sequential and can depend on previous responses.

In the Half-Lie Game Carole has a further restriction. If the correct answer is yes then Carole *must* say yes. We show that this increases the maximal n for which Paul wins by a factor of 2^k.

The Half-Lie Game has two useful interpretations. The state of a game may be given by a vector $\vec{x} = (x_0, \ldots, x_k)$ where x_i is the number of possibilities for which Carole has already lied i times. Linear algebra methods yield natural weight functions. More combinatorially: we examine the set of possible response sequences (in $\{Y, N\}^q$) for Carole with a particular answer. For example, if $w_1 \cdots w_q$ is her "honest" response and $w_i = N$ then there is a child sequence identical on the first $i - 1$ coordinates but with i-th coordinate Y. This yields combinatorial structures (subsets of $\{Y, N\}^q$) called k-sets and the problem becomes one of packing k-sets.

In Coding Theory language the Half-Lie game is played on the Z-channel. The analysis readily extends to arbitrary channels. For example: let Paul ask ternary queries with answers A, B, C. We allow Carole to "lie" with B when the answer is A and with C when the answer is B, but otherwise she cannot lie. For any fixed channel and fixed k we find (asymptotically) the largest n for which Paul will win.

Joint work with Ioana Dumitriu and Catherine Yan.

The Strong Perfect Graph Theorem

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Abstract

A graph is perfect if for every induced subgraph, the chromatic number is equal to the maximum size of a complete subgraph. The Strong Perfect Graph Conjecture (SPGC) of Berge from 1960 asserts that a graph is perfect if and only if it has no induced subgraph isomorphic to an odd cycle of length at least five, or the complement of such a cycle.

The class of perfect graphs is important for several reasons. For instance, many problems of interest in practice that are intractable in general can be solved efficiently when restricted to the class of perfect graphs. Also, the question of when a certain class of linear programs always have an integer solution can be answered in terms of the perfection of an associated graph.

I will explain why perfect graphs are interesting, and then will discuss a proof of the SPGC, obtained in joint work with M. Chudnovsky, N. Robertson and P. Seymour.

Contributed talks

An efficient algorithm to find tight families of additive bases*

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Abstract

An *additive bases* of length Δ for $N \in \mathbb{N}$ is a set of integers

$$A_{\Delta,N} = \{1 \leq s_1 < s_2 < ... < s_\Delta\}$$

which covers $\mathbb{Z}_N$, that is, $\forall n \in \{0, 1, ..., N-1\}$, $\exists \alpha_{1,n}, ..., \alpha_{\Delta,n} \in \mathbb{N}$ and $\lambda_n \in \mathbb{Z}$ such that

$$\sum_{k=1}^{\Delta} \alpha_{k,n} s_k + \lambda_n N = n.$$

The *order* of $A_{\Delta,N}$ is defined by

$$h(A_{\Delta,N}) = \max_{0 \leq n \leq N-1} \min_{\substack{\alpha_{1,n},...,\alpha_{\Delta,n} \in \mathbb{N} \\ \sum_{k=1}^{\Delta} \alpha_{k,n} s_k + \lambda_n N = n}} \alpha_{1,n} + \cdots + \alpha_{\Delta,n}.$$

A sufficient condition on $A_{\Delta,N}$ to cover $\mathbb{Z}_N$ is known to be $\gcd(N, s_1, ..., s_\Delta) = 1$. When $\Delta = 2$, a sharp lower bound is known for the order

$$h(A_{2,N}) \geq \mathrm{lb}(N) = \left\lceil \sqrt{3N} \right\rceil - 2.$$

We say that $A_{2,N}$ is *tight* if $h(A_{2,N}) = \mathrm{lb}(N)$.

In this work an efficient symbolic algorithm on the following problem is proposed: Given $N_0 \in \mathbb{N}$, find if possible an infinite family of tight additive bases $A_{2,N(x)} = \{s_1(x), s_2(x)\}$ for integral $x \geq x_0$ with $N(x_0) = N_0$.

The algorithm does not solve the problem for all N_0, however for $N_0 \leq 300000$ it have found a solution in 99.8% of times. Its time cost is $O(\sqrt{N_0} \log N_0)$, in the worst case. Moreover, when it fails to find a solution it is possible to generate an infinite family of tight additive bases without the condition $N(x_0) = N_0$ for some x_0.

*Work supported by the Ministry of Science and Technology of Spain, the European Regional Development Fund (ERDF) under project TIC2001-2171 and by the Catalan Research Council under project 2000SGR00079.

1 Graph theory point of view

The problem of computing the number $h(A_{2,N})$ and finding the elements of $A_{2,N}$, as stated in the abstract, can be studied from the point of view of Graph Theory through the *double-loop* digraphs.

A *double-loop* digraph $G = G(N; s_1, s_2)$, with $1 \le s_1 < s_2 < N$ and $\gcd(N, s_1, s_2) = 1$, have the vertex set $V = \mathbb{Z}_N$ and the adjacencies are defined by $v \to v + s_i \pmod N$ for $v \in V$ and $i = 1, 2$. The diameter of G is denoted by $D(N; s_1, s_2)$ and we define $D(N) = \min_{\substack{1 \le s_1 < s_2 < N,\\ \gcd(N,s_1,s_2)=1}} D(N; s_1, s_2)$. It has been proved that $D(N) \ge \mathrm{lb}(N) = \left\lceil \sqrt{3N} \right\rceil - 2$ and the sharpness of the lower bound $\mathrm{lb}(N)$. A double-loop $G(N; s_1, s_2)$ is called *tight* if $D(N; s_1, s_2) = \mathrm{lb}(N)$.

There is an equivalence between a double-loop $G(N; s_1, s_2)$ and the bases $A_{2,N} = \{s_1, s_2\}$. We have $D(N; s_1, s_2) = h(A_{2,N})$. A tight double-loop $G(N; s_1, s_2)$ gives a tight additive bases $A_{2,N} = \{s_1, s_2\}$. So we can work in the set of double-loop digraphs.

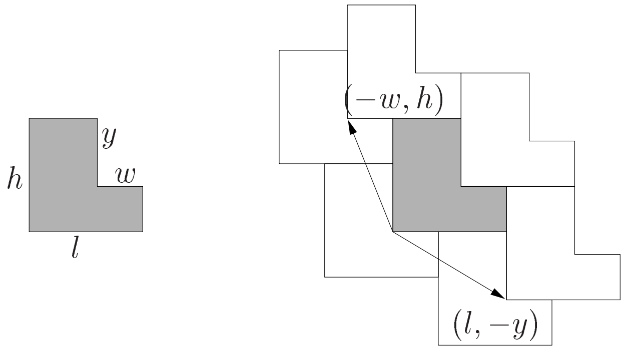

Figure 1: Generic dimensions of an L-shaped tile and its related tessellation.

Each double-loop $G(N; s_1, s_2)$ is related to a plane L-shaped tile of area N which fully describes its metrical properties and periodically tessellates the plane. See [5, 4, 3] for details. Figure 1 describes how we denote the dimensions of a generic L-shaped tile and how the resulting tilling of the plane can be fully described from the integral matrix $M = \begin{pmatrix} l & -w \\ -y & h \end{pmatrix}$, whose entries are the (column) vectors $\boldsymbol{u} = (l, -y)^\top$ and $\boldsymbol{v} = (-w, h)^\top$. In particular, the computation of the diameter of G can be done from the dimensions of $L(l, h, w, y)$ by $d(L) = d(L(l, h, w, y)) = \max\{l + h - w - 2, l + h - y - 2\}$.

For obvious reasons, the value $d(L)$ is called the diameter of the tile L. Also from a given *tight* tile $L(l, h, w, y)$ of area N, with $\gcd(l, h, w, y) = 1$, we can recover a double-loop $G(N; s_1, s_2)$ with diameter $D(N; s_1, s_2) = d(L) = \mathrm{lb}(N)$. The computation of s_1 and s_2 from the matrix M is described in the following proposition which is proved in [4].

Proposition 1 *Let G be a double-loop with linked tile $L = L(l, h, w, y)$. Let M be the matrix defining the tilling related to L, with $\gcd(l, h, w, y) = 1$ and area $N = lh - wy$. Let $S(M) =$diag$(1, N)$ be the Smith normal form of M, with related unimodular matrices*

U and V such that $S(M) = UMV$. Then the pair of steps $s_1 \equiv U_{2,1} \pmod N$, $s_2 \equiv U_{2,2}$ $\pmod N$ defines $G'(N; s_1, s_2)$ which is isomorphic to the original digraph G.

2 Tight tiles

In this section we provide some known results on tight tiles which will be used later in the algorithm (for the details of the proof see [4].) More precisely, a complete parameterization of tight tiles, the *Tight Tiles Table* (Table 1,) will be used in this work. For symmetry reasons, we can suppose $0 \leq w \leq y$ in the dimensions of a tight tile because if $G(N; s_1, s_2)$ is related to (l, h, w, y), then $G(N; s_2, s_1)$ is also related to (h, l, y, w).

Theorem 1 *Let $L(l, h, w, y)$ be a tight tile with $0 \leq w \leq y$. Then $y - 2 \leq w \leq y$ except for the tiles $L(3, 3, 0, 3)$ and $L(2, 3, 0, 3)$.*

This theorem let us to find the characterization of tight tiles using the three cases $y = w$, $y = w + 1$ and $y = w + 2$ which correspond to the second entry of the nine families in the Tight Tile Table, that is the family $\mathcal{T}[*, j]$ corresponds to tiles with $y = w + 3 - j$ for $j = 1, 2, 3$. This characterization is given by the following theorem, provided the parameterization of natural numbers $\mathbb{N} = \bigcup_{x \geq 0} I(x)$, where $I(x) = [3x^2 + 1, 3(x + 1)^2]$.

Theorem 2 *Let $L = (l, h, w, y)$, $y - 2 \leq w \leq y$, be a tight tile with area $N = 3x^2 + Ax + B$. Then, the values of A, B, w, y are given by Table 1 with $l = 2x + a$ and $h = 2x + b$.*

Given any N_0, the existence of a tight $G(N_0; s_1, s_2)$ is equivalent to the existence of a tight L-shaped tile $L(l, h, w, y)$ with area N_0, $w \leq y$ and $\gcd(l, h, w, y) = 1$. This tile will be detected in Table 1. Also all the tight tiles of area N_0 in this table can be found in time cost $O(\sqrt{N_0})$, in the worst case. See [2] or [1] for a detailed proof. Now we give the idea of *procreation* which will be used later to obtain infinite families of tight tiles in the symbolic algorithm.

Definition 1 *Let $L(l, h, w, y)$ be a tight tile. We say that L procreates or it is a procreating tile if the tiles $L(t) = L(l + 2t, h + 2t, w + t, y + t)$ are also tight for $t = 0, 1, 2, \ldots$*

Theorem 3 *Let $L = L(l, h, w, y)$, $0 \leq w \leq y$, be a tight tile different from $(3, 3, 0, 3)$ and $(2, 3, 0, 3)$. Then L procreates.*

3 Algorithm

The algorithm for finding tight families of additive bases can be stated in terms of tight double-loop digraphs. More precisely, we propose the following two problems:

P1 Given $N_0 \in \mathbb{N}$, find if possible a family of integral values $N(x)$ for $x \geq x_0$ and $N(x_0) = N_0$.

Family	$N(x)$	w	y
Tiles with area in $I_1(x) = [3x^2+1, 3x^2+2x]$ and diameter $d(L) = 3x-1$			
$\mathcal{T}[1,1]$	$3x^2+1,\ x \geq 1$ $a = b = 0$	 $x-1$	 $x+1$
$\mathcal{T}[1,2]$	$3x^2+x+B_{1,2}$ $B_{1,2} = ab-(a+b-1)(a+b)$ $x \geq \max\{1-B_{1,2}, B_{1,2}\}$	$x+a+b-1$	$x+a+b$
$\mathcal{T}[1,3]$	$3x^2+2x+B_{1,3}$ $B_{1,3} = ab-(a+b-1)(a+b-1)$ $B_{1,3} \leq 0,\ x \geq \lceil (1-B_{1,3})/2 \rceil$	$x+a+b-1$	$x+a+b-1$
Tiles with area in $I_2(x) = [3x^2+2x+1, 3x^2+4x+1]$ and diameter $d(L) = 3x$			
$\mathcal{T}[2,1]$	$3x^2+2x+1,\ x \geq 1$ $a=1, b=0$ $a=0, b=1$ $a=1, b=1$	 $x-1$ $x-1$ x	 $x+1$ $x+1$ $x+2$
$\mathcal{T}[2,2]$	$3x^2+3x+B_{2,2}$ $B_{2,2} = ab-(a+b-2)(a+b-1)$ $x \geq \lvert 1-B_{2,2} \rvert$	$x+a+b-2$	$x+a+b-1$
$\mathcal{T}[2,3]$	$3x^2+4x+B_{2,3}$ $B_{2,3} = ab-(a+b-2)(a+b-2)$ $B_{2,3} \leq 1,\ x \geq \lceil (1-B_{2,3})/2 \rceil$	$x+a+b-2$	$x+a+b-2$
Tiles with area in $I_3(x) = [3x^2+4x+2, 3x^2+6x+3]$ and diameter $d(L) = 3x+1$			
$\mathcal{T}[3,1]$	$3x^2+4x+2,\ x \geq 1$ $a=1, b=1$ $a=2, b=1$ $a=1, b=2$	 $x-1$ x x	 $x+1$ $x+2$ $x+2$
$\mathcal{T}[3,2]$	$3x^2+5x+B_{3,2}$ $B_{3,2} = ab-(a+b-3)(a+b-2)$ $x \geq \max\{2-B_{3,2}, B_{3,2}-3\}$	$x+a+b-3$	$x+a+b-2$
$\mathcal{T}[3,3]$	$3x^2+6x+B_{3,3}$ $B_{3,3} = ab-(a+b-3)(a+b-3)$ $B_{3,3} \leq 3,\ x \geq \lceil (2-B_{3,3})/2 \rceil$	$x+a+b-3$	$x+a+b-3$

Table 1: Tight Tiles different from $(3,3,0,3)$ and $(2,3,0,3)$.

P2 Given $N_0 \in \mathbb{N}$, find if possible a tight family $G(x) = G(N(x); s_1(x), s_2(x))$ for $x \geq x_0$ and $N(x_0) = N_0$.

An $O(\sqrt{N_0}\log N_0)$ algorithm, in the worst case, for **P1** is given in [1]. The explicit computation of $s_1(x)$ and $s_2(x)$ makes problem **P2** to be more difficult than problem **P1**. Here we propose a partial solution of **P2** in the sense of the following result.

Theorem 4 (Algorithm for modified P2 problem) *Let be $N_0 \in \mathbb{N}$. We can solve the **P2** problem for N_0 when there is a tight tile $L = L(l_0, h_0, w_0, y_0)$, $0 \leq w_0 \leq y_0$, with area N_0 in the case of $\gcd(l_0, w_0) = 1$ or $\gcd(y_0, h_0) = 1$ or $\gcd(l_0, y_0) = 1$ or $\gcd(w_0, h_0) = 1$. The time cost of computing an explicit solution is $O(\sqrt{N_0}\log N_0)$, in the worst case.*

We can find all tight tiles of area N_0 with time cost $O(\sqrt{N_0})$ as it is proved in [2]. Hence we can find a tight tile of area N_0 and $\gcd(l_0, w_0) = 1$ or $\gcd(y_0, h_0) = 1$ or $\gcd(l_0, y_0) = 1$

or $\gcd(w_0, h_0) = 1$, in time cost $O(\sqrt{N_0}\log N_0)$, in the worst case, if it exists.

By the idea of procreation, the expression of $N(x)$ is found in constant time. Also we use the Tight Tile Table and the Smith normal form to compute the explicit expression of $s_1(x)$ and $s_2(x)$ in constant time, provided that the additional condition of the modified **P2** problem is fulfilled.

A computer search for $N_0 \leq 300000$ shows that this additional condition is fulfilled in 99.8% of times. When this additional condition is not fulfilled and there is a tight tile $L(l_0, , h_0, w_0, y_0)$ with area N_0 and $\gcd(l_0, h_0, w_0, y_0) = 1$, the algorithm can give a tight family of digraphs $G(x)$, however the condition $N(x_0) = N_0$ is not satisfied. In this case, the value N_0 acts like a seed for generating this family of tight digraphs.

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Constructing d-regular graphs whose subgraphs tend to be acyclic

Noga Alon * Eitan Bachmat †

1 Introduction

The moment of the appearance of the first cyclic component in an evolving random graph has been studied extensively in [3]. It is known that the first cyclic component appears on average when the graph has approximately 0.44N edges where N is the number of vertices, however, this random variable has huge variance and the probability of containing a cycle when there are εN edges is bounded away from zero for all $\varepsilon > 0$.

Random graphs may be viewed as random subgraphs of the family of complete graphs. In this paper we consider the appearance of the first cycle in the evolution of random subgraphs of a family of d regular graphs where d is fixed. In particular we are interested in constructing graphs (or families of graphs) in which the appearance of the first cycle is postponed as much as possible.

This problem is motivated by the problem of designing efficient configurations for mirrored storage systems. The details will appear in the full version of the paper.

Let G be a graph. Our model for picking a random edge subset A will be the standard random subgraph model in which each edge of G is chosen independently with some fixed probability ρ. We let $G(\rho)$ denote the probability space on the subgraphs of G induced by this model.

Consider a family $\bar{G} = (G_n)$ of d regular graphs. Consider the random variable $\chi_{n,\rho} : G_n(\rho) \to \{0, 1\}$ whose value on A is one iff the subgraph of G induced by the edge set A is acyclic.

We define the *Exclusive bandwidth* of the family $\bar{G}$ as
$EB(\bar{G}) = Sup\left\{\frac{1}{2}d\rho \mid LimSup_{n\to\infty}E(\chi_{n,\rho}) = 1\right\}$. Note that $\frac{1}{2}d\rho|V(G_n)|$ is the expected number of edges in a subgraph in $G_n(\rho)$ therefore $EB(\bar{G})$ represents the maximum possible asymptotic ratio of edges to vertices when the first cycle is expected not to occur with high probability.

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It is easy to show, using standard results about branching processes, that for any family of d-regular graphs $\bar{G}$ we have $EB(\bar{G}) \leq \frac{d}{2(d-1)}$.

In section 2 we show that the families of graphs of number theoretic origin constructed by Lubotzky, Phillips and Sarnak [5] achieve this upper bound.

In section 3 we show that by deleting a few edges (to eliminate small cycles) from a random d regular graph we obtain with high probability almost regular graphs which asymptotically achieve the upper bound as well.

1.1 Some definitions and Notation

A *cycle* in G is a connected 2 regular subgraph of G. The length of a cycle, denoted $l(C)$, is the number of edges in it. A (self avoiding) *walk* W in G from vertex v to vertex u is a sequence of vertices $v = v_0, ..., u = v_m$ in G such that $(v_i, v_{i+1}) \in E(G)$ and $v_i \neq v_{i+2}$ for all $i = 0, ..., m-1$. The length $l(W)$ of such a walk is m. We note that if C is a cycle and v is a vertex of C then C induces two walks from v to itself of length $l(C)$ by going around the cycle in either direction, hence the number of walks from v to itself of length k is at least twice the number of cycles of length k containing v. The girth of a graph, denoted $g(G)$, is the minimal cycle length in G.

2 The exclusive bandwidth of number theoretic graphs

2.1 The LPS construction

Let p, q be distinct primes satisfying $p, q = 1(Mod\ 4)$. In [5] Lubotzky, Philips and Sarnak constructed $d = (p+1)$-regular graphs $X_{p,q}$ some of whose properties are summarized in the following theorem

Theorem 1 *([5]) $X_{p,q}$ has the following properties*

A) $|V(X_{p,q})| \leq q^3$

B) $g(X_{p,q}) \geq \frac{2}{3}\log_p(|V(X_{p,q})|)$.

C) Let W_v be the number of walks of length k that begin and end with a given vertex v. Let $r_{p,q}(k)$ be the number of integer solutions of the equation

$$(2) \quad x_0^2 + 4q^2(x_1^2 + x_2^2 + x_3^2) = p^k$$

then $W_v = (1/2)(r_{p,q}(k) - r_{p,q}(k-2))$ independently of v.

We have the following theorem on the exclusive bandwidth of an LPS family of graphs.

Theorem 2 *Fix a prime $p = 1\ Mod\ 4$, let q_n be an increasing sequence of primes all equal to 1 modulo 4, and let $\bar{G}$ be the family of graphs $G_n = X_{p,q_n}$ then*

$$EB(\bar{G}) = \frac{d}{2d-2}$$

Proof. Let $c_{k,n}$ denote the number of cycles in G_n of size k. Consider $r_{k,n} = r_{p,q_n}(k)$ the number of integer solutions to the equation $p^k = x_0^2 + 4q_n^2(x_1^2 + x^2 + x_3^2)$.

By theorem 1 parts a and c we have $c_{k,n} \le r_{k,n}|V(G_n)| \le r_{k,n}q_n^3$. Any x_0 in a solution to equation (2) must satisfy

$$(3) \quad x_0^2 = p^k \ Mod \ q_n^2$$

It is well known that the multiplicative group of invertible elements $(\mathbf{Z}/q_n^2\mathbf{Z})^*$ is cyclic hence this equation has at most two solutions. As a result x_0 has to be in the union of two arithmetic progressions, $c + jq_n^2$ and $d + jq_n^2$, where c and d denote solutions to (3). We also have $|x_0^2| \le p^k$, hence $|x_0| \le p^{k/2}$. There are thus at most $4p^{k/2}/q_n^2$ possible values for x_0. Given one of the x_0 as above and assuming that $4|(p^k - x_0^2)$ we consider $m = (p^k - x_0^2)/(4q_n^2)$. Obviously $|m| \le p^k/q_n^2$. We also have $x_1^2 \le m$ so given a fixed x_0 there are at most $2\sqrt{m} \le 2p^{k/2}/q_n$ possible choices for x_1. Having chosen x_1 as well we need to consider all solutions of the equation $x_2^2 + x_3^2 = m - x_1^2 \le p^k$. By theorem 338 of [4] for any $\varepsilon > 0$ there is a constant A_ε such that there are at most $A_\varepsilon p^{\varepsilon k}$ solutions to this equation. We thus have

$$(4) \quad r_{p,q_n}(k) \le (4p^{k/2}q_n^{-2})(2p^{k/2}q_n^{-1})(A_\varepsilon p^{\varepsilon k}) = B_\varepsilon p^{(1+\varepsilon)k}/q_n^3$$

solutions to equation (2), where B_ε is a constant which depends on ε. We deduce that $c_{k,n} \le B_\varepsilon p^{(1+\varepsilon)k}$. If edges of the graph G_n are chosen with probability ρ then the probability of a given cycle of length k is ρ^k. Let $\rho = p^{-(1+\delta)} = (d-1)^{-(1+\delta)}$ for some $\delta > 0$ and let ε be such that $\delta > \varepsilon > 0$. Let $C_{n,\rho}$ be the random variable which counts the number of cycles in a graph in the $G_n(\rho)$ model. By the first moment method $E(C_{n,\rho})$ is an upper bound on the probability that a graph in the $G_n(\rho)$ model contains a cycle. Let $V_n = |V(X_{p,q_n})|$, We estimate as follows

$$E(C_{n,\rho}) = \sum_{k=3}^{\infty} c_{k,n}\rho^k = \sum_{k=(2/3)log_p(|V_n|)}^{\infty} c_{k,n}\rho^k \le$$
$$\sum_{k=(2/3)log_p(V_n)}^{\infty} B_\varepsilon(p^{\varepsilon-\delta})^k = O(V_n^{(2/3)(\varepsilon-\delta)})$$

for all $\varepsilon > 0$ proving the theorem. If $(\frac{p}{q_n}) = -1$ the same argument yields the estimate $O(V_n^{(4/3)(\varepsilon-\delta)})$, as it is known that in this case the girth is at least $\frac{4}{3}\log_p V_n$. ∎

3 Families of random graphs

We showed that the families of LPS graphs have asymptotically optimal exclusive bandwidth, however these families exist only for a sparse set of values of d and $|V|$. In some applications one would like to construct graphs whose subgraphs tend to be acyclic for all d and $|V|$.

Looking at the proof of theorem 2 we notice that there are two properties which allowed us to prove optimality. The first is high girth and the second is that for any k there are at most $O((d-1)^{k(1+\varepsilon)})$ cycles of size k in the graph. Random d regular graphs have the second property but not the first. One can construct efficiently "Pseudo Random" d

regular graphs with high girth of given size n, see [2] for example, however it seems difficult to prove that they retain the second property. The solution is to consider random d regular graphs on N labeled vertices using the well studied configuration model (see, e.g. [1], [6]) and to eliminate the few small cycles they contain by deleting a few edges.

Motivated by the discussion above we describe a simple, efficient, randomized procedure to generate, for every fixed integer d, and fixed real $\epsilon > 0$, and for any large integer n, a graph G on n vertices, with maximum degree d and with $nd/2 - O(1)$ edges, such that a random subgraph of G obtained by keeping each edge of G, randomly and independently, with probability $\frac{1-\epsilon}{d-1}$ is acyclic with high probability.

The precise statement of the result is the following.

Theorem 3 *Let d be a fixed integer, and let $\epsilon > 0$, $\delta > 0$ be positive reals. Define $k_0 = k_0(\epsilon, \delta) = \lceil \frac{1}{\epsilon} \ln(\frac{1}{\delta}) \rceil$ and $K = K(d, \epsilon, \delta) = 8d^{k_0+1}$. Then, for every integer $n > K$, there exists a graph G on n vertices, with maximum degree d and with at least $\frac{nd}{2} - K$ edges, such that a random subgraph of G obtained by keeping each edge of G, randomly and independently, with probability $\frac{1-\epsilon}{d-1}$, is acyclic with probability at least $1-\delta$. Moreover, there is a linear time randomized algorithm that produces a graph that has this property with probability at least $1/4$.*

Remark: The constants above can be improved, we make no attempt to optimize them. In the theorem above, both ϵ and δ may be functions of n. It is also easy to replace the success probability of the randomized algorithm above by any constant smaller than 1, by changing the definition of k_0.

Proof. We may and will assume that n is even, since for odd n we can take the construction for $n+1$ and omit a vertex. We further assume that $d \geq 3$, as the result for $d < 3$ is trivial. Let $H = (V, E)$ be a random d-regular graph on n labeled vertices generated according to the configuration model described in [1], [6]. This is done by taking a uniform random perfect matching on the set $W = \{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq d\}$ and then by collapsing each group $V_i = \{v_{ij}, 1 \leq j \leq d\}$ to a vertex v_i. The number of configurations in this model, that is, the number of perfect matchings on W, is $(nd-1)!! = (nd-1)(nd-3)(nd-5)\cdots 1$. For every integer k satisfying $1 \leq k \leq n$, the number of these configurations leading to a graph in which the vertices $v_1, v_2, \ldots, v_k$ form a simple cycle of length k in this order is at most

$$2(\binom{d}{2})^k 2^{k-1}(nd - 2k - 1)!!$$

As there are $\frac{1}{2k} n(n-1)(n-2)\cdots(n-k+1)$ potential simple cycles of length n on the vertices $v_1, v_2, \ldots, v_n$, this implies that the expected number of simple cycles of length k in H is at most

$$c(n, d, k) = \frac{1}{2k} \frac{n(n-1)\cdots(n-k+1)(d(d-1))^k}{(nd-1)(nd-3)\cdots(nd-2k+1)}.$$

This number is at most $\frac{1}{2k}(d-1)^k$ for all $k \geq 3$, and at most $\frac{1}{2k}(d-1)^k(1 + O(\frac{1}{nd}))$ for $k = 1, 2$.

Let C_k denote the number of cycles of length k in H, and define $\rho = \frac{1-\epsilon}{d-1}$. By linearity of expectation, the expected number of cycles of length at most k_0 in H is at most $\sum_{k=1}^{k_0} c(n,d,k) < 2d^{k_0+1} = K/4$, where K is as defined in the statement of the theorem. Therefore, with probability at least 3/4, the number of such cycles is at most K.

Let $X = X(H)$ be the random variable defined as follows: $X = \sum_{k>k_0} C_k \rho^k$. Again, linearity of expectation gives that the expectation of X is at most

$$\sum_{k>k_0} \frac{1}{2k}(d-1)^k \rho^k < \frac{1}{2k_0} \sum_{k>k_0} (1-\epsilon)^k \leq \frac{\delta}{2}.$$

Therefore, with probability at least 1/2, the value of X is at most δ. It follows that with probability at least 1/4, H has at most K simple cycles of length at most k_0, and the random variable X computed at H is at most δ. Fix such a graph H, and let G be a graph obtained from H by omitting an arbitrarily chosen edge from each simple cycle of length at most k_0 in H. If G_ρ is a random subgraph of G obtained by picking each edge of G, randomly and independently, with probability ρ, then the expected number of simple cycles in G_ρ is at most $\sum_{k>k_0} C_k \rho^k = X(H) \leq \delta$. This is because, by construction, G has no cycles of length at most k_0. It follows that the probability that G_ρ is acyclic is at least $1-\delta$, as needed.

The randomized algorithm to generate G is simple: generate H, find all cycles of length at most k_0 in it using, for example, dynamic programming, and omit an edge from each of them. □

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The number of distinct values in a geometrically distributed sample

Margaret Archibald, Arnold Knopfmacher and Helmut Prodinger*

We consider words $x_1x_2\dots x_n$ with letters $x_i \in \{1,2,\dots\}$. The letter i occurs with (geometric) probability pq^{i-1} where $p+q=1$, and the letters are considered to be independent, so that $x_1x_2\dots x_n$ appears with probability $(p/q)^n q^{x_1+\cdots+x_n}$. The combinatorics of sets of such words has recently been studied with respect to various parameters, partially motivated by some computer science contexts; see [9, 10, 11, 12, 8, 13].

In this paper, we address the following natural question: How many different letters appear in words of length n, generated by geometric random variables? For this parameter, we derive expectation and variance and establish the Gaussian limiting distribution.

We then generalise this question as follows: How many letters appear at least b times, where $b \geq 1$ is a design parameter. Clearly, $b=1$ means the previous parameter.

As an example, we find that the average number of letters that appear at least b times in random words of length n, is

$$\mathbb{E}(d_n^{<b>}) \sim \log_Q n + \frac{\gamma}{L} + \log_Q(Q-1) - \frac{1}{2} - \frac{1}{L}H_{b-1} + \delta(\log_Q(n(Q-1)))$$

where $Q = q^{-1}$, $L = \log Q$ and $H_{b-1} = \sum_{k=1}^{b-1}\frac{1}{k}$ is the $(b-1)$-st harmonic number. Also $\delta(x)$ is a periodic function of period one, small amplitude and mean zero.

We then investigate how the expectation and variance depends on the parameter b. This leads to interesting questions as the series that appear belong to the realm of modular functions; various identities, such as the one for Dedekind's η–function, play a rôle here.

In the asymptotic formulæ that we derive, there appear ubiquitous periodic oscillations, due to poles of certain functions at $z = \chi_k := \frac{2k\pi i}{L},\ k \in \mathbb{Z},\ k \neq 0$. They are usually tiny, but play an essential role especially in the variance. This phenomenon occurred frequently when analyzing digital trees, see e. g., [6, 5].

Our analysis uses in a prominent way the so-called *Rice's method,* a technique from complex analysis which is adapted to the asymptotic evaluation of alternating sums, [2].

In the following we want to give a few details in order to describe the flavour of our investigations.

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The Distinct Value Problem. When looking at the number of distinct values in a word, we make use of exponential generating functions. Our function of interest is the probability generating function

$$F(z,u) := \prod_{i\geq 1} \bigl(1 + u(e^{zpq^{i-1}} - 1)\bigr),$$

whose coefficients (multiplied by n) are the probability that a word of length n has k distinct values (we look at the coefficients of z^n and u^k, where z counts the length of the word and u counts all values occurring at least once). If letter i occurs at least once, then this will be accommodated by the presence of the u in front of the expression $(e^{zpq^{i-1}} - 1)$ which represents all non-empty 'sets' of letter i which occur in the word. The problem of the letters appearing at different places in the word is overcome by the use of the exponential generating function.

Note that substituting $u = 1$ into this function gives e^z (since all probabilities sum to 1), which is to be expected because this reduces it to a generating function whose coefficients represent the probability that a word of length n has no restrictions.

Theorem

- *The expected value $\mathbb{E}(d_n)$ of distinct letters occuring in a word of length n is given by*

$$\mathbb{E}(d_n) \sim \log_Q n + \frac{\gamma}{L} + \log_Q(Q-1) - \frac{1}{2} + \delta_E(\log_Q(n(Q-1))),$$

where

$$\delta_E(x) := -\frac{1}{L}\sum_{k\neq 0}\Gamma(-\chi_k)e^{2k\pi i x}.$$

- *The variance of this quantity is*

$$\text{Variance} = \frac{\pi^2}{6L^2} - \frac{2\alpha}{L} + \frac{1}{12} + \delta_V(\log_Q(n(Q-1)))$$

where

$$\alpha = \sum_{k\geq 1}\frac{(-1)^k}{k}\frac{q^k}{1-q^k},$$

and $\delta_V(x)$ is a periodic function of period one, small amplitude and mean zero.

To find the expected value, we use the formula ([3])

$$\mathbb{E}(d_n) = n![z^n]\frac{\partial}{\partial u}F(z,u)\Big|_{u=1},$$

to get

$$\mathbb{E}(d_n) = n![z^n]\frac{\partial}{\partial u}\prod_{i\geq 0}\bigl(1 + u(e^{zpq^i} - 1)\bigr)\Big|_{u=1} = \sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{-p^k}{1-q^k}. \tag{1}$$

However, we cannot easily see how the number of distinct values grows from this form. To get a better idea we approximate this alternating sum by an integral using *Rice's method.* For this residues of the integrand are calculated (in this case at $z = 0$ and $z = \chi_k$ where $k \in \mathbb{Z}$ for all $k \neq 0$). To do this we often need to make use of the expansion of the quantity

$$[n; z] = \frac{(-1)^{n-1} n!}{z(z-1)\cdots(z-n)} \sim -\frac{1}{z}(1 + zH_n),$$

around $z = 0$. By consulting [1], we see that

$$[n; \chi_k] = \frac{\Gamma(-\chi_k)\Gamma(n+1)}{\Gamma(n+1-\chi_k)} \sim \Gamma(-\chi_k) n^{\chi_k}.$$

Both of these quantities are used to get the final result for the expected value. The variance is treated in a similar manner and we use the formula

$$\text{Variance } = n![z^n] \frac{\partial^2}{\partial u^2} F(z,u)\Big|_{u=1} + \mathbb{E}(d_n) - \mathbb{E}^2(d_n).$$

In this case the first term is

$$n![z^n] \frac{\partial^2}{\partial u^2} F(z,u)\Big|_{u=1} = 2 \sum_{j<k} \big(1 - (1-pq^j)^n - (1-pq^k)^n + (1 - pq^j - pq^k)^n\big),$$

which we split up (preserving convergence) as follows in order to be dealt with in two parts:

$$n![z^n] \frac{\partial^2}{\partial u^2} F(z,u)\Big|_{u=1} = 2 \sum_{j<k} \big[1 - (1-pq^k)^n\big] + 2 \sum_{j<k} \big[(1 - pq^j - pq^k)^n - (1-pq^j)^n\big].$$

The reason for this is that now the summand of the first sum is independent of j and can be dealt with separately from the second sum which requires a slightly different approach. (This technique is also used in the general case.) The separate results are lengthy, and thus are not included here.

It is interesting to note the extreme cases of α (see statement of theorem). As $q \to 0$, $\alpha \to 0$. If $q \to 1$, then by letting $q = e^t$, we must investigate $t \to 0$. Rewriting, we have the quantity

$$\sum_{k \geq 1} \frac{(-1)^k}{k} \frac{e^{-tk}}{1 - e^{-tk}} = -\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \frac{1}{e^{tk} - 1}.$$

This can be found in the appendix of [7], and by calling it $g(t)$ we get the following result from that paper, which makes use of Mellin transforms to get:

$$\alpha = g(t) = -\frac{\pi^2}{12t} + \frac{\log 2}{2} - \frac{t}{24} + g\Big(\frac{2\pi^2}{t}\Big).$$

This identity holds for $0 < t < 2\pi^2$. We are interested in what happens as $t \to 0$. Since

$$g(t) = -\sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \frac{1}{e^{tk} - 1},$$

it can be seen that $g(\frac{2\pi^2}{t}) \to 0$ as $t \to 0$, and thus the last term in the expression for α is small enough to be insignificant. The remaining three terms provide a very good approximation for α near $q = 1$ where $t = \log q$.

The Gaussian limiting distribution will be a consequence of Hwang's quasi-power theorem [4].

Distinct values occurring more than once. To consider all letters which occur at least twice, we change this generating function to

$$F_2(z, u) := \prod_{i \geq 0} \left(1 + zpq^i + u(e^{zpq^i} - 1 - zpq^i)\right),$$

by disallowing u from counting those letters which only appear once (corresponding here to the term zpq^i). This process can continue indefinitely using the infinite expansion of e^z. In general the generating function which counts distinct values which occur a minimum of b times is

$$F_b(z, u) := \prod_{i \geq 0} \left(\sum_{k=0}^{b-1} \frac{(zpq^i)^k}{k!} + u\left(e^{zpq^i} - \sum_{k=0}^{b-1} \frac{(zpq^i)^k}{k!}\right)\right).$$

In both cases (at least once and at least b times), we calculate the expected value (average) and variance by means of partial derivatives with respect to u, subsequently replacing u by 1 and examining the coefficient of z^n, in the resulting exponential generating functions.

At this point we usually have an alternating sum as a result of this differentiation. For example (1) gives the exact alternating sum formula for the expected value of the number of distinct values.

By applying Rice's method [2], these alternating sums are transformed into contour integrals as follows: Let C be a curve surrounding the points $1, 2, \ldots, n$ in the complex plane, and let $f(z)$ be analytic inside C. Then

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_C [n; z] f(z) dz,$$

where

$$[n; z] = \frac{(-1)^{n-1} n!}{z(z-1)\cdots(z-n)} = \frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)}.$$

By extending the contour of integration, it turns out that under suitable growth conditions the asymptotic expansion of such an alternating sum is given by

$$\sum \operatorname{Res}([n; z] f(z)) + \text{smaller order terms},$$

where the sum is taken over all poles different from $1, \ldots, n$.

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The functions over a finite field that determine few directions

Simeon Ball

Let f be a function from $GF(q)$ to $GF(q)$, let

$$D := \left\{ \frac{f(y) - f(x)}{y - x} \;\middle|\; x, y \in GF(q),\ x \neq y \right\}$$

be the set of directions determined by the function f and $N = |D|$. Rédei and Megyesi [7] proved that either $N = 1$, $N \geq (q+3)/2$ or $(q+s)/(s+1) \leq N \leq (q-1)/(s-1)$ for some $s = p^e \leq \sqrt{q}$. In 1995 Blokhuis, Brouwer and Szőnyi [3] simplified their proof and in 1999 together with myself and Storme [2] sharp bounds on N were proven in all cases except $s = 2$ and $s = 3$.

In this talk (and in the article [1]) I present a proof of the following theorem using Hasse derivatives, which also covers the unresolved cases.

Theorem 1 *Let f be a function from $GF(q)$ to $GF(q)$, $q = p^h$ for some prime p, and let N be the number of directions determined by f. Let $s = p^e$ be maximal such that any line with a direction determined by f that is incident with a point of the graph of f is incident with a multiple of s points of the graph of f. One of the following holds:*

(i) $s = 1$ *and* $(q+3)/2 \leq N \leq q+1$*;*

(ii) $GF(s)$ *is a subfield of* $GF(q)$ *and* $q/s + 1 \leq N \leq (q-1)/(s-1)$*;*

(iii) $s = q$ *and* $N = 1$*.*

Moreover if $s > 2$ then the graph of f is $GF(s)$-linear.

This theorem implies that any function that determines less than half the directions is a linearised polynomial. In other word it is of the form

$$f(x) = \sum_{i=0}^{n-1} a_i x^{s^i},$$

where $q = s^n$ and $a_i \in GF(q)$.

All the bounds in Theorem 1 can be attained. The function $f(X) = X^{(q+1)/2}$ determines $(q+3)/2$ directions when q is odd. The function $f(X) = Tr_{q \to s}(X) = X + X^s + X^{s^2} + \ldots +$

$X^{q/s}$ determines $q/s+1$ directions and the function $f(X)=X^s$ determines $(q-1)/(s-1)$ directions whenever $GF(s)$ is a subfield of $GF(q)$.

The following theorem is from Lovász and Schrijver [5].

Theorem 2 *Any function from $GF(q)$ to $GF(q)$ that determines $(q+3)/2$ directions when q is prime is affinely equivalent to $f(X)=X^{(q+1)/2}$.*

The following theorem is from Gács [4].

Theorem 3 *Any function from $GF(q)$ to $GF(q)$ that determines more than $(q+3)/2$ directions when q is prime determines at least $\lfloor 2(q-1)/3 \rfloor + 1$ directions.*

There are examples of Megyesi [7] that determine $q+1-d$ directions for every $d|q-1$ with $1<d<q-1$.

The following is due to Gács, Lovász and Szőnyi [4].

Theorem 4 *Let $q=p^2$ where p is prime. Any function from $GF(q)$ to $GF(q)$ that determines $(q+3)/2$ directions is affinely equivalent to $f(X)=X^{(q+1)/2}$. Moreover any function that determines more than $(q+3)/2$ directions determines at least $(q+p)/2+1$ directions.*

There are examples of Polverino, Szőnyi and Weiner [6] that determine $(q+\sqrt{q})/2+1$ directions when q is square.

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Constructions of Sparse Asymmetric Connectors

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1 Introduction

A major task in the design of communication networks is to establish sparse connections between n inputs and N outputs that allow all the inputs to send information to arbitrary distinct outputs simultaneously. In the usual graph model, the problem can be stated as follows.

> Given $n, N \in \mathbb{N}$ $(n \leq N)$, construct a digraph $G = (V, E)$, where $V = I \cup L \cup O$ is partitioned into input vertices, link vertices and output vertices such that
>
> – $|I| = n$, $|O| = N$,
> – for every injective mapping $f : I \to O$ there are vertex disjoint paths connecting i to $f(i)$ for all $i \in I$,
> – $|E|$ is small, or even minimum.

We call a digraph as above an (n, N)-*connector*. An (n, N, k)-*connector* (or (n, N)-connector of *depth* k) is an (n, N)-connector where any output can be reached by any input via a path of length k. Let $e(n, N, k)$ denote the minimum number of edges sufficient for building an (n, N, k)-connector.

Previous Work
The size of $e(n, N, k)$ in the symmetric case $N = n$ is well-studied. Pippenger and Yao [6] proved that $e(n, n, k) = \Omega(n^{1+1/k})$ and showed by a probabilistic argument that $e(n, n, k) = O(n^{1+1/k}(\log n)^{1/k})$. The best explicit construction of sparse symmetric connectors with odd depth is also due to Pippenger [5] who showed how to build $(n, n, 2j+1)$-connectors with $O(n^{1+1/(j+1)})$ edges. Richards and Hwang [3] gave a construction of an $(n, n, 2)$-connector with $O(n^{5/3})$ edges that can be extended by Pippenger's method [2] to yield $(n, n, 2j)$-connectors with $O(n^{1+2/(3j-1)})$ edges $(j \geq 2)$. Less results are known for asymmetric connectors. Oruç [4] was the first who devised (n, N)-connectors for arbitrary n and N. In particular, he gave constructions of $(n, N, \Omega(\log_2 N + \log_2 n))$-connectors and $(n, N, \Omega(\log_2 N + \log_2^2 n))$-connectors with $O((N + n)\log_2 n)$ and $O(N + n\log_2 n)$ edges,

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respectively, relying on recursive Clos networks [1] and concentrators as building blocks.

Our Results
We focus on (n, N, k)-connectors with $k = 2$. Note that for $k = 1$ the only solution is the complete bipartite graph $K_{n,N}$ (so-called crossbar). In practical applications (design of electronic switches) one typically has $n \le \sqrt{N}$. We will determine the size of an optimal (n, N)-connector of depth 2 for $n \le N^{1/2-\varepsilon}$ as $\Theta(N)$ and propose explicit constructions for $(n, N, 2)$-connectors with $O(Nn^{1/2} + N^{1/2}n^{3/2})$ edges.

2 Sparse $(n, N, 2)$-Connectors

We propose connectors with the following general structure: the outputs are connected to the link vertices via few edges, whereas the link vertices are completely connected to the input vertices.
This structure reduces the problem of finding vertex disjoint paths to that of finding a perfect matching between link and output vertices. Moreover, Hall's theorem provides a handy connector condition.

Proposition 1 (connector condition) *A digraph with vertices partitioned into sets I, L and O, $|I| = n$, $|O| = N$ is an $(n, N, 2)$-connector, if I and L are completely connected and $|\Gamma(S)| \ge |S|$ for all $S \subseteq O$ with $|S| \le n$.*

Let us assume that $\boxed{(1) \quad \deg(x) = d \text{ for all } x \in O,}$ where $d \in \mathbb{N}$ is constant.

3 An Optimal $(n, N, 2)$-connector

Theorem 1 *Let $n \le N^{\frac{1}{2}-\varepsilon}$ for some $0 < \varepsilon < 1/2$. An optimal (n, N)-connector of depth 2 has $\Theta(N)$ edges.*

Proof: The proof of the lower bound is trivial, since each of the N output vertices must at least have degree 1. For the proof of the upper bound let $|L| = \sqrt{N}$. For each output vertex pick at random d link vertices (we will determine d such that the construction is successful). The probability that Hall's condition is not satisfied can be bounded by $\sum_{i=1}^{n} \binom{N}{i}\binom{\sqrt{N}}{\sqrt{N}-i} \cdot (1 - \frac{\sqrt{N}-i}{\sqrt{N}})^{di} \le \sum_{i=1}^{n} (N\sqrt{N})^i \cdot (\frac{n}{\sqrt{N}})^{di} \le \sum_{i=1}^{n} N^{2i} \cdot (\frac{n}{\sqrt{N}})^{di}$. We have to ensure that the latter term is strictly smaller than 1, which is true for $N^2 \cdot (\frac{n}{\sqrt{N}})^d = \frac{1}{2}$ and thus for $d = \frac{\log(2N^2)}{\log\sqrt{N} - \log n} = \frac{2\log N + \log 2}{\log\sqrt{N} - \log n} \le \frac{2\log N + \log 2}{\frac{1}{2}\cdot\log N - (\frac{1}{2} - \varepsilon)\log N} = O(1)$. So, there is a connector with $N \cdot d + \sqrt{N} \cdot N^{\frac{1}{2}-\varepsilon} = O(N)$ edges. $\square$

4 Explicit Construction I: Diagonal Method

The connector condition suggests that the neighborhoods of the output vertices should be "almost distinct". We propose the following additional condition, meaning that two elements of O may share at most one neighbor, $\boxed{(2) \quad |\Gamma(x) \cap \Gamma(x')| \ \le \ 1 \quad \text{for all } x \neq x' \in O.}$ Let us prove that conditions (1) and (2) imply the connector condition, if d is large enough, i.e. $d \geq \sqrt{n}$.

Lemma 1 *The connector condition is true if (1), (2) and $d \ \geq \ \sqrt{n}$ hold.*

Proof: Let $1 \leq l \leq n$ and $S \subseteq O$ with $|S| = l$ be given.
For the sake of a contradiction we assume $|\Gamma(S)| < l$. There are ld edges between S and $\Gamma(S)$, so by the pigeon-hole principle there is a $y \in \Gamma(S) \subseteq L$ with $|\Gamma(y) \cap S| \geq d+1$. Because of (1) and (2) the neighbors of y in S are connected to $(d+1)(d-1)$ link vertices in addition to y, and hence, $n > |\Gamma(S)| \geq |\Gamma(\Gamma(y) \cap S)| \geq (d+1)(d-1)+1 = d^2$, contradicting $d \geq \sqrt{n}$. $\square$

The Diagonal Method

1. Let $O = \{0, \ldots, N-1\}$,
2. $d := \lceil \sqrt{n} \rceil$,
3. $q := \min\{q' \in \mathbb{N}_{\geq \sqrt{N}} \mid \text{ for all } p \in \{2, 3, \ldots, d-1\} : p \nmid q'\}$,
4. $L := \{0, \ldots, qd-1\}$ (we assume L and O to be disjoint, though!),
5. for $x \in O$ let $x = x_1 + q \cdot x_2$ be the q-ary decomposition of x ($x_1, x_2 \in \{0, \ldots, q-1\}$), choose $\Gamma(x) := \{(i-1)(1 + x_2 \cdot d) + x_1 \cdot d \pmod{qd} \mid i \in \{1, \ldots, d\}\}$.

Theorem 2 *The diagonal method yields connectors with $(1 + o(1))(Nn^{1/2} + 2N^{1/2}n^{3/2})$ edges.*

Proof: We have to show that condition (2) is satisfied. Consider $x, x' \in O$ with $|\Gamma(x) \cap \Gamma(x')| \geq 2$. By step 5 we have

$$\begin{aligned} (i_1 - 1)(1 + x_2 d) + x_1 d &\equiv (i_1 - 1)(1 + x_2' d) + x_1' d \pmod{qd} \\ (i_2 - 1)(1 + x_2 d) + x_1 d &\equiv (i_2 - 1)(1 + x_2' d) + x_1' d \pmod{qd} \end{aligned}$$

for some $i_1 \neq i_2 \in \{1, \ldots, d\}$. Consequently, $qd \mid (i_2 - i_1)(x_2 - x_2')d$ and thus $q \mid (x_2 - x_2')$, since $\gcd(q, i_2 - i_1) = 1$ by the choice of q. This implies $x = x'$. The claim follows by Lemma 1 using that $|E| = Nd + |L|n$ and the fact that $q \leq 2\sqrt{N}$ (by Bertrand's theorem there is prime between $\sqrt{N}$ and $2\sqrt{N}$). $\square$

Remark 1 *The diagonal method's name reflects the fact that the neighborhoods in step 5 can be obtained by successively taking diagonals of a matrix with d columns where the set $\{0, \ldots, qd-1\}$ is arranged into the first q rows, the diagonal starting at position $(1,1)$ is the $(q+1)$-st row, the diagonal starting at $(2,1)$ is the $(q+2)$-nd row, etc.*

5 Explicit Construction II: Copy Method

The following method assigns each element two neighbors in some small subset of L, and subsequently copies this neighborhood several times in order to satisfy the connector condition. For odd d an additional neighbor is added to the last copy.

The Copy Method

1. Let $O = \{0, \dots, N-1\}$,
2. $d := \lceil \sqrt{n} \rceil$,
3. $q := \left\lceil \sqrt{N} \right\rceil$,
4. $L := \{0, \dots, qd-1\}$ (still, we assume L and O to be disjoint!),
5. for $x \in O$ let $x = x_1 + q \cdot x_2$ be the q-ary decomposition of x ($x_1, x_2 \in \{0, \dots, q-1\}$) and choose $\Gamma(x) := \bigcup_{i \in \{0,\dots,\lfloor d/2-1 \rfloor\}} \Gamma_i(x)$, where

$$\Gamma_i(x) := \begin{cases} \{x_1, x_2+q, (x_1+x_2) \pmod q + 2q\} + \frac{d-3}{2} \cdot 2q, & \text{if } i = \frac{d-3}{2}, d \text{ odd}, \\ \{x_1, x_2+q\} + i \cdot 2q & \text{otherwise.} \end{cases}$$

To prove the connector condition we need the following lemma.

Lemma 2 *Let $l \in \mathbb{N}$, $S \subseteq \mathbb{N}_0$ with $|S| = l$.*
a) Let $f = (f_1, f_2) : S \to \mathbb{N}_0^2$ be injective, then $|f_1(S)| + |f_2(S)| \geq 2\sqrt{l}$.
b) If $f = (f_1, f_2, f_3) : S \to \mathbb{N}_0^3$ is injective, $|f_1(S)| + |f_2(S)| + |f_3(S)| \geq 3\sqrt{l}$.

Proof: a) By the pigeon-hole principle, there is an $x \in f_1(S)$ with $|f_1^{-1}(x)| \geq \frac{l}{|f_1(S)|}$. As f is injective, we have $|f_2(S)| \geq \frac{l}{|f_1(S)|}$, and thus $|f_1(S)| + |f_2(S)| = (\sqrt{|f_1(S)|} - \sqrt{|f_2(S)|})^2 + 2\sqrt{|f_1(S)| \cdot |f_2(S)|} \geq 2\sqrt{l}$.
b) follows from a), since $|f_1(S)| + |f_2(S)| + |f_3(S)| = \frac{1}{2}(|f_1(S)| + |f_2(S)| + |f_1(S)| + |f_3(S)| + |f_2(S)| + |f_3(S)|)$. □

Theorem 3 *The copy method yields connectors with $(1 + o(1))(Nn^{1/2} + N^{1/2}n^{3/2})$ edges.*

Proof: We have to show that the neighborhoods constructed by the copy method satisfy Hall's condition. Let $S \subseteq O$ with $|S| \leq n$. The function $f = (f_1, f_2) : S \to \mathbb{N}_0^2$, $x \mapsto (x_1, x_2)$, where $x_1 \cdot q + x_2$ is the q-ary decomposition of x, is injective. By construction of $\Gamma_0(x)$, $|\Gamma_0(S)| = |f_1(S)| + |f_2(S)| \geq 2\sqrt{|S|}$ (Lemma 2). For d even, we obtain $|\Gamma(S)| = \frac{d}{2}|\Gamma_0(S)| \geq \frac{d}{2} \cdot 2\sqrt{|S|} \geq |S|$. For d odd, we have $|\Gamma(S)| = \frac{d-3}{2}|\Gamma_0(S)| + |\Gamma_{\frac{d-3}{2}}(S)| \geq (d-3)\sqrt{|S|} + 3\sqrt{|S|} \geq |S|$. The number of edges can be estimated as in the proof of Theorem 2. □

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Dividing the edges of a graph into connected subgraphs of prescribed size

János Barát and Carsten Thomassen

In a classical paper [1] E. Győri proved the following.

Theorem 1 *The vertices of a k-connected graph can be partitioned into k classes of prescribed size such that each part spans a connected subgraph.*

Let us consider the analogous question for the edges. In the same paper Győri remarks that via line graphs roughly the same theorem holds for edge-partitions. We show that instead of connectivity k, much less is enough to be required.

Let a graph G and positive integers $m_1, m_2, \dots m_k$ be given such that $|E(G)| = m_1 + m_2 + \dots m_k$. We say that G has the edge-partition property if for any proper $m_1, m_2, \dots m_k$ there is a partition $E_1, E_2, \dots E_k$ of $E(G)$ such that $G|_{E_i}$ is connected for all i and $m_i = |E_i|$ for all i.

In the view of Theorem 1 our task is to decide what kind of connectivity assumption is necessary for G to have the edge-partition property.

If G has connectivity 1, then for most of the sequences $m_1, m_2, \dots m_k \geq 3$ there is a 'bad' G. In particular the following holds.

Lemma 1 *Let $|E(G)| = m_1 + m_2$, where $m_i \geq 3$. If $m_1 = 3$ and m_2 is even or $(m_1, m_2) = (4, 6)$, then any G with connectivity one has the edge-partition property. For any other pair m_1, m_2 there is a tree G not having the edge-partition property.*

In contrast to vertex-partitions, if G is 4-edge-connected, then G always has the edge-partition property for any $m_1, m_2, \dots m_k$.

Theorem 2 *Any 4-edge-connected graph G has the edge-partition property.*

Also if G is cubic and planar, then cyclically 4-edge-connectedness suffices.

Theorem 3 *Any planar, cubic, cyclically 4-edge-connected graph G has the edge-partition property.*

Notice that 2-edge-connected is not enough In Theorem 2. This can be seen using the graph on Figure 1, which does not have an edge-partition with $m_1, \dots m_7 = 4, \dots 4$.

Raising the connectivity makes the partition into $4, 4, \dots$ accessible.

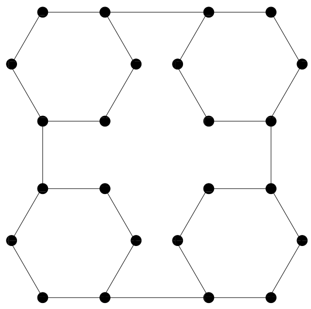

Figure 1:

Lemma 2 *Let G be a 3-connected cubic graph with $4m$ edges. Then $E(G)$ can be partitioned into connected sets of size four.*

Another extremal case we settle is the partition into sets with three edges.

Lemma 3 *Let G be a 2-edge-connected graph. Then $E(G)$ can be divided into connected sets with three edges.*

We state the following conjecture.

Conjecture 1 *If G is any 3-edge-connected graph, then it has the edge-partition property.*

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Extension of Arrow's theorem to symmetric sets of tournaments

Eyal Beigman *

Abstract

Arrow's impossibility theorem [1] shows that the set of acyclic tournaments on more then two vertices is not closed to non dictatorial Boolean aggregation. We extend the notion of aggregation to general tournaments and show that for tournaments with four vertices or more any proper symmetric (closed to vertex permutations) subset cannot be closed to non dictatorial monotone aggregation and to non neutral aggregation. We also demonstrate a proper subset of tournaments that is closed to parity aggregation for an arbitrarily large number of vertices. This proves a conjecture of Kalai [2] for the non neutral and the non dictatorial and monotone cases and gives a counter example for the general case.

An acyclic tournament is a binary relation that defines a linear order; such relations are thought to represent individual preferences between alternatives. It is a basic problem in economics to find functions that map voter preference profiles to group preferences satisfying two natural conditions. The first condition is the independence of irrelevant alternatives (condition I), which requires that the group preference between any two alternatives depend only on the voters' preference between them. Thus if the individuals' preferences between two alternatives are Boolean variables then the group preference between these alternatives is determined by a Boolean function on the variables. The second condition is unanimity also called the Pareto Principle (condition P). It requires that if all voters have the same preference then the group have this preference. This implies that the Boolean functions that determine the aggregation are such that if all the variables are 0 then the function is 0 and the same for 1.

This notion of aggregation can be generalized to general tournaments.

Definition 1 *Let Δ_X denote the set of all tournaments on X. An m-**place generalized aggregation** or simply **aggregation** is a function $f = \Delta_X^m \to \Delta_X$ satisfying for any $R_1, \ldots, R_m \in \Delta_X$ and $R = f(R_1, \ldots, R_m)$:*

condition I *aRb depends only on $aR_1b, \ldots, aR_mb$ for all $a, b \in X$.*

condition P *If aR_ib for $i = 1, \ldots, m$ then aRb.*

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An aggregation f is *dictatorial* if for $1 \leq i \leq m$ fixed $f(R_1, \ldots, R_m) = R_i$ for any $R_1, \ldots, R_m \in \Delta_X$. Arrow's impossibility theorem shows that if $|X| > 2$ then for all non dictatorial aggregation functions there exists a profile of acyclic tournaments that cannot be aggregated into an acyclic tournament.

Definition 2 *A set of tournaments $\mathfrak{C} \subset \Delta_X$ is* ***closed*** *to an aggregation f if and only if $f(R_1, \ldots, R_m) \in \mathfrak{C}$ for $R_1, \ldots, R_m \in \mathfrak{C}$.*

Thus Arrow's theorem shows that the set of acyclic tournaments on more than two vertices is not closed to any non dictatorial aggregation satisfying conditions I and P. From a combinatorial point of view it is interesting to ask whether a similar result would hold on other sets of tournaments. It is well known that there are sets of even acyclic tournaments that are closed to aggregation but it can be shown that in all of these cases the alternatives are not interchangeable. A set of tournaments is *symmetric* if it is closed to permutations of the vertices. For example, the set of i-nearly acyclic tournaments $0 \leq i \leq \binom{n}{2}$, tournaments which disagree with acyclic tournament on no more then i pairs $a, b \in X$, is symmetric. Are there any symmetric sets of tournaments that are closed under some aggregation? For $|X| = 3$ the set of cyclic tournaments is symmetric and closed to majority. It was conjectured by Kalai [2] that for $|X| \geq 4$ the only symmetric set of tournaments closed to non dictatorial aggregation is the set of all tournaments. We explore the conjecture in this paper.

An aggregation is *neutral* if $R = f(R_1, \ldots, R_m)$ implies $R^\pi = f(R_1^\pi, \ldots, R_m^\pi)$ for $R, R_1, \ldots, R_m \in \Delta_X$ where R^π is the tournament obtained from R by permuting the vertices by π.

We prove the following theorem:

Theorem 1 *If $\mathfrak{C}$ is a symmetric set of tournaments on at least four vertices closed to an aggregation function that is either non neutral or non dictatorial and monotone then $\mathfrak{C}$ is the set of all tournaments.*

The proof is by induction on the number of vertices. We deal separately with non neutral and neutral aggregations. In the first case we show that a set of tournaments is closed to non neutral aggregation iff it is closed to a two voter non neutral aggregation. In this case the Boolean functions are AND, OR or the two dictator functions. Then the idea is to show that if the aggregation is non dictatorial then for at least one vertex it is possible to increase and decrease the out degree. This enables application of the induction assumption to finish the proof.

In the second case we show that a set of tournaments is closed to neutral aggregation iff it is closed to a three voter non neutral aggregation. In this case the only non dictatorial monotone Boolean function is Majority. We once again show that the out degree may be increased or decreased by aggregation and use induction to finish.

We also prove that the conjecture is not true for non monotone neutral aggregations through an explicit construction.

Proposition 1 *For X arbitrarily large there exists a proper symmetric subset of Δ_X that is closed to non monotone aggregation.*

The idea is to take a set of tournaments on an odd number of vertices such that all the out degrees are odd. Since the aggregation function is parity it follows that the out degree mod 2 of the group tournament equals the sum of the out degree mod 2 of the voters' tournaments, for any odd number of voters it follows that the group tournament belongs to the set.

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The Cyclic and Path Multi-peg Tower of Hanoi

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Variants of the classical Tower of Hanoi problem evolved in various directions. Two of these are:

i) Allowing more than 3 pegs,

ii) Imposing limitations on the possible moves among the pegs.

Unlike the original problem, where the complexity is exponential, the unrestricted multi-peg problem for $h \geq 4$ pegs is known to be sub-exponential. A natural question is to what extent this is the case when we disallow direct moves between some pairs of pegs. In other words, the pegs are the vertices of some directed graph. In such a representation, the unrestricted problem corresponds to K_h- the complete graph on h vertices. Here we examine two other variants: Cyclic_h and Path_h, for $h \geq 4$.

Cyclic_h is the uni-directional circuit graph on h vertices. Thus, the pegs are arranged in a circle, and moves are allowed only from a peg to its successor (in the clockwise direction, say). It turns out that the complexity of this variant is exponential. More precisely, we have

Theorem 1 *The complexity of the Tower of Hanoi problem for* Cyclic_h *behaves regularly as* λ^n *for some* λ *depending on* h*. Moreover,* λ *satisfies*

$$1 + \frac{6}{7h^2} \leq \lambda_h \leq (h-1)^{\frac{2}{h-2}} \qquad h \geq 3\,.$$

Note that $(h-1)^{\frac{2}{h-2}} \approx 1 + \frac{2\ln h}{h}$ for large h, so that λ approaches 1 as h becomes large. For any specific value of h, it is possible in principle to estimate λ arbitrarily well, but the rate of convergence of λ to 1 is unknown. The theorem answers a question of Stockmeyer(1994).

Path_h denotes a path on h vertices. That is, the pegs are arranged linearly, and moves are allowed only between a peg and its nearest neighbour(s). Despite the significant reduction of allowed moves compared with the unrestricted multi-peg problem, the complexity is sub-exponential, as the following theorem shows in particular.

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Theorem 2 *The complexity of the Tower of Hanoi problem for $Path_h$ is $C_h n^{\alpha_h} 3^{\theta_h n^{\frac{1}{h-2}}}$ for $n \geq 2$, where:*

$\theta_h = ((h-2)!)^{\frac{1}{h-2}}$,
$\alpha_h = \frac{h-3}{h-2}$,
$C_h = \frac{(h-2)\cdot 6^{h-3}}{\theta_h}$.

It is worthwhile to note that the upper bound provided by Theorem 2 is not much larger than the bound $O(n^{\frac{h-3}{h-2}} 2^{n^{\frac{1}{h-2}}})$, obtained from the best known algorithm for K_h.

Keywords: **Tower of Hanoi, directed graphs, cyclic graphs, path graphs**

On perfect 2-codes in the odd graphs

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The Odd graph $O(k)$ has as its vertex set all $(k-1)$-element subsets of $\{1, \ldots, 2k-1\}$, with two vertices adjacent if and only if their intersection is empty. A perfect e-code in $O(k)$ is a subset $\mathcal{C}$ of the vertex set such that every vertex of $O(k)$ is at distance at most e from exactly one vertex in $\mathcal{C}$. Only two perfect codes in the odd graphs are known: the Fano plane in $O(4)$ and the Witt design $S(4,5,11)$ in $O(6)$. Both are 1-perfect, and it has been shown in [1] and [4] that a perfect 1-code in $O(k)$ is the same thing as a Steiner system $S(k-2, k-1, 2k-1)$. In [3] it is proved that there are no perfect 4-codes in the odd graphs. Here we will study perfect 2-codes.

Nonexistence proofs for perfect codes (in various graphs, most prominent the Hamming graphs) almost always used the following two results: the phere packing condition and Lloyd's Theorem. Let there be a perfect e-code in a distance-regular graph G. The sphere packing condition says that the size of a ball $\mathcal{B}_e$ with radius e around a vertex must divide the number of vertices of the graph. Lloyd's Theorem says that a certain polynomial has exactly e integer zeros in the set $\{1, 2, \ldots, d\}$, where d is the diameter of G. The polynomial in question is given by a sum $\sum_{i=0}^{e} P_i(x)$, where $P_i(x)$ is a polynomial of degree i in the eigenvalues of the graph G such that the numbers $P_i(j)$, $j = 0, \ldots, d$, are the the eigenvalues of the i-th distance graph of G. In the cases we are interested in these eigenvalues $P_i(j)$ will be integers.

We will make use of the observation that the sphere packing condition can be splitted into several divisibility conditions. These are most easily stated in the case $e = 1$: Given a distance-regular graph, let v_i be the number of vertices at distance i from a given vertex.

Theorem 1 *Let there be a perfect* 1*-code in a distance-regular graph of diameter d. Let $s \in \{1, \ldots, d\}$ be the integer zero given by Lloyd's Theorem. Then $|\mathcal{B}_1|$ divides $v_i - P_i(s)$ for all integers $i = 0, \ldots, d$.*

Note that since all eigenvalues $P_i(s)$, $i = 0, \ldots, d$, add up to zero, the divisibility conditions of Theorem 1 add up to the sphere packing condition. Similar "local sphere packing conditions" hold for all e. Combinatorially, these conditions express the integrality of the number of vertices of the code which are at fixed distance from a choosen vertex of the graph, and it is known that this number depends only on the distance of the choosen vertex from the code [2, 5].

Returning to the odd graph, let us mention the related Johnson graph $J(n, k)$, which has all k-element subsets of $\{1, \ldots, n\}$ as its vertices, with two vertices adjacent if and only if their intersection has size $k-1$. A famous conjecture of Delsarte [2] says that no nontrivial perfect code exist in $J(n, k)$ (the trivial ones consist of at most two or all vertices). An easy argument shows that the existence of a perfect 2-code in $O(k)$ implies the existence of a perfect 1-code in $J(2k, k)$ and conversly, making it possible to apply the conditions for $e = 1$ of Theorem 1 .

It turns out that for perfect 1-codes in $J(2k, k)$ the number of violated divisibility conditions is related to the number of prime factors of the Lloyd zero s: the more prime factors s has, the more local sphere packing conditions are violated, but, curiously, the more difficult it is also to prove that just one single of these conditions is not satisfied. Up to now we were not able to prove the nonexistence of perfect 2-codes in $O(k)$ for all k, but a strong number theoretical condition obtained from Lloyd's Theorem and Theorem 1 is the following:

Theorem 2 *If there is a perfect* 2*-code in the odd graph* $O(k)$ *then* $k = q^2 + q + 1$ *for some odd integer* q *such that* $s = q^2 + 1$ *is square free and all prime divisors* p *of* $q^2 + 1$ *satisfy* $(q + 1)^2 \equiv 2r_p - p \pmod{p^2}$*, where* r_p *is the smallest positive residue of* $q \pmod{p}$*.*

The condition of Theorem 2 is not satisfied for all $k \leq 10^{11}$, and we conjecture that it does not hold for any k.

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Topological properties of active orders for matroid bases

Rieuwert J. Blok* and Bruce E. Sagan

The external and internal orders Let M be a matroid on the base set E. We denote the set of bases of M by $\mathcal{B} = \mathcal{B}(M)$. We say that M is *ordered* if E is linearly ordered. ¿From now on all matroids will be ordered.

Given a set $A \subseteq E$ we say that $e \in E$ is *active with respect to* A if there is a circuit $C(A; e) \subseteq A \cup \{e\}$ in which e is minimal with respect to the ordering on E. Let

$$\mathrm{Act}(A) = \{e \mid \ e \text{ is active with respect to } A\}.$$

Note that we do not exclude the possibility that $e \in A$.

For $A \subseteq E$ we define

$$\mathrm{Ext}(A) = \mathrm{Act}(A) - A.$$

The elements of $\mathrm{Ext}(A)$ are called *externally active with respect to* A. This coincides with the usual notion of externally active edges with respect to a an element of $\mathcal{B}$.

Las Vergnas [4] introduced the *external order* on $\mathcal{B}$ in connection with the Tutte Polynomial and the Orlik-Solomon algebra of a matroid. Using Las Vergnas [4, Proposition 3.1] we can define this order as follows: For $A, B \in \mathcal{B}$, define

$$A \leq_{\mathrm{ext}} B \iff A \subseteq B \cup \mathrm{Ext}(B).$$

It was proved in [4] that, when augmented with a minimal element, the resulting order is in fact a graded lattice with rank function $\rho(B) = |\,\mathrm{Ext}(B)| + 1$. We will denote this lattice by $L(M, \leq_{\mathrm{ext}}^{M})$ or simply $L(M)$ or even just L.

Let us give two simple examples of these lattices by using the cycle matroid of a multigraph $G = (V, E)$. If G is the circuit on n vertices, then L consists of an antichain of $n - 1$ elements with a minimum and maximum adjoined. At the other extreme, if G consists of two vertices with n edges between them then L is a chain of $n + 1$ elements.

Following Las Vergnas [4], for a spanning subset $A \subseteq E$ we define

$$\mathrm{MaxBas}\, A = A - \mathrm{Act}(A).$$

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Alternatively, one can define this as the lexicographically maximum base of M contained in A (where the elements of a base are listed in increasing order). We obtain the maximum element of $L = L(M, \leq_{\text{ext}}^{M})$ as

$$T = \operatorname{MaxBas} E$$

and reserve the notation T for this top element.

The following formula for the join operator $\vee$ on several elements is a slight improvement on the original formula for two elements by Las Vergnas.

Corollary 1 *The join of elements $B_i \in \mathcal{B}(M)$ $(i = 1, 2, \ldots, m)$ in $L(M)$ is given by*

$$\bigvee_{i=1}^{m} B_i = \operatorname{MaxBas}\left(\bigcup_{i=1}^{m} B_i\right)$$

Homology It was empirically observed by Las Vergnas that the Möbius function μ of the external lattice $L(M)$ often satisfies $\mu(L(M)) = 0$. It is known that, given any lattice L with Möbius function μ, one has

$$\mu(L) = \tilde{\chi}(\Delta) = \sum_{k=0}^{\infty} (-1)^k \dim \tilde{\mathrm{H}}_k(\Delta) \tag{1}$$

where Δ is the order complex of L, $\tilde{\chi}$ is the reduced Euler characteristic, and $\tilde{\mathrm{H}}_k(\Delta)$ is the reduced k-dimensional homology group of Δ with coefficients in $\mathbb{Z}$ (see e.g. Stanley [7, Ch.3]). We study the reduced homology of the order complex of the lattice $L(M)$ in order to explain Las Vergnas' observation.

Let L be a lattice with minimum and maximum elements $\hat{0}$ and $\hat{1}$, respectively. We denote by $\Delta(L)$, or simply Δ the *order complex* of L, that is, the abstract simplicial complex on the set $L - \{\hat{1}, \hat{0}\}$ whose faces are the nonempty chains in $L \setminus \{\hat{0}, \hat{1}\}$ ordered by inclusion. If $L = L(M)$ for some matroid, then we will also use the notation $\Delta(M) = \Delta(L(M))$.

There is another abstract simplicial complex associated with a matroid. The *matroid complex* of M, denoted $[M]$, is the simplicial complex of nonempty independent subsets of M. Let T' be the elements of T that are independent in M^*. Then $[M^*|T] = [M^*|T']$. The following theorem relates the two complexes we have defined.

Theorem 1 *We have*

$$\Delta(M) \simeq [M^*|T]$$

(homotopy equivalence). So, for all $k \geq 0$,

$$\tilde{\mathrm{H}}_k(\Delta(M)) \cong \tilde{\mathrm{H}}_k([M^*|T]).$$

Note that this result implies that the homotopy type of the order complex depends only on the maximum base T.

We will prove Theorem 1 using the next two propositions.

Let L be an arbitrary lattice with atom set $\mathcal{A}$. Let $\mathcal{J} = \mathcal{J}(L)$ be the abstract simplicial complex of all subsets of $\mathcal{A}$ whose join is not $\hat{1}$. The following is a special case of a theorem of Segev [6].

Proposition 1 *For any lattice L*

$$\Delta(L) \simeq \mathcal{J}(L). \qquad \blacksquare$$

Let $\mathcal{F}$ be an abstract simplicial complex on a set F with facets $F_0, F_1, \dots, F_n$. Define its *intersection complex* $\mathsf{S}(\mathcal{F})$ as follows. The vertices of $\mathsf{S}(\mathcal{F})$ are the facets of $\mathcal{F}$ and $\{F_{i_0}, F_{i_1}, \dots, F_{i_m}\}$ is a face of $\mathsf{S}(\mathcal{F})$ if and only if $F_{i_0} \cap F_{i_1} \cap \dots \cap F_{i_m} \neq \emptyset$ in $\mathcal{F}$.

Using the Nerve Theorem of Borsuk and Folkman (see (10.6) in Björner [2]) we obtain our second proposition.

Proposition 2 *Let $\mathcal{F}$ be a simplicial complex on a set F. Then*

$$\mathcal{F} \simeq \mathsf{S}(\mathcal{F}). \qquad \blacksquare$$

In order to prove Theorem 1 one combines Propositions 1 and 2 to obtain

$$\Delta \simeq \mathcal{J} \simeq \mathsf{S}(\mathcal{J})$$

and one then shows that $\mathsf{S}(\mathcal{J})$ and $[M^*|T]$ are isomorphic as simplicial complexes. Here we use Corollary 1 to determine $\mathsf{S}(\mathcal{J})$.

Examples We will use the notations $\tilde{H}_k(M)$ and $\mu(M)$ for $\tilde{H}_k(\Delta(M))$ and $\mu(L(M))$, respectively. The following corollary to Theorem 1 shows that a number of external lattices have Möbius function zero. To state the result we will use the matroid rank $r(F)$ of a subset $F \subseteq E$, that is, the maximum size of an independent set contained in F.

Corollary 2 *Let M be an ordered matroid with maximum base T and let $n = |T| - 1 \geq 1$.*

(i) Suppose there is $t \in T$ such than $r(E - T) = r((E - T) \cup t)$. Then

$$\tilde{H}_k(M) = \{0\} \text{ for all } k \geq 0 \quad \text{and} \quad \mu(M) = 0.$$

(ii) Suppose that $M|(E - S)$ is spanning for all proper subsets $S \subset T$ but is not spanning for $S = T$. Then

$$\tilde{H}_k(M) = \begin{cases} \mathbb{Z} & \text{if } k = n - 1, \\ \{0\} & \text{otherwise,} \end{cases} \qquad \text{and} \quad \mu(M) = (-1)^{n-1}.$$

As a result of this corollary, we can see that $L(M)$ is not, in general, shellable (even though $[M^*|T]$ always is, see Björner [1, Theorem 7.3.3]). If X is a shellable simplicial complex pure of dimension d, then X is topologically a wedge of d-spheres and so only has homology in dimension d. So if a finite lattice L graded of rank ρ is shellable, then it only has homology in dimension $\rho - 2$ (since we remove $\hat{0}$ and $\hat{1}$). But in $L(M)$ we have

$$\rho(L(M)) = \rho(T) = |\operatorname{Ext}(T)| = |E - T|.$$

In particular for the cycle matroid of the complete graph K_n we get

$$\rho(L(K_n)) = \binom{n}{2} - (n - 1) = \binom{n-1}{2}.$$

But from the previous corollary, if T has an internal vertex then $L(M(K_n))$ has homology in dimension $n - 3 < \binom{n-1}{2} - 2$ for $n \geq 5$.

Another homological result A theorem of Björner [1, Theorem 7.8.1] characterizes the reduced homology of $[M]$ for any matroid M and so will be useful in conjunction with Theorem 1 for computations. To state it, we will need the lattice of flats of M which will be denoted $L_F = L_F(M)$ to distinguish it from the external activity lattice. Also, define the *reduced Möbius function* of M to be

$$\tilde{\mu}(M) = \begin{cases} |\mu(L_F(M))| & \text{if } M \text{ is loopless,} \\ 0 & \text{else.} \end{cases}$$

Theorem 2 (Björner) *If matroid M has rank r then*

$$\tilde{H}_k([M]) \cong \begin{cases} \mathbb{Z}^{\tilde{\mu}(M^*)} & \text{if } k = r-1, \\ \{0\} & \text{else.} \end{cases} \qquad \blacksquare$$

Now if $F \subseteq E$, the ground set of M, then let $M.F$ denote the contraction of M to F. Our interest in this operation stems from the fact that $(M^*|F)^* = M.F$. An immediate corollary of the previous theorem and Theorem 1 is as follows.

Theorem 3 *Let r be the rank of $M^*|T$. Then*

$$\tilde{H}_k(\Delta(M)) \cong \begin{cases} \mathbb{Z}^{\tilde{\mu}(M.T)} & \text{if } k = r-1, \\ \{0\} & \text{else.} \end{cases} \qquad \blacksquare$$

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Tilings on the butterfly lattice

Olivier Bodini*

Abstract

Let F be a simply connected figure constituted of cells of the butterfly lattice (fig.1). We show that there exists a linear algorithm which says whether F is tilable by the three tiles described below (fig.2). Moreover, we point out some specific geometrical properties, especially on the notion of *flip*.

1 Introduction

We use flows theory in graphs to extend a result by Thurston on tiling [3] and a result by T. Chaboud [1] about the problem of tiling with two types of tiles. In his paper, Thurston deals with some geometrical interpretations of tilings in terms of relevant 3-dimensional surfaces. We show here that there exists a bijection between tilings and some specific tensions in a special graph (see Theorem 2.1). Moreover, we point out some specific geometrical properties, especially on the notion of *flip*. In this paper, we prove that if F is a tilable B-figure, then the two following quantities are equal:

- the number of connected components in the *flip-accessibility graph of the tilings of* F
- the number of minimum tilings.

Precise definitions are given in the sequel. However, we can already say that this theorem is related to the following classical result: All the domino tilings of a polyomino are accessible by flip.

The *butterfly lattice* is the subdivision of the plane into hexagons and equilateral triangles with same side length and such that the sides are always shared by one hexagon and one triangle (see fig.1). A *B-figure* is a finite union of cells of the butterfly lattice. Now, let G be the infinite graph on the nodes of the butterfly lattice, such that the edges join two nodes in a same triangle. In other words, G is the 1-skeleton of the butterfly lattice. Let F be a B-figure, we denote by G_F the partial subgraph of G constituted by the vertices and the edges of G belonging (geometrically) to F.

The following important notion comes from Thurston [3]. The *cellular orientation* of a B-figure F is an orientation of G_F such that these edges (which become arcs) are oriented

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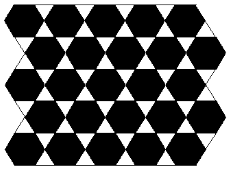

Fig.1 The butterfly lattice.

to travel along the boundary of the hexagons (resp. triangles) in the trigonometric (resp. clockwise) orientation. This oriented graph is denoted by H_F.

The union of one hexagon and two adjacent triangles is called a *prototile* (see fig.2). A tiling Q of a B-figure F is a partition of the cells of F into prototiles. These prototiles are the *tiles* of the tiling Q. We say that a B-figure is *tilable* if it has at least one tiling (fig.3).

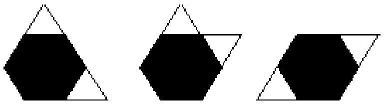

Fig.2 The three types of prototiles up to isometry.

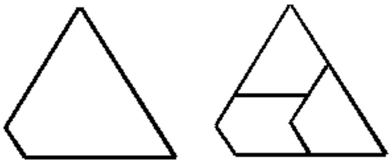

Fig.3 A B-figure and a tiling.

Let F be a tilable B-figure and Q a tiling of F. A *tiling arc of* Q is a common arc of two tiles of Q or a boundary arc.

2 flows, flux and potentials.

Let G be an oriented graph, a *flow* on G is a map C from $E(G)$ into $\mathbb{C}$. A *travel* from a vertex v to a vertex v', is a chain linking v to v' provided with a way of travel. The *flux* of the flow C on the travel T is $\sum_{e\in T^+} C(e) - \sum_{e\in T^-} C(e)$ (where T^+ denotes the arcs travelled with respect of their orientation and T^- the others). In simple words, we begin with 0 in v and we follow T, if we travel along an arc e of T in the good direction, we add $C(e)$ otherwise we subtract $C(e)$. The flux is the number that we obtain in v'. We denote by $F_T(C)$ the flux along T. We verify immediately that the flux is additive by concatenation of travels : $F_{T_1T_2}(C) = F_{T_1}(C) + F_{T_2}(C)$ where T_1 and T_2 are two travels such that the extremity of T_1 coincides with the origin of T_2 and where T_1T_2 is obtained by joining T_1 and T_2 end to end.

As for polyominoes (finite set of squares in infinite chessboard), the *boundary* of a B-figure F is the set of edges of G which belong to a unique cell of F and we denote it by ∂F.

We say that a B-figure is *full* if its boundary is a topological circle. Let F be a B-figure with a tiling Q, we are going to associate a flow C_Q on H_F to the tiling Q as follows : $C_Q(e) = 1$ when e is a tiling arc of Q and $C_Q(e) = -2$ otherwise (see fig.4). For every cell c, let T_c be the travel defined by turning around c in the trigonometric way. We can notice that $F_{T_c}(C_Q) = 0$.

If for every *closed* travel (i.e. its origin and its extremity coincide) on the boundary of a B-figure F we have $F_T(C) = 0$ where C is a flow such that $C(e) = 1$ for all $e \in \partial F$, then we say that F has a *balanced boundary.* In particular, we can notice that a full balanced B-figure has a balanced boundary. More specifically, if for all closed travel (not necessary on the boundary), we have $F_T(C) = 0$, then the flow C is called a *tension.*

If F is a B-figure with a balanced boundary and Q is a tiling of F, we observe (because we can contract all closed paths by succession of contraction of cells and cycles on the boundary) that the flow C_Q is a tension and so, it derives from a *potential* φ_Q on the vertices of H_F. In other words, there exists a potential φ_Q defined by : for all $(x, y) \in V(H_F)^2$, $\varphi_Q(y) - \varphi_Q(x) = F_{T_{x,y}}(C_Q)$ where $T_{x,y}$ is a travel linking x to y, and $\min\limits_{x \in \partial F} \varphi_F(x) = 0$. The arbitrary condition $\min\limits_{x \in \partial F} \varphi_F(x) = 0$ is necessarily, otherwise φ_Q is just defined up to a constant.

Theorem 1 *Let F be a B-figure with balanced boundary. There exists a one-to-one correspondence between its tilings and tensions C on H_F which satisfies :*

- $\forall e \in \partial F, C_Q(e) > 0.$
- $\forall e \in F, C_Q(e) \in \{-2, 1\}.$

Proof: We already shown that for each tiling of F, we can associate a tension which satisfies the hypotheses. Conversely, if we consider a tension which satisfy the hypotheses, we can notice that each hexagon of F has exactly four positive arcs. They are the tiling arcs of a tiling of F. $\square$

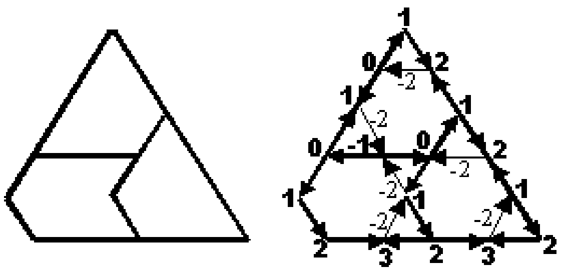

Fig.4 A tiling and its associated tension (in bold, the associated potential).

We say that x is a *(local) maximum vertex* if for every adjacent vertex v in H_F we have $\varphi(x) \geq \varphi(v)$.

Lemma 1 *Let F be a B-figure with a balanced boundary (F is not necessarily a full B-figure). There exists a tiling Q such that φ_Q is only maximum on some vertices on the boundary of F.*

Proof: Let Q be a tiling such that, for every tiling Q' of F, we have $\sum_{x\in V(H_F)} \varphi_Q(x) \leq \sum_{x\in V(H_F)} \varphi_{Q'}(x)$, we are going to prove that Q satisfies the conditions of the lemma. Suppose that x_m is a vertex which is not on the boundary and such that $\varphi_Q(x_m)$ is maximum. The vertices in

$$\delta(x_m) = \{y; (x_m, y) \in E(H_F) \text{ or } (y, x_m) \in E(H_F)\}$$

have a smaller potential than x_m, so the arcs in

$$E^+(x_m) = \{(x_m, y); (x_m, y) \in E(H_F)\}$$

are not tiling arcs of Q and the arcs in $E^-(x_m) = \{(y, x_m); (y, x_m) \in E(H_F)\}$ are tiling arcs. Let h_1, h_2, t_1, t_2 be the two hexagons and the two triangles respectively which contain x_m. We can suppose that h_1, t_1 (resp. h_2, t_2) belong to the same tile $\{h_1, t_1, t_a\}$ (resp. $\{h_1, t_1, t_b\}$). Now, we can consider the tiling Q' composed of the tiles of Q but where we have replaced the tiles $\{h_1, t_1, t_a\}$ and $\{h_2, t_2, t_b\}$ by the tiles $\{h_1, t_2, t_a\}$ and $\{h_2, t_1, t_b\}$. Then $\sum_{x\in V(H_F)} \varphi_{Q'}(x) = \sum_{x\in V(H_F)} \varphi_Q(x) - 3$ because for every arc e in $E^-(x_m)$ we have $C_{Q'}(e) = -2$, for every arc e in $E^+(x_m)$ we have $C_{Q'}(e) = 1$, for all the other arcs e, $C_{Q'}(e) = C_Q(e)$ so $\varphi_{Q'}(x_m) = \varphi_Q(x_m) - 3$ and for all $x \in V(H_F)\backslash\{x_m\}, \varphi_{Q'}(x) = \varphi_Q(x)$. This is in contradiction with the hypotheses. □

Let $T_1 = \{h_1, t_1, t_a\}$ (h always denotes an hexagon) and $T_2 = \{h_2, t_2, t_b\}$ two tiles with t_1 adjacent to h_2 and t_2 adjacent to h_1 (as in the proof of the lemma 1), then we can transform this two tiles into $T_1' = \{h_2, t_1, t_a\}$ and $T_2' = \{h_1, t_2, t_b\}$ which cover the same cells. This operation is called a *flip*. Let A_F be the simple graph where $V(A_F)$ is the set of the tilings of F and $\{Q, Q'\}$ is an edge if Q and Q' differ from only one flip. We call $Tmin_F$ the set of the tilings such that the maximum of their potential are on the boundary.

Lemma 2 *Let F be a tilable B-figure with a balanced boundary. For every tiling Q, there exists a unique tiling in $Tmin_F$ obtained from Q by a succession of flips.*

Proof: We must prove that the tiling in $Tmin_F$ obtained by flips around successive maxima does not depend on the successive choice of the maxima. But, in the lemma 1, we can observe that this two facts are equivalent:

- to make a flip around a maximum vertex x_m (which transforms the tiling Q into Q')
- to transform $\varphi_Q(x_m)$ into $\varphi_{Q'}(x_m) = \varphi_Q(x_m) - 3$ and keep equal the other values.

So, if we have a choice between two maximum vertices x_m and x'_m, it is clear that we have to make a flip on each of them and the result does not depend of the order to do it. □

Lemma 3 *Let F be a tilable B-figure with a balanced boundary. A tiling Q in $Tmin_F$ is totally determined by the values of φ_Q on ∂F.*

Proof: By induction on the number of cells in F. We refer to the following figure (Fig.5). Let v_1 be an absolute maximum vertex (on the boundary) and set $\varphi_Q(v_1) = M$. firstly, we can remark that v_1 is in the middle of a segment of length 2 belonging to the boundary. The arcs in $E^+(v_1)$, denoted by e_1, satisfy $C_Q(e_1) = -2$ (otherwise $\varphi_Q(v_1)$ will not be maximum). So, they do not belong to the boundary of F and $\varphi_Q(v_2) = M - 2$. Now, we are going to separate the problem in four cases wich depend on the position of the second negative arc.

- (v_3, v_4) is the second negative arc of the black hexagon. In this case, $\varphi_Q(v') = M$ or $\varphi_Q(v') = M - 3$. In fact, $\varphi_Q(v')$ is equal to M, otherwise v_2 is a (local) maximum and this is in contradiction with $Q \in Tmin_F$. Thus, v' is on the boundary of F and it is maximum.
- (v_4, v_5) is the second negative arc of the black hexagon. In this case $\varphi_Q(v_4) = M$. So, v_4 is on the boundary of F and it is maximum.
- (v_5, v_6) is not the second negative arc of the black hexagon. Indeed, in this case $\varphi_Q(v_5) = M + 1$. This is in contraction with the maximality of v_1.
- (v_2, v_3) is the second negative arc of the black hexagon.

Now, if F is tilable. We can delete a tile of Q which is totally determined. Indeed, we have three exclusive possibilities: if v' (resp. v_4) is a maximum vertex on the boundary, then (v_3, v_4) (resp. (v_4, v_5)) is a negative arcs and we have determined the tile of Q which covers v_1. Otherwise, (v_2, v_3) is the second negative arc. By induction, the other tiles are also totally determined. □

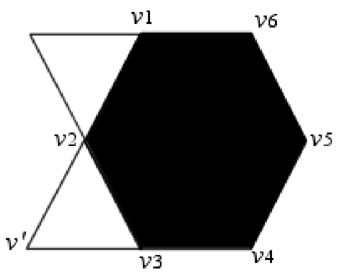

Fig.5 Around a maximum.

Theorem 2 *Let F be a tilable B-figure. The number of connected components in A_F is equal to the cardinal of $Tmin_F$.*

Proof: By lemma 1, for every tiling Q in a same connected component, the values of the potential φ_Q on ∂F do not depend on the tiling Q. Moreover, by lemma 3, there is a unique $T \in Tmin_F$ compatible with this values. □

Lemma 4 *Let F be a full tilable B-figure, there is a unique element in $Tmin_F$, in other words A_F is connected.*

Proof: Since ∂F is a topological circle, the values of the potential φ_Q on ∂F are integrally determined by the cellular orientation of the arcs of ∂F and do not depend on any tiling Q. So, there is a unique tiling in $Tmin_F$. □

As a consequence, if F is a full tilable B-figure, then for every vertex x on the boundary of F, we simply denote by $\varphi(x)$ the common value of all the potentials φ_Q in x. This restriction of φ_Q on the boundary of F is called - in the case of full polyominoes - *height on the boundary* by J-C Fournier [2] and was introduced by Thurston [3]. The reader can notice that this potential on the boundary can be defined if F has a balanced boundary. Since φ can be calculated without knowing a tiling of F, for the full tilable B-figure, the lemma 3 obviously gives us, by induction, a linear time algorithm to build the unique minimum tiling (tiling in $Tmin_F$) :
1) We build (or complete) φ.
2) We take the maximum vertices and then we deduce by lemma 3 a tile of the minimum tiling.
3) We delete this tile and then we go back to 1) with this new B-figure.

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Generating Unlabeled Cubic Planar Graphs Uniformly at Random

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Abstract

We present an expected polynomial time algorithm to generate a 2-connected unlabeled cubic planar graph uniformly at random. We derive recurrence formulas to count all such graphs with n vertices, based on a decomposition along the connectivity structure. For 3-connected planar graphs we use the fact that they have a unique embedding, but special care has to be taken for graphs having a sense-reversing automorphism. In the last stage we switch to the dual and apply counting techniques for triangulations. We also obtain an expected polynomial time algorithm to generate the graph of a polyhedron where all facets are triangles uniformly at random.

1 Introduction

While there is an exact recursive counting formula and an efficient algorithm that generates labeled planar graphs uniformly at random [1], there is no such counting formula and generation procedure for *unlabeled* planar graphs, i.e. graphs that are considered up to isomorphism. As a step in this direction, here we show how to generate *2-connected unlabeled cubic planar graphs*.

To generate cubic planar graphs, we first have to *root* them. The root is a distinguished directed edge, and rooted cubic planar graphs are counted up to isomorphisms which leave the vertices of this edge fixed. Clearly generating a random rooted cubic planar graph and then simply ignoring the root edge does not yield the uniform distribution, since unlabeled graphs generally correspond to different numbers of rooted graphs. But this imbalance can be compensated by *rejection sampling*, i. e. the generation algorithm is restarted with a probability that is inverse proportional to the size of the orbit of the root. This way we can generate unlabeled 2-connected cubic planar graphs in expected polynomial time.

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We *decompose* rooted planar graphs along their connectivity structure. In principle our approach here is similar to the one described in [1], but for unlabeled structures the details are not straightforward. A classical theorem of Whitney (see e. g. [3]) says that a rooted 3-connected cubic planar graph can have either one or two embeddings in which the root edge is embedded at the outer face. In the first case we say that it has a *sense-reversing automorphism*. We account for this difficulty by a new bijective correspondence to *colored planar networks* (defined below) and a decomposition of these objects.

The dual of an embedded 3-connected cubic planar graph is a triangulation. Hence our final ingredient is a counting formula for rooted triangulations: Using ideas of Tutte [5], we count triangulations with a distinguished vertex u, a distinguished vertex v adjacent to u and a distinguished face f incident to uv, where the face f is bounded by a specified number of edges, and the vertex u has a specified degree.

In the full version of this paper we extend our technique to connected cubic planar graphs [2]. Note that the above bijection also yields an expected polynomial time algorithm to generate the graph of a polyhedron where all facets are triangles uniformly at random. Schaeffer [4] obtained an almost uniform random generator for them.

2 Graph decompositions

In this section we introduce the concept of a network. Basically, one can think of a network as a rooted 2-connected graph minus the root edge. Formally, a *network* N is a simple unlabeled graph with two distinguished vertices 0 and 1, called the *poles*, such that if an edge is inserted from the 0-pole to the 1-pole, the resulting multigraph N^* is 2-connected. (Thus we allow another edge between the poles – but note that the root edge itself is not considered to be a part of the network.) A network N is called *k-connected* if N^* is a k-connected multigraph, and similarly we define *k-cuts* of networks. A 2-cut is also called a *split pair*. A network N is said to be *cubic* (resp. *planar*) if and only if N^* is.

The *core* of a network N is the network $\mathrm{core}(N)$ defined by iterating the following process until convergence: Every split pair $\{k_1, k_2\}$ in N induces a partition of the edge set of N^*. A partition class which does not contain the root of N induces a *subnetwork*. The poles of the subnetwork are the vertices $\{k_1, k_2\}$. (In this context we do not need to specify their order.) We pick one subnetwork, delete its non-root edges, and insert an edge between k_1 and k_2. A pair of parallel edges is replaced with a single one. The result is again a network with the original poles. The process defining $\mathrm{core}(N)$ terminates if the current network is 3-connected or has at most 3 vertices. Thus it holds:

Theorem 1 *The core of a cubic planar network N is unique and has of one of the following three types: p:* $\mathrm{core}(N)$ *is a triangle, s:* $\mathrm{core}(N)^*$ *is a triangle, or h:* $\mathrm{core}(N)^*$ *is 3-connected.*

The three types of cores are shown in Figure 1. The corresponding cubic planar networks are called *p-networks*, *s-networks* and *h-networks*, respectively.

3 Counting networks

In this section we consider the decomposition of cubic planar networks further and derive recurrence formulas for counting them. Let $N(n)$ be the number of cubic planar networks on n vertices. Note that n is an even number for cubic graphs. According to Theorem 1 we have $N(n) = H(n) + P(n) + S(n)$, where the functions $H(n)$, $P(n)$, $S(n)$ count the number of h-, p-, and s-networks on n vertices, respectively.

***p*-networks.** If a p-network N has an edge between its poles, then the path over the third core vertex represents an arbitrary network N' (see Fig. 2, upper part). Otherwise, N can be split into two subnetworks N', N'', which can be of any types (see Fig. 2, lower part). If these two networks have the same size, we must not double count the case where the two networks are isomorphic, and this explains the following formula:

$$P(n) \;=\; N(n-2) \;+\; \sum_{i=2}^{\frac{n-2}{2}-1} N(i)N(n-i-2) \;+\; \binom{N(\frac{n-2}{2})+1}{2}.$$

***s*-networks.** Here we split off an h- or p-network at the 0-pole, and count the remaining networks (see Fig. 3).

$$S(n) \;=\; \sum_{i=2} (H(i)+P(i))N(n-i)\,.$$

***h*-networks.** We call an h-network H *symmetric* if it has a *sense-reversing automorphism* φ, i. e., $\varphi \neq \mathrm{id}$, but $\varphi(0) = 0$ and $\varphi(1) = 1$, and *asymmetric* otherwise. If H is asymmetric, one can order the edges of core(H) uniquely. If H is symmetric, there is a unique ordering of the pairs of core edges which are orbits of φ [2]. Let $A_{k,l}(n)$ be the number of asymmetric h-networks H on n vertices such that core(H) has k edges and the first l edges of core(H) are also edges of H. Similarly, let $B_{k,2l}(n)$ denote the number of symmetric h-networks whose first l pairs of core edges are edges in the network. An asymmetric h-network counted by $A_{k,l}$ can arise in two ways. The first is to take a symmetric h-network H counted by $B_{k,l+2}$ and substitute two different subnetworks for the $(l+1)$-th and $(l+2)$-th core edge. The second is to take an asymmetric h-network counted by $A_{k,l+1}$ and insert a subnetwork for the $(l+1)$-th core edge. Thus for $l < k$:

$$A_{k,l}(n) = \sum_i N(i)\Big(\sum_{j<i} N(j)B_{k,l+2}(n-i-j) + \frac{N(i)-1}{2}B_{k,l+2}(n-2i) + A_{k,l+1}(n-i)\Big)$$

$$B_{k,l}(n) \;=\; \sum_i N(i)B_{k,l+2}(n-2i)$$

With these numbers we can compute $H(n) = \sum_k (A_{k,0}(n) + B_{k,0}(n))$. The basic case $l = k$ counts the 3-connected h-networks, and we deal with it in the next chapter.

4 Triconnected h-networks

Let H be a 3-connected h-network, i. e., H^* is 3-connected and $H = \mathrm{core}(H)$. Since H^* has a planar embedding we can count the dual objects, namely *rooted triangulations*. A closed counting formula for face-rooted triangulations is due to Tutte [5]. But as the embedding of H^* is unique only if H is symmetric, the number of all rooted triangulations is $2A_{k,k}(n) + B_{k,k}(n)$. Hence we know $H(n)$ if we determine $B_{k,k}(n)$.

Now assume that H is symmetric and φ is a sense-reversing automorphism. A vertex v of H is called *blue* if $\varphi(v) = v$, and *red* if v is connected to $\varphi(v)$ by an edge. Thus a vertex is either red, blue, or uncolored, and the poles are blue. We can think of H as being embedded in such a way that φ corresponds to a reflection, the blue vertices are aligned on the reflection axis, and the red edges cross this axis perpendicularly (see Fig. 4, left part). Our argumentation does not 'rely' on such a representation, however.

If we remove from H the blue vertices and the edges between red vertices (i. e., we cut H along the symmetry axis), then the resulting graph has exactly two connected components. The graphs induced by these components and the blue vertices are isomorphic and will be called H_1 and H_2 (see Fig. 4, right part). As the poles of H do not have a common neighbor in H_1 for $|H| > 4$, the unique non-pole neighbors of 0 and 1 can be taken as the new poles of H_1. In [2] we show that H_1^* is 2-connected, and hence H_1 is a network. Note that the poles of H_1 are never blue in H (but may be red). Next we formalize these properties.

A *colored network* is a network N in which some vertices are colored *red* and *blue* such that: (1) the colored vertices have degree 2 in N^*, and all other non-pole vertices have degree 3, (2) N^* has a planar embedding such that all colored vertices and the poles lie at the outer face, (3) every subnetwork of N contains at least one colored vertex, and (4) no subnetwork of N (including N itself) has two blue poles. Let $N'_b(n)$ denote the number of colored networks with b blue vertices, and n vertices altogether. For $n > 4$ there is a bijection between the colored networks counted by $N'_b(n + b - 2)$ and the symmetric h-networks on n vertices having b blue vertices [2]. Hence for $6 < k = 3n/2$ we have $B_{k,k}(n) = \sum_b N'_b(n + b - 2)/2$.

A colored network is either an h-, p-, or s-network, so we introduce the corresponding counting functions $H'_b(n)$, $P'_b(n)$ and $S'_b(n)$ for colored networks with n vertices and b blue vertices: $N'_b(n) = H'_b(n) + P'_b(n) + S'_b(n)$. For technical reasons we need one more function: Let $S''_b(n)$ be the number of colored s-networks which have no blue cut-vertex.

Colored s-networks. Let S be a colored s-network. Consider the cut-vertex u in S which is closest to the 0-pole. This vertex might be red (but not blue). If u is uncolored, then 0 and u define a colored h- or p-network. There is a unique neighbor v of u such that uv is a bridge in S. Either v and 1 define a colored network (uncolor v if it was red), or v is a blue vertex. In the latter case v is adjacent to another blue vertex, which in turn has another unique neighbor w in S. The colored networks between w and 1 are counted by

$N''_b(n) := S''_b(n) + HP'_b(n)$, where $HP'_b(n) := H_b(n) + P_b(n)$. Then

$$S'_b(n) = N'_b(n-1) + \sum_{i,b'} HP'_{b'}(i)\big(N'_{b-b'}(n-i) + N''_{b-b'-2}(n-i-2)\big)\,.$$

We can also compute the number $S''_b(n) = N'_b(n-1) + \sum_{i,b'} HP'_{b'}(i) N''_{b-b'}(n-i)$.

Colored p-networks. The colored vertices of a colored p-network must all lie in one of its two parts. It is possible that one (but not both) of the unique neighbors of 0 and 1 in this subnetwork is blue. The other part of the p-network is just an edge. Hence

$$P'_b(n) = N'_b(n-2) + 2N''_{b-2}(n-4)\,.$$

Colored h-networks. There is a unique embedding of the core of a colored h-network H into the plane where the root edge and the core edges that are replaced by colored networks lie on the outer face. Let $H'_{b,k,l}(n)$ be the number of colored h-networks having k edges on the outer face of the embedding of the core, of which the l closest edges to the 0-pole are also edges in H; moreover we require that the $(l+1)$-th edge is a split pair in H. Similar as in the s-case, the vertices of the split pair are adjacent to unique vertices that are the poles of a colored network, or they are blue. For $k = l$ we need the number of rooted cubic 3-connected planar maps with k vertices on the outer facet, which is $T_{k,3}(n)$, see the next section. Then $H'_b(n) = \sum_{k\geq 2, l\geq 1} H'_{b,k,l}(n)$ and

$$H'_{b,k,l}(n) = \sum_{b',i}\Big(T_{k,3}(i) + \sum_{l'=l+1}^{k} H'_{b',k,l'}(i)\Big)\Big(N'_{b-b'}(n-i) + 2N''_{b-b'-2}(n-i-2)\Big)\,.$$

Rooted cubic planar graphs and triangulations The dual of a rooted 3-connected cubic planar graph is a rooted triangulation with three vertices on the outer face, where the outer face of the graph corresponds to the 0-pole of its dual. We count triangulations by removing edges from the outer face (Fig. 5) or cutting them along chords (Fig. 6). This is similar to Tutte [5], but we need to specify more parameters. The details are given in the full version of this paper [2].

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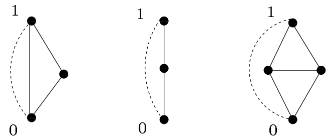

Figure 1: The p-core, the s-core, and an h-core

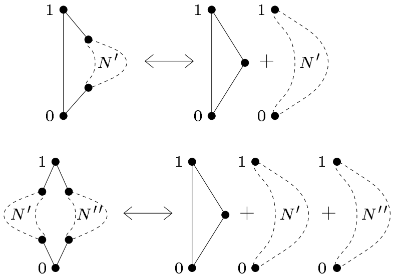

Figure 2: Decomposition of a p-network

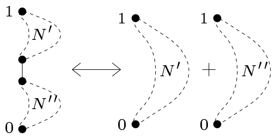

Figure 3: Decomposition of an s-network

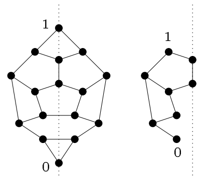

Figure 4: Decomposition of a symmetric h-network

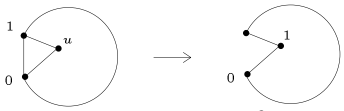

Figure 5: Decomposition of a triangulation

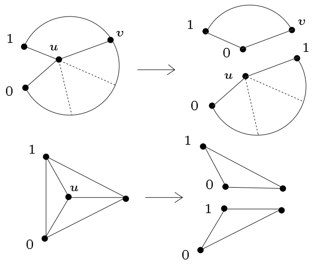

Figure 6: Decomposition of a triangulation

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Properties of expanding graphs and digraphs

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This is a survey about joint work with Hajo Broersma, Reinhard Diestel, Matthias Kriesell [3] (undirected case, see also [4] for an extended version) and Jørgen Bang-Jensen [1] (directed case).

A lot of research was performed on the construction of expanding graphs and on the evaluation of their expansion properties. Our aim is to investigate graph theoretical properties mainly related to cycles under a certain expansion concept and observe some similarities and differences between the undirected and the directed case.

There are numerous different expansion concepts, the majority of them being a condition that the neighbourhood of every "not too large" subset S of the vertex set of the graph has at least $f(|S|)$ neighbours outside S, f typically being a linear function. The neighbourhood $N(S)$ can be viewed as the (disrete) boundary of S. In fact, methods from many different fields of mathematics have been applied to construct expanding graphs and analyze their expansion.

Let $N(S)$ denote the neighbourhood of a set S in $V(G)\setminus S$. For a function $f : \mathbb{N} \to \mathbb{R}$ we call a graph G *f-connected*, if every nonempty subset $S \subseteq V(G)$ satisfies $|N(S)| \geq f(|S|)$ whenever $|S| \leq |V(G) \setminus (S \cup N(S))|$. Similarly, we call a digraph D *f-expanding*, if every nonempty subset $S \subseteq V(D)$ satisfies both $|N^+(S)| \geq f(|S|)$ whenever $|S| \leq |V(G) \setminus (S \cup N^+(S))|$ and $|N^-(S)| \geq f(|S|)$ whenever $|S| \leq |V(G) \setminus (S \cup N^-(S))|$. The terms $N^+(S)$ and $N^-(S)$ denote the out- and the in-neighbourhood of S, respectively.

The concept of f-connectedness (f-expansion, resp.) allows to formulate connectivity statements for f being a constant function, separator type statements for $f(k) = \Theta(\sqrt{k})$ and classical expansion statements for f being a linear function. Moreover, it allows to measure larger expansion by using functions f growing faster than linear. In our investigations we will mainly deal with functions f that grow at least linearly. Such functions are needed to imply really long cycles in the graph.

We show that a quadratic expansion function is sufficient to imply a hamiltonian cycle while we have an exponential function to imply hamiltonicity in digraphs. In both cases possibly a suitable linear function is sufficient. Moreover we show that linear expansion implies linear length (directed) cycles, as well as a 2-factor and a cycle factor for a suitable linear function in graphs and digraphs. For a fairly general subclass of digraphs we prove a

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tight expansion bound for hamiltonicity. With the use of our results for undirected graphs we can, e.g., prove some graph theoretical properties of so-called Ramanujan graphs.

The results

A major question and much of the motivation of this research is the question for an expansion condition guaranteeing a hamiltonian cycle. We shall prove that quadratic f-connectedness can force a graph to be hamiltonian.

Theorem 1 *If $f(k) \geq 2(k+1)^2$ for all $k \in \mathbb{N}$, then every f-connected graph of order $n \geq 3$ is hamiltonian.*

We do not believe that quadratic growth of f is needed in Theorem 1, but expect that some linear growth will do. In fact, perhaps the following special linear bound holds:

Conjecture 1 *If $f(k) \geq 2k+1$ for all $k \in \mathbb{N}$, then every f-connected graph of order $n \geq 3$ is hamiltonian.*

Since complete bipartite graphs $K_{n,n-1}$ are not hamiltonian but f-connected for $f\colon k \mapsto 2k$ when n is odd, this bound would be best possible. Using the observation that large expansion implies large toughness (but not vice versa), the truth of Chvátal's t-tough conjecture [5] for any fixed value t would imply a linear function f that ensures a hamiltonian cycle. So Conjecture 1 can be viewed as a relaxation of Chvátal's conjecture.

Very recently, Krivelevich and Sudakov [6] proved a sufficient condition for a hamiltonian cycle in regular graphs based on the spectrum of the adjacency matrix. The condition applies to a special class of expanding graphs of large order n with polylogarithmic degree in n. According to the authors, their methods might also apply to almost regular graphs. Note that even for fast growing functions $f(k)$ there can be several small degree vertices in an f-connected graph. Krivelevich and Sudakov [6] also offer a conjecture for hamiltonian cycles based on the spectrum of the adjacency matrix that would follow from Conjecture 1. See also [7] for further spectrum related results.

Random graphs tend to be expanding and hamiltonian if they are not too sparse. E.g., d-regular random graphs with $d \geq 3$ are almost surely expanding for a linear function f and hamiltonian. Conjecture 1 (or any linear bound on f) would imply that they are hamiltonian just because they are expanding if d is sufficiently large.

The proof of Theorem 1 makes use of a variant of Posa's Lemma (see, e.g. Bondy [2, Lemma 6.3] for another variant) that guarantees long paths in a well-expanding graph. The method used to prove Posa's Lemma by constructively flipping over parts of the path is not compatible with directed paths in digraphs. Therefore a completely different method is needed to prove the existence of a directed hamiltonian cycle in an f-expanding digraph. Unfortunately, instead of a quadratic function f in the undirected case we only succeeded to establish an exponential function f for digraphs. Note that (di-)graphs with $\mathcal{O}(n^{\frac{3}{2}} \log n)$ edges can be f-connected (f-expanding, resp.) with a quadratic function f, while quadratically many edges are needed in the case of an exponential function.

Theorem 2 *Every f-expanding digraph D with $f(k) \geq 3^{k+1}$ of sufficiently large order n has a directed hamiltonian cycle.*

From this we can derive a sufficient degree sequence condition for a directed hamiltonian cycle starting with constant terms in the sequence which seemed to be missing so far.

We have no indication that the optimal functions f are substantially different in the undirected and the directed case. In fact, we believe that a polynomial bound should suffice in the directed case as well.

Since the bound for digraphs is huge, we tried to establish better results for more restricted classes of digraphs. In fact, for a fairly general class, the so-called semicomplete multipartite digraphs, we succeeded to obtain a bound for f, which is best possible for this class to imply a directed hamiltonian cycle.

Since the bounds we could prove in the case of hamiltonian cycles for general graphs and digraphs are probably far from being optimal, we can investigate concepts that are easier to achieve like long cycles and 2-factors. It turns out that in the case of long cycles we get that every linear function f implies a cycle of length linear in n in the directed and in the undirected case by almost the same proof, which is best possible up to a constant factor. More precisely, the bound $f(k) \geq ck$ implies a (directed) cycle of length at least $(\frac{c}{c+2})^2 n$ in an f-connected graph (f-expanding digraph, resp.). Note that $(\frac{c}{c+2})^2 \to 1$ as $c \to \infty$. So for large linear functions f we get cycles containing all but a small fraction of the vertices of the graph. If $c \geq 2$ we can further improve the bound to guarantee cycles of length at least $\frac{c+1}{c+2}n$ in the undirected case with the use of Posa's Lemma.

For 2-factors and for the corresponding concept of cycle factors in digraphs we even get the optimal linear function $f(k) = 2k + 1$ in the directed and the undirected case by two completely different proofs. Note that a (directed) hamiltonian cycle is a special case of a 2-factor (cycle factor, resp.).

Finally, with some standard reasoning we can relate f-connectedness for undirected regular graphs to the spectrum of the adjacency matrix of the graph. The spectrum can be used to bound expansion properties of the graph from below. As an example, we can derive properties of the so-called Ramanujan graphs [8], which are graphs that are in some sense optimally expanding from the view of the spectrum. Infinite sequences of d-regular Ramanujan graphs are known for an infinite number of values d. Using our previous results we can show that non-bipartite d-regular Ramanujan graphs with $d \geq 35$ contain a 2-factor, a cycle of length at least $3n/4$, and, if Conjecture 1 is valid, even a hamiltonian cycle.

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Paths and Cycles Spanning Hypercubes *

Rostislav Caha, Václav Koubek †

This talk continues on papers [1] and [2]. T. Dvořák has proved the following

Statement 1 [2] *For every hypercube* $\mathbf{Q}$ *and for every pair of vertices* $\{A, B\}$ *and* $\{C, D\}$ *such that both* $\operatorname{dist}(A, B)$ *and* $\operatorname{dist}(C, D)$ *are odd, and* $\{A, B\} \cap \{C, D\} = \emptyset$ *there exist two vertex-disjoint paths* $\mathbf{P}_1$ *between* A *and* B *and* $\mathbf{P}_2$ *between* C *and* D *spanning* $\mathbf{Q}$ *(i.e.* $V(\mathbf{P}_1) \cup V(\mathbf{P}_2) = V(\mathbf{Q})$*). Moreover, if* $\operatorname{dist}(A, B) = 1$ *and either* $\dim(\mathbf{Q}) \neq 3$ *or* $\operatorname{dist}(\{A, B\}, \{C, D\}) \neq 2$ *then we can assume that length of* $\mathbf{P}_1$ *is* 1.

The first aim is to generalize this result. It is easy to see that there exists a triple $\{A, B\}$, $\{C, D\}$, $\{E, F\}$ of vertex-pairs in a hypercube $\mathbf{Q}$ with $\dim(\mathbf{Q}) = 3$ and $\operatorname{dist}(A, B) = \operatorname{dist}(C, D) = \operatorname{dist}(E, F) = 1$ such that there exist no three vertex-disjoint paths $\mathbf{P}_1$ between A and B, $\mathbf{P}_2$ between C and D and $\mathbf{P}_3$ between E and F spanning $\mathbf{Q}$. But there exists a greater hypercube $\mathbf{Q}'$ such that $\mathbf{Q}$ is a subcube of $\mathbf{Q}'$ and there exist three vertex-disjoint paths $\mathbf{P}_1$ between A and B, $\mathbf{P}_2$ between C and D and $\mathbf{P}_3$ between E and F spanning $\mathbf{Q}'$. We generalize this observation as follows:

Theorem 1 *For every hypercube* $\mathbf{Q}$ *and for every* n*-tuple* $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ *of pairwise disjoint vertex-pairs such that* $\operatorname{dist}(A_i, B_i)$ *is odd for all* $i = 1, 2, \ldots, n$ *there exist a hypercube* $\mathbf{Q}'$ *containing* $\mathbf{Q}$ *as an induced subgraph and a family* $\{\mathbf{P}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint paths spanning* $\mathbf{Q}'$ *such that* $\mathbf{P}_i$ *is a path between* A_i *and* B_i *for all* $i = 1, 2, \ldots, n$.

There exists a function π_1 *from* $\mathcal{N}$ *(the set of all natural numbers) into itself such that for every* $n \in \mathcal{N}$ *we have:*

> *if* $\mathbf{Q}$ *is a hypercube with* $\dim(\mathbf{Q}) \geq \pi_1(n)$ *then for every* n*-tuple* $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ *of pairwise disjoint vertex-pairs such that* $\operatorname{dist}(A_i, B_i)$ *is odd for all* $i = 1, 2, \ldots, n$ *there exists a family* $\{\mathbf{P}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint paths spanning* $\mathbf{Q}$ *such that* $\mathbf{P}_i$ *is a path between* A_i *and* B_i *for all* $i = 1, 2, \ldots, n$.

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This result can be modified for cycles

Corollary 1 *For every hypercube* $\mathbf{Q}$ *and every* n*-tuple* $\{\mathcal{E}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise disjoint edges there exist a hypercube* $\mathbf{Q}'$ *containing* $\mathbf{Q}$ *as an induced subgraph and a family* $\{\mathbf{C}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint cycles spanning* $\mathbf{Q}'$ *such that* $\mathbf{C}_i$ *contains the edge* $\mathcal{E}_i$ *for all* $i = 1, 2, \ldots, n$.

There exists a function σ_1 *from* $\mathcal{N}$ *into itself such that for every* $n \in \mathcal{N}$ *we have:*

> *if* $\mathbf{Q}$ *is a hypercube with* $\dim(\mathbf{Q}) \geq \sigma_1(n)$ *then for every* n*-tuple* $\{\mathcal{E}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise disjoint edges there exists a family* $\{\mathbf{C}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint cycles spanning* $\mathbf{Q}$ *such that* $\mathbf{C}_i$ *contains the edge* $\mathcal{E}_i$ *for all* $i = 1, 2, \ldots, n$.

Clearly, length l_i of $\mathbf{P}_i$ must be odd and $l_i \geq \operatorname{dist}(A_i, B_i)$ for all $i = 1, 2, \ldots, n$, and $n + \sum_{i=1}^{n} l_i = 2^m$ where $m = \dim(\mathbf{Q}')$. On the other hand for every hypercube $\mathbf{Q}$ with $\dim(\mathbf{Q}) \geq 3$ there exists a triple $\{\{A, B\}, \{C, D\}, \{E, F\}\}$ of pairwise disjoint vertex-pairs such that there exists no family $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ of pairwise disjoint paths spanning $\mathbf{Q}$ such that $\mathbf{P}_1$ is a path between A and B of length $\operatorname{dist}(A, B)$, $\mathbf{P}_2$ is a path between C and D of length $\operatorname{dist}(C, D)$ and $\mathbf{P}_3$ is a path between E and F. This motivates the following theorem.

Theorem 2 *There exist functions* π_2, σ_2 *from* $\mathcal{N}$ *into itself such that for every* $n \in \mathcal{N}$ *we have:*

> *if* $\mathbf{Q}$ *is a hypercube with* $\dim(\mathbf{Q}) \geq \pi_2(n)$ *then for every* n*-tuple* $\{(A_i, B_i, m_i) \mid i = 1, 2, \ldots, n\}$ *such that* $\{A_i, B_i\}$ *for* $i = 1, 2, \ldots, n$ *are pairwise disjoint vertex-pairs,* $\operatorname{dist}(A_i, B_i)$ *and* $m_i \in \mathcal{N}$ *are odd and* $m_i > \operatorname{dist}(A_i, B_i)$ *for all* $i = 1, 2, \ldots, n$, *and* $n + \sum_{i=1}^{n} m_i = 2^{\dim(\mathbf{Q})}$ *there exists a family* $\{\mathbf{P}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint paths spanning* $\mathbf{Q}$ *such that* $\mathbf{P}_i$ *is a path between* A_i *and* B_i *of length* m_i *for all* $i = 1, 2, \ldots, n$;
>
> *if* $\mathbf{Q}$ *is a hypercube with* $\dim(\mathbf{Q}) \geq \sigma_2(n)$ *then for every* n*-tuple* $\{(\mathcal{E}_i, m_i) \mid i = 1, 2, \ldots, n\}$ *such that* $\mathcal{E}_i$ *for* $i = 1, 2, \ldots, n$ *pairwise disjoint edges there exists a family* $\{\mathbf{C}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint cycles spanning* $\mathbf{Q}$ *such that* $\mathbf{C}_i$ *contains the edge* $\mathcal{E}_i$ *for all* $i = 1, 2, \ldots, n$.

For a cube with $\dim(\mathbf{Q}) > 2$ it is easy to find pairs $\{A, B\}$, $\{C, D\}$ such that $\operatorname{dist}(A, B)$ and $\operatorname{dist}(C, D)$ are even and there exist disjoint paths $\mathbf{P}_1$ between A and B, $\mathbf{P}_2$ between C and D spanning $\mathbf{Q}$. This fact leads to the following results.

An n-tuple $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ of pairwise disjoint vertex-pairs from a hypercube $\mathbf{Q}$ is called *path-building* if the number of $i = 1, 2, \ldots, n$ such that size of A_i is even and $\operatorname{dist}(A_i, B_i)$ is even is equal to the number of $i = 1, 2, \ldots, n$ such that size of A_i is odd and $\operatorname{dist}(A_i, B_i)$ is even.

Theorem 3 *For an n-tuple* $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ *of pairwise disjoint vertex-pairs from a hypercube* $\mathbf{Q}$ *there exist a hypercube* $\mathbf{Q}'$ *containing* $\mathbf{Q}$ *as an induced subgraph and a family* $\{\mathbf{P}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint paths spanning* $\mathbf{Q}'$ *such that* $\mathbf{P}_i$ *is a path between* A_i *and* B_i *for all* $i = 1, 2, \ldots, n$ *if and only if* $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ *is path-building.*

There exist functions π_3 *and* π_4 *from* $\mathcal{N}$ *into itself such that for every* $n \in \mathcal{N}$*:*

if $\mathbf{Q}$ *is a hypercube with* $\dim(\mathbf{Q}) \geq \pi_3(n)$ *then for every path-building n-tuple* $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ *there exists a family* $\{\mathbf{P}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint paths spanning* $\mathbf{Q}$ *such that* $\mathbf{P}_i$ *is a path between* A_i *and* B_i *for all* $i = 1, 2, \ldots, n$*;*

if $\mathbf{Q}$ *is a hypercube with* $\dim(\mathbf{Q}) \geq \pi_4(n)$ *then for every n-tuple* $\{(A_i, B_i, m_i) \mid i = 1, 2, \ldots, n\}$ *such that* $\{(A_i, B_i) \mid i = 1, 2, \ldots, n\}$ *is path-building,* $m_i \in \mathcal{N}$ *has the same parity as* $\mathrm{dist}(A_i, B_i)$ *and* $m_i > \mathrm{dist}(A_i, B_i)$ *for all* $i = 1, 2, \ldots, n$*, and* $n + \sum_{i=1}^{n} m_i = 2^{\dim(\mathbf{Q})}$ *there exists a family* $\{\mathbf{P}_i \mid i = 1, 2, \ldots, n\}$ *of pairwise vertex-disjoint paths spanning* $\mathbf{Q}$ *such that* $\mathbf{P}_i$ *is a path between* A_i *and* B_i *of length* m_i *for all* $i = 1, 2, \ldots, n$*.*

The second aim of this talk is to strengthen a result from [1]. Let $\mathbf{Q}$ be a hypercube, $\mathcal{D}$ be a set of vertices from $\mathbf{Q}$, A and B be vertices from $\mathcal{D}$ and $\mathcal{F}$ be a set of edges from $\mathbf{Q}$ over $\mathcal{D}$. We say that

$\mathcal{F}$ is *star-free* if $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 = \emptyset$ for any triple of distinct edges from $\mathcal{F}$;

$\mathcal{F}$ is $\mathcal{D}$-*cycle-free* if any cycle $\mathbf{C}$ of length greater than 2 consisting of edges from $\mathcal{F}$ contains all edges from $\mathcal{F}$ and it is $\mathcal{D}$-spanning cycle;

$\mathcal{F}$ is $(\mathcal{D}, A, B)$-*path-free* if there exists no cycle $\mathbf{C}$ of length greater than 2 consisting of edges from $\mathcal{F}$ and any path $\mathbf{P}$ between A and B consisting of edges from $\mathcal{F}$ contains all edges from $\mathcal{F}$ and it is $\mathcal{D}$-spanning path;

a vertex $D \in \mathcal{D}$ is called $\mathcal{F}$-*potential end-vertex* if there exists at most one edge $\mathcal{E} \in \mathcal{F}$ containing D.

We say that $\mathcal{F}$ satisfies a $\mathcal{D}$-*cycle conditions* if $\mathcal{F}$ is star-free and $\mathcal{D}$-cycle-free. We say that $\mathcal{F}$ satisfies $(\mathcal{D}, A, B)$-*path conditions* if size of $\Delta(A, B)$ is odd, $\mathcal{F}$ is star-free, $\mathcal{D}$-path-free, and A and B are $\mathcal{F}$-potential end-vertices.

Obviously, if $\mathcal{D}$ is a set of vertices of a hypercube $\mathbf{Q}$ and $\mathcal{F}$ is a set of edges over $\mathcal{D}$ such that there exists a $\mathcal{D}$-spanning path between A and B containing edges from $\mathcal{F}$ for some vertices $A, B \in \mathcal{D}$ then $\mathcal{F}$ satisfies $(\mathcal{D}, A, B)$-path condition. If there exists a $\mathcal{D}$-spanning cycle containing edges from $\mathcal{F}$, then $\mathcal{F}$ satisfies the $\mathcal{D}$-cycle condition. The authors have proved

Theorem 4 [1] *There exist functions* π_0 *and* σ_0 *from* $\mathcal{N}$ *into itself such that for every* $n \in \mathcal{N}$

1. *a hypercube* $\mathbf{Q}$ *satisfies* $\dim(\mathbf{Q}) \geq \pi_0(n)$ *if and only if for every pair of distinct vertices* A *and* B *from* $\mathbf{Q}$ *and for every family* $\mathcal{F}$ *of edges from* $\mathbf{Q}$ *with* $|\mathcal{F}| \leq n$

   there exists a hamiltonian path $\mathbf{P}$ *from* A *to* B *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $(V(\mathbf{Q}), A, B)$*-path condition;*

2. *a hypercube* $\mathbf{Q}$ *satisfies* $\dim(\mathbf{Q}) \geq \sigma_0(n)$ *if and only if for every family* $\mathcal{F}$ *of edges from* $\mathbf{Q}$ *with* $|\mathcal{F}| \leq n$

   there exists a hamiltonian cycle $\mathbf{C}$ *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $V(\mathbf{Q})$*-cycle condition.*

T. Dvořák has reckoned the precise value of functions σ_0 and π_0.

Theorem 5 [2] *The following is true*

$$\pi_0(n) = \begin{cases} 0 & \text{if } n \leq 1, \\ \lceil \frac{n+4}{2} \rceil & \text{if } n > 1 \end{cases} \quad \text{and} \quad \sigma_0(n) = \begin{cases} 0 & \text{if } n \leq 3, \\ \lceil \frac{n+3}{2} \rceil & \text{if } n > 4. \end{cases}$$

For a natural number n, let $\lambda(n)$ denote the number of 1 in the binary expression of n and $\delta(n)$ denote the greatest natural number such that n is divided by $2^{\delta(n)}$. It was shown for dense sets (for definition see [1]):

Theorem 6 [1] *There exist functions* π_5 *and* σ_5 *from* $\mathcal{N}$ *into itself such that for every* $n \in \mathcal{N}$

1. *if* $\mathcal{D}$ *is a dense set in a hypercube* $\mathbf{Q}$ *with* $\delta(|\mathcal{D}|) \geq \pi_5(n)$ *then for every pair of distinct vertices* A *and* B *from* $\mathcal{D}$ *and for every family* $\mathcal{F}$ *of edges from* $\mathbf{Q}$ *over* $\mathcal{D}$ *with* $|\mathcal{F}| \leq n$ *there exists a* $\mathcal{D}$*-spanning path* $\mathbf{P}$ *between* A *and* B *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $(\mathcal{D}, A, B)$*-path condition;*

2. *if* $\mathcal{D}$ *is a dense set in a hypercube* $\mathbf{Q}$ *with* $\delta(|\mathcal{D}|) \geq \sigma_5(n)$ *then for every family* $\mathcal{F}$ *of edges from* $\mathbf{Q}$ *over* $\mathcal{D}$ *with* $|\mathcal{F}| \leq n$ *there exists a* $\mathcal{D}$*-spanning cycle* $\mathbf{C}$ *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $\mathcal{D}$*-cycle condition.*

We strengthened this result as follows

Theorem 7 *For every natural number* $n > 1$*, a dense set* $\mathcal{D}$ *satisfies* $\delta(|\mathcal{D}|) + \lambda(|\mathcal{D}|) \geq \pi_0(n) - 1$ *if and only if for every pair of distinct vertices* A *and* B *from* $\mathcal{D}$ *and for every family* $\mathcal{F}$ *of edges over* $\mathcal{D}$ *with* $|\mathcal{F}| \leq n$

there exists a $\mathcal{D}$*-spanning path* $\mathbf{P}$ *between* A *and* B *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $(\mathcal{D}, A, B)$*-path condition.*

For every natural number $n > 3$*, a dense set* $\mathcal{D}$ *satisfies* $\delta(|\mathcal{D}|) + \lambda(|\mathcal{D}|) - 1 \geq \pi_0(n)$ *if and only if for every family* $\mathcal{F}$ *of edges over* $\mathcal{D}$ *with* $|\mathcal{F}| \leq n$

there exists a $\mathcal{D}$*-spanning cycle* $\mathbf{C}$ *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $\mathcal{D}$*-cycle condition.*

The results in [1] complete Theorem 9.

Statement 2 [1] *Let* $n = 0$ *or* $n = 1$*. For a dense set* $\mathcal{D}$*, for every pair of distinct vertices* A *and* B *from* $\mathcal{D}$ *and for every family* $\mathcal{F}$ *of edges over* $\mathcal{D}$ *with* $|\mathcal{F}| \leq n$

there exists a $\mathcal{D}$*-spanning path* $\mathbf{P}$ *between* A *and* B *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $(\mathcal{D}, A, B)$*-path condition*

just when $\delta(|\mathcal{D}|) > 1$ *or* $\lambda(|\mathcal{D}|) \neq 2$*.*

Let $n \leq 3$ *be a natural number. For a dense set* $\mathcal{D}$ *and for every family* $\mathcal{F}$ *of edges over* $\mathcal{D}$ *with* $|\mathcal{F}| \leq n$

there exists a $\mathcal{D}$*-spanning cycle* $\mathbf{C}$ *containing all edges from* $\mathcal{F}$ *if and only if* $\mathcal{F}$ *satisfies the* $\mathcal{D}$*-cycle condition*

just when $\delta(|\mathcal{D}|) > 1$ *or* $\lambda(|\mathcal{D}|) \neq 2$*.*

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On a generalization of the stable roommates problem

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Abstract

We consider two generalizations of the stable roommates problem: we allow multiple partnerships and parallel edges in the underlying graph. We reduce both problems to the stable roommates problem, extend Irving's algorithm and deduce a structural result of stable b-matchings.

1 Introduction

We study a generalization of the stable marriage problem of Gale and Shapley [6]. Let $G = (V, E)$ be a finite simple graph and for each vertex v a linear order $<_v$ on the set of neighbours $\Gamma(v)$ of v in G is given. It is convenient to denote by $\mathcal{O}$ the set of linear orders $<_v$, and we say that $(G, \mathcal{O})$ is an instance of the *stable roommates problem* (or SR, for short). A *stable matching of SR instance* $(G, \mathcal{O})$ is a set M of disjoint edges of E in such a way that for each edge e of E either $e \in M$ or e has a vertex (say) v and M has an edge (say) m such that $m <_v e$ holds. It is usual to call the vertices of G agents and the linear order $<_v$ the preference order of agent v. A partnership situation is stable if no two agents would prefer to quit there eventual partnership to become partners.

Gale and Shapley in [6] have described the so called *stable marriage problem*, a special case of the stable roommates problem with G bipartite. They exhibited an efficient algorithm that finds a stable matching for this case. In contrast to this, a SR instance might admit no stable matching. The first efficient algorithm to decide if a given SR instance has a stable matching is due to Irving [7]. Such algorithms were found by Feder [3], Subramanian [9], and Tan and Hsueh [10].

A common feature of the stable marriage and stable roommates problems is that graph G is simple. This might not be a reasonable assumption. In Section 2, we describe a more general model, in which several different partnerships may be possible between two agents,

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and and agents can participate in multiple partnerships. We reduce this more general problem to the stable matching problem. Our motivation is a result of Cechlárová and Ferková [1] on a manpower scheduling problem for airplanes, and the result of Irving and Scott [8] on so called stable fixtures. In Section 3, we extend Irving's algorithm and show a structural result of stable b-matchings. EGRES report [2] is a detailed version of this work.

2 Generalized stable matching problems

Let finite graph $G = (V, E)$ and quota function $b : V \to \mathbb{N}$ be given and for each vertex v of V we have a linear order $\prec_v$ on the set $D(v)$ of edges incident with v. If $\mathcal{O}$ denotes the set of linear orders $\prec_v$ then $(G, \mathcal{O})$ and $(G, \mathcal{O}, b)$ are called instances of the *stable activities problem*, (or SA) and the *stable multiple activities problem* (or SMA), respectively. We say that subset F of E (b-)dominates edge e of E if there is a vertex v of V and an edge f (different edges $f_1, \dots f_{b(v)}$) of F such that $f \prec_v e$ ($f_i \prec_v e$ for $1 \leq i \leq b(v)$). Let $\mathcal{D}(F)$ ($\mathcal{D}^b(F)$) denote the set of edges that are (b-)dominated by F. A *stable (b-)matching* of SA instance $(G, \mathcal{O})$ (SMA instance $(G, \mathcal{O}, b)$) is a subset M of E with the property that $\mathcal{D}(M) = E \setminus M$ ($\mathcal{D}^b(M) = E \setminus M$). Clearly, a stable matching of a SR instance is a special case of a stable matching of a SA instance. We shall reduce the SMA to the SR. Fix SA instance $(G, \mathcal{O})$. We define a new SR instance $(G', \mathcal{O}')$ where $G' = (V', E')$ is a simple graph and $\mathcal{O}'$ is a set of linear orders $\prec_{v'}$.

To define G', substitute each edge $e = uv$ of G by a 6-cycle with two hanging edges as in Figure 1. The linear orders on the neighbours are represented by the numbers on the edges around a vertex, so if edge e was the ith best in $<_u$ and jth best in $<_v$ then u_0^e is ith best in $<_u$ and v_0^e is jth best in $<_v$.

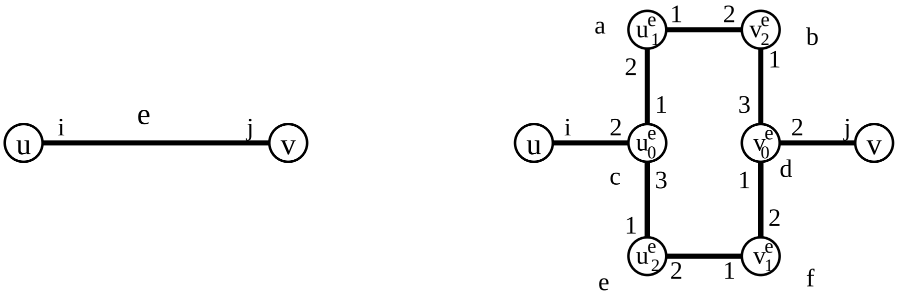

Figure 1: The gadget to be introduced along edge e for the definition of G'

Theorem 1 *Let $(G, \mathcal{O})$ be an instance of the SA, and let SR instance $(G', \mathcal{O}')$ be constructed from $(G, \mathcal{O})$ as described above. There is a stable matching of $(G, \mathcal{O})$ if and only if there is a stable matching of $(G', \mathcal{O}')$. Moreover, any stable matching of $(G', \mathcal{O}')$ induces a stable matching of $(G, \mathcal{O})$ and each stable matching of $(G, \mathcal{O})$ can be induced by some stable matching of $(G', \mathcal{O}')$.*

The college admission problem can be reduced to the stable marriage problem by the *b-expansion* of the underlying instance defined below. Let $(G, \mathcal{O}, b)$ be an instance of the SMA. Define instance $(G^b, \mathcal{O}^b)$ of the SA by $V(G^b) := \{v^i : v \in V(G) \text{ and } i = 1, 2, \ldots, b(v)\}$, $E(G^b) := \{e^{i,j} = u^i v^j : e = uv \in E(G) \text{ and } u^i, v^j \in V(G^b)\}$ and $\mathcal{O}^b := \{\prec_{v^i} : v(i) \in V(G^b)\}$, where $u^i v^j \prec_{u^i} u^i w^k$ iff $uv \prec_u uw$ or $(v = w$ and $j < k)$. Roughly speaking, we introduce $b(v)$ equivalent copies instead of each vertex v of G, and the linear orders are inherited from the SMA instance. SMA instance $(G, \mathcal{O}, b)$ has the *many-to-one property* (or $(G, \mathcal{O}, b)$ *is mto*, for short) if $b(u) = 1$ or $b(v) = 1$ for each edge $e = uv$ of $E(G)$.

Lemma 1 *If $(G, \mathcal{O}, b)$ is mto then there is a sb-m of $(G, \mathcal{O}, b)$ if and only if there is a stable matching of $(G^b, \mathcal{O}^b)$. Any stable matching of $(G^b, \mathcal{O}^b)$ induces a sb-m of $(G, \mathcal{O}, b)$ and each sb-m of $(G, \mathcal{O}, b)$ can be induced this way.*

Hence, we can reduce any mto SMA to SA. Moreover, the general SMA can be reduced to mto SMA by our previous construction. For SMA instance $(G, \mathcal{O}, b)$, define SMA instance $(G', \mathcal{O}', b')$ such that G' and $\mathcal{O}'$ are as before, and $b'(v) := b(v)$ if v is a vertex of G and $b'(v) := 1$ otherwise. Clearly, SMA instance $(G', \mathcal{O}', b')$ is mto. Lemmas 1 and 2 reduce the SMA to SR.

Lemma 2 *There is a sb-m of SMA instance $(G, \mathcal{O}, b)$ if and only if there is a sb′-m of $(G', \mathcal{O}', b')$. Moreover, any sb′-m of $(G', \mathcal{O}', b')$ induces a sb-m of $(G, \mathcal{O}, b)$ and each sb-m of $(G, \mathcal{O}, b)$ can be induced by some sb′-m of $(G', \mathcal{O}', b')$.*

3 An extension of Irving's algorithm

In this Section, we extend Irving's algorithm [7] to an algorithm for SMA. The input of the extension of Irving's algorithm (the *EI algorithm*, for short) is an instance $(G, \mathcal{O}, b) = (G_0, \mathcal{O}_0, b)$ of the SR, and its output is either a conclusion that no sb-m of $(G, \mathcal{O}, b)$ exists or it exhibits a sb-m of $(G, \mathcal{O}, b)$. In the ith step, the EI algorithm transforms $(G_i, \mathcal{O}_i, b)$ to instance $(G_{i+1}, \mathcal{O}_{i+1}, b)$ such that

$$\begin{gathered} G_{i+1} \text{ is a proper subgraph of } G_i, \\ \text{if } (G_i, \mathcal{O}_i, b) \text{ has a s}b\text{-m then } (G_{i+1}, \mathcal{O}_{i+1}, b) \text{ has one,} \\ \text{and any s}b\text{-m of } (G_{i+1}, \mathcal{O}_{i+1}, b) \text{ is a s}b\text{-m of } (G_i, \mathcal{O}_i, b). \end{gathered} \tag{1}$$

Define $B(u, G_i)$ as the set of the $b(u)$ best edges of G_i in $\prec_u$ and let $D(u, G_i)$ denote those edges f of G_i that can only be b-dominated at u:

$$D(u, G_i) := \{f = ux \in E(G_i) : f \in B(x, G_i)\} .$$

SMA instance $(G_i, \mathcal{O}_i, b)$ has the *first-last property* if for each vertex u of G_i and for each edge $e \in E(G_i)$ incident with u

$$|\{f \in D(u, G_i) : f \prec_u e\}| < b(u) \tag{2}$$

holds. If $(G_i, \mathcal{O}_i, b)$ has the first-last property then either $E(G_i)$ is a b-matching M, hence a sb-m of $(G_0, \mathcal{O}_0, b)$ by (1); or the algorithm finds and eliminates a so called rotation to get $(G_{i+1}, \mathcal{O}_{i+1}, b)$. A *rotation* of $(G_i, \mathcal{O}_i, b)$ is a pair of edge sets $R = (\{e_1, e_2, \ldots, e_k\}, \{f_1, f_2, \ldots, f_k\})$ in such a way that $e_j = u_j v_j$, $f_j = u_j v_{j+1}$ (addition is modulo k), e_j is maximal in $\prec_{v_j}$ and f_j is the $(b(u_j)+1)$st least element of $\prec_{u_j}$. The following pseudocode summarizes the EI algorithm.

Input: SMA instance $(G_0, \mathcal{O}_0, b)$. **Output**: sb-m of $(G_0, \mathcal{O}_0, b)$, if exists

```
begin
  i:=0
  while E(G_i) is not a b-matching do
    begin
      if (G_i, O_i, b) does not have first-last property
      then find and delelte e ∈ E(G_i) that violate (2) to get (G_{i+1}, O_{i+1}, b)
      else find rotation ({e_1, e_2, ..., e_k}, {f_1, f_2, ..., f_k})
           if {e_1, e_2, ..., e_k} = {f_1, f_2, ..., f_k} then STOP: No sb-m exists
           else delelte {e_1, e_2, ..., e_k} to get (G_{i+1}, O_{i+1}, b)
           end if
      end if
      increase i by 1
    end
  STOP: Output stable b-matching E(G_i) of (G_0, O_0, b)
end
```

Theorem 2 *Let $(G_0, \mathcal{O}_0, b)$ be an instance of SMA and m be the number of edges of G_0. The EI algorithm finds a sb-m of $(G_0, \mathcal{O}_0, b)$ if exists in $O(m^2)$ time.*

At last, we deduce an interesting splitting property of sb-m's directly from the EI algorithm. This property was recently observed by Fleiner [5] and Teo *et. al.* [11] for bipartite instances, and Fleiner has reduced the general property to the bipartite case in [4]. We say that SA instance $(G^{(b)}, \mathcal{O}^{(b)})$ is a *b-split* of SMA instance $(G, \mathcal{O}, b)$ if G is a contraction minor of $G^{(b)}$ in such a way that each vertex v of G is contracted out of $b(v)$ vertices of $G^{(b)}$, moreover, the linear ordes in $\mathcal{O}^{(b)}$ are restricted orders of $\mathcal{O}$ to the supervertices. Rougly speaking, the b-split and the b-expansion has the same vertex set, but for each edge of G the b-expansion has many copies while the b-split has only one. There is a natural bijection between edges of G and that of $G^{(b)}$, hence for any subset F of edges of G we can talk about the corresponding subset $F^{(b)}$ of edges of $G^{(b)}$. The following theorem is formulated for bipartite SMA instances in [5] and for bipartite SMA instances with the many-to-one property in [11].

Theorem 3 *Any SMA instance $(G, \mathcal{O}, b)$ has a b-split $(G^{(b)}, \mathcal{O}^{(b)})$ in such a way that for any sb-m M of $(G, \mathcal{O}, b)$, $M^{(b)}$ is a stable matching of $(G^{(b)}, \mathcal{O}^{(b)})$.*

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The d-precoloring problem on k-degenerate graphs*

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Definitions. Let $G = (V, E)$ be a finite simple graph. For a set of colors S, a proper coloring of G is a mapping $f : V \to S$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$. An S-*precoloring* f' of G is any proper coloring $f' : V' \to S$ of the subgraph of G induced by $V' \subseteq V$. We say that a (proper) coloring f of G, $f : V \to S$, is an *extension* of the S-precoloring f' if $f(v) = f'(v)$ for each vertex $v \in V'$. The subset $V' \subseteq V$ is the set of *precolored* vertices and $V \setminus V'$ is the set of *precolorless* vertices. For each precolorless vertex v, the *admissible set* of colors for v is the set of all colors of S which do not occur on precolored neighbors of v. An S-precoloring of G is called $d[S]$-precoloring, if each color of S is used to at most d ($d \geq 0$) precolored vertices. In such case G is called d-precolored graph. A graph G is k-*degenerate* if each subgraph of G has a vertex of degree at most k. k-trees (introduced in [1]) are defined recursively as follows: a clique with $(k+1)$ vertices is a k-tree and a k-tree with $(n+1)$ vertices can be obtained from a k-tree with n vertices by adding a new vertex adjacent to all vertices of a k-clique. A partial k-tree is a subgraph of a k-tree.

We investigate the following problem:

d-Precoloring Extension Problem (d-PrExt), $d \geq 0$ is a fixed integer
Instance: A set of colors S, a graph G and a $d[S]$-precoloring of G.
Question: Is there an extension of the $d[S]$-precoloring to the entire graph G (using only colors of S)?

The d-PrExt problem is a subproblem of the Precoloring Extension problem (PrExt). In the PrExt problem there is no restriction of the number of precolored vertices for any color of S. PrExt and d-PrExt arise in practical applications such as scheduling or VLSI-layout and are initiated in [2].

Obviously, 0-PrExt is the problem to decide whether $\chi(G) \leq |S|$. Hence the chromatic number problem, which is NP-complete for general graphs ([4]), is a subproblem of the d-

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PrExt problem. But d-PrExt is NP-complete even for restricted graph classes: e.g. 1-PrExt problem for bipartite graph (and $|S| = 3$) [3], 1-PrExt problem for permutation graphs [7] and 2-PrExt problem for interval graphs [2]. On the other hand, PrExt can be solved in polynomial time for (a) split graphs, (b) complements of bipartite graphs [5], (c) for partial k-trees [6], [8] (if $|S|$ is a constant in linear time [6]) and in linear time also for trees. Further research was continued in the series of papers, see [9] and also [10] for a survey and references therein.

In this contribution we consider some combinatorial and algorithmic aspects of the d-PrExt problem restricted to the class of k-degenerate graphs which includes some well studied classes of graphs such as partial k-trees, planar (5-degenerate) and outer-planar graphs (2-degenerate).

Structural results

An interesting combinatorial question relevant to d-PrExt is the following: if there is an extension of a $d[S]$-precoloring to the entire graph G (using only colors of S), how many colors of S are necessary to the precolorless vertices of G? We define the d-chromatic number of G, $\chi_d(G)$, as the minimum number m such that for every set of colors S and every extendible $d[S]$-precoloring of G, there is an extension to the entire graph G that uses at most m colors of S to the precolorless vertices. For a class of graphs $\mathcal{G}$, let $\chi_d = \sup_{G\in\mathcal{G}} \chi_d(G)$ ($+\infty$ if unbounded). We prove that for the class of k-degenerate graphs χ_d can be bounded by a function of d and k. We note, that e.g. for the class of bipartite graphs, χ_d is unbounded.

Theorem 1 *Let $k \geq 1$, $d > 0$ be fixed integers and let G be a d-precolored k-degenerate graph. If there is an extension of given d-precoloring to the entire graph G, then there is also an extension which uses at most $k(d+2)$ colors, i.e. $\chi_d(G) \leq k(d+2)$.*

The previous theorem implies that $\chi_d \leq 5(d+2)$ for planar graphs and $\chi_d \leq 2(d+2)$ for outer planar graphs. For planar graphs a stronger result can be proved:

Theorem 2 *Let $d > 0$ be a fixed integer and let G be a d-precolored planar graph. If there is an extension of given d-precoloring to the entire graph G, then there is also an extension which uses at most $3(d+2)$ colors, i.e. $\chi_d(G) \leq 3(d+2)$.*

In the following theorem we establish the optimal value χ_d for trees and lower bounds for some subclasses of k-degenerate graphs.

Theorem 3 *Let $d \geq 0$ be a fixed integer. Then:*

(a) $\chi_d = d+2$ *for the class of d-precolored trees,*

(b) $\chi_1 \geq 4$ *and* $\chi_d \geq d+4$, $d \geq 2$, *for the class of outer-planar graphs,*

(c) $\chi_d \geq 2d+4$, $d \geq 1$, *for the class of planar graphs,*

(d) $\chi_d \geq k(d+1)+1$, $d \geq 1$, *for the class of d-precolored partial k-trees,* $k \geq 1$.

Algorithm results

In this section we present a linear time algorithm for the d-PrExt problem and its search version (i) restricted to the class of k-degenerate graphs with a sufficient large set S of available colors, (ii) restricted to the class of partial k-trees without any size restriction on the set of available colors. The result for the d-PrExt problem restricted to the class of partial k-trees improves the best known result for this class with running time $O(|V||S|^{k+1})$ ([6]). Linear time algorithms for many intractable problems over the class of partial k-trees follow from a general result: all graph properties definable in monadic second-order logic with quantification over vertex and edge sets can be decided in linear time for partial k-trees. However, PrExt problem does not appear to be directly expressible by these means. It is an open problem, whether the property "a precoloring of a graph is extendible to all vertices" is an EMS-property when a set of colors is a part of the input. Even though we do not know the answer to the previous problem, due to some structural results for the d-PrExt problem on k-degenerate graphs (and hence also partial k-trees), we derive a linear time algorithm for the d-PrExt problem and its search version.

Our result is based on a *safe* reduction, which reduces a precolored k-degenerate graph to a bounded irreducible instance. The basic property of the safe reduction is keeping the same answer to the PrExt problem during reduction.

Definition 1 *Let a set S of colors be fixed and G=(V,E) be a graph with a fixed S-precoloring. For a precolorless vertex $v \in V$, let C_v be the set of colors occurring on precolored neighbors of v and let N_v be the set of precolorless neighbors of v. A vertex v is called safe, if v is precolorless and $|C_v| + |N_v| < |S|$. Deleting the safe vertex from G is called a safe reduction for the S-precolored graph G. An ordering $Q = [v_1, \ldots, v_m]$ of a subset of precolorless vertices from V is called a safe elimination scheme of G if for all i, $i = 1, \ldots, m$, v_i is safe in the graph $G \setminus \{v_1, \ldots, v_{i-1}\}$. An S-precolored graph is called safe-free (or irreducible) if it contains no safe vertex.*

Observation 1 *Let $G = (V, E)$ be an instance of* PrExt *and $Q = [v_1, \ldots, v_m]$ be a safe elimination scheme of G. Then obviously the answer to the* PrExt *problem is the same for G as for any $G \setminus \{v_1, \ldots, v_i\}$, $i = 1, \ldots, m$.*

First, we consider the following task which is solvable in linear time for the class of k-degenerate graphs.

Task A: $k \geq 1$ is a fixed integer (k is not a part of the input)
Input: A set of colors S, a k-degenerate graph G and an S-precoloring of G.
Output: A safe-free subgraph $G \setminus Q$ of G and a safe elimination scheme Q for G.

The reduction of an instance of the PrExt problem (hence also d-PrExt) is crucial for a linear time algorithm for the d-PrExt problem restricted to the class of k-degenerate graphs due to the following theorem. It is also easy to see that a similar theorem cannot hold for the PrExt problem.

Theorem 4 *Let $d, k \geq 1$ be fixed integers and G be a safe-free $d[S]$-precolored k-degenerate graph, $|S| \geq 2k+1$. Then the following hold:*

(a) G contains at most $\alpha(k,d)$ precolorless vertices, where $\alpha(k,d) \leq 2k^2d+kd$, if $2k+1 \leq |S| < 2k(kd+1)$ and $\alpha(k,d) \leq kd$, if $|S| \geq 2k(kd+1)$.

(b) The size of the admissible set for each precolorless vertex is at most $\alpha(k,d)-1$.

Especially, for $|S| \geq 2k(kd+1)$, the number of precolorless vertices in G is at most kd and the size of the admissible set of any precolorless vertex is at most $kd-1$.

Using an algorithm for Task A we can reduce a k-degenerate d-precolored graph G into a safe-free subgraph $G \setminus Q$ (with the safe elimination scheme Q) such that the answers to d-PrExt problem for G and the safe-free subgraph $G \setminus Q$ are the same. Assuming $|S| \geq 2k+1$, a safe-free $d[S]$-precolored subgraph $G \setminus Q$ contains at most $\alpha(k,d)$ precolorless vertices and the size of the admissible set of any of them is at most $\alpha(k,d)-1$. For each precolorless vertex $v \in G \setminus Q$ we construct the admissible set of colors $S \setminus C_v$ (of the cardinality at most $\alpha(k,d)-1$). Then we check by 'brute force' all possible assignments of admissible colors for precolorless vertices of the safe-free graph $G \setminus Q$. Hence we can obtain a linear time algorithm for the d-PrExt problem restricted to the class of k-degenerate graphs.

Obviously, a search version of the d-PrExt problem for k-generate graphs can be solved using a safe elimination scheme Q in linear time.

Theorem 5 *Let d, $k \geq 1$ be fixed integers and let G be a $d[S]$-precolored k-degenerate graph with n vertices, $|S| \geq 2k+1$. Then the d-PrExt problem and its search version can be solved in linear time $O(n)$.*

Theorem 6 *Let $d \geq 1$ be a fixed integer and let G be a $d[S]$-precolored planar graph with n vertices, $|S| \geq 7$. Then d-PrExt problem and its search version can be solved in linear time $O(n)$.*

For 0-PrExt problem the assumption $|S| \geq k+1$ is sufficient to obtain a linear time algorithm restricted to the class of k-degenerate graphs. It implies $|S| \geq 6$ for planar graphs.

Using a linear time algorithm for the PrExt problem with a constant number of colors restricted to partial k-trees [6, Theorem 2.4], we obtain

Theorem 7 *Let $d, k \geq 1$ be fixed integers and let G be a d-precolored partial k-tree with n vertices. Then d-PrExt problem and its search version can be solved in linear time $O(n)$.*

Conclusion

We have presented a linear time algorithm for the search version of d-PrExt for k-degenerate graphs, which uses all colors of S. As follows from Theorem 1 there exists also an extension which uses at most $k(d+2)$ colors to all precolorless vertices. The following natural question arises: Is there a linear time algorithm to extend the $d[S]$-precoloring (if there exists any) with bounded number of colors to the precolorless vertices? Other interesting questions are to tighten the bounds for χ_d for any pair of $k \geq 2$ and $d \geq 1$ for subclasses of k-degenerate graphs.

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Construction of a family of graphs with a small induced proper subgraph with minimum degree 3

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Abstract

We investigate the following question proposed by Erdös; *Is there a constant* c *such that, for each n, if G is a graph with n vertices, 2n-1 edges, and $\delta \geq 3$, then G contains an induced proper subgraph H with at least cn vertices and $\delta(H) \geq 3$?*

Previously we showed that there is no such c by constructing a family of graphs whose induced proper subgraph contains at most $\sqrt{n}+1$ vertices. In this talk we introduce a construction of a family of graphs whose largest induced proper subgraph is K_4. Also a similar construction of graphs with n vertices and $\alpha n+\beta$ edges is given.

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Enumerating Satisfiable Propositional Formulæ

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Abstract

It is known experimentally that there is a threshold for satisfiability in 3-CNF formulæ around the value 4.25 for the ratio of variables to clauses. It is also known that the threshold is sharp [Fri99], but that proof does not give a value for the threshold.

We use purely combinatorial techniques to count the number of satisfiable boolean formulæ given in conjunctive normal form. The intention is to provide information about the relative frequency of boolean functions with respect to statements of a given size, and to give a closed-form formula for any number of variables, literals and clauses. We describe a correspondence between the syntax of propositions to the semantics of functions using a system of equations and show how to solve such a system.

1 Introduction

The purpose of this paper is to apply combinatorial techniques to count the number of satisfiable boolean formulæ for a given syntax and provide information about the relative frequency of boolean functions with respect to statements of a given size. This in turn may help one understand the performance of algorithms that decide problems such as satisfiability and validity, and may aid in finding bounds on the threshold between satisfiability and unsatisfiability. The method we use is an explicit counting of those formulæ that are satisfiable. We restrict ourselves to k-CNF form, and describe a correspondence between this syntax of propositions and the semantics of the boolean functions they represent, using a system of equations. Then we show how to solve such a system, giving a general closed-form solution. For the traditional counting model for literals within a clause (unordered without replacement, no contradictory literals), we simplify it to a more specific solution.

Dershowitz and Lindenstrauss [DL89] use generating function techniques for counting with boolean formulæ; that method is extended here to solving systems of equations counting boolean functions generated from a given syntax. Chauvin, Flajolet et al. [CFGG02]

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form a similar set of equations for the case of unrestricted syntax. It is expected that further analysis of the closed form enumeration found here will result in establishing an analytic solution to the satisfiability threshold problem ([JSV00]).

2 Syntax and Semantics

The counting problem we address corresponds to the logical problem "k-CNF-SAT". We have a set V of v independent propositional variables, and a set $\overline{V}$ of their v negations. Variables and their negations are called *literals*. A *clause* is a disjunction of a sequence of k literals, and a *boolean formula* is a conjunction of a sequence of c clauses. Most of the literature on k-CNF views a clause as a set of literals (unordered without replacement), but a formula as a sequence of clauses. We use sequences instead of sets implying that a literal or clause may repeat within a clause or formula, respectively, and that the order of the literals or clauses matter. For example,

$$(p \vee q \vee p) \wedge (\overline{p} \vee q \vee q) \wedge (p \vee \overline{p} \vee \overline{q}) \wedge (q \vee p \vee p)$$

is in 3-CNF form with 4 clauses and 2 variables.

Because of the restricted syntax, the set of k-CNF formulæ is straightforward to specify as a regular language: $((V + \overline{V})^k)^c$. So the total number of formulæ over v variables, k literals, and c clauses is $(2v)^{kc}$. From this one can quickly derive the number of satisfiable formulæ for degenerate and small values of k, c, and v. For arbitrary (positive integral) values of all three parameters and for every one of the 2^{2^v} boolean functions, we will count the number of formulæ (by number of literals and clauses) that evaluate to each function. For $v = 1$, there are 4 functions, $\mathbf{F}$, $\overline{\mathbf{P}}$, $\mathbf{P}$, and $\mathbf{T}$. For an arbitrary number of variables, we index a function by the binary representation of the integer corresponding to its truth table.

For any given boolean operator (here we restrict ourselves to $\vee$ and $\wedge$, but the method can be applied to any operator), the function produced depends only on the functions represented by the two operands. For example, $\vee$ has the behavior shown below. The binary function $\wedge$ has the obvious dual behavior.

$\vee$	$\mathbf{F}$	$\overline{\mathbf{P}}$	$\mathbf{P}$	$\mathbf{T}$
$\mathbf{F}$	$\mathbf{F}$	$\overline{\mathbf{P}}$	$\mathbf{P}$	$\mathbf{T}$
$\overline{\mathbf{P}}$	$\overline{\mathbf{P}}$	$\overline{\mathbf{P}}$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{P}$	$\mathbf{P}$	$\mathbf{T}$	$\mathbf{P}$	$\mathbf{T}$
$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$

3 The enumeration

From the table of a logical operator acting on boolean functions, one can construct a system of equations whose solution is the number of formulæ for each function.

3.1 A system of equations

For the moment, let us consider the functions that can be represented with a single clause over one variable.

Let $\mathbf{F}_k$ be the number of formulæ for the boolean function f using k literals, $\vee$, and a single variable. Given a clause of length $k-1$, we can compute $\mathbf{F}_k$ by constructing a recurrence from the operator table above:

$$\begin{array}{rcl}
\mathbf{F}_k & = & \mathbf{F}_1\mathbf{F}_{k-1} \\
\overline{\mathbf{P}}_k & = & \mathbf{F}_1\overline{\mathbf{P}}_{k-1} + \overline{\mathbf{P}}_1(\mathbf{F}_{k-1} + \overline{\mathbf{P}}_{k-1}) \\
\mathbf{P}_k & = & \mathbf{F}_1\mathbf{P}_{k-1} + \mathbf{P}_1(\mathbf{F}_{k-1} + \mathbf{P}_{k-1}) \\
\mathbf{T}_k & = & \mathbf{F}_1\mathbf{T}_{k-1} + \overline{\mathbf{P}}_1(\overline{\mathbf{P}}_{k-1} + \mathbf{T}_{k-1}) + \\
 & & \quad \mathbf{P}_1(\mathbf{P}_{k-1} + \mathbf{T}_{k-1}) + \mathbf{T}_1(\mathbf{F}_{k-1} + \overline{\mathbf{P}}_{k-1} + \mathbf{P}_{k-1} + \mathbf{T}_{k-1})
\end{array}$$

with base cases

$$\begin{array}{rcl}
\mathbf{F}_1 & = & 0 \\
\overline{\mathbf{P}}_1 & = & 1 \\
\mathbf{P}_1 & = & 1 \\
\mathbf{T}_1 & = & 0
\end{array}$$

the ones produced by the literals, zeroes elsewhere. Note that a solution for this system is also a solution for a system based on $\wedge$, but with functions ordered in reverse (the complement of the bitwise representation).

The linear system can be represented as a matrix of coefficients for the recurrence:

$$\begin{bmatrix} \mathbf{F}_k \\ \overline{\mathbf{P}}_k \\ \mathbf{P}_k \\ \mathbf{T}_k \end{bmatrix} = \begin{bmatrix} f_{00} & f_{01} & f_{10} & f_{11} \\ 0 & f_{00}+f_{01} & 0 & f_{10}+f_{11} \\ 0 & 0 & f_{00}+f_{10} & f_{01}+f_{11} \\ 0 & 0 & 0 & f_{00}+f_{01}+f_{10}+f_{11} \end{bmatrix}^k \cdot \begin{bmatrix} \mathbf{F}_1 \\ \overline{\mathbf{P}}_1 \\ \mathbf{P}_1 \\ \mathbf{T}_1 \end{bmatrix}$$

Letting $\bar{f}_k$ be the vector for the set of all boolean functions formed by a clause of length k, and $\mathbf{OR}(1)$ be the above matrix, the equation

$$\bar{f}_k = \mathbf{OR}(1)^k \cdot \bar{f}_1$$

can be solved by simply multiplying out the matrix for fixed k, or, symbolically, by Gaussian elimination. But we would like to solve the system symbolically for an arbitrary number of variables. That the system is linear is a direct consequence of the grammar for k-CNF being regular.

The linear system for an arbitrary number of variables can be described recursively.

3.2 Solving the system

Let $\mathcal{B}_n$ be the Boolean lattice with n generators, with partial order $\preceq$, where $a \preceq b$ if $a \vee b = b$. The symbol $\vee$ is conveniently overloaded for both the lattice's least upper bound and the bitwise-or operation on the binary representation of integers from 0 to $2^{2^n}-1$. For v variables, we are concerned with $\mathcal{B}_{2^v}$.

Theorem 1 *The number of formulæ of length n equivalent to f_i (generated as a disjunction of atomic function symbols $f_0, f_1, \ldots$) is given by:*

$$\mathbf{OR}(v)_i^n = \sum_{s \preceq i} \left[(-1)^{2^v - d(s)} \left(\sum_{t \preceq s} f_t^1 \right)^n \right] \tag{1}$$

For example, $\mathbf{OR}(1)_3^n = (f_0 + f_1 + f_2 + f_3)^n$ - $(f_0 + f_1)^n -$ $(f_0 + f_2)^n +$ f_0^n. If the base cases are as above, then $\mathbf{OR}(1)_3^n = 2^n - 2$. Note that to reduce the symbolic complexity of Equation 1, the base case f_m^1 is implied.

4 Conclusion

Using elementary arguments, we were able to count the number of k-CNF formulæ for every boolean function, and specifically the number of satisfiable formulæ for a given number of variables, clauses, and literals per clause. The method of creating a system of equations to count the functions can be applied to any formula syntax using any set of operators. In this case, CNF syntax and the dual boolean operators simplify the analysis considerably.

Once the result is refined to give the exact function for the number of satisfiable k-CNF formulæ, asymptotic analysis will give better bounds on the SAT/UNSAT threshold phenomenon.

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Counting List H-Colorings and Variants*

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1 Problem definitions and preliminary results

All the graphs in this paper are undirected, have no multi-edges, but may have loops. For a fixed graph H, we say that a mapping $\chi : V(G) \to V(H)$ is an *H-coloring* of G if for all $\{u, v\} \in E(G)$, $\{\chi(u), \chi(v)\} \in E(H)$.

The *H-coloring* problem asks whether, given a graph G, there is an H-coloring of G. Given a graph G and, for every $v \in V(G)$, a set $L(v) \subseteq V(H)$, the *list H-coloring* problem asks whether there is an H-coloring χ of G such that for every $u \in V(G)$ we have $\chi(u) \in L(u)$. Notice, that making every list $L(u) = V(H)$, the list H-coloring is the H-coloring problem. The complexity of the H-coloring and the list H-coloring problems is described by dichotomy conditions separating the polynomial time cases from those NP-complete in [6] and [5] respectively.

We will also consider two variants of the list H-coloring, the *C-list H-coloring*: For every vertex $v \in V(G)$ the graph $H[L(v)]$ is connected, and the *OA-list H-coloring*: the list is either $V(H)$ or a single vertex, both variants were introduced in [4].

We are interested in counting problems and, as usual, we will prefix the H in the problem name by #. A basic result on the complexity of the #H-coloring problem is the following.

Theorem 1 ([3]) *The #H-coloring problem is #P-complete if H has a connected component which is not a complete reflexive graph or a complete irreflexive bipartite graph. Otherwise the counting problem is in* P.

A stronger dichotomy result is known for the list #H-coloring problem.

Theorem 2 ([7]) *The list #H-coloring is #P-complete if H has a connected component which is not a complete reflexive graph or a complete irreflexive bipartite graph. Otherwise the counting problem is in* P*. Moreover, the same theorem holds even if G is a graph of degree bounded by some constant.*

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Following [1] the H-coloring problems can be *parameterized* as follows: Let $C = \{a_1, \ldots, a_r\}$ be a set of r vertices of H and $K = (k_1, \ldots, k_r)$ an r-tuple of non negative integers. We call the triple (H, C, K) a *partial weighted assignment* on H. We say that a mapping $\chi : V(G) \to V(H)$ is an (H, C, K)*-coloring* of G if χ is an H-coloring of G such that, for all i, $1 \leq i \leq r$, $|\chi^{-1}(v_i)| = k_i$. The same type of parameterization generates the *list* (H, C, K)*-coloring* and, the counting versions of both problems, and variants.

In the case that (C, K) forms part of the input, the above problem is known as the *restrictive* H-coloring problem, dichotomy results for this variant and its counting version can be found in [2].

2 Results

We can extend the hardness results in Theorem 2 to the variants of the list H-coloring, by reducing the $\#H$-coloring to the OA-list $\#H$-coloring and the C-list $\#H$-coloring problems. So, we get

Theorem 3 *The C-list and the OA-list $\#H$-coloring problems are $\#\mathsf{P}$-complete if H has a connected component which is not a complete reflexive graph or a complete irreflexive bipartite graph. Otherwise the counting problems are in P.*

To prove a dichotomy theorem for the list $\#(H, C, K)$-coloring and its variations, we first show that for any parameter assignment (H, C, K) where the problem of counting list-$(H - C)$-colorings is in P, the problem of counting list $\#(H, C, K)$-coloring is also in P. The second step is to show that the list $\#(H, C, K)$-coloring problem is $\#\mathsf{P}$-complete whenever the $\#(H - C)$-coloring problem is $\#\mathsf{P}$-complete. These facts together with Theorem 1 give

Theorem 4 *The list $\#(H, C, K)$-coloring is $\#\mathsf{P}$-complete if $H - C$ has a connected component which is not a complete reflexive graph or a complete irreflexive bipartite graph. Otherwise the counting problem is in P.*

The reduction used in the proof of the previous Theorem generates an instance in which the list are either $V(H)$ or a single vertex. However it does not generate a connected list instance, we can replace the list $V(H)$ by the set of vertices in the connected component of $H - C$ that is not a complete reflexive graph or a complete irreflexive bipartite graph we get the same $\#\mathsf{P}$-completeness result. Therefore, we have

Corollary 1 *The OA-list and the C-list $\#(H, C, K)$-coloring are $\#\mathsf{P}$-complete if $H - C$ has a connected component which is not a complete reflexive graph or a complete irreflexive bipartite graph. Otherwise the counting problem is in P.*

As any H-coloring problem can be seen as a list H-coloring in which the list for each vertex is $V(H)$, all the positive results for the list $\#(H, C, K)$-coloring problem translate as well to the $\#(H, C, K)$-coloring problem. Hardness results seems more difficult to obtain, we have been able to obtain them only for a particular type of parameter assignment.

Let (H, C, K) be a partial weighted assignment and let a_i be a vertex in C. We call it *i-reducible* if H is H'-colorable where $H' = H - \{a_i\}$. We say that (H, C, K) is *reducible* if it is i-reducible for some $1 \leq i \leq r$, and we say that it is *irreducible* when it is not reducible.

Lemma 1 *For any irreducible parameter assignment (H, C, K), the problem of counting the $\#(H, C, K)$-colorings of a graph G is $\#\mathsf{P}$-complete whenever the $\#(H - C)$-coloring problem is $\#\mathsf{P}$-complete.*

This last result combined with Theorem 4.

Theorem 5 *For any irreducible partial weight assignment (H, C, K). The problem of counting the $\#(H, C, K)$-coloring problem is $\#\mathsf{P}$-complete if H has a connected component which is not a complete reflexive graph or a complete irreflexive bipartite graph. Otherwise the counting problem is in P.*

Finally, we provide a sufficient condition for the list $\#(H, C, K)$-coloring problem to be solvable in $O(kn + f(k, h))$ steps. The main idea of our algorithm is to create an equivalent problem whose size does not depend in the size of G, and then apply a brute force approach to count the number of H-colorings.

Lemma 2 *There exists an $O(kn + f(k, h))$ time algorithm that returns the number of (H, C, K)-colorings of G when $E(H - C) = \emptyset$*

Notice, the above result can also be used to solve the (H, K, C)-colorings. However, there exists faster algorithm, covering more general cases, for the decisional version [1].

We can define other parameterization by relaxing the condition of the number of vertices in C. If in the definition of (H, C, K)-coloring we replace "$=$" with "$\leq$" we say that χ is a *$(H, C, \leq K)$-coloring* of G. That way we can define the "$\leq$"-versions of all the problems mentioned so far.

We state that all the presented results hold also if we replace (H, C, K) by $(H, C, \leq K)$.

For the positive results it is enough to observe that the $\#(H, C, \leq K)$-coloring problem can be reduced to the computation of at most k^r instances of the $\#(H, C, K')$-problem where $K' \leq K$.

For the negative results, notice that the number of (H, C, K)-colorings is the difference of the number of $(H, C, \leq K)$-colorings minus the number of $(H, C, \leq K^-)$-colorings where K^- is obtained selecting arbitrarily a non-zero component of K, and decreasing it by one.

Finally we mention that the positive results of Theorem 2 can be easily adapted to solve their "$\leq$"-versions. That way, we can maintain the same complexity bounds without using the reduction described above.

3 Open problems

It remains open to prove the negative part of Theorem 5 for reducible partial weighted assignments. A natural question is to know whether the $O(f(K, H)n^\alpha)$-time complexity bounds of Theorem 2 can be extended to a wider class of graphs.

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Non-independent Randomized Rounding, Linear Discrepancy and an Application to Digital Halftoning

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Abstract

Motivated by an application from image processing (Asano et al., SODA 2002), we investigate the problem to round a given $[0, 1]$–valued matrix to a $0, 1$ matrix such that the L_1 rounding error with respect to all 2×2 boxes is small. We present a randomized algorithm computing roundings with expected error at most 0.5463 per box. Our algorithm is the first one solving this problem fast enough for practical application, namely in linear time. We use a modification of randomized rounding. Instead of independently rounding the variables, we impose a number of suitable dependencies. This reduces the rounding error significantly compared to independent randomized rounding, which leads to an expected error of 0.8294 per box. Finally, we give a characterization of realizable dependencies.

1 Introduction

In this paper, we are concerned with rounding problems. In general form, these problems are of the following type: Given some numbers $x_1, \ldots, x_n$, one is looking for roundings $y_1, \ldots, y_n$ such that some given error measures are small. By rounding we always mean that $y_i = \lfloor x_i \rfloor$ or $y_i = \lceil x_i \rceil$. Since there are 2^n possibilities, such rounding problems are good candidates for hard problems. In fact, even several restricted versions like the discrepancy problem are known to be NP–hard.

On the other hand, there are cases that can be solved optimally in polynomial time. Knuth [4] for example has shown that given a permutation π there exists a rounding such that the errors $|\sum_{i=1}^{k}(y_i - x_i)|$ and $|\sum_{i=1}^{k}(y_{\pi(i)} - x_{\pi(i)})|$ are at most $\frac{n}{n+1}$ for all $1 \leq k \leq n$. Such roundings can be obtained by computing a maximum flow in a network. A recent generalization to arbitrary totally unimodular rounding problems can be found in [2].

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2 The Digital Halftoning Problem: A Matrix Rounding Problem

The rounding problem regarded in this paper is motivated by an application from image processing, namely the digital halftoning problem. The digital halftoning problem is to convert a continuous-tone intensity image (each pixel may have an arbitrary 'color' on the white-to-black scale) into a binary image (only black and white dots are allowed). An intensity image can be represented by a $[0,1]$–valued $m \times n$ matrix A. Each entry a_{ij} corresponds to the brightness level of the pixel with coordinates (i, j). Since many devices, e.g., laser printers, can only output white and black dots, we have to round A towards a $0, 1$ matrix. Naturally, this has to be done in a way that the resulting image looks similar to the original one.

This notion of similarity is a crucial point. From the viewpoint of application, similarity is defined via the human visual system: A rounding is good, if an average human being can retrieve most of the original information from the rounded image. What was missing so far from the theoretical point of view is a good mathematical formulation of similarity. So far, the most widely accepted criterion for a good halftoning algorithm is that it has the 'blue noise' property (cf. the surveys Ulichney [9] and Lau and Arce [5]). This refers not to a similarity measure comparing two images, but an analysis on how the algorithm performs on constant grey level areas. Thus, on the other hand, this criterion gives little information on how changing intensities, in particular, shapes, are reproduced. On last year's SODA conference, Asano et al. [1] reported that they made progress towards finding a similarity measure. Their experimental results indicate that good digital halftonings have small error with respect to all 2×2 subregions. More formally, we end up with this problem:

Let $A \in [0,1]^{m \times n}$ denote our input matrix. The set $R_{ij} := \{(i,j), (i+1,j), (i,j+1), (i+1,j+1)\}$ for some $i \in [m-1], j \in [n-1]$ is called a 2×2 subregion (or box) in $[m] \times [n]$.[1] Denote by $\mathcal{R}$ the set of all these boxes. We write $A_{R_{ij}}$ for the 2×2 matrix $\begin{pmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{pmatrix}$ induced by R_{ij}. For any matrix A put $\Sigma A := \sum_{i,j} a_{ij}$.

For a matrix $B \in \{0,1\}^{m \times n}$ — which by definition is a rounding of A — we define the rounding error of A with respect to B by

$$d_{\mathcal{R}}(A, B) := \sum_{R \in \mathcal{R}} |\Sigma A_R - \Sigma B_R| .$$

We usually omit the subscript $\mathcal{R}$ when there is no danger of confusion.

3 Results

Asano et al. [1] exhibited that roundings B such that $d(A, B)$ is small, yield good digital halftonings. They showed that for any A an optimal rounding B^* satisfies $d(A, B^*) \leq$

[1] For an arbitrary number r we denote by $[r]$ the set of positive integers not exceeding r.

$\frac{3}{4}|\mathcal{R}|$. They also gave a polynomial time algorithm computing a rounding B such that $d(A,B) \leq d(A,B^*) + 0.5625|\mathcal{R}|$. It is easy to see that there are matrices A such that all roundings (and in fact all integral matrices) B have $d(A,B) \geq \frac{1}{2}|\mathcal{R}|$.

A major draw-back of the algorithm given in [1] is that it is not very practical. It requires the solution of an integer linear program with totally unimodular constraint matrix. This leads to a run-time bound which is at least quadratic in the number nm of pixels. As pointed out by the authors, this is too slow for digital halftoning applications. We obtain the following result.

Theorem 1 *There is a randomized algorithm computing in linear time a rounding B of $A \in [0,1]^{m \times n}$ such that*

(i) the expected rounding error satisfies $E(d(A,B)) \leq 0.5463|\mathcal{R}|$;

(ii) if B^* is an optimal rounding, then $E(d(A,B)) \leq d(A,B^*) + 0.3125|\mathcal{R}|$;

(iii) for all $\varepsilon > 0$, $\Pr\big(d(A,B) > E(d(A,B)) + \varepsilon|\mathcal{R}|\big) < 3\exp(-\frac{1}{6}\varepsilon^2(m-3))$.

Our algorithm can be derandomized. This yields a deterministic linear-time algorithm computing roundings with error guarantee at most the expected one of the randomized version. Using the language of discrepancy theory, our result states that the L_1 linear discrepancy of a set $\mathcal{R}$ of 2×2 boxes in the grid is at most $0.5463|\mathcal{R}|$. For an experimental study of how well our algorithm performs in practise, we refer to Schnieder [8]. Some of these results are also contained in [3].

4 Non-independent Randomized Rounding

The key idea of our algorithm might also be of a broader interest. We develop a randomized rounding scheme where the individual roundings are not independent. The classical approach of randomized rounding due to Raghavan and Thompson [6, 7] is to round each variable x_i independently with probabilities given by the fractional part $\{x_i\}$ of its value, i.e., $P(y_i = \lfloor x_i \rfloor + 1) = \{x_i\} = 1 - \Pr(y_i = \lfloor x_i \rfloor)$. Independence of the roundings allows to use Chernoff-type large deviation inequalities showing that a sum of independent random variables is highly concentrated around its expectation.

In addition to the randomized rounding condition for single variables, we impose dependencies of the type

$$P\left(\sum_{i \in I_k} y_i = \left\lfloor \sum_{i \in I_k} x_i \right\rfloor + 1\right) = \left\{\sum_{i \in I_k} x_i\right\} = 1 - P\left(\sum_{i \in I_k} y_i = \left\lfloor \sum_{i \in I_k} x_i \right\rfloor\right) \qquad (1)$$

for some sets I_k. Hence our roundings not only are randomized roundings with respect to single variables, but also with respect to the sums $\sum_{i \in I_k} x_i$. The roundings B computed in Theorem 1 satisfy

(i) for all $i \in [\frac{m}{2}]$, $j \in [n-1]$, $R := \{2i-1, 2i\} \times \{j, j+1\}$, $\Sigma B_{|R}$ is a randomized rounding of $\Sigma A_{|R}$,

(ii) for all $i \in [\frac{m}{2}]$, $j \in [n]$, $b_{2i-1,j} + b_{2i,j}$ is a randomized rounding of $a_{2i-1,j} + a_{2i,j}$,

(iii) for all $i \in [m]$, $j \in [n-1]$, $b_{i,j} + b_{i,j+1}$ is a randomized rounding of $a_{i,j} + a_{i,j+1}$,

(iv) for all $i \in [m]$, $j \in [n]$, $b_{i,j}$ is a randomized rounding of $a_{i,j}$.

(v) $b_{i,j}$ is mutually independent from all $b_{i',j'}$ such that $\lceil i/2 \rceil \neq \lceil i'/2 \rceil$.

Such roundings can be computed in linear time (in mn) and yield the bounds claimed in Theorem 1.

We also regard the question what dependencies can be realized. More formally, we ask how the sets I_k have to be chosen, such that a randomized rounding satisfying (1) exists for all values of the variables. Surprisingly, there is a simple characterization:

Theorem 2 *Let $\mathcal{H} = (I, \mathcal{E})$ be a hypergraph. The following two properties are equivalent:*

(i) For all $(x_i)_{i \in I}$ there is a randomized rounding $(y_i)_{i \in I}$ satisfying (1) for all $E \in \mathcal{E}$.

(ii) $\mathcal{H}$ is totally unimodular.

Thus, in particular, roundings B such that $\Sigma B_{|R}$ is a randomized rounding of $\Sigma A_{|R}$ for all $R \in \mathcal{R}$ — which would be perfect for our purposes — do not exist for all A.

5 Open Problems

From the work of Asano et al. [1] and this work, several open problems and areas for future research arise. Both motivate a closer investigation of L_p, $p < \infty$, discrepancy problems, which attracted less attention than the L_∞ discrepancy. They also suggest the study of discrepancy problems where the rounding error depends on small regions (like here, 2×2 boxes). The latter is particularly interesting since many of the commonly used methods like randomization together with large deviation bounds fail.

A problem simple to state but seemingly hard to solve in this context is the one of the L_∞ linear discrepancy of the 2×2 boxes, i.e., what is the smallest k such that for any $A \in [0,1]^{m \times n}$ there is a $B \in \{0,1\}^{m \times n}$ such that $|\Sigma A_R - \Sigma B_R| \le k$ holds for all 2×2 boxes R. Asano (private communication) has shown $1 \le k \le 5/3$. Since greedy rounding trivially yields an upper bound of 2, one has the feeling that this problem is not too well understood.

Of course, the same problem for the L_1 discrepancy, i.e., the approximation problem regarded in this paper, is also not completely understood, and it seems to be an interesting question whether the lower bound of $0.5|\mathcal{R}|$ is sharp or not.

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Tree-Decomposition of Graphs with small Diameter Bags *

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Abstract

This paper deals with the *length* of a Robertson-Seymour's tree-decomposition. The *tree-length* of a graph denotes the largest distance between two nodes of a bag of a tree-decomposition, minimized over all tree-decompositions of the graph. The study of this invariant may be interesting in its own right as the class of bounded tree-length graphs includes (but is not reduced to) bounded chordality graphs (like chordal graphs, permutation graphs, AT-free graphs, etc.). For instance, we show that the tree-length of any outerplanar graph is $\lceil k/3 \rceil$, where k is the chordality of the graph.

The introduction of this parameter is motivated by the design of compact routing tables with near-optimal route length.

Keywords: tree-decomposition, tree-width, chordality, small separator.

1 Introduction

The notions of tree-decomposition and of tree-width are very rich concepts with many algorithmic implications. Many NP-complete problems are polynomial for graphs of bounded tree-width [1, 2]. In addition, the notion of tree-width is in the kernel of the graph minor theory [4]. Intuitively tree-width can be seen as a measure of "how far" a graph is from a tree. The smaller the tree-width, the more tree-like the graph is.

However many problems have still polynomial solutions on large class of graphs even when the size of the separator is large (or large tree-width), like interval graphs or more generally chordal graphs. Actually, the "shape" of the separator plays an important role. Introducing *well-separated* class of graphs Katz, Katz and Peleg [7] showed that the graphs of this class enjoy distance labeling with short labels, namely the property that all the nodes of the graph can be labeled with $O(\log^2 n)$ bit binary strings so that the distance between two nodes can be retrieved from the two node labels, without any other source of information [8]. They showed that interval and permutation graphs fall into that class.

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The same phenomenon occurs for bounded clique-width graphs, for which Courcelle and Vanicat [3] showed that $O(\log^2 n)$ bit distance labeling scheme exists, whereas the tree-width of these graphs is unbounded in general.

An application different from distance computation which is also related to global structure of the graph is the field of routing with succinct routing tables. Tree-width k graphs have shortest-path routing scheme with addresses and routing tables of $O(k \log^2 n)$ bits per nodes [9]. However, when diameter of the separator is small better results exist. For instance [5] proposed for chordal graphs (a class including interval graphs, and whose all separators are cliques) a routing scheme independent of separator size (unbounded for these graphs). Addresses and routing tables are of $O(\log^3 n / \log\log n)$ bits per nodes. This scheme guarantees a one-sided error of 2 on the route length. Decreasing the error or reducing the memory requirements is still an open problem even for this class of graphs.

1.1 Our results

The *tree-length* of a graph denotes the largest distance between two nodes of a bag of a tree-decomposition, minimized over all Robertson-Seymour's tree-decompositions of the graph. In Section 2 we discuss about the gap between the tree-length and the chordality of a graph. We prove that the tree-length of any outerplanar graph of chordality k is exactly $\lceil k/3 \rceil$ and we calculate the tree-length of the mesh. A natural question is: "for every graph G does there exist a tree-decomposition T of G such that the length and the width of T are close to the tree-length and the tree-width of G respectively?". More formally a tree-decomposition T is α-optimal if the length and the width of T are both at most α times the tree-length and the tree-width of the graph. In Section 3 we show that there are graphs with n nodes for which every α-optimal tree-decomposition requires $\alpha = \Omega(n^{1/5})$. In the long version, we show a polynomial 3-approximation algorithm running in linear time.

1.2 Definition

We need the notion of *tree-decomposition* used by Robertson and Seymour in their work on graph minors [10].

Definition 1 *A tree-decomposition of a graph G is a tree T whose nodes, called* bags, *are subsets of $V(G)$ such that:*

1. $\bigcup_{X \in V(T)} X = V(G)$;
2. *for all $\{u,v\} \in E(G)$, there exists $X \in V(T)$ such that $u,v \in X$; and*
3. *for all $X,Y,Z \in V(T)$, if Y is on the path from X to Z in T then $X \cap Z \subseteq Y$.*

For every induced subgraph H of G, the *diameter of H in G* is $\mathrm{diam}_G(H) = \max_{x,y \in V(H)} d_G(x,y)$, where $d_G(x,y)$ denotes the distance in G between x and y. Observe that $\mathrm{diam}_G(H)$

can be constant even if H is not connected. By extension, $\mathrm{diam}_G(X) = \mathrm{diam}_G(G[X])$ for every subset $X \subseteq V(G)$, where $G[X]$ denotes the subgraph of G induced by X.

Let T be a tree-decomposition of a graph G. The *width of* T is $\mathrm{width}(T) = \max_{X \in T} \{|X| - 1\}$. The *length of* T is $\mathrm{length}(T) = \max_{X \in T} \mathrm{diam}_G(X)$. The *tree-width* and the *tree-length of* G are respectively $\min_T \mathrm{width}(T)$ and $\min_T \mathrm{length}(T)$, where the minimum is taken over all tree-decompositions of G.

A graph is k-chordal if the length of the longest induced cycle is at most k. The *chordality of* G is the smallest integer k such that G is k-chordal. Graphs of tree-length 1 are exactly 3-chordal graphs.

2 Tree-length and chordality

An *isometric subgraph* H of G is a connected subgraph of G such that $d_H(x, y) = d_G(x, y)$, for all $x, y \in V(H)$. It is a natural generalization of induced subgraph (any isometric subgraph is clearly an induced subgraph). In particular, in the case H is a cycle of length k, then k must be at least the girth of G and at most the chordality of G. Isometric subgraph is a tool for tree-length as minor subgraph is for tree-width.

Proposition 1 *For every isometric subgraph H of G, the tree-length of H is no more than the tree-length of G.*

Proposition 2 *Every cycle of length k has tree-length $\lceil k/3 \rceil$.*

More generaly we have:

Theorem 1 *Every outerplanar graph of chordality k has tree-length $\lceil k/3 \rceil$. Moreover, a tree-decomposition of width 2 and of length $\lceil k/3 \rceil$ can be constructed in linear time.*

The gap between the chordality and the tree-length can be large. For instance, a $p \times q$ mesh (p rows, q columns) has chordality $\Omega(pq)$. However,

Theorem 2 *The tree-length of the mesh with p rows and q columns is $p - 1$ if $p = q$ is odd, or is $\min\{p, q\}$ otherwise.*

However chordality is an upper bound for the tree-length:

Theorem 3 **[6]** *Every k-chordal graph has tree-length at most $k/2$.*

3 Optimal tree-decomposition

Let us define an α-*optimal* tree-decomposition for a graph G as a tree-decomposition of G whose width is at most αk and whose length is at most $\alpha\delta$, where k and δ are respectively the tree-width and the tree-length of G. Theorem 1 says nothing else that every outerplanar graph has an 1-optimal tree-decomposition, that is a tree-decomposition optimizing both parameters, the width and the length.

At this step, an interesting question is to know whether every graph has a $O(1)$-optimal tree-decomposition? We answer this question in the negative.

Theorem 4 *There exists a graph with at most n nodes on which any α-optimal tree-decomposition requires $\alpha \geqslant \frac{1}{2}n^{1/5} - \frac{1}{2}$.*

An interesting question would be to know whether an $o(n)$ bound on α exists for every n-node graphs. Before proving Theorem 4, let us denote by $M(p,q,r)$ the graph composed of a $p \times q$ mesh (p rows, q columns), $p \leqslant q$, in which every edge of a column is replaced by a path of length r. The number of nodes of $M(p,q,r)$ is $pqr - q(r-1)$.

Lemma 1 *Let T be any tree-decomposition of $M(p,q,r)$. Then, if* $\mathrm{length}(T) < (p-1)r$*, then* $\mathrm{width}(T) \geqslant q$.

By choosing $p = n^{1/5}$ and $q = r = n^{2/5}$, Lemma 1 implies Theorem 4.

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On a coloring problem for the integer grid

Adrian Dumitrescu* Radoš Radoičić†

Abstract

János Pach asked whether for any 2-coloring of the points of the integer grid in the plane, there exist arbitrary long sequences of consecutive collinear points of the same color. We answer this question in the negative for the plane, and more generally for any number of dimensions. More precisely, we show that for any $d \geq 1$, there exists a *maximum* positive integer $k = k(d)$ such that for any 2-coloring of $\mathbb{Z}^d$, there exist k consecutive collinear points of the same color. We also give upper and lower bounds on $k(d)$ for all d.

1 Introduction

Consider the set of points $\mathbb{Z}^d \subset \mathbb{R}^d$ having integer coordinates, where $d \geq 1$, which we refer to as the *integer grid* in d dimensions (or d-grid). Van der Waerden's theorem states that if the set of integers $\mathbb{Z}$ is partitioned into two classes then at least one of the classes must contain arbitrary long arithmetic progressions (see e.g. [GRS90]), a result which holds for any fixed number of classes.

Gallai's theorem provides the following generalization of van der Waerden's theorem in two dimensions: Let $V = \{(i, j) : 0 \leq i, j \leq t-1\}$. If $\mathbb{Z}^2$ is finitely colored, there exist x_0, y_0, h so that all t^2 points of the form $(x_0 + ih, y_0 + jh)$, $0 \leq i, j \leq t-1$, are of the same color (see [GRS90]).

The following statement, due to E. Witt, is also known: If $\mathbb{Z}^2$ is finitely colored, and if $M \subset \mathbb{Z}^2$ is a finite subset, then at least one of the color classes contains a subset that is affinely equivalent, or even more generally, homothetic to M (cf. [HD64], page 31).

Consider again a 2-coloring of $\mathbb{Z}$. If one tries instead to find long arithmetic progressions consisting of consecutive integers in the same class, the maximum length over all partitions is one (as determined by the partition into even and odd numbers). What happens in higher dimensions? János Pach asked whether for any 2-coloring of $\mathbb{Z}^2$, there exist arbitrary long sequences of *consecutive collinear* points of the same color. He also noticed that one can always find three such points, a result which we show is best possible (see Theorem 2 below), and thus answer his question in the negative. We further show that the answer

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is negative for any number of dimensions, by exhibiting a 2-coloring of the grid obtained by tiling. The coloring of the tile itself is found using the probabilistic method. From the opposite direction, we prove that for any 2-coloring of the d-grid there exist $k(d)$ consecutive collinear points of the same color, where $k(d)$ is bounded from below by a function growing slowly to ∞. This result is a simple consequence of the Hales-Jewett theorem [HJ63, GRS90].

A *combinatorial line* in $C_t^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, t-1\}\}$, the n-cube over t elements, is a sequence of points, $\mathbf{x_0}, \ldots, \mathbf{x_{t-1}}$, $\mathbf{x_i} = (x_{i1}, \ldots, x_{in})$ so that in each coordinate j, $1 \le j \le n$, either $x_{0j} = x_{1j} = \ldots = x_{t-1,j}$ or $x_{sj} = s$, for $0 \le s \le t-1$, the latter occurring for at least one j.

Theorem 1 (Hales-Jewett Theorem) *For all r, t there exists $N = HJ(r,t)$ so that for $n \ge N$, the following holds: If the vertices of C_t^n are r-colored there exists a monochromatic combinatorial line.*

A sequence of consecutive collinear points (on some line in $\mathbb{R}^n$) is said to form a *segment* in $\mathbb{R}^n$. Note that a combinatorial line in C_t^n is a segment of t points in C_t^n (and thus in $\mathbb{R}^n$), although the opposite does not hold.

The best upper bound on $HJ(2,t)$ [GRS90] can be obtained from Shelah's proof of the Hales-Jewett theorem [Sh88], in terms of a rapidly growing function from the Ackerman hierarchy. The Ackerman hierarchy is a sequence of functions $f_1, f_2, f_3, \ldots$, having the set of positive integers as domain and range [GRS90], where f_{i+1} can be defined inductively by $f_{i+1}(1) = 2$, $f_{i+1}(x+1) = f_i[f_{i+1}(x)]$. Let β be the functional inverse of f_4 defined by $\beta(n) = \min\{x : f_4(x) \ge n\}$. Clearly $\lim_{n\to\infty} \beta(n) = \infty$, although its growth rate is very slow.

Theorem 2 *For any $d \ge 1$, there exists a maximum positive integer $k = k(d)$ such that for any 2-coloring of $\mathbb{Z}^d$, there exist k consecutive collinear points of the same color. Furthermore,*

(i) $k(1) = 1$ *and* $k(2) = 3$.

(ii) *For* $d \ge 3$, $\max(3, \beta(d) - 2) \le k(d) \le d \log d(1 + o(1))$.

2 Proof of Theorem 2

(i) We start with the lower bound $k(2) \ge 3$, and refer to Fig. 1 (the case $d = 1$, is immediate from the earlier comment). Consider a 2-coloring of the grid points in the plane by white or black. If for each point p, all its four neighbors in the grid (in the N,S,E,W directions) have a color different from that of p, any diagonal is monochromatic, thus one can select three consecutive collinear points of the same color. Assume therefore that there exist two adjacent points on a horizontal line of the same color, say a and b in Fig.1(left) are white. If point d is white, the following colors are forced when one has to avoid a monochromatic segment of three points: c black, h black, f white, e black. No matter what the color of g is, three points (a, d, g or c, e, g) as desired are found. Finally assume that no triplet of points (as a, b, d above) forms a monochromatic L shape, as in Fig.1(right) . By symmetry, if a and b are white, j and k are black, f and g are black, and the coloring is completely determined as a 2×1 domino tiling: l, m white, c, d black, etc. Then f, c, n form a monochromatic (black) segment of three points. This concludes the proof of the lower bound in the plane.

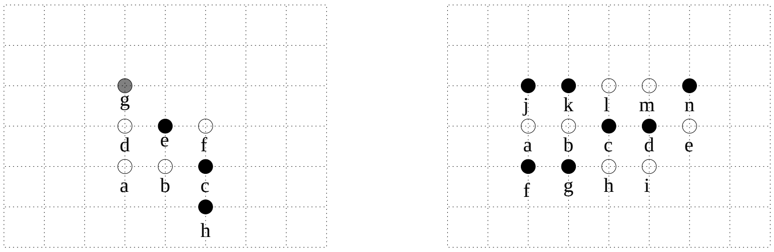

Figure 1: Proof of the lower bound $k(2) \geq 3$.

Before showing that $k(2) \leq 3$, we introduce some notation regarding periodic colorings and monochromatic segments in $\mathbb{Z}^d$.

For a positive integer q, consider the $q \times \ldots \times q$ square tile in $\mathbb{Z}^d$: $V = \{(x_1, \ldots, x_d) : 0 \leq x_i \leq q-1\}$, and some 2-coloring of V. Then extend the coloring of V to the whole grid, so that for each grid point $p = (x_1, \ldots, x_d)$, the grid point $\tau(p) = (x_1 \bmod q, \ldots, x_d \bmod q)$ of V has the same color. We call $\tau(p)$ the *trace* of p.

For any segment S of k points in the grid, let $\tau(S) = \{\tau(p) : p \in S\}$, and call it the trace of S. Since the grid coloring is a tile coloring, S is bichromatic if and only if the trace of S is bichromatic. A segment of k points in the d-grid is of the form $S = \{a, a + r, a + 2r, \ldots, a + (k-1)r\}$, for some $a, r \in \mathbb{Z}^d$. We also say that S is defined by the pair (a, r). Since the grid coloring is a tile coloring, we can replace a and r by their traces $\tau(a)$ and $\tau(r)$. Denote by $\gcd(u, v)$ the greatest common divisor of u and v, and in general, denote by $\gcd(u_1, \ldots, u_n)$ the greatest common divisor of $u_1, \ldots, u_n$. Since S consists of consecutive points on some line in $\mathbb{R}^d$, we must have $\gcd(r_1, \ldots, r_d) = 1$, where $\tau(r) = (r_1, \ldots, r_d)$.

We next show that $k(2) \leq 3$. For this purpose, consider the coloring of the grid obtained by extending the coloring of a 4×4 tile $V = \{(i, j) : 0 \leq i, j \leq 3\}$, as shown in Fig. 2.

We have to prove that any segment of four points is bichromatic. The condition $\gcd(r_1, r_2) = 1$ implies that the trace of any segment of four points in the grid is one

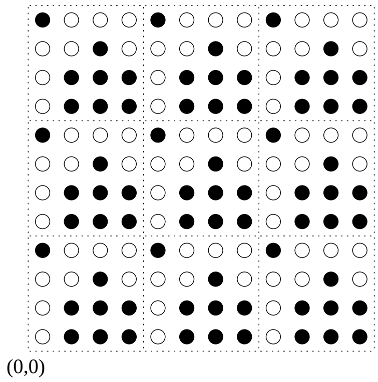

Figure 2: Proof of the upper bound $k(2) \leq 3$: a 3×3 portion of the tiling; tile V is at the bottom left corner.

of the following:

$$\begin{array}{lll}
\{(0,0),(1,0),(2,0),(3,0)\}, & \{(0,1),(1,1),(2,1),(3,1)\}, & \{(0,2),(1,2),(2,2),(3,2)\}, \\
\{(0,3),(1,3),(2,3),(3,3)\}, & \{(0,0),(0,1),(0,2),(0,3)\}, & \{(1,0),(1,1),(1,2),(1,3)\}, \\
\{(2,0),(2,1),(2,2),(2,3)\}, & \{(3,0),(3,1),(3,2),(3,3)\}, & \{(0,0),(1,1),(2,2),(3,3)\}, \\
\{(0,0),(2,1),(0,2),(2,3)\}, & \{(0,0),(3,1),(2,2),(1,3)\}, & \{(0,0),(1,2),(2,0),(3,2)\}, \\
\{(1,0),(2,1),(3,2),(0,3)\}, & \{(1,0),(0,1),(3,2),(2,3)\}, & \{(1,0),(3,1),(1,2),(3,3)\}, \\
\{(1,0),(3,0),(0,2),(2,2)\}, & \{(2,0),(0,1),(2,2),(0,3)\}, & \{(2,0),(1,1),(0,2),(3,3)\}, \\
\{(2,0),(3,1),(0,2),(1,3)\}, & \{(3,0),(0,1),(1,2),(2,3)\}, & \{(3,0),(1,1),(3,2),(1,3)\}, \\
\{(3,0),(2,1),(1,2),(0,3)\}, & \{(0,1),(2,1),(1,3),(3,3)\}, & \{(1,1),(3,1),(0,3),(2,3)\}.
\end{array}$$

It is easy to check that all the traces above are bichromatic. Therefore, any segment of four points in the grid is bichromatic as well.

(ii) We first show the upper bound on $k(d)$. We describe a 2-coloring of the grid points, so that there are no more than $d \log d(1 + o(1))$ consecutive collinear points of the same color.

Let q be a positive integer (which will be selected later large enough), and a random 2-coloring of a $q \times \ldots \times q$ square tile in $\mathbb{Z}^d$, $V = \{(x_1, \ldots, x_d) : 0 \leq x_i \leq q - 1\}$: each integer point in the tile is colored white or black, independently with probability 1/2. Then extend the coloring of V to the whole grid.

Now use the Lovász Local Lemma [EL75] stated below.

Let $A_1, \ldots, A_n$ be events in a probability space Ω. A graph G on $[n]$ is said to be a dependency graph of $\{A_i\}$ if, for all i, the event A_i is *mutually independent* of $\{A_j : \{i,j\} \notin E(G)\}$.

Theorem 3 (LLL) *Let $A_1, \ldots, A_n$ be events with $P(A_i) \leq p$ for all i and with a dependency graph G of maximum degree at most Δ, that is, for all i, $|\{j : \{i,j\} \in E(G)\}| \leq \Delta$. If $e \cdot p \cdot (\Delta + 1) \leq 1$, then $P(\bigcap \overline{A_i}) > 0$.*

For a given segment S of q points let A_S be the event that S is monochromatic. First note that the lemma can be applied since we have a finite number of events — as every segment is defined by (a, r) where $a, r \in \{0, 1, \ldots, q-1\}^d$, and thus their number is finite. Let S and T be adjacent in G if their traces $\tau(S)$ and $\tau(T)$ intersect (i.e. $\tau(S) \cap \tau(T) \neq \emptyset$). We have

(*i*) $p = 2^{1-q}$ and

(*ii*) $\Delta \leq q^{d+2} - 1$ (this may be slightly improved),

as we show below.

(*i*) Let S be a segment of q points defined by the pair (a, r). Recall that $\gcd(r_1, \ldots, r_d) = 1$, where $\tau(r) = (r_1, \ldots, r_d)$. This implies that $|\tau(S)| = q$. Assume for contradiction that $|\tau(S)| < q$. Then there exist two points of S which have the same trace, in other words, there exists some $i \in \{1, \ldots, q-1\}$ so that $ir_1 \equiv 0 \pmod q, ir_2 \equiv 0 \pmod q, \ldots, ir_d \equiv 0 \pmod q$. Since $\gcd(r_1, \ldots, r_d) = 1$, there exist integers $y_1, \ldots, y_d$ so that (see e.g. [Le77]) $\sum_{j=1}^{j=d} y_j r_j = 1$. For each $j = 1, \ldots, d$, multiply the j-th congruence in the above system by y_j and add them up. We obtain: $i\left(\sum_{j=1}^{j=d} y_j r_j\right) \equiv 0 \pmod q$, or equivalently $i \equiv 0 \pmod q$. This contradicts the assumption $i \in \{1, \ldots, q-1\}$, and thus proves (*i*).

(*ii*) Let S (resp. T) be defined by the pair (a, r) (resp. (b, s)). There are at most q choices for specifying the point of $\tau(S)$ in which the intersection occurs, not more than $q^d - 1$ choices for s, and at most q choices for specifying the point of $\tau(T)$ in which the intersection occurs (we view now $\tau(T)$ as an ordered sequence). Consequently, $\Delta \leq q^2(q^d - 1) \leq q^{d+2} - 1$.

For a given d, $\lim_{q \to \infty} q^{d+2}/2^{q-1} = 0$. Hence by the lemma, if $e \cdot \frac{1}{2^{q-1}} \cdot q^{d+2} \leq 1$, there exists a 2-coloring of V, whose extension to the entire grid does not allow for a monochromatic segment of q points. This proves that $k(d) \leq q - 1$ exists. Furthermore, a straightforward calculation shows that the above inequality holds for $q = d \log d(1 + o(1))$.

Next we show the lower bound on $k(d)$. Let $S(t)$ be the number $N = HJ(2, t)$ obtained from Shelah's proof of the Hales-Jewett theorem [Sh88]. Conform with [GRS90], for $t \geq 1$, $S(t) \leq f_4(t+1)$. This gives a lower bound on $k(d)$ in terms of the functional inverse of f_4. For $d \geq 3$, $\beta(d) \geq 2$, thus by the definition of $\beta(\cdot)$, for $d \geq 3$ we have $d \geq f_4(\beta(d) - 1) \geq S(\max(\beta(d) - 2, 1))$, from which the lower bound follows. $\square$

3 Concluding remarks

Since our construction is probabilistic, an explicit constructive proof of the upper bound would be more enlightening, and perhaps also yield a better bound. For $d = 3$, the first case where we do not have an exact value for $k(d)$, $k(3) \in [3, 12]$, where the lower bound follows from the monotonicity of $k(\cdot)$, and the upper bound was computer generated using a program we have written.

The question of obtaining a better lower bound on $k(d)$ remains an interesting question to be answered. In particular, what is the minimum d for which $k(d) \geq 4$? It may be that such an improved bound could lead to further improved upper bounds on van der Waerden numbers.

The goal of obtaining many consecutive points of the same color in a line is precisely the goal in (the many variants of) the two-player game of tic-tac-toe. In the (t, n) version of the positional game tic-tac-toe, the "board" is the n-cube over t elements C_t^n [HJ63, JT95]. In this game, two players alternately select a previously un-selected vertex (point) of C_t^n, and the goal of each player is to collect all t vertices of C_t^n on some line in $\mathbb{R}^n$ (i.e. not only combinatorial lines form winning configurations). [BCG82] presents other variants and interesting results regarding tic-tac-toe. This game has been the object of research (and play!) for many years, and its study has lead to significant advances in the theory of positional games on hypergraphs [HJ63, ES73, Pa78, Be81].

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Hamiltonian cycles and paths with prescribed edges in hypercubes

Tomáš Dvořák

The *n-dimensional hypercube* Q_n is a graph whose vertex set consists of all binary vectors of length n, two vertices being adjacent whenever the corresponding vectors differ in exactly one coordinate. It is well-known that Q_n is hamiltonian for every $n \geq 2$ and the research on hamiltonian cycles satisfying certain additional properties has received a considerable attention ([7]). The applications in parallel computing inspired study of hypercubes with faulty links, which lead to the investigation of hamiltonian cycles of Q_n avoiding certain set of forbidden edges ([2, 3, 8]).

A problem in a sense complementary has been proposed recently by R. Caha and V. Koubek: Given a set of prescribed edges in a hypercube, under which conditions there always exists a hamiltonian cycle, passing through every edge of this set? In [1] they observed that *any proper subset $\mathcal{P}$ of edges of a hamiltonian cycle necessarily induces a subgraph consisting of pairwise vertex-disjoint paths* and showed that in case $|\mathcal{P}| \leq n-1$, $n \geq 2$, this condition is also sufficient to guarantee the existence of a hamiltonian cycle of Q_n, passing through every edge of $\mathcal{P}$. On the other hand, for any $n \geq 3$ there is a set of $2n-2$ edges, satisfying the above condition, but not contained in any hamiltonian cycle. Indeed, let v be an arbitrary vertex of Q_n and $\mathcal{P}$ a set of edges incident with neighbors of v so that each but one neighbor of v is incident with exactly two edges of $\mathcal{P}$, but no edge of $\mathcal{P}$ is incident with v. It is not difficult to see that this can be always done in such a way that the above condition is preserved. Since Q_n is a regular graph of degree n, it follows that $|\mathcal{P}| = 2n-2$ and apparently, any cycle passing through all edges of $\mathcal{P}$ avoids v.

The main purpose of this contribution is to show that the upper bound obtained in this way is really sharp:

Theorem 1 *Let $\mathcal{P} \subseteq E(Q_n)$, $n \geq 2$, $|\mathcal{P}| \leq 2n-3$. Then there is a hamiltonian cycle in Q_n passing through every edge of $\mathcal{P}$ iff the subgraph induced by $\mathcal{P}$ consists of pairwise vertex-disjoint paths.*

The existence of hamiltonian cycles passing through prescribed edges is apparently related to the above mentioned problem of hamiltonicity of graphs with faulty edges. Indeed, if two prescribed edges are incident with the same vertex v, any cycle passing through them must avoid all the remaining edges, incident with v. In particular, it was observed by P. Gregor ([4]) that this argument can be used to show how the Theorem implies a classical result of [3] on the existence of hamiltonian cycles of Q_n avoiding $n-2$ forbidden edges.

In may be of interest to explore further this connection comparing the complexity of both problems. The problem of hamiltonicity of hypercubes with faulty edges is known to be NP-complete ([2]). Does a similar result hold for the variant with prescribed edges?

Another generalization would lead to considering more then $2n-3$ prescribed edges in Q_n and looking for conditions sufficient for an existence of a hamiltonian cycle. A natural way to strengthen the condition of the Theorem is to consider only prescribed edges forming a matching. There is a related conjecture of G. Kreweras saying that *any perfect matching of Q_n ($n \geq 2$) is contained in a hamiltonian cycle*. It is easy to see to be true for $n = 2, 3$, and it was verified in [6] for $n = 4$.

A classical result of I. Havel ([5]) says that *for any $x, y \in V(Q_n)$ ($n \geq 1$), there exists a hamiltonian path between x and y in Q_n iff $d(x, y)$ is odd.* A result similar to that of the Theorem holds for hamiltonian paths between given pairs of endvertices at odd distance, passing through prescribed edges of Q_n.

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Eulerian colorings and the Bipartizing matchings conjecture of Fleischner

Zdeněk Dvořák [1,3], Tomáš Kaiser [2,3] and Daniel Král' [1,3]

Abstract

A trigraph is a multigraph with half-edges colored in three colors. We introduce the notion of an eulerian coloring of a trigraph and show that the existence of two orthogonal eulerian colorings in a special class of trigraphs is closely related to the Bipartizing matchings conjecture of Fleischner, and hence to the Cycle double cover conjecture and Tutte's 5-flow conjecture. We prove that every trigraph has an eulerian coloring and that a rainbow cubic trigraph has a pair of orthogonal eulerian colorings if and only if it has a perfect matching.

1 Introduction

Fleischner [2] has proposed a novel approach to several deep graph-theoretic conjectures, such as the Cycle double cover conjecture [6, 7] or the 5-flow conjecture of Tutte [8]. A key ingredient in this approach is the Bipartizing matchings conjecture (Conjecture 2 below, abbreviated BMC). We introduce the notion of an eulerian coloring of a trigraph, providing a setting that allows a generalization of the Bipartizing matchings conjecture and inspires new questions. Extending a result from [4], we show that every trigraph has an eulerian coloring. We investigate the existence of two orthogonal eulerian colorings in cubic trigraphs.

We describe in detail the connection between BMC and the Cycle double cover conjecture which asserts that any bridgeless graph has a *cycle double cover (CDC)*, i.e., a (multi)set of circuits which covers each edge exactly twice. It is well known [9, Theorem 7.2.4] that it is enough to prove the CDC conjecture for 3-connected cubic graphs, in fact, even for *snarks* (3-connected, cyclically 4-edge-connected cubic graphs with no 3-edge-coloring, containing no 4-cycles).

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It is easy to find a CDC of a hamiltonian cubic graph. Although no snark has a Hamilton cycle, they are all conjectured to be rather close to being hamiltonian:

Conjecture 1 (Dominating cycle conjecture [1]) *Every snark has a dominating cycle, i.e., a cycle incident with every edge.*

If a cubic graph has a dominating cycle, does it help in finding a CDC? Not in an obvious way. However, with the help of another conjecture of Fleischner [2] (Conjecture 2 below), it is indeed possible to construct a CDC. Let C be a dominating cycle in a cubic graph G. A *bipartizing matching* in G (with respect to C) is a matching M which is edge-disjoint from C, such that M covers all vertices not on C and $G - M$ is a subdivision of a cubic bipartite graph. Fig. 1 shows a bipartizing matching M in the Petersen graph with respect to the "outside" dominating cycle. Indeed, the removal of M yields a subdivision of $K_{3,3}$.

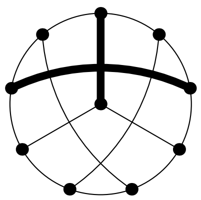


Figure 1: A bipartizing matching in the Petersen graph (bold).

Conjecture 2 (Bipartizing matchings conjecture [2]) *Every snark with a dominating cycle C has a pair of (edge-)disjoint bipartizing matchings with respect to C.*

Fleischner [2, 3] showed that Conjecture 2 implies the existence of a CDC and a nowhere-zero 5-flow for any snark with a dominating cycle. Thus, if Conjectures 1 and 2 are true, then the CDC conjecture and the 5-flow conjecture follow. Speaking about two bipartizing matchings, it is natural to ask whether at least one bipartizing matching (with respect to a given dominating cycle) always exists. Indeed, Fleischner and Stiebitz [4] showed that this is the case.

Conjecture 2 holds for any cubic graph with a dominating cycle C which is in fact a Hamilton cycle [2]. An alternative proof of this, which is the starting point of our work, is as follows. Two chords of a Hamilton cycle C in G *intersect* if one of them separates the end-vertices of the other. Let us define the *intersection graph* $I(G)$ of G to be the graph whose vertices are the chords of C in G, with edges joining intersecting pairs of chords. Throughout this paper, an *eulerian* graph is a graph that has even vertex degrees and may or may not be connected.

Proposition 1 *Let C be a Hamilton cycle in the cubic graph G. Then a matching $M \subset E(G) - E(C)$ is bipartizing if and only if $I(G) - M$ is an eulerian graph.*

Thus, there are two disjoint bipartizing matchings in G if and only if $I(G)$ can be partitioned into two eulerian induced subgraphs. Not only is the latter condition always true; somewhat surprisingly, the following stronger assertion holds for all graphs. The theorem appears as Problem 5.17(a) in [5]:

Theorem 1 *The vertices of any graph H can be partitioned into two sets, each of which induces an eulerian subgraph of H.*

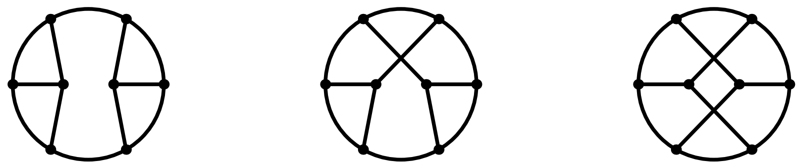
By Proposition 1, the partition in Theorem 1 yields a partition of the chords of C in G into two bipartizing matchings, thus providing a new proof of Conjecture 2 in the hamiltonian case. In what follows, we modify the definition of the intersection graph for the case that C is just a dominating cycle.

2 Eulerian colorings of trigraphs

In this section, we present our own results in the direction indicated by Proposition 1. The key approach is to transfer questions about bipartizing matchings to a more general level of eulerian colorings of trigraphs (defined below).

Let C be a dominating cycle of a cubic graph G. We can assume that C has no chords; this is harmless, since each chord can be replaced by an internal vertex in a way preserving all the relevant properties. Thus, every edge not on C is incident with a vertex not on C (an *internal vertex* as we shall call it).

Let us define an analogue of the intersection graph from Section 1. In contrast to the previous case, there are essentially 3 possible configurations of two internal vertices x and y, as shown in Fig. 2. To tell them apart, one can form a cyclic sequence σ that contains the symbol x for each neighbor of x, and the symbol y for each neighbor of y, as they appear in the clockwise order around C. Then, internal vertices x and y form a *disjoint* pair if $\sigma = xxxyyy$, a *crossing* pair if $\sigma = xxyxyy$ and an *alternating* pair if $\sigma = xyxyxy$. Now, a *trigraph*, an analogue of the intersection graph, is a loopless multigraph whose



A disjoint pair. A crossing pair. An alternating pair.

Figure 2: The three configurations of two internal vertices.

edges are viewed as composed of two half-edges, and each half-edge is colored in one of the colors $1, 2, 3$. If an edge e joins vertices x and y, then the color of its half-edge incident with x is also referred to as the *color of e at x*.

The *trigraph $T(G)$ of G* (with respect to C, which is omitted from the notation) is defined as follows. Its vertices are the internal vertices of G. Two vertices are joined by

0, 1 or 3 edges according to whether they form a disjoint, crossing or alternating pair, respectively.

The coloring of the half-edges of $T(G)$ reflects the special role of certain edges, e.g., in a crossing pair of internal vertices. Choose a labelling ℓ of $V(C)$ by 1, 2 and 3 such that the neighbors of each internal vertex get all three labels. For a pair of vertices $x, y \in V(T(G))$, let G_{xy} be the graph obtained from G by removing all internal vertices except x and y, and suppressing all vertices of degree 2 this creates. Thus, G_{xy} has 8 vertices, and is isomorphic to one of the graphs in Fig. 2.

If x and y form a crossing pair, let x_1 and y_1 be the unique neighbors of x and y, respectively, that are not contained in any triangle in G_{xy}. The (single) edge xy in $T(G)$ gets the color $\ell(x_1)$ at the vertex x and the color $\ell(y_1)$ at y.

If x and y form an alternating pair, then for each neighbor x' of x, there is a unique vertex, y', such that x' and y' are not contained in any 4-cycle in G_{xy}. Correspondingly, one of the 3 edges joining x to y in $T(G)$ will have the color $\ell(x')$ at x and $\ell(y')$ at y. The other choices of x' yield the colorings of the remaining two edges. This completes the definition of $T(G)$.

Not all trigraphs arise as $T(G)$ for a cubic graph G. A trivial obstruction is the presence of more than 3 parallel edges joining two vertices. It is, however, not difficult to construct such examples without any multiple edges as well.

The notion of a bipartizing matching translates nicely to trigraphs. Matchings covering all internal vertices (let us call them *full matchings*) are clearly in 1-1 correspondence with (not necessarily proper) vertex colorings of $T(G)$ by the colors $1, 2, 3$. For a trigraph T and a coloring $f : V(T) \to \{1, 2, 3\}$, let T_f be the graph on $V(T)$ that consists of all edges e such that the color of e at x differs from $f(x)$, and the color of e at y differs from $f(y)$. The coloring f is *eulerian* if T_f is an eulerian graph. Eulerian colorings are related to bipartizing matchings:

Proposition 2 *Let M be a full matching in a cubic graph G with a dominating cycle C, and let f_M be the corresponding vertex coloring of $T(G)$. M is a bipartizing matching if and only if f_M is an eulerian coloring.*

The following generalization of a result from [4] gives some hope that trigraphs provide a reasonable setting for the study of Conjecture 2:

Theorem 2 *Every trigraph has an eulerian coloring.*

A pair of disjoint bipartizing matchings in G corresponds to a pair of eulerian colorings of $T(G)$ that differ at each vertex; we call such colorings *orthogonal*. There are trigraphs with no pair of orthogonal eulerian colorings, since there are examples of cubic graphs with no pair of disjoint bipartizing matchings for a given dominating cycle. For instance, an example due to Fleischner yields the trigraph shown in Fig. 3(a). There are also 2-connected such trigraphs, see Fig. 3(b). We have investigated in detail the existence of two orthogonal eulerian colorings in *cubic* trigraphs, i.e., ones in which each vertex has degree 3. Such a trigraph is *rainbow* if the three half-edges incident with any vertex have all three colors. The existence of a perfect matching in such a trigraph plays a key role:

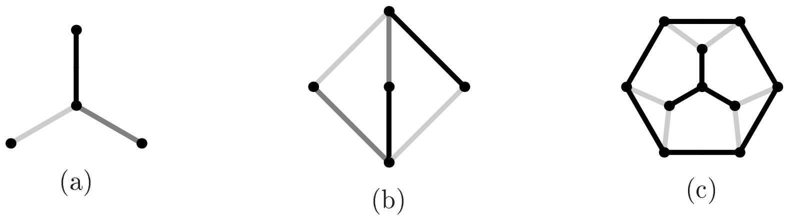


Figure 3: Trigraphs with no orthogonal pair of eulerian colorings (the colors $1, 2, 3$ are represented by grey shades).

Theorem 3 *A rainbow cubic trigraph T has a pair of orthogonal eulerian colorings if and only if T has a perfect matching.*

Theorem 3 cannot be extended to non-rainbow cubic trigraphs, see Fig. 3(c).

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On the construction of small sumsets

S. Eliahou*

This is joint work with Michel Kervaire. Let G be a group. How small can the size of the sumset $A+B$ (or AB) be, when A, B are finite subsets of G of given cardinality? And how can one construct such small sumsets?

Let $\mu_G(r,s)$ denote the minimum of $|A+B|$ over all $A, B \subset G$ with $|A| = r$ and $|B| = s$. The classical Cauchy-Davenport Theorem determines the function μ_G for G the cyclic group of prime order p, namely $\mu_G(r,s) = \min\{p, r+s-1\}$. Later work by Yuzvinsky [5], Bollobás-Leader [1] and Eliahou-Kervaire [2] dealt with elementary abelian groups. And more recently, together with Alain Plagne [4], we have determined μ_G for G an arbitrary finite abelian group. Our solution reads

$$\mu_G(r,s) = \min_{d|n}(\lceil r/d \rceil + \lceil s/d \rceil - 1)d,$$

where $n = |G|$. A similar result can be formulated for infinite abelian groups.

The non-abelian case remains largely open. In [3], we have determined $\mu_G(r,s)$ whenever $r \leq 3$ or $r+s \geq |G|$. As for other values of r, s, we have obtained upper bounds on $\mu_G(r,s)$ by suitable constructions of small sumsets. Namely, we have proved the inequality $\mu_G(r,s) \leq r+s-1$ for all $r, s \leq |G|$, provided G is a finite solvable group. In the non-solvable case, we get the same inequality $\mu_G(r,s) \leq r+s-1$ but only when $r \leq 12$ or $r+s \geq |G|-11$. Our proof in that case involves the classification of simple groups whose order is bounded above by $27720 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$.

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$L(p, q)$ Labeling of d-Dimensional Grids

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Abstract

In this paper, we address the problem of λ labelings, concerning frequency assignment for telecommunication networks. In this model, stations within a given radius r must use frequencies that differ at least by a value p, while stations that are within a larger radius $r' > r$ must use frequencies that differ by at least another value q. The aim is to minimize the span of frequencies used in the network. This can be modeled by a graph coloring problem, called the $L(p, q)$ labeling, where one wants to label vertices of the graph G modeling the network by integers in the range $[0; M]$, while minimizing the value of M. M is then called the λ number of G, and is denoted by $\lambda_q^p(G)$.

We study here the $L(p, q)$ labeling problem, where, for a specific class of networks, namely the d-dimensional grid $G_d = G[n_1, n_2 \dots n_d]$, we give bounds on the value of the λ number of an $L(p, q)$ labeling for any $p, q \geq 0$ in any d-dimensional grid G_d. Some of these results are optimal (namely, in the following cases: (1) $p = 0$, (2) $q = 0$, (3) $q = 1$ and (4) $p \geq 2dq + 1$) ; when the results we obtain are not optimal, we observe that the bounds differ by an additive factor never exceeding $2q - 2$. The optimal result we obtain in the case $q = 1$ answers an open problem stated by Dubhashi et al. [DMP$^+$02], and generalizes results from [BPT00] and [DMP$^+$02].

1 Introduction

In this paper, we study the *frequency assignment problem*, originally introduced in [Hal80], where radio transmitters that are geographically close may interfere if they are assigned close frequencies. This problem arises in mobile or wireless networks. Generally, this problem is modeled by a graph coloring problem, where the transmitters are the vertices, and an edge joins two transmitters that are sufficiently close to potentially interfere. The aim here is to color (i.e. give a value, corresponding to the frequency) the vertices of the graph in such a way that:

- any two neighbors (transmitters that are very close) are assigned colors (frequencies) that differ by a parameter at least p ;
- any two vertices at distance 2 (transmitters that are close) are assigned colors (frequencies) that differ by a parameter at least q ;
- the greatest value for the colors is minimized.

It has been proved that under this model, we could assume the colors to be integers, starting at 0 [GY92]. In that case, the minimum range of frequencies that is necessary to assign to the vertices of a graph G is denoted $\lambda_q^p(G)$, and the problem itself is usually called the $L(p, q)$ labeling problem. The frequency assignment problem has been studied in many different specific topologies [GY92, Sak94, WGM95, BPT00, BKTvL00, CKK$^+$02, MS02, BPT02]. The case $p = 2$ and $q = 1$ is the most widely studied (see for instance [CK96, JNS$^+$00, Jha00, CP01]). Some variants of the model also exist, such as the following generalization where one gives k constraints on the k first distances (any two vertices at distance $1 \leq i \leq k$ in G must be assigned colors differing by at least δ_i). One of the issues also considered in the frequency assignment problem is the *no-hole property*, where one wants to know whether a given coloring uses all the possible colors in the range $[0; \lambda_q^p(G)]$.

In this paper, we mainly focus on the $L(p, q)$ labeling problem. More precisely, in Section 2 we study the case of the $L(p, q)$ labeling in the d-dimensional grid G_d. We first address in Section 2.1 the cases where $p = 0$ or $q = 0$. In Section 2.2, we give results for the $L(p, q)$ labeling of G_d for any $p, q, d \geq 1$. We give lower and upper bounds on $\lambda_q^p(G_d)$, and show that in some cases, these bounds coincide. Notably, in the case $q = 1$, the results we obtain are optimal ; this answers an open problem stated by Dubhashi et al. in [DMP$^+$02], and generalizes results from [BPT00] and [DMP$^+$02]. The results we give are also optimal when $p = 0$, $q = 0$, and $p \geq 2dq + 1$. We also prove that in some cases (namely, when $1 \leq p \leq 2dq$), the coloring we propose satisfies the no-hole property.

2 $L(p, q)$ labeling of G_d

We study here the λ labeling problem with two constraints on the distances, in a particular network topology, namely the d-dimensional grid $G_d = G[n_1, n_2 \ldots n_d]$. We first address the special cases $p = 0$ (resp. $q = 0$) in Section 2.1. We then address the more general case where $p, q \geq 1$ in Section 2.2.

Due to space limitations, only a very restricted number of proofs will be given here. However, we note that they very often rely on the same process, i.e. (1) a deep study of the constraints induced by the grid topology, in order to derive lower bounds for $\lambda_q^p(G_d)$, and (2) a specific coloring for any vertex $u \in V(G_d)$, for which we show that it is a valid $L(p, q)$ labeling of G_d (this gives an upper bound for $\lambda_q^p(G_d)$).

2.1 $L(p, q)$ labeling when $p = 0$ or $q = 0$

Property 1 *For any $p \geq 0$ and $d \geq 1$, $\lambda_0^p(G_d) = p$.*

Property 2 *For any $q \geq 0$ and $d \geq 1$, $\lambda_q^0(G_d) = (2d-1)q$.*

We note that except in specific cases, the colorings we have given above do not satisfy the no-hole property (we recall that the no-hole property holds when all colors in the range $[0; \lambda_q^p]$ are used). Indeed, in the case $q = 0$, only colors 0 and p are used, thus the coloring cannot be no-hole, except when $p = 1$. Similarly, in the case $p = 0$, the colors used are taken in the set $\{0, q, 2q \ldots (2d-1)q\}$, thus the coloring cannot be no-hole, except in the specific case $q = 1$.

2.2 $L(p, q)$ labeling when $p, q \geq 1$

We now address the $L(p, q)$ labeling of G_d, for any values of $p, q \geq 1$ and $d \geq 1$. First, we note that we can obtain two trivial upper bounds on $\lambda_q^p(G_d)$. The first one relies on an existing $L(p', 1)$ labeling of G_d.

Observation 1 *For any $p, q, d \geq 1$, $\lambda_q^p(G_d) \leq q \cdot \lambda_1^{\lceil \frac{p}{q} \rceil}(G_d)$.*

There exists another upper bound for $\lambda_q^p(G_d)$, relying on an existing $L(1, 1)$ labeling of G_d.

Observation 2 *For any $p, q, d \geq 1$, $\lambda_q^p(G_d) \leq \max\{p, q\} \cdot 2d$.*

The two above mentioned simple observations present the disadvantage to be based upon an existing labeling of G_d. In the following, we study the problem in more details, and define upper and lower bounds on $\lambda_q^p(G_d)$ for all values of $p, q, d \geq 1$. These results are summarized in Theorem 1.

Theorem 1 ($L(p, q)$ labeling of G_d, for any value of $p, q, d \geq 1$) *Let $p \geq 1$ and $d \geq 1$. Then:*

1. $2p + (2d-2)q \leq \lambda_q^p(G_d) \leq 2dq$ *when* $2 \leq 2p < q$
2. $2p + (2d-2)q \leq \lambda_q^p(G_d) \leq 2p + (2d-1)q - 1$ *when* $1 \leq q \leq 2p \leq 4dq$
3. $\lambda_q^p(G_d) = p + (4d-2)q$ *when* $p \geq 2dq + 1$

When $q = 1$, the bounds we get coincide, thus yielding an optimal $L(p, 1)$ labeling of G_d for any value of p. We note that this generalizes Lemma 5 of [BPT00] and Theorem 3 of [DMP$^+$02], and also answers an open problem stated in [DMP$^+$02]. We also note that when the above bounds do not coincide, they differ by an additive factor equal to $q - 1$ when $1 \leq q \leq 2p \leq 4dq$, and equal to $2q - 2p \leq 2q - 2$ when $2 \leq 2p < q$.

Moreover, in the case $1 \leq q \leq 2p \leq 4dq$, for sufficiently large grids (that is, when the x_is are large enough for each $1 \leq i \leq d$), the coloring we propose to achieve an $L(p, q)$ labeling satisfies the *no-hole* property, that is all the colors in the range $[0; 2p+2(d-1)q-1]$ are used. This is the purpose of the following Proposition 1 below.

Proposition 1 (No-hole property when $1 \leq q \leq 2p \leq 4dq$) *Let $d \geq 1$. In that case, when $1 \leq q \leq 2p \leq 4dq$, there exists a no-hole $L(p, q)$ labeling of G_d such that $\lambda_q^p(G_d) = 2p + (2d-1)q - 1$.*

In the other cases, the proposed $L(p, q)$ labelings cannot be no-hole labelings, because some colors are forbidden. Indeed, in the case $2 \leq 2p < q$, colors are taken in the set $\{0, q, 2q, \ldots 2dq\}$, thus it cannot be a no-hole coloring. In the case $p \geq 2dq + 1$, the colors ranging in the interval $[(2d-1)q + 1; p + (2d-1)q - 1]$ are forbidden, thus the coloring we suggest cannot be no-hole.
Table 1 summarizes the results obtained in Section 2 concerning bounds for the $L(p, q)$ coloring of G_d, for all the possible cases. In this table, we give the lower and upper bounds for $\lambda_q^p(G_d)$; they are given in bold characters when the bounds coincide. We also mention the gap between the upper and lower bounds, when they do not coincide. Finally, in the rightmost column, we state whether the no-hole property holds for the colorings suggested in this paper.

Values of p and q	$\lambda_q^p(G_d) \geq$	$\lambda_q^p(G_d) \leq$	Gap	No-hole
$p \neq 0$; $q = 0$	$\boldsymbol{p}$		0	No
$p = 0$; $q \neq 0$	$\boldsymbol{(2d-1)q}$		0	No
$p, q \geq 1$; $2p < q$	$2p + (2d-2)q$	$2dq$	$2q - 2p$	No
$q = 1$; $1 \leq p \leq 2d$	$\boldsymbol{2p + 2d - 2}$		0	Yes (Prop 1)
$p, q \geq 1$; $q \leq 2p \leq 4dq$	$2p + (2d-2)q$	$2p + (2d-1)q - 1$	$q - 1$	Yes (Prop 1)
$p, q \geq 1$; $p \geq 2dq + 1$	$\boldsymbol{p + (4d-2)q}$		0	No

Table 1: $L(p, q)$ labeling of G_d: Summary of the results

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Brooks-type Theorem for Generalized List T-Coloring

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Abstract

We introduce the generalized list T-coloring as a common generalization of the channel assignment problem and the T-coloring. We state and prove a Brooks-type theorem for this new problem. This unifies the previously known Brooks-type theorems for the channel assignment problem and the T-coloring. In particular, we show that if the problem is balanced and its underlying graph is connected and distinct from a Gallai tree, then it has a good labeling.

1 Introduction

We introduce and study a common generalization of the graph coloring, the list coloring, the T-coloring, the channel assignment problem and the list channel assignment problem. We call it the *generalized list T-coloring*. Our approach unifies several previously known Brooks-type results, in particular [3, 5, 8, 14]. The generalized list T-coloring is a more flexible model for problems from practice compared to the channel assignment problem because, in addition, you may prevent two transmitters from assigning frequencies with prescribed differences.

An instance of the *generalized list T-coloring* is a triple (G, Λ, t) where G is a graph, $\Lambda : V(G) \to 2^{\mathbb{N}_0}$ and $t : E(G) \to 2^{\mathbb{N}_0}$ where $\mathbb{N}_0$ is a set of all non-negative integers. We write $V(G)$ and $E(G)$ for the vertex set and the edge set of a graph G, respectively. Elements of the sets $\Lambda(v)$ are called *colors* and elements of the sets $t(e)$ *forbidden differences*. The function t must satisfy $0 \in t(e)$ for every edge $e \in E(G)$. An instance of the generalized list T-coloring is also called a *generalized list T-coloring problem*. The goal of the problem is to find a mapping $c : V(G) \to \mathbb{N}_0$ such that $c(v) \in \Lambda(v)$ for each $v \in V(G)$ and $|c(x) - c(y)| \notin t(xy)$ for each $xy \in E(G)$. Such a mapping is called a *good labeling*.

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The *t-degree* $\deg_t(v)$ of a vertex v is equal to the sum $\sum_{vw \in E(G)} |t(vw)|$. An instance (G, Λ, t) is called *balanced* if $|\Lambda(v)| = \deg_t(v)$ for each $v \in V(G)$. The problem is called *overbalanced* if $|\Lambda(v)| \geq \deg_t(v)$ for each $v \in V(G)$ and at least at one vertex the inequality is strict. Our main results are the following: Each overbalanced generalized list T-coloring problem has a good labeling (Theorem 1). We completely describe balanced generalized list T-coloring problems which do not have good labelings (Theorem 4). In particular, we show that if G is a connected graph which is not a Gallai tree, then (G, Λ, t) has a good labeling.

Our results translate to the above mentioned concepts:

• **The graph coloring**

The generalized list T-coloring problems, where Λ is equal to $\mathbb{N}_0$ at every vertex and t is constantly equal to $\{0\}$ at every edge, correspond to the usual graph coloring. Theorem 1 translates to the well-known inequality $\chi(G) \leq \Delta(G) + 1$ and Theorem 4 to Brooks' theorem [3].

• **The list coloring**

The list coloring is a graph coloring where each vertex has to be assigned a color from its list [6, 12]. In our setting, Λ is just the function assigning lists of colors to the vertices and t is again a constant mapping equal to $\{0\}$. Theorem 1 translates to the claim that each graph G is $(\Delta + 1)$-choosable and Theorem 4 coincides with Brooks-type theorems for the list coloring from [2, 4, 13].

• **The T-coloring and the list T-coloring**

In the T-coloring, one has to assign numbers to the vertices of a graph in such a way that the difference between the numbers assigned to two adjacent vertices is not from a certain set of integers T, see [1, 7, 15]. It is required that $0 \in T$. In the list T-coloring, each vertex is equipped with a list of available numbers and the number assigned to it must be from its list. The generalized list T-coloring restricts to the (list) T-coloring when the function t is a constant function equal to the set T. Theorem 4 for such a function t is just the Brooks-type theorem for the list T-coloring proved by Waller [14]: G is not $|T|\Delta(G)$-T-choosable if and only if G is either a complete graph or an odd cycle and T is an arithmetic set. A set A of integers of size k is *arithmetic* with a difference d if $A = \{0, d, 2d, \ldots, (k-1)d\}$.

• **The (list) channel assignment problem**

In the channel assignment problem, a graph with edge weights is given. The difference between the numbers assigned to the end-vertices of an edge must be at least its weight [9]. Theorem 1 translates to a counterpart of the equality "$\chi \leq \Delta + 1$" for the channel assignment problem proven by McDiarmid [10] and Theorem 4 extends the Brooks-type theorem from [5].

2 2-connected graphs

In what follows, we write $A \uplus B$ for the union of disjoint sets A and B; this notation is used only to emphasize that the sets are disjoint. Arithmetic sets are also often considered in the paper, so we define $\mathrm{Ar}_d(k)$ to be the set $\{0, d, 2d, \ldots, d(k-1)\}$. First, a counterpart of the well-known graph inequality $\chi \leq \Delta + 1$ is stated:

Theorem 1 *An overbalanced generalized list T-coloring problem (G, Λ, t) has a good labeling whenever G is a connected graph.*

Most of balanced problems have also a good labeling:

Proposition 1 *Let (G, Λ, t) be a balanced problem such that G is 2-connected. If G is neither a complete graph nor an odd cycle, or there are two vertices whose lists have distinct minimal or maximal colors, then the problem (G, Λ, t) has a good labeling.*

The following proposition is a corollary of the Brooks-type theorem for T-coloring mentioned in the introduction which was proved by Waller [14]:

Proposition 2 *Let (K_2, Λ, t) be a balanced problem with $K_2 = uv$. The problem (K_2, Λ, t) does not have a good labeling if and only if $t(uv)$ is arithmetic and $\Lambda(u) = \Lambda(v) = k + t(uv)$ for some k.*

Before the analysis of the case of odd cycles, we define three special types of vertices. Let (C_n, Λ, t) be a balanced problem where C_n is a cycle $v_1 \ldots v_n$. Let Λ_{min} and Λ_{max} be the minimal and the maximal color from the lists $\Lambda(v)$, $v \in V(G)$. If the i-th of the following conditions holds ($i = 1, 2, 3$), then the vertex v_i of C_n is of the *first, second and third type*, respectively:

1. $t(v_{i-1}v_i) = t(v_iv_{i+1})$ and $\Lambda(v_i) = (\Lambda_{\mathrm{min}} + t(v_{i-1}v_i)) \uplus (\Lambda_{\mathrm{max}} + t(v_{i-1}v_i))$.

2. The sets $t(v_{i-1}v_i)$ and $t(v_iv_{i+1})$ are arithmetic and they share the same difference d but $t(v_{i-1}v_i) \neq t(v_iv_{i+1})$. The list $\Lambda(v_i)$ is $\Lambda_{\mathrm{min}} + \mathrm{Ar}_d(k)$ where $k = |t(v_{i-1}v_i)| + |t(v_iv_{i+1})|$. In particular, $\Lambda_{\mathrm{max}} - \Lambda_{\mathrm{min}} = d(k-1)$.

3. Both $t(v_{i-1}v_i)$ and $t(v_iv_{i+1})$ are arithmetic sets such that $t(v_{i-1}v_i) = \mathrm{Ar}_d(k)$ and $t(v_iv_{i+1}) = \mathrm{Ar}_d(k')$ where $d \neq d'$, $k, k' \geq 2$ and $kd = k'd' = \mathrm{lcm}(d, d')$. In addition, $\Lambda_{\mathrm{max}} - \Lambda_{\mathrm{min}} = \mathrm{lcm}(d, d')$ and $\Lambda(v_i) = (\Lambda_{\mathrm{min}} + \mathrm{Ar}_d(k)) \uplus (\Lambda_{\mathrm{max}} - \mathrm{Ar}_{d'}(k')) = (\Lambda_{\mathrm{min}} + \mathrm{Ar}_{d'}(k')) \uplus (\Lambda_{\mathrm{max}} - \mathrm{Ar}_d(k))$.

Theorem 2 *A balanced generalized list T-coloring problem (C_n, Λ, t) does not have a good labeling if and only if:*

- *n is odd and all the sets $t(e)$ for $e \in E(C_n)$ are arithmetic,*
- *all the lists $\Lambda(v)$ for $v \in V(C_n)$ have the same minimal and maximal color,*
- *each vertex is one of the three types described above, and*
- *there is at least one vertex of the first or of the second type.*

Next, we deal with complete graphs:

Theorem 3 *Let (K_n, Λ, t) be a balanced generalized list T-coloring problem with $n \geq 4$. Let $V(K_n) = \{v_1, \ldots, v_n\}$. The problem (K_n, Λ, t) does not have a good labeling if and only if one of the following holds:*

- *There exist integers $1 \leq a, d$ and $k_1 < \cdots < k_{n-1}$ such that:*
  - $t(e) = \mathrm{Ar}_d(a)$ *for all* $e \in E(K_n)$ *and*
  - $\Lambda(v_i) = \biguplus_{1 \leq j \leq n-1} (k_j + \mathrm{Ar}_d(a))$ *for all* $1 \leq i \leq n$
- *There exist integers $1 \leq a < b$, $1 \leq d$ and k such that (possibly after an appropriate permutation of the vertices):*
  - $t(e) = \begin{cases} \mathrm{Ar}_d(a) & \text{if an edge } e \text{ is incident with the vertex } v_n, \\ \mathrm{Ar}_d(b) & \text{otherwise.} \end{cases}$
  - $\Lambda(v_i) = \begin{cases} k + \mathrm{Ar}_d(a + b(n-2)) & \text{if } i \neq n, \\ \bigcup_{0 \leq j \leq n-2} (k + b \cdot j \cdot d + \mathrm{Ar}_d(a)) & \text{if } i = n. \end{cases}$

3 General Graphs

Theorem 4 *Let (G, Λ, t) be a balanced generalized list T-coloring problem where G is a connected graph. Let $B_1, \ldots, B_l$ be the blocks of G. The problem (G, Λ, t) does not have a good labeling if and only if there exists $\Lambda_i : V(B_i) \to 2^{\mathbb{N}_0}$ and $t_i : E(B_i) \to 2^{\mathbb{N}_0}$, $1 \leq i \leq l$ such that:*

1. *For each $v \in V(G)$,* $\Lambda(v) = \biguplus_{\substack{1 \leq i \leq l \\ v \in V(B_i)}} \Lambda_i(v)$.
2. $t(e) = t_i(e)$ *for the unique i such that $e \in E(B_i)$.*
3. *Each generalized list T-coloring problem (B_i, Λ_i, t_i) is balanced and it does not have a good labeling.*

In particular, G is a Gallai tree and each (B_i, Λ_i, t_i) is as described in Proposition 2 or in Theorems 2 and 3.

All our proofs are algorithmic and hence we may conclude:

Corollary 1 *There is a polynomial-time algorithm which for each overbalanced generalized list T-coloring problem finds a good labeling. And, there is also a polynomial-time algorithm which for each balanced problem decides whether the problem has a good labeling and if so, then it finds such a labeling.*

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Cantor-Bernstein type theorem for locally constrained graph homomorphisms

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Abstract

In the paper we show that the simultaneous existence of a locally surjective and of a locally injective graph homomorphisms between a connected graph G and a connected and finite graph H assures that all such homomorphisms are in fact locally bijective.

We give a short proof of this assertion which unifies previously known partial results of this form. We utilize the notion of universal cover, and relate its properties to the notion of degree refinement, which was used as a principal tool in other works.

In this paper we consider so called locally constrained graph homomorphisms. For graphs G and H a (graph) homomorphism is a vertex mapping $f : V_G \to V_H$ satisfying the property that $(f(u), f(v))$ is an edge of H whenever vertices u and v are adjacent in G.

In addition, we consider three types of "local" restrictions. We may request that for every vertex u of G the set of its neighbors $N_G(u)$ is mapped

- *bijectively* onto $N_H(f(u))$, then the mapping f is called *locally bijective*, or also a *full covering projection*, or
- *injectively* into $N_H(f(u))$, then f is called *locally injective*, or also a *partial covering projection*, or
- *surjectively* onto $N_H(f(u))$, then f is called *locally surjective*, or also a *surjective covering projection*.

(The symbol $N_H(f(u))$ denotes the set of all neighbors of $f(u)$ in the graph H.)

All these three sorts of mappings have interesting theoretical and practical aspects. Full covers were first considered in algebraic graph theory [3]. As special cases of covering spaces

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from algebraic topology [14], graph covers are used in many applications in topological graph theory [9].

Partial covers were applied in a hardness proof for the existence of distance constrained labelings of graphs [8], a notion stemming from a highly practical problem of interference-free frequency assignment for wireless networks.

Surjective covers have been introduced by Everett and Borgatti [6], who called them role colorings. They originated in the theory of social behavior. The graph H, i.e. the *role graph*, models roles and their relationships, and we ask whether roles (i.e. the vertices of the role graph) can be assigned to individuals of a given society such that the relationships are preserved: Each person playing a particular role has among its neighbors exactly all necessary roles as they are prescribed by the model.

Computational applications of graph covers were used by Angluin [2] who studied "local knowledge" in distributed computing environments, and by Courcelle and Métivier [5] who showed that nontrivial minor-closed graph classes cannot be recognized by local computations.

In [4] Bodlaender raised the question of computational complexity of H-cover problems. The H-(partial) cover problem asks if a given input graph G (partially) covers H (here H is considered a fixed parameter of the problem). Similarly we define the H-role assignment problem. In [1] Abello *et al.* showed that there are both polynomial-time solvable (*easy*) and NP-complete (*difficult*) versions of this problem depending on the parameter graph H. The complexity of the H-cover problem was further studied in [10, 11]. Several infinite classes of both polynomial and NP-complete instances were recognized, however, currently no good conjecture concerning a good characterization of graphs H, for which the H-cover problem is polynomially solvable, is at hand (assuming, of course, $P \neq$ NP).

At this point we would like to highlight two structural theorems that provide an important tool for proving NP-hardness reduction from the H-cover problem to the H-partial cover and the H-role assignment problems.

Theorem 1 ([7]) *Let G be a finite graph and let H be a finite connected graph. Suppose there is a locally bijective homomorphism from G to H. Then any locally injective homomorphism from G to H is locally bijective, i.e. a full cover.*

An similarly:

Theorem 2 ([12]) *Let G be a finite graph and let H be a finite connected graph. Suppose there is a locally bijective homomorphism from G to H. Then any locally surjective homomorphism from G to H is locally bijective, i.e. a full cover.*

Let us remark that both of these theorems can be strengthened such that instead of the existence of a full covering projection it suffices to assume that both graphs have the same degree refinement matrix (see [7, 12]) which is a weaker condition (see e.g. [10]). The notion of degree refinement and of its matrix originated in the graph isomorphism theory and was very useful in proving relationship between the existence of full and other covering projections. Since this structure can be computed in polynomial time, it frequently participated in NP-hardness constructions for various graph covering problems.

In our study we involve the notion of the universal cover instead of the degree refinement to link all these theorems together. This notion was defined by Angluin in [2] in a study on common covers. We show, that by use of this structure we can generalize these two structural theorems in a unified one, which is moreover valid also for infinite graphs G:

Our main result can be expressed as follows:

Theorem 3 *Let G be a graph and let H be a finite connected graph. Suppose there is a locally surjective homomorphism $g : G \to H$ and a locally injective homomorphism $h : G \to H$. Then both g and h are locally bijective, i.e. full covers.*

The proof of this assertion is based on a study of locally constrained homomorphisms of universal covers derived from the accordingly constrained homomorphisms between the underlying graphs. The finiteness of H is required to encompass a size argument and it is shown in latter dicussion that this condition is necessary. We further discuss relationship of our method to previously known approach employing the notion of degree refinement.

Due to space restrictions we omit further details here.

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The Firefighter Problem For Graphs of Maximum Degree Three

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1 Introduction

We consider a dynamic problem introduced by B. Hartnell in 1995 [6]. Let (G, r) be a connected rooted graph (where $r \in V(G)$). At time 0, a fire breaks out at r. At each subsequent time interval, the firefighter *defends* some vertex which is not on fire, and then the fire spreads to all undefended neighbours of each *burning* (i.e., on fire) vertex. Once defended, a vertex remains so for all time intervals. The process ends when the fire can no longer spread. The firefighter (optimization) problem is to determine the maximum number of vertices that can be *saved*, i.e., that are not burning when the process ends.

Papers investigating the firefighter problem have appeared in the literature. Algorithms for two- and three-dimensional grid graphs are presented in [10]. These lead to bounds on the maximum number of vertices that can be saved. NP-completeness of the firefighter problem on bipartite graphs is established in [9]. This paper also establishes improved bounds and exact values for two-dmensional grids, and considers the restriction of the problem to trees. The results include exponential algorithms for solving the firefighter problem on trees (one of these runs in linear time for binary trees), and a polynomial-time algorithms for a subclass of trees related to perfect graphs. It is proved in [8] that the greedy algorithm is a 2-approximation algorithm on trees, that is, the maximum number of vertices saved is never more than twice the number saved using the greedy algorithm. (It need not be the case that the number of vertices burned under a greedy strategy is at

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most twice the number of vertices burned under an optimum strategy.) Other aspects of the firefighter problem are studied in [2]. Related topics are examined in [1, 5, 7, 4].

Perhaps the most interesting related open problem is to determine the complexity of the firefighter problem for trees. A formal conjecture has never appeared in the literature, but it has been widely believed for some time that the problem is NP-complete for trees. We prove that this is indeed the case, and more. A sequence of transformations with increasing expressive power is used to show that the problem is NP-complete for trees of maximum degree three. This result is then used to show that the firefighter problem is NP-complete for cubic graphs. By contrast, we show that if the fire breaks out at a vertex of degree two, the problem can be solved in polynomial time for graphs of maximum degree three.

2 NP-Completeness Results

We describe a sequence of transformations involving the problems listed below (in the reverse order they are used).

3T-FIRE
INSTANCE: A triple (T, r, k), where (T, r) is a rooted tree with $\Delta \leq 3$ and k is a positive integer.
QUESTION: When the fire begins at r, is there a strategy such that at most k vertices burn?

3T′-FIRE
INSTANCE: A triple (T, r, k), where (T, r) is a rooted tree such that $d(r) = 2^m + 2$ for some positive integer m, every other vertex in T has degree at most 3, and k is a positive integer.
QUESTION: When the fire begins at r, is there a strategy such that at most k vertices burn?

3-FIRE
INSTANCE: A triple (G, r, k), where (G, r) is a rooted cubic graph, and k is a positive integer.
QUESTION: When the fire begins at r, is there a strategy such that at most k vertices burn?

The *level* (or *depth*) of a vertex x in a rooted tree (T, r) is its distance from r. The *depth of the tree* (T, r) is the maximum depth of a vertex of T. A rooted tree is called *full* if all of its leaves occur at the same level.

3FL-FIRE
INSTANCE: A full rooted tree (T, r) with maximum degree $\Delta(T) \leq 3$.
QUESTION: When the fire begins at r, is there a strategy such that no leaf burns?

3NN-SAT (Not All Equal 3-SAT without negated literals)
INSTANCE: An ordered pair (B, C) where B is a set of boolean variables and C is a set

of logical clauses over B in conjunctive normal form, each containing three non-negated literals.
QUESTION: Is there a truth assignment for B such that every clause in C contains at least one true literal and at least one false literal?

3NN′-SAT
INSTANCE: An ordered pair (B, C) where B is a set of boolean variables and C is a set of logical clauses over B in conjunctive normal form, where $|B| = 2^m$ for some integer $m \geq 2$, exactly $|C|/2$ clauses in C contain only non-negated literals, and the remaining clauses are negations of these.
QUESTION: Is there a truth assignment for B such that exactly $|C|/2$ clauses in C contain exactly one true literal and the rest contain exactly two true literals?

The following straightforward lemma plays an important role in our proofs.

Lemma 1 *Suppose (G_1, r) is a spanning subgraph of (G_2, r). Let σ be a strategy for the firefighter problem on (G_2, r). Then σ is also a strategy for the firefighter problem on (G_1, r), and will save at least as many vertices in G_1 as it does in G_2.*

Our polynomial transformations are as follows:

Theorem 1 *3NN-SAT $\rightarrow$ 3NN′-SAT.*

Theorem 2 *3NN′-SAT $\rightarrow$ 3T′-FIRE.*

This reduction is the central and most substantial proof of the paper. The remaining theorems in the section are proved by continuting this reduction in a straightforward manner.

Theorem 3 *3NN′-SAT $\rightarrow$ 3T-FIRE.*

Theorem 4 *3NN′-SAT $\rightarrow$ 3′-FIRE.*

Theorem 5 *3NN′-SAT $\rightarrow$ 3FL-FIRE.*

The NP-completeness of 3NN-SAT (Not All Equal 3-SAT without negated literals, see [3]) thus implies that all aforementioned problems are NP-complete.

3 Polynomiality Results

Let (G, r) be a connected rooted graph with $\Delta(G) \leq 3$ and $1 \leq deg(r) \leq 2$. We consider the nontrivial case where $deg(r) = 2$. We define a function $f : V \rightarrow \mathbb{Z}^+$, computable in polynomial time, which is used to determine an optimal strategy for (G, r). The results below culminate in a polynomial time algorithm for this case.

Lemma 2 *Let (G, r) be a connected rooted graph, where $1 \leq deg(r) \leq 2$. Then there is an optimal solution to the firefighter problem in which, at each time, the vertex defended is adjacent to a burning vertex.*

Theorem 6 *Our strategy yields an optimal solution to the firefighter problem on a rooted graph (G, r) with $\Delta \leq 3$ and $1 \leq deg(r) \leq 2$.*

Corollary 1 *Let (G, r) be a connected rooted graph with $\Delta \leq 3$ and $deg(r) = 2$. The maximum number of vertices that can be saved by any strategy is $MVS(G, r) = |V(G)| - min\{f(x) \mid x \in (V_1 \cup V_2 \cup V_c)\}$.*

Corollary 2 *The firefighter problem is in P for graphs with maximum degree 3 in which the fire starts at a vertex of degree at most two.*

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A method for proving polynomial enumeration formulas

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Abstract

On the base of a refinement of the Bender-Knuth (ex-)Conjecture I present an elementary method for proving enumeration formulas which are polynomial in at least one parameter. The Bender-Knuth (ex-)Conjecture gives the generating function of column-strict plane partitions with parts in $\{1, 2, \ldots, n\}$ and at most c columns. In our refinement of this result the number of parts equal to n is fixed in addition.

1 Introduction

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition, i.e. $\lambda_i \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$. A *strict plane partition* of shape λ is an array $\pi_{1 \leq i \leq r, 1 \leq j \leq \lambda_i}$ of non-negative integers such that the rows are weakly decreasing and the columns are strictly decreasing. The *norm* $n(\pi)$ of a strict plane partition is defined as the sum of its parts and π is said to be a strict plane partition of the non-negative integer $n(\pi)$. For example

$$\begin{matrix} 7 & 5 & 5 & 4 & 3 & 2 \\ 6 & 4 & 3 & 2 & & \\ 5 & 2 & & & & \\ 3 & 1 & & & & \end{matrix}$$

is a strict plane partition of shape $(6, 4, 2, 2)$ with norm 52. In [2, p.50] Bender and Knuth had conjectured that the generating function of strict plane partitions with at most c columns whose parts lie in the set $\{1, 2, \ldots, n\}$ and with respect to this norm is equal to

$$\sum q^{n(\pi)} = \prod_{i=1}^{n} \frac{[c+i;q]_i}{[i;q]_i},$$

where $[n;q] = 1 + q + \cdots + q^{n-1}$ and $[a;q]_n = \prod_{i=0}^{n-1} [a+i;q]$. This conjecture was independently proved by Andrews [1], Gordon [5], Macdonald [6, Ex. 19, p.53] and Proctor [7, Prop. 7.2]. In our refinement of this result the number of parts equal to n in the strict plane partition is fixed in addition.

Theorem 1 *The generating function of strict plane partitions with parts in* $\{1, 2, \ldots, n\}$, *at most* c *columns and* k *entries equal to* n *is*

$$\frac{q^{kn}[k+1;q]_{n-1}[1+c-k;q]_{n-1}}{[1;q]_{n-1}} \prod_{i=1}^{n-1} \frac{[c+i+1;q]_{i-1}}{[i;q]_i}.$$

If we sum this generating function over all k's, $0 \leq k \leq c$, we easily obtain the Bender-Knuth (ex)-Conjecture. Probably this detour over Theorem 1 is so far the easiest and most elementary possibility to prove the Bender-Knuth (ex-)Conjecture. For the full version of this paper see [3].

In this extended abstract I explain how to prove the case $q = 1$ of Theorem 1, i.e. how to compute the number of strict plane partitions with at most c columns, parts in $\{1, 2, \ldots, n\}$ and k parts equal to n. However, the method can be extended in order to prove the more general result. Note that in the special case $q = 1$ the formula in Theorem 1 is a polynomial in k, which factorises into linear factors over $\mathbb{Z}$. In the following I give a recipe which might be suitable for proving polynomial enumeration formulas of that type in general. It is divided into three steps.

1. **Extension of the combinatorial interpretation.** Suppose we are given combinatorial objects we want to enumerate which depend on an integer parameter k and we suspect that there exists an enumeration formula for these objects which is polynomial in k and factorises into distinct linear factors over $\mathbb{Z}$. Typically the admissible domain of k is a set S of non-negative integers. In the first step of our method we have to find (most likely new) combinatorial objects indexed by an *arbitrary* integer k which are in bijection with the original objects for $k \in S$.

2. **The extension is enumerated by a polynomial.** The extension of the combinatorial interpretation in the previous step has to be chosen such that we are able to prove that the new objects are enumerated by a polynomial in k. Mostly this is done with the help of a recursion. Moreover the degree of this polynomial has to be computed.

3. **Exploring 'natural' linear factors.** Finally one has to find the parameters k's for which there exist none of these objects, i.e. one has to compute the (integer) zeros of the enumeration polynomial. Typically these zeros will not lie in S, which made the extension in Step 1 necessary. Moreover one has to find one non-zero evaluation of the polynomial which is easy to compute, and together with the zeros the polynomial is computable.

In the following I sketch the proof of Theorem 1 for the special case $q = 1$. The three steps refer to the three steps in the description of our method. Throughout our considerations we use the extended definition of the summation symbol, namely,

$$\sum_{i=a}^{b} f(i) = \begin{cases} f(a) + f(a+1) + \ldots + f(b) & \text{if } a \leq b \\ 0 & \text{if } b = a-1 \\ -f(b+1) - f(b+2) - \ldots - f(a-1) & \text{if } b+1 \leq a-1 \end{cases}.$$

This assures that for any polynomial $p(X)$ over an arbitrary integral domain I there exists a unique polynomial $q(X)$ over I such that $\sum_{x=0}^{y} p(x) = q(y)$ for all integers y.

Step 1: From strict plane partitions to generalised $(n-1, n, c)$-Gelfand-Tsetlin-patterns

Let r, n, c be integers, r non-negative and n positive. In this paper a *generalised* (r, n, c)-*Gelfand-Tsetlin-pattern* (short: (r, n, c)-pattern) is an array $(a_{i,j})_{1 \leq i \leq r+1, i-1 \leq j \leq n+1}$ of integers with

- $a_{i,i-1} = 0$ and $a_{i,n+1} = c$,
- if $a_{i,j} \leq a_{i,j+1}$ then $a_{i,j} \leq a_{i-1,j} \leq a_{i,j+1}$
- if $a_{i,j} > a_{i,j+1}$ then $a_{i,j} > a_{i-1,j} > a_{i,j+1}$.

Thus

$$\begin{array}{ccccccccccccccc}
 & & & 0 & & 3 & & -5 & & 10 & & 4 & & & \\
 & & 0 & & 2 & & -2 & & 3 & & 8 & & 4 & & \\
 & 0 & & 2 & & -1 & & 2 & & 4 & & 7 & & 4 & \\
0 & & 0 & & 0 & & 1 & & 2 & & 5 & & 6 & & 4
\end{array}$$

is an example of an $(3, 6, 4)$-pattern. Note that a generalised $(n-1, n, c)$-Gelfand-Tsetlin pattern with $0 \leq a_{n,n} \leq c$ is what is said to be a Gelfand-Tsetlin pattern, see [8, p. 313] or [4, (3)] for the original reference. It is crucial to our considerations that $(n-1, n, c)$-patterns with $0 \leq a_{n,n} =: k \leq c$ are in bijection with strict plane partitions with parts in $\{1, 2, \dots, n\}$, at most c columns and k parts equal to n: Given an $(n-1, n, c)$-pattern, the corresponding strict plane partition is such that the shape filled by entries greater than i corresponds to the partition given by the $(n-i)$-th row of the $(n-1, n, c)$-pattern, the topmost row being the first row.

Therefore it suffices to enumerate the $(n-1, n, c)$-patterns with $0 \leq a_{n,n} = k \leq c$. A pair $(a_{i,j}, a_{i,j+1})$ with $a_{i,j} > a_{i,j+1}$ and $i \neq 1$ is called an *inversion* of the (r, n, c)-pattern and $(-1)^{\#\text{ of inversions}}$ is said to be the *sign* of the pattern, denoted by $\operatorname{sgn}(a)$. The $(3, 6, 4)$-pattern in the example above has altogether 6 inversions and thus its sign is 1. We define the following expression

$$F(r, n, c; k_1, k_2, \dots, k_{n-r}) = \sum_{a} \operatorname{sgn}(a),$$

where the sum runs over all (r, n, c)-patterns $(a_{i,j})$ with $k_i = a_{r+1,r+i}$ for $i = 1, \dots, n-r$. Note that the number of $(n-1, n, c)$-patterns with $0 \leq a_{n,n} = k \leq c$ is equal to $F(n-1, n, c; k)$. This is because an $(n-1, n, c)$-pattern with $0 \leq a_{n,n} \leq c$ has no inversions.

Clearly it makes no sense to ask for the number of strict plane partitions with parts in $\{1, 2, \ldots, n\}$, at most c columns and k parts equal to n if $k \notin \{0, 1, \ldots, c\}$, since by the column-strictness parts equal to n can only occur in the first row of a strict plane partition. However, $F(r, n, c; k)$ is well-defined for all integers k and therefore what we have done so far is 'naturally' extended the combinatorial interpretation of the number of these strict plane partitions to arbitrary integers k. In the following we see that 'naturally' here stands for the fact that the extension was chosen such that the extended interpretation is enumerated by a polynomial in k.

Step 2: $F(n-1, n, c; k)$ is a polynomial in k of degree $2n-2$

The definition of an (r, n, c)-pattern immediately implies the following recursion

$$F(r, n, c; k_1, k_2, \ldots, k_{n-r}) = \sum_{l_1=k_0}^{k_1} \sum_{l_2=k_1}^{k_2} \cdots \sum_{l_{n-r+1}=k_{n-r}}^{k_{n-r+1}} F(r-1, n, c; l_1, l_2, \ldots, l_{n-r+1}),$$

for $r > 0$ and with $k_0 = 0$, $k_{n-r+1} = c$. This recursion and the fact that

$$F(0, n, c; k_1, k_2, \ldots, k_n) = 1$$

shows inductively (with respect to r) that $F(r, n, c; k_1, \ldots, k_{n-r})$ can be expressed by a polynomial in $k_1, k_2, \ldots, k_{n-r}$. In the following $F(r, n, c; k_1, k_2, \ldots, k_{n-r})$ will be identified with this polynomial. It can be shown that this polynomial is of degree $2r$ in every k_i. However, this is the most difficult part in the proof of Theorem 1 and therefore we omit it here. It can be found in [3]. Consequently $F(n-1, n, c; k)$ is a polynomial in k of degree $2n-2$.

Step 3: Exploring 'natural' linear factors

Now observe that there exists no (r, n, c)-pattern with fixed first row $0, k_1, \ldots, k_{n-r}, c$ such that $k_1 \in \{-1, -2, \ldots, -r\}$ by induction with respect to r: For $r = 0$ there is nothing to prove, otherwise the southwest neighbour of k_1 in an (r, n, c)-pattern must be in $\{-1, -2, \ldots, -r+1\}$ and the assertion follows by induction. Analogously there exists no such (r, n, c)-pattern with $k_{n-r} \in \{c+1, c+2, \ldots, c+r\}$.

This implies that $(k+1)_{n-1}(1+c-k)_{n-1}$ is a factor of $F(n-1, n, c; k)$. But $(k+1)_{n-1}(1+c-k)_{n-1}$ is of degree $2n-2$ in k as well as $F(n-1, n, c; k)$ and therefore we have determined the enumeration polynomial $F(n-1, n, c; k)$ up to a constant. The constant can finally be determined by computing the evaluation $F(n-1, n, c; c)$, which turns out to be manageable.

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On the Ramsey degrees of posets

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In this talk all partially ordered sets (posets) and other combinatorial structures will be finite.

In the 1970s and 1980s remarkable progress was made with the problem of determining the Ramsey objects in various classes of combinatorial configurations, largely due to the fundamental papers by Nešetřil and Rödl [7, 8]. In the second half of the eighties the proofs were simplified by Nešetřil and Rödl [10] through the introduction of new amalgamation techniques.

Recall that if $\mathcal{C}$ is a class of finite structures for which we have a notion of a copy (or image under an embedding) of any object A in any object B of $\mathcal{C}$, then A is said to be a Ramsey object in $\mathcal{C}$ if for each B in $\mathcal{C}$ and $r < \omega$ there is some object C such that for each partition (or r-colouring) $\chi : [C, A] \to [r]$, where $[C, A]$ is the set of copies of A in C and $[r] = \{1, \ldots, r\}$, there is a copy B' of B in C such that all the elements of $[B', A]$ are in one block of the partition (or monochromatic).

In the papers [2, 3, 4] one of the present authors found, for various classes $\mathcal{C}$, for each A in $\mathcal{C}$, the smallest natural number, $t(A)$, with the following property: For each $r < \omega$, for each B in $\mathcal{C}$, there is some C in $\mathcal{C}$, such that, for each partition $\chi : [C, A] \to [r]$, there is some copy B' of B in $\mathcal{C}$ such that χ assumes at most $t(A)$ values on $[B', A]$. The number, $t(A)$, if it exists, is called the *Ramsey degree* of A. Thus A is a Ramsey object iff $t(A) = 1$.

Let $\mathbf{P} = (X, P)$ be a poset, and L a total order on the underlying set X of $\mathbf{P}$. The total order L is a *linear extension* of $\mathbf{P}$ if, for all x and y in X we have: $x < y(P) \Rightarrow x < y(L)$, i.e. if $P \subseteq L$.

For a finite poset $\mathbf{P}$, let $e(\mathbf{P})$ denote the number of linear extensions of $\mathbf{P}$ and write $A(P)$ for the automorphism group of $\mathbf{P}$. Set

$$t(\mathbf{P}) = e(\mathbf{P})/|A(\mathbf{P})|.$$

Since $t(\mathbf{P})$ is the number of non-equivalent linear extensions of $\mathbf{P}$ under the action of $A(\mathbf{P})$, it is a natural number. An *embedding* of a poset $\mathbf{P}$ into a poset $\mathbf{Q}$ is a one-to-one map $\lambda : \mathbf{P} \to \mathbf{Q}$ such that, for all $x, y \in \mathbf{P}$, we have that $x < y$ iff $\lambda(x) < \lambda(y)$. We refer to an image under an embedding of $\mathbf{P}$ in $\mathbf{Q}$ as a copy of $\mathbf{P}$ in $\mathbf{Q}$. The set of copies of $\mathbf{P}$ in $\mathbf{Q}$ is denoted by $[\mathbf{Q}, \mathbf{P}]$. In [2] one of the authors proved the following theorem.

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Theorem 1 *For posets* $\mathbf{P}$ *and* $\mathbf{Q}$ *and a natural number* r*, there exists a poset* $\mathbf{R}$ *such that, for any* r*-colouring of the copies of* $\mathbf{P}$ *in* $\mathbf{R}$*, there is a copy,* $\mathbf{Q}'$*, of* $\mathbf{Q}$ *in* $\mathbf{R}$ *such that* χ *assumes at most* $t(\mathbf{P})$ *values on* $[\mathbf{Q}', \mathbf{P}]$*. Conversely, for any* $\mathbf{P}$ *and* $r \geq t(\mathbf{P})$*, there is a poset* $\mathbf{Q}$ *such that for any* $\mathbf{R}$ *containing a copy of* $\mathbf{Q}$*, an* r*-colouring of* $[\mathbf{R}, \mathbf{P}]$ *can be found which assumes, on any set of the form* $[\mathbf{Q}', \mathbf{P}]$*, with* $\mathbf{Q}'$ *a copy of* $\mathbf{Q}$ *in* $\mathbf{R}$*, at least* $t(\mathbf{P})$ *values.*

It follows that $t(\mathbf{P})$ is the Ramsey degree of the poset $\mathbf{P}$. The direct sum of anti-chains has Ramsey degree one, and is therefore a Ramsey object in the class of finite posets. It is readily seen that if $\mathbf{P}$ is not an ordinal sum of anti-chains, one can find two non-equivalent linear extensions of $\mathbf{P}$, and the Ramsey degree is greater than one. In this way we recover the result of Nešetřil and Rödl [9] that the ordinal sums of anti-chains are the only Ramsey objects in the class of finite posets.

The proof of Theorem 1 revolves around the observation that a partial order is the intersection of all its linear extensions. The proof then uses the partition theorems of parameter words of Graham and Rothschild [6]. This has the implication that it does not yield a primitive recursive bound for the size of the poset $\mathbf{R}$ in terms of the sizes of $\mathbf{P}$, $\mathbf{Q}$ and r in the formulation of Theorem 1.

Using techniques from Nešetřil and Rödl [10], originally formulated in terms of set systems, one of the authors showed (using primitive recursive arithmetic) that finite ordinal sums of finite anti-chains are Ramsey objects in the category of finite posets and height-preserving embeddings, while restricting the heights of the posets for which the homogeneous structures can be found to the best possible in terms of the classical Ramsey numbers $R(r, h, H)$ [1]:

Theorem 2 *Suppose* $\mathbf{P}$ *is a complete poset of height* h *and* $\mathbf{Q}$ *is a poset of height* H*. Then, for every* r*, there is a poset* $\mathbf{R}$ *of height* $R(r, h, H)$ *such that, for any colouring* $\chi : [\mathbf{R}, \mathbf{P}]_h \to [r]$ *of the height-preserving copies of* $\mathbf{P}$ *in* $\mathbf{R}$*, there is a height-preserving embedding* λ *of* $\mathbf{Q}$ *into* $\mathbf{R}$ *such that* χ *is monochromatic on* $[\mathbf{Q}', \mathbf{P}]_h$ *where* $\mathbf{Q}'$ *is the image of* $\mathbf{Q}$ *under* λ*.*

The proof of this theorem uses techniques of [10]. We also proved the following result for binary posets using primitive recursive arithmetic [5].

Theorem 3 *Suppose* $\mathbf{P}$ *and* $\mathbf{Q}$ *are posets of height 2. Then, for every* $r \geq 1$*, there is a poset* $\mathbf{R}$ *of height 2 such that for any partition* $\chi : [\mathbf{R}, \mathbf{P}] \to [r]$ *there is a copy* $\mathbf{Q}'$ *of* $\mathbf{Q}$ *in* $\mathbf{R}$ *such that the restriction of* χ *to* $[\mathbf{Q}', \mathbf{P}]$ *assumes at most* $t(\mathbf{P})$ *values, where* $t(\mathbf{P})$ *is the Ramsey degree of* $\mathbf{P}$*.*

The generalisation of Theorem 3 to posets with arbitrary height (in primitive recursive arithmetic) is currently under investigation.

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On Interval Routing in Random Meshes*

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Abstract

In this paper we consider routing with compact tables in random meshes. More precisely, we study interval routing on random graphs $\mathbb{G}(B,p)$ obtained from a base graph B by independently removing each edge of B with a failure probability $1-p$. We focus on additive stretched routing for n-node random graphs for which the base B is a square mesh and with $p = 0.5$, that is the percolation model at the critical phase. We show a lower bound of $\Omega(\sqrt{\log n}/(\delta+1))$ on the number of intervals required per edge for every additive stretch $\delta \geqslant 0$. On the other side, our experimental results show that the size of the largest biconnected component is $\Theta(n^{0.827})$, and thus that there exists a trivial shortest-path routing scheme using at most $O(n^{0.827})$ intervals per edge. On the other hand we prove that the size of the largest biconnected component is $\Omega(n^c)$ for some constant $c > 0$.

1 Introduction

Research in the area of routing algorithms on computer networks became of great interest by many researchers in recent years. This topic is interesting both from the theoretical and also from the application point of view. One aspect of research in this field is to compact routing tables, by maintaining the smallest amount of routing information (or knowledge) locally in each router while guaranteeing that the routes are near the shortest paths.

Many results concern the design of *universal* routing strategies in the sense that they are applicable to all networks. In particular, the proposed schemes give trade-offs between the memory requirements (the size of the local routing tables) and the stretch factor, namely the maximum ratio between the length of the route between any two nodes and their distance in the network.

Above strategies apply to all the networks, however is it natural to wonder whether other more efficient techniques can be applied on realistic networks. Although there is still no answer to the question "what a realistic network is?", many models consider that such

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networks are based on some structured underlying topology (which is certainly not the complete network) with some random extra connections or some random link failures (cf. the augmented grid Kleingberg's model of Small World[1]).

2 Model and Problem Statement

2.1 Reliability Networks

In this paper we consider a point-to-point communication network modeled by a simple connected graph $G = (V, E)$, where V is the set of nodes (or processors or routers) and E is a set of edges (or bidirectional communication links). We focus on random graphs $\mathbb{G}(B, p)$ obtained from a graph B, with node set $\{1, \ldots, |V(B)|\}$, by independently removing each edge of B with a failure probability $1 - p$. So $G \in \mathbb{G}(B, p)$ is a uniform (labeled) random spanning subgraph of B as $V(G) = V(B)$ and $E(G) \subseteq E(B)$. More precisely,

$$\Pr(G) \ = \ p^{|E(G)|}(1-p)^{|E(B)|-|E(G)|} \ .$$

The graph B is called the *base* graph, and the value $1 - p$ the *failure probability*. This model, called the *random B-graph model*, appears in Bollobás[2] and is described in more detail by Janson, Łuczak and Ruciński[3]. The random B-graph is a natural generalization of the binomial random graph model of Erdös-Rényi, denoted hereafter $\mathbb{G}(n, p)$, for which $B = K_n$ is the complete graph on n nodes. The motivation of using random B-graph model comes from the several areas of computer science. This model is also called *reliability network model* because it describes computer (or processor) interconnection network with faulty communication links satisfying probabilistic distribution. The random B-graph model based on infinite square mesh belongs to percolation theory (cf. Grimmett[4]). This model can be generalized further by allowing different probabilities of failure at different edges. It is also related to other problems of computer science such as grid-computing, fault-tolerant distributed computing, effective data structures, etc.

2.2 Routing Tables and Interval Routing

Our paper is concerning *additive* stretched routing schemes, rather than multiplicative stretch (or stretch factor). A path of a graph is *δ-stretched* if the length of the path is at most the length of a shortest path between its extremities plus δ. A *δ-stretched routing scheme* is a scheme for which all the routes are δ-stretched paths. It is provable that even a small additive stretch allows better optimizations for spanner construction (Elkin and

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Peleg[5]), distance computation, and yields also compact routing tables (Dourisboure and Gavoille[6]).

We focus on *interval routing scheme*, a particular way of implementing standard routing tables. Recall that a routing table on a graph G consists of the set of distinct addresses ranging in $\{1, \ldots, |V(G)|\}$ and a set of *local routing tables* associated with each node. When a source u sends a message to a destination v, it attaches to the message the address of v, say the integer i, and transmits the message through the output port number q which is computed by u by looking at the ith entry of its local table. So the route is computed in a distributed fashion by all the nodes along the route between u and v. Obviously it is required that for every source-destination pair u, v a route connects u to v. Interval routing implements local routing tables as follows: u stores a d entry tables, d being the degree of u. The qth entry is the list of destination addresses v for which the route from u to v uses output port number q. If for all the nodes, the destination addresses using the same output port can be grouped into k sets of consecutive integers (consecutive modulo n), we say that the routing scheme is a *k-interval routing scheme* (k-IRS for short). For more precise formulations and other details see surveys of Gavoille[7] and Ružička[8].

The main difficulty in the design of interval routing schemes for a given graph is to find out a suitable address assignment for the nodes and a suitable system of routes for all the pairs of nodes such that the number of intervals per output port (equivalently per outgoing edge) is minimum while keeping the routes near-shortest paths. Whenever shortest paths are required, the problem to decide whether a graph supports a 1-IRS is already NP-complete (see Eilam, Moran and Zaks[9]. A δ-stretched k-IRS, denoted by (k, δ)-IRS, is simply a k-IRS that is a δ-stretched routing scheme. A $(k, 0)$-IRS is also called a shortest-path k-IRS.

3 Previous Works on Random Graphs

The main advantage of k-IRS concerns the size of the memory requirements. In an n-node graph supporting a k-IRS, a node of degree d has to store $O(kd \log n)$ bits of information whereas $O(n \log d)$ bits are required for a standard routing table implementation. In particular, interval routing is efficient for structured graphs like cycles, complete graphs, meshes, trees, outerplanar graphs, tori, hypercubes, etc. All these graphs support shortest-path $O(1)$-IRS.

Flammini, van Leeuwen and Marchetti-Spaccamela[10] proved a non-constant lower bound on k for shortest-path k-IRS on random graphs of $\mathbb{G}(n, p)$, the Erdös-Rényi model. It is

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proved therein that, with high probability, a graph $\mathbb{G}(n,p)$ requires $\Omega(n^{1-1/\Theta\sqrt{\log n}})$ intervals per outgoing edge for some specific value of p, namely for $p = 1/n^{1-1/\Theta\sqrt{\log n}}$. On the other hand, Gavoille and Peleg[11] proved that almost all graphs (that is graphs of $\mathbb{G}(n,p)$ for $p = 0.5$ and with high probability) support a shortest-path 2-IRS. Actually, they constructed a routing scheme such that every node has at most $O(\log^3 n)$ outgoing edges with 2 intervals, all the other one having 1 interval, leaving open the question of whether almost all graphs support shortest-path 1-IRS. Finally, Nehéz[12] proved a preliminary lower bound of $k = \Omega(\sqrt{\log n}\,)$ for shortest-path k-IRS on random tori.

4 Our Results

Our contribution is a lower bound on k for (k,δ)-IRS on random n-node square meshes with constant failure probability. We show that asymptotically almost surely, $k = \Omega(\sqrt{\log n}\,)$, for every additive stretch $\delta \geqslant 0$. This result has been generalized by the same authors to higher dimensional meshes[13].

For upper bounds, we have studied random square meshes with $p = 0.5$, that is the percolation threshold probability. Recall that when the size of the mesh becomes infinite, $p = 0.5$ is precisely the probability where the mesh contains a unique infinite connected components, cf. Grimmett. Unfortunately, as many interesting problems in Percolation Theory (and as suggested by Benjamini and Schramm[14], we are reduced to make experiments. Based on the expected size of the largest biconnected components of a random square meshes, our results suggest that random square n-node meshes support shortest-path k-IRS with $k = O(n^{0.827})$. However, we can show that almost surely the size of the largest biconnected components is at least n^c for some constant $c > 0$.

To prove Result 1 we combine probabilistic and graph-theoretic proof techniques covering also an inductive construction of a proper "worst-case" graph. Result 2 is obtained by some computer experiments and simulations, while the lower bound on the largest biconnected components is obtained by constructing a cycle of length n^c observing that with constant probability, a random square mesh contains a path connecting a node of the first column to a node of the last column.

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Orbits on cycles and circulant objects

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In this talk the fundamentals of oligomorphic permutation groups, a class of infinite permutation groups with connections with (enumerative) combinatorics, and their actions, are first recalled. A permutation group G acting on a countable set Ω is said to be *oligomorphic* if, for each natural number k, G has only finitely many orbits on Ω^k.

We are interested in counting orbits on cycles of elements of the group; the *Parker sequence*, or *Parker vector*, of an oligomorphic permutation group G is the sequence $\mathbf{p}(G) = (p_1, p_2, p_3, \ldots)$, where p_k is the number of orbits of G on the set of k-cycles appearing in elements of G, with G acting by conjugation. For instance, for the symmetric group S acting on a countable set, the Parker sequence is just $(1, 1, 1, \ldots)$. A less trivial example is the group C preserving a circular order on a countable set; here, for the Parker sequence one has $p_k = \varphi(k)$.

It will be shown how to work out the Parker sequences for several known classes of oligomorphic groups, particularly those related to classes of relational structures (say, graphs, tournaments, treelike structures).

If we are dealing with a group G defined as the automorphism group of the limit of a Fraïssé class $\mathcal{F}$ of relational structures, the Parker sequence of G has an interpretation as the sequence enumerating the finite circulant structures in that class. More precisely, the kth component of the Parker sequence counts the relational structures in (the age of) $\mathcal{F}$ on the set $\{1, 2, \ldots, k\}$ admitting as an automorphism the permutation $(1\ 2\ \ldots\ k)$.

This mirrors what happens with the L-sequence (F_k) of the same group, which is defined as the number of orbits on k-tuples of distinct elements, and is equal to the number of labelled structures on k points. The same holds for the U-sequence f_k of the number of orbits on k-sets, giving the number of unlabelled structures.

On the complexity of computing the Tutte polynomial of bicircular matroids

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1 Introduction

The Tutte polynomial $T(M; x, y)$ of a matroid M is a well-known invariant that has been studied from many different points of view. A thorough exposition of its properties and relationship to other areas can be found in [4] and [17]; see also [18] for a more recent survey.

A question that has received much attention is whether the evaluation of $T(M; x, y)$ at a particular point of the (x, y) plane can be done in polynomial time. The seminal paper in the area is by Jaeger, Vertigan and Welsh [7], who showed that evaluating the Tutte polynomial of a graph is #P-hard at every point except those lying in the hyperbola $(x-1)(y-1) = 1$ and the eight special points $(1, 1), (-1, 0), (0, -1), (-1, -1), (i, -i), (-i, i), (j, j^2), (j^2, j)$, where i is the imaginary unit and $j = e^{2\pi i/3}$. In each of the exceptional cases the evaluation can be done in polynomial time. In a later paper Vertigan and Welsh [16] showed that the same statement is true when the problem is restricted to bipartite graphs.

Another result relevant to this paper is that the evaluation of $T(M; x, y)$ for the class of transversal matroids is #P-hard for all points except those in the hyperbola $(x-1)(y-1) = 1$. This was proved by Colbourn, Provan and Vertigan in [5] using the construction of free extensions and coextensions of matroids. The situation for the class of matroids representable over a given field is more complex and is fully studied by Vertigan in [15].

In a different direction, it has been shown that the full Tutte polynomial can be computed in polynomial time in several cases. First it was proved for matroids of bounded width by Oxley and Welsh in [12]; this includes the class of series-parallel graphs, which are precisely the graphs of tree-width at most two. The extension to graphs of bounded tree-width was obtained independently by Andrzejak [1] and Noble [10]. Recently Hliněný [6] has obtained the same result for matroids of bounded branch-width representable over a given finite field, which is a substantial generalization of the previous results. The few

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other cases we know where $T(M;x,y)$ can be computed in polynomial time include matroids with a high degree of symmetry (see for instance [9]) and the class of lattice-path matroids introduced in [2] (this has been extended recently to a larger class in [3]).

In this paper we study the complexity of evaluating $T(M;x,y)$ for the class of bicircular matroids. If $G=(V,E)$ is a graph, the bicircular matroid $B(G)$ has E as ground set and has as bases the edge sets of spanning subgraphs of G in which every connected component has a unique cycle. We refer to [8] for additional properties of bicircular matroids, including the definition in terms of circuits.

We show that evaluating $T(M;x,y)$ for the class of bicircular matroids is #P-hard at every point (x,y) except those in the hyperbola $(x-1)(y-1)=1$ and possibly those on the lines $x=0$ and $x=-1$. For these two lines we are unable to prove either that it is #P-hard or that it can be done in polynomial time. Since bicircular matroids form a rather restricted subclass of transversal matroids, our results can be seen as a strengthening of the results in [5] (except for the two lines $x=0$ and $x=-1$ where we must leave the complexity issue undecided).

The proof has three ingredients. In Section 2 we show that if G is a graph with exactly one loop at every vertex, then

$$T(B(G);x,0)=x\,T(G;x+1,1),$$

where $T(G;x,y)$ is the Tutte polynomial of the cycle matroid associated to G. This, together with the results in [7] imply that computing the Tutte polynomial of bicircular matroids at $(a,0)$ is #P-hard for every $a\neq 0$. In Section 3 we prove that computing the number of bases in a bicircular matroid is #P-hard. This is done by reduction from the problem of computing the number of perfect matchings in a graph, a problem well-known to be #P-hard [14]. Since the number of bases in a matroid M is equal to $T(M;1,1)$, this covers the point $(1,1)$. Finally in Section 4 we use the matroid operation of stretching and polynomial interpolation to obtain the complexity results for the remaining points of the (x,y) plane.

We refer to [17] for an introduction to the complexity of counting and the definition of #P-hardness, and to [11] for general background on matroids.

2 The line $\mathbf{y=0}$

Let M be a matroid whose ground set E is linearly ordered. For a basis A of M, the internal activity $i(A)$ is the number of elements e in A such that e is the minimum element in the unique cocircuit contained in $(E-A)\cup\{e\}$; the external activity $e(A)$ is the number of elements e in $E-A$ such that e is the minimum element in the unique circuit contained in $A\cup\{e\}$. A classical result of Tutte says that

$$T(M;x,y)=\sum_{A}x^{i(A)}y^{e(A)}, \tag{1}$$

where the sum is over the set of all bases A of M.

Now let $G = (V, E)$ be a graph with exactly one loop at every vertex. If G has n vertices, we order the edges of E so that those corresponding to the n loops have the smallest labels. We also label the vertices of G with the same labels as the loops attached to them.

Lemma 1 *Let G be as above, and let A be a basis of $B(G)$. Then $e(A) = 0$ if and only if A is the edge-set of a spanning forest of G having one loop attached to the minimum vertex of every component of the forest. Moreover, the internal activity $i(A)$ is equal to the number of components.*

Since $T(G; x+1, 1)$ is the generating function for forests of G according to the number of components [4], as a corollary we obtain the following.

Theorem 1 *If G is as above, then*

$$T(B(G); x, 0) = x\, T(G; x+1, 1).$$

3 The number of bases

Here we prove that counting the number of bases in a bicircular matroid is a #P-hard problem. This is important in proving our main result, since the number of bases of a matroid M is precisley $T(M; 1, 1)$. Our proof is by means of a two step reduction from the problem of counting the number of perfect matchings in graphs, which is known to be #P-hard [14]. Next we state the relevant counting problems in detail.

Problem: #PM (Perfect matchings)
Input: A graph G.
Output: The number of perfect matchings of G.

Problem: #BB (Bicircular bases)
Input: A graphical representation of a bicircular matroid B.
Output: The number of bases of B.

Problem: #SBB (Signed Bicircular bases)
Input: A graphical representation of a bicircular matroid B plus an assignment of a sign $+1$ or -1 to every element of B.
Output: The weighted sum of the bases of B, where the weight of a basis is the product of the sign of its elements.

The following two propositions imply that #PM is polynomial-time reducible to #BB, hence #BB is a #P-hard problem.

Proposition 1 *#PM is polynomial-time reducible to #SBB.*

This is proved by taking an input graph G and defining a weighted graph G' such that $\#PM(G) = |\#SBB(G')|$.

Proposition 2 *$\#SBB$ is polynomial-time reducible to $\#BB$.*

The proof in this case is more complex and uses several auxiliary (non-weighted) graphs and polynomial interpolation.

4 Complexity results

The following is a direct corollary of Theorem 1 and the results in [7], and is a key result in what follows.

Lemma 2 *Computing the Tutte polynomial of bicircular matroids at $(a, 0)$ is #P-hard for any $a \neq 0$.*

A pointed matroid N_e consists of a matroid N and a distinguished element e of N which is neither a loop nor an isthmus. If M is any matroid, the tensor product $M \otimes N_e$ is the matroid formed from M by performing a 2-sum of N_e and M at every element of M. When N is the uniform matroid $U_{k,k+1}$ (a circuit of size $k+1$), the corresponding tensor product is called the k-stretching of M and is denoted by $s_k(M)$. The same operation can be defined for graphs and we have the following elementary lemma.

Lemma 3 *For any graph G, we have $s_k(B(G)) \cong B(s_k(G))$.*

The key fact about the stretching operation (proved by Brylawski; see [7]) is that evaluating $T(s_k(M); x, y)$ is, up to a trivial factor, equivalent to evaluating $T(M; X, Y)$, where

$$X = x^k, \qquad Y = \frac{y + x + \cdots + x^{k-1}}{1 + x + \cdots + x^{k-1}}.$$

The relation $(X-1)(Y-1) = (x-1)(y-1)$ allows then interpolation along the hyperbolas $(x-1)(y-1) = a$. Using interpolation, Lemma 3 and the results in the previous section, we obtain our main result.

Theorem 2 *Computing the Tutte polynomial of bicircular matroids is #P-hard at every point (a, b), except those points lying on the trivial hyperbola $(a-1)(b-1) = 1$, and possibly those on the lines $a = 0$ and $a = -1$.*

Note. Unfortunately, the two lines in the statement above are still undecided. Those familiar with [7] may wonder why we do not use the operation of thickening (dual to stretching); the reason is that the thickening of a bicircular matroid is in general not a bicircular matroid, hence we cannot use the interpolation technique in the theorem above.

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On Hamiltonian decompositions of 4-regular planar graphs

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Abstract

We study the existence of Hamiltonian cycles and Hamiltonian decompositions in graphs of vox-solids. Intuitively a vox-solid can be thought as a solid obtained by gluing 3-dimensional unit cubes by entire faces in such a way that the faces on the boundary give a surface. The vertices and edges on the boundary can be seen as a tiling of the surface. A 4-regular graph G admits a Hamiltonian decomposition if the edge set of G is the union of two edge-disjoint Hamilton cycles. We study Hamilton decompositions of certain families of 4-regular planar graphs. The families studied arise as dual graphs of spherical vox-solids. An $a \times b \times c$ prism is the product of three mutually orthogonal segments of lengths a, b and c. It consists of abc unit cubes, and the boundary has $2(ab + ac + bc)$ faces. A planar representation of the $a \times b \times c$ prism is obtained by drawing three concentric families of mutually crossing circles, where the number of circles in each family is, respectively, a, b and c. We show that the graph of a prism has a Hamiltonian decomposition. We introduce some other infinite families of vox-solids, namely, tree vox-solids and anti-prisms. We also show that the graphs of these vox-solids admit a Hamiltonian decomposition. Finally we present some results towards the characterization of 4-regular graphs which admit a Hamiltonian decomposition.

Caterpillar arboricity of planar graphs

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Abstract

We solve a conjecture of Y. Roditty [2] on the caterpillar arboricity of planar graphs. We prove that for every planar graph $G = (V, E)$, E can be partitioned into 4 subsets $(E_i)_{1 \le i \le 4}$ such that $G[E_i]$ is a forest of caterpillars.

1 Introduction

The *arboricity* $a(G)$ of a graph G is defined to be the minimum number of forests containing all the edges of G. In a similar way, the *linear arboricity* $la(G)$ (resp. *star arboricity* $sa(G)$, *caterpillar arboricity* $ca(G)$) of G is the smallest number of forests containing all the edges of G such that each component of each forest is a path (resp. a star, a caterpillar). A *star* is a tree where all the edges are adjacent to a vertex of degree 1. A *caterpillar* is a tree with the property that removal of all the end vertices and the edges containing them yields a path.

Since stars are caterpillars (resp. caterpillars are trees), a decomposition of a graph G into k forests of stars (resp. caterpillars) can be considered as a decomposition of G into k forests of caterpillars (resp. trees). So, for any G we have :

$$a(G) \le ca(G) \le sa(G) \tag{1}$$

Gyárfás and West [4] showed that the lower bound is thight for complete bipartite graphs. Indeed, they show that $ca(K_{m,n}) = a(K_{m,n}) = \lceil mn/(m + n - 1) \rceil$. Here, we show that the upper bound of this inequality is also tight. To achieve that, we exhibit a family of graphs for which the inequality holds.

Theorem 1 *For any $k > 0$, there is a k-tree T_k such that $sa(T_k) = ca(T_k) = k + 1$.*

Nash-Williams [6] proved that $a(G) = \max_{H \subset G} \lceil |E_H|/|V_H| - 1 \rceil$. Since a subgraph of a planar graph is planar and since planar graphs with n vertices have at most $3n - 6$ edges, planar graphs have arboricity at most 3. Hakimi, Mitchem and Schmeichel [5] showed

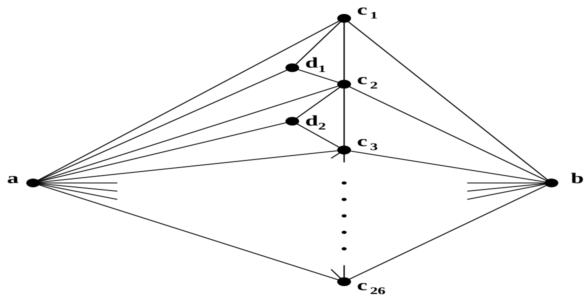

Figure 1: The graph H.

using a deep result of Borodin [1] on the acyclic coloring of planar graphs, that planar graphs have star arboricity at most 5. We are interested in determining such upper bound for the caterpillar arboricity of planar graphs. For a graph G with arboricity 3 we have $3 = a(G) \leq ca(G) \leq sa(G) \leq 5$, so this upper bound is either 3, 4 or 5. But the graph H in figure 1 reduces the possible values to 4 and 5.

Theorem 2 $ca(H) > 3$.

The main result of this paper is the following theorem :

Theorem 3 *For any planar graph G, we have $ca(G) \leq 4$.*

In the next section, we define the caterpillar partition of graphs and use it to prove theorems 1 and 2. In section 3 we focus on planar graphs and refine the notion of caterpillar partition to prove theorem 3. In section 4, we give an application of this theorem in monotone paths. Then we finally conclude.

2 The caterpillar partition

In a caterpillar, the set of *feet vertices* is a stable of vertices of degree one such that deleting all them we obtain a path. The other vertices are the *spine vertices*. Note that a spine vertex can also have degree one and that there is not a unique subset of vertices which could be the set of the feet vertices. For instance K_2 can have 0 or 1 foot vertex (figure 2).

A *caterpillar partition* of $G = (V, E)$ is a pair $\left(\overrightarrow{G}, (E_i)_{1 \leq i \leq k}\right)$ where $(E_i)_{1 \leq i \leq k}$ is a partition of E such that $G[E_i]$ is a forests of caterpillars and $\overrightarrow{G} = (V, \overrightarrow{E})$ is the graph G where some edges have been oriented. This partial orientation of G aim to distinguish the feet vertices from the spine vertices in $G[E_i]$. The edges of $\overrightarrow{G}$ are denoted $\{u, v\}$ if they are unoriented and (u, v) if they are oriented from u to v. If $\{u, v\} \in E_i$ and u and v are spine vertices in $G[E_i]$ then the edge remains unoriented, and $\{u, v\} \in \overrightarrow{E}$. If $\{u, v\} \in E_i$, u being a spine vertex and v a foot vertex in $G[E_i]$ then the edge $\{u, v\}$ is oriented from u to v, and $(u, v) \in \overrightarrow{E}$. Remark that a vertex can be a foot vertex (or spine vertex) in the

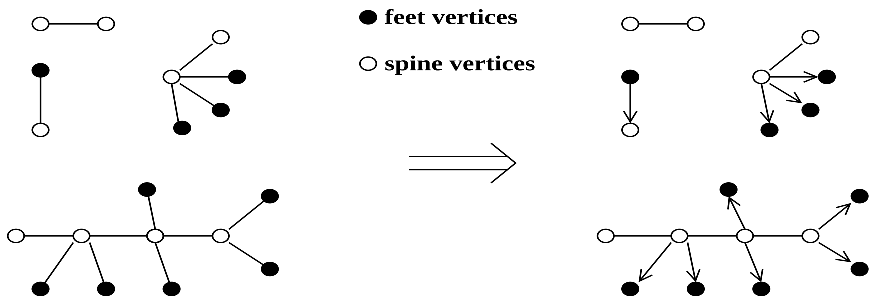

Figure 2: Partial orientation of the edges in a forest of caterpillars.

ith forest of caterpillars and have any status, foot or spine vertex in the others forests of caterpillars.

More formally, the caterpillar partition is a partial orientation of G and a partition of E that have to respect tow rules :

- The *feet preserving rule* ensures that the feet vertices of the ith forest of caterpillars have degree one in $G[E_i]$. The feet vertices of the ith forest of caterpillars are the vertices v that have a neighbor $u \in V$ such that $(u, v) \in \overrightarrow{E_i}$ (u being a spine vertex for the ith forest of caterpillar). If w is a neighbor of v in G, then we have either $w = u$ or $\{v, w\} \notin E_i$.

- The *spine preserving rule* ensures that after removing all the feet vertices in $G[E_i]$ we obtain a forest of paths. To achieve that, the unoriented edges of E_i have to form a forest of paths.

One can easily check that the caterpillar arboricity of a graph G, is the minimum k such that G has a caterpillar partition $(\overrightarrow{G}, (E_i)_{1 \le i \le k})$.

Proof of theorem 2. If $ca(H) \le 3$, there is a caterpillar partition $(\overrightarrow{H}, (E_i)_{1 \le i \le 3})$ of H. According to the previous rules there is at most one oriented edge $(x, a) \in \overrightarrow{E_i}$ with $x \in \{c_i | 1 \le i \le 26\}$ or at most two unoriented edges $\{x, a\}$ and $\{y, a\}$ in $\overrightarrow{E_i}$ for each i. So there are at most 6 edges in $\overrightarrow{H}$ adjacent to vertex a (resp. vertex b) that are not oriented from a (resp. b) to the other end of the edge. This implies that there is a j such that (a, c_j),(a, c_{j+1}),(b, c_j) and $(b, c_{j+1}) \in \overrightarrow{E}$. Given the feet preserving rule, the partition $(E_i)_{1 \le i \le 3}$ separates $\{a, c_j\}$, $\{b, c_j\}$ and $\{c_j, c_{j+1}\}$. Let $\{a, c_j\} \in E_1$, $\{b, c_j\} \in E_2$ and $\{c_j, c_{j+1}\} \in E_3$. Since $\{a, c_{j+1}\}$, $\{b, c_{j+1}\}$ and $\{c_j, c_{j+1}\}$ are also separated we have $\{a, c_{j+1}\} \in E_1$ and $\{b, c_{j+1}\} \in E_2$, or $\{a, c_{j+1}\} \in E_2$ and $\{b, c_{j+1}\} \in E_1$. By the feet preserving rule $\{c_j, d_j\}$ and $\{c_{j+1}, d_j\}$ cannot belong to E_1 or E_2. So there is a cycle $(\{c_j, c_{j+1}\}, \{c_{j+1}, d_j\}, \{d_j, c_j\})$ in $H[E_3]$, which is impossible in a forest of caterpillar.

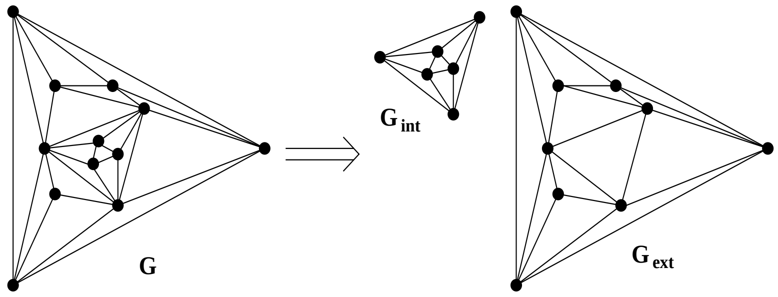

Figure 3: G splited into G_{int} and G_{ext}.

Outline of the proof of theorem 1. k-trees have acyclic chromatic number at most $k+1$. Since [5] proved that the star arboricity of a graph is bounded by its acyclic chromatic number we obtain that k-trees have star arboricity at most $k+1$. Since $ca(G) \leq sa(G)$ we just have to show a k-tree T_k such that $ca(T_k) > k$. We will do so in the full version of the paper, using similar arguments of theorem 2.

3 The planar case

A *maximal planar graph* is a simple graph that is not a spanning subgraph of another planar graph. Every planar graph is the subgraph of some maximal planar graph. Since a subgraph of a forest of caterpillars is also a forest of caterpillars we restrict our work to maximal planar graphs. We now consider that these maximal planar graphs are embedded in the plane. Because of the maximality of the graph, all the faces are bounded by three edges. We say that this graph is *splitable* if there is an induced K_3 which do not delimit a finite face or the infinite face. As expected, a splitable graph can be splited into two maximal planar graphs preserving the embedding. Figure 3 gives an exemple of such spliting. We denote by G_{ext} the maximal planar subgraph having the same infinite face as G, and G_{int} be the other one.

The proof of theorem 3 works by induction on the number of vertices of G and needs a refinement of the notion of caterpillar partition. We refine it adding new rules that are not detailed in this abstract.

Outline of the proof of theorem 3. We start the induction by proving that K_3 and K_4 admit a caterpillar partition with 4 subsets. Then we deal with two cases, G is splitable or it is not. If G is splitable, by the hypothesis of induction G_{ext} has a 4-caterpillar partition. This caterpillar partition implies some constraints to find a 4-caterpillar partition of G_{int} such that these two partitions form a 4-caterpillar partition of G. We show that this is possible. If G is not splitable, we obtain G' by contracting a given edge and fusioning the multiple edges created. By the hypothesis of induction G' have a 4-caterpillar partition. This partition is then extended to G and we are done.

This is a constructive proof based on a planar embedding of G. We design a quadratic algorithm that divides the edges of an embedded planar graph into 4 forests of caterpillars. Since it is linear to find a planar embedding for a planar graph, dividing a planar graph into 4 forests of caterpillars remains quadratic.

4 Monotone paths

An *edge-ordered graph* is a pair (G, f), where $G = (V, E)$ is graph and f is a bijective function, $f : E \to \{1, 2, \dots, |E|\}$. f is called an *edge ordering* of G. A *monotone path of length* k in (G, f) is a simple path $P_k : v_0, v_1, \dots, v_k$ in G such that $f(\{v_i, v_{i+1}\}) < f(\{v_{i+1}, v_{i+2}\})$ for $i = 0, 1, \dots, k-2$. To bound the length of monotone paths, we bound the length of monotone trials, this is paths not necessarily simples. Given a graph G denote by $\alpha'(G)$ the minimum of the maximum length of a monotone trial over all edge orderings of G. For a partition of $E = \cup_{i=1}^{l} E_i$ we have $\alpha'(G) \leq \sum_{i=1}^{l} \alpha'(G(V, E_i))$. For a tree T and a caterpillar C we have $\alpha'(T) \leq 3$ and $\alpha'(C) \leq 2$. This implies that $\alpha'(G) \leq 3 \times a(G)$ and that $\alpha'(G) \leq 2 \times ca(G)$. Given that planar graphs have arboricity at most 3, for any planar graph G, $\alpha'(G)$ is bounded by $3 \times 3 = 9$. Roditty's conjecture, about the caterpillar arboricity of planar graphs, was raised [7] to improve this bound to $2 \times 4 = 8$. There is still a gap since we know no planar graphs having a monotone path of length at least 6 for all edge ordering.

5 Conclusion

These results raise new questions. Could we for instance determine the graphs with caterpillar arboricity close to $a(G)$ or to $sa(G)$? Maybe, using a similar proof we could extend theorem 3 to graphs embedded in surface of genus k.

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Fast Matrix Multiplication via Matrix-Covers

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Abstract

We show how one can use non-prime-power, composite moduli for computing representations of the product of two $n \times n$ matrices using only $n^{2+o(1)}$ multiplications.

1 Introduction

The matrix multiplication is a basic operation in mathematics in applications in almost every branch of mathematics itself, and also in the science and engineering in general. An important problem is finding algorithms for fast matrix multiplication. The natural algorithm for computing the product of two $n \times n$ matrices uses n^3 multiplications. The first, surprising algorithm for fast matrix multiplication was the recursive method of Strassen [9], with $O(n^{2.81})$ multiplications. The best known algorithm today was given by Coppersmith and Winograd [4], requiring only $O(n^{2.376})$ multiplications. Some of these methods can be applied successfully in practice for the multiplication of large matrices [1].

The best lower bounds for the number of needed multiplications are between $2.5n^2$ and $3n^2$, depending on the underlying fields (see [2], [3], [8]). A celebrated result of Raz [7] is an $\Omega(n^2 \log n)$ lower bound for the number of multiplications, if only bounded scalar multipliers can be used in the algorithm.

The main result of the present paper is an algorithm with $n^{2+o(1)}$ multiplications for computing a *representation* of the matrix product modulo non-prime power composite numbers (e.g., 6). The algorithm is a straightforward application of a method of computing the representation of the dot-product of two length-n vectors with only $n^{o(1)}$ multiplications.

2 Preliminaries

2.1 Why do we count only the multiplications?

In algebraic algorithms it is quite usual to count only the multiplications in a computation. The reason for this is that the multiplication is *considered* to be a harder operation than the addition in most practical applications, and moreover, the multiplication is *proven* to be

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harder in most theoretical models of computation: (e.g., multiplying two length-n binary sequences needs super-polynomial size, constant-depth Boolean circuits with unbounded fan-in gates, or computing multiplication is hard in the communication-game model).

2.2 Covering the non-diagonal elements with few rectangles

Suppose that we would like to cover the elements of an $n \times n$ matrix by rectangles (i.e., submatrices) modulo m in the following way: The covering multiplicities of the non-diagonal elements are not 0 modulo m, while the covering multiplicities of the diagonal elements are equal to 0 mod m.

It turned out that for a prime modulus $m = p$, one needs at least

$$n^{1/(p-1)} - p$$

rectangles for such a cover, but for non-prime-power, composite moduli (say 6)

$$n^{o(1)}$$

are enough [5].

Why is this interesting for computational complexity of the dot-product and for the matrix-product?

Let us consider the matrix, where in row i and column j there is product $x_i y_j$, $i = 1, 2, \ldots, 10, j = 1, 2, \ldots, 10$. Any submatrix - that is, a rectangle - in this matrix corresponds to a product; for example the upper right 5 times 4 rectangle corresponds to the product $(x_1 + x_2 + x_3 + x_4 + x_5)(y_7 + y_8 + y_9 + y_{10})$. Consequently, a cover of the non-diagonal elements with $n^{o(1)}$ rectangles with non-zero multiplicity modulo m corresponds to a polynomial:

$$\sum_{\substack{i,j \in \{1,2,\ldots,n\} \\ i \neq j}} a_{ij} x_i y_j,$$

where coefficients a_{ij} are non-zero modulo m and this polynomial can be computed modulo m with $n^{o(1)}$ multiplication.

2.3 Matrix-cover and matrix product

Moreover, one can also get a cover with rectangles satisfying a somewhat more demanding constraint: one can cover the non-diagonal elements - say, modulo 6 -with $n^{o(1)}$ rectangles, such that the covering multiplicity is 1 either mod 2 or mod 3, or possibly both. Suppose, that

$$g(x, y) = \sum_{\substack{i,j \in \{1,2,\ldots,n\} \\ i \neq j}} a_{ij} x_i y_j \tag{1}$$

is the polynomial, resulting from that cover. Since the cover contained only $n^{o(1)}$ rectangles, (1) can be computed with $n^{o(1)}$ multiplications. Then consider the following polynomial:

$$h(x, y) = (x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots + y_n) - g(x, y).$$

In h, the coefficient of x_iy_i is 1, for all i's, and the coefficient of x_iy_j $i \neq j$ is 0 either mod 2 or mod 3. Moreover, $h(x,y)$ can be computed using only $n^{o(1)}$ multiplications. We say that this h 1-a-strong represents the dot-product of vectors x and y:

$$\sum_{i=1}^{n} x_i y_i,$$

modulo 6, since the coefficients of the x_iy_i are 1 in h, and the coefficients of the monomials which are not needed in the dot-product are 0, modulo at least one of 2 and 3. (See Definition 1 for details). Our main result is that a similar representation of the matrix-product can be computed by using this idea n^2 times, with $O(n^{2+o(1)})$ multiplication.

2.4 A-strong representations

In [6] we gave the definition of the *a-strong (i.e., alternative-strong) representation* of polynomials. Here we define the *1-a-strong* representation of polynomials. Note, that for prime or prime-power moduli, polynomials and their representation (defined below), coincide. Perhaps that is the reason that such representation were not defined before.

Definition 1 *Let m be a composite number $m = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$. Let Z_m denote the ring of modulo m integers. Let f be a polynomial of n variables over Z_m:*

$$f(x_1, x_2, \ldots, x_n) = \sum_{I \in \{0,1,2,\ldots,d\}^n} a_I x_I,$$

where $a_I \in Z_m$, $x_I = \prod_{i=1}^{n} x_i^{\nu_i}$, where $I = \{\nu_1, \nu_2, \ldots, \nu_n\} \in \{0,1,2,\ldots,d\}^n$. Then we say that

$$g(x_1, x_2, \ldots, x_n) = \sum_{I \in \{0,1,2,\ldots,d\}^n} b_I x_I,$$

is a 1-a-strong representation *of f modulo m, if $\forall I \in \{0,1,2,\ldots,d\}^n$ $\exists j \in \{1,2,\ldots,\ell\}$: $a_I \equiv b_I \pmod{p_j^{e_j}}$, and, furthermore, if for some i, $a_I \not\equiv b_I \pmod{p_i^{e_i}}$, then $a_I \equiv 0 \pmod{m}$.*

Example 1 *Let $m = 6$, and let $f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3$, then $g(x_1, x_2, , x_3) = x_1x_2 + x_2x_3 + x_1x_3 + 3x_1^2 + 4x_2$ is a 1-a-strong representation of f modulo 6.*

In other words, for modulus 6, in the 1-a-strong representation, the non-zero coefficients of f are correct for both moduli in g, but the zero coefficients of f can be non-zero either modulo 2 or modulo 3 in g, but not both.

2.5 Our main theorem

Theorem 1 *Suppose that m has at least two distinct prime divisors. Then a 1-a-strong representation modulo m of AB can be computed using*

$$n^2 2^{O(\sqrt{\log n (\log\log n)})} = n^{2+o(1)}$$

multiplications.

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Introduction to Domination Analysis

Gregory Gutin*

1 Introduction

In the recently published book on the Traveling Salesman Problem, half of Chapter 6 [12] is devoted to domination analysis (DA) of heuristics for the Traveling Salesman Problem. Another chapter [10] is a detailed overview of the whole area of DA. Both chapters are of considerable length. The purpose of this talk is to give a short introduction to results and applications of DA. Some open problems are also raised.

Research on combinatorial optimization (CO) heuristics has produced a large variety of heuristics especially for well-known CO problems. Thus, we should develop ways to select the best ones among them. In most of the literature, heuristics are compared in computational experiments. While experimental analysis is of definite importance, it cannot cover all possible families of instances of the CO problem at hand and, in particular, it usually does not cover the hardest instances.

The theoretical performance of CO heuristics is normally studied by the means of approximation analysis (AA) [3]. Usually, upper or lower bounds for the worst case performance ratio are obtained.

DA was introduced by Glover and Punnen in [6]. DA provides an alternative and a complement to AA. In DA, we are usually interested in the minimum possible number (proportion, respectively) of feasible solutions that are worse or equal in quality to the heuristic one. This is called the domination number (domination ratio, respectively) of the heuristic solution. In many cases, domination analysis is very useful. For example, the greedy algorithm has domination number 1 for many CO problems [8]. In other words, the greedy algorithm, in the worst case, produces the unique worst possible solution. This is reflected in one of the latest computational experiments with the greedy algorithm, see, e.g., [13], where it was concluded that the greedy algorithm 'might be said to be self-destruct'.

Local Search (LS) is one of the most successful approaches in the design of CO heuristics. Recently, several researchers carried out computational and theoretical investigation of LS with Very Large Scale Neighborhoods (see, e.g., [1, 4, 12]). The hypothesis 'justifying' this approach says that the larger the neighborhood the better solution is expected to be found [1]. However, some computational experiments do not support this hypothesis. This

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means that some other parameters are responsible for the relative power of a neighborhood. Computational experiments with TSP indicate that one such parameter may well be the domination ratio of the corresponding LS.

2 Basic Terminology

Let P be a CO problem, and let H be a heuristic for P. The **domination number** $\mathrm{domn}(H, n)$ is the maximum integer $m(n)$ such that the solution $x(I)$ obtained by H for any instance I of P of size n is not worse than at least $m(n)$ feasible solutions of I (including $x(I)$). The **domination ratio** $\mathrm{domr}(H, n)$ is the maximal $q(n)$ such that the solution $x(I)$ obtained by H for any instance I of P of size n is not worse than at least the fraction $q(n)$ of the feasible solutions of I. Clearly, domination ratio belongs to the interval $(0, 1]$ and exact algorithms are of domination ratio 1.

We illustrate the above definitions using the Asymmetric TSP (ATSP): given an n-vertex complete digraph D with weights on its edges, find a Hamilton cycle of D of minimum total weight. The fact that the nearest neighbor algorithm (NN) for ATSP is of domination number 1 (first proved in [11]) means that for every $n \geq 2$, there is an instance of ATSP on n vertices, for which NN finds the *unique* worst possible Hamilton cycle. Since the number of distinct Hamilton cycles in an n-vertex complete digraph is $(n-1)!$, we see that NN is of domination ratio $1/(n-1)!$. There are many ATSP algorithms of domination number at least $(n-2)!$ [12], i.e., in the worst case they guarantee that their Hamilton cycle is at least as good as $(n-2)! - 1$ others.

3 Results

The *Minimum Partition Problem* (MPP) is the following CO problem: given n nonnegative numbers $a_1, a_2, \ldots, a_n$, find a bipartition of the set $\{1, 2, \ldots, n\}$ into sets X and Y such that $d(X, Y) = |\sum_{i\in X} a_i - \sum_{i\in Y} a_i|$ is minimum. For simplicity, we assume that solutions X, Y and X', Y' are different as long as $X \neq X'$, i.e. even if $X = Y'$ (no symmetry is taken into consideration). Thus, the MPP has 2^n solutions.

Consider the following greedy-type algorithm $\mathcal{G}$ for the MPP: $\mathcal{G}$ sorts the numbers such that $a_{\pi(1)} \geq a_{\pi(2)} \geq \cdots \geq a_{\pi(n)}$, initiates $X = \{\pi(1)\}$, $Y = \{\pi(2)\}$, and, for each $j \geq 3$, puts $\pi(j)$ into X if $\sum_{i\in X} a_i \leq \sum_{i\in Y} a_i$, and into Y, otherwise. It is easy to see that any solution X, Y produced by $\mathcal{G}$ satisfies $d(X, Y) \leq a_{\pi(1)}$.

Consider any solution X', Y' of the MPP for the input $\{a_1, a_2, \ldots, a_n\} - \{a_{\pi(1)}\}$. If we add $a_{\pi(1)}$ to Y' if $\sum_{i\in X'} a_i \leq \sum_{i\in Y'} a_i$ and to X', otherwise, then we obtain a solution X'', Y'' for the original problem with $d(X'', Y'') \geq d(X, Y)$. Thus, the domination number of $\mathcal{G}$ is at least 2^{n-1} and we have the following:

Proposition 1 *[7] The domination ratio of $\mathcal{G}$ is at least 0.5.*

In fact, a slight modification of $\mathcal{G}$ is of domination ratio very close to 1. Consider the following greedy-type algorithm $\mathcal{H}$. Initially sort the numbers such that $a_{\pi(1)} \geq a_{\pi(2)} \geq$

$\cdots \geq a_{\pi(n)}$. Choose an integral constant $p > 0$ and fix $k = \lfloor p \log_2 n \rfloor$. Solve MPP to optimality for $a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(k)}$. (This can be trivially done in time $O(n^p)$.) Now for each $j > k$, use the algorithm $\mathcal{G}$.

The proof in [2] of the following result is based on a lemma proved by Erdős [5] viz the one-dimensional Littlewood-Offord problem.

Theorem 1 *The domination ratio of* $\mathcal{H}$ *is at least* $1 - \binom{k}{\lfloor k/2 \rfloor}/2^k = 1 - \Theta(\frac{1}{\sqrt{k}})$.

We mentioned above that there are many ATSP heuristic of domination ratio $\Omega(1/n)$. However, it is not known yet whether there exists a polynomial time ATSP heuristic of domination ratio $\Omega(1)$. Recently, Alon, Gutin and Krivelevich [2] proved that some polynomial time algorithms for Max Cut, Max k-SAT and several of its generalizations are of domination ratio $\Omega(1)$.

Notice that we were not able to show that some CO problems admit polynomial time algorithms of domination ratio $\Omega(1/p(n))$ for any polynomial $p(n)$. One such problem is the Quadratic Assignment Problem (QAP). In [9], a polynomial time QAP algorithm is considered; it is proved that the algorithm is of domination ratio $\Omega(1/n^2)$ when n is a prime power. The result seems to be non-extendable to every n using the approach of [9]. Not every CO problem admits a polynomial time algorithm of domination ratio $\Omega(1/p(n))$ for any polynomial $p(n)$. Using a very powerful non-approximability result on Max Clique (see [3]) the following theorem was proved in [7].

Theorem 2 *The Max Clique and Min Vertex Cover do not admit polynomial time algorithms of domination ratio* $\Omega(1/p(n))$ *for any polynomial* $p(n)$.

We mentioned above that the nearest neighbor algorithm for ATSP is of domination number 1. The same result holds for the greedy algorithm for ATSP [11]. The last result is extended to a wide class of CO problems in [8]. It turns out that the domination number of the greedy algorithm for (even) the Assignment Problem is 1.

Another striking result is obtained in [15], where the authors prove that the well-known Double Spanning Tree (DST) heuristic for Symmetric TSP with triangle inequality is of domination number 1. This is despite the fact that DST is of performance ratio 2 for Symmetric TSP with triangle inequality.

DA has been used for several other problems not mentioned above; for example, in [14], two algorithms for the Frequency Assignment Problem are compared using DA.

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Sandwich Problems for Posets

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1 Introduction

A graph $G_t = (V, E_t)$ is a *spanning subgraph* of $G = (V, E)$ if $E_t \subseteq E$. A graph $G_s = (V, E_s)$ is a *sandwich graph* for the pair (G, G_t) if $E_t \subseteq E_s \subseteq E$. The GRAPH SANDWICH PROBLEM for a property Π is defined as follows: *given two graphs $G_t = (V, E_t)$ and $G = (V, E)$ such that $E_t \subseteq E$, does there exist a sandwich graph $G_s = (V, E_s)$ for the pair (G, G_t) satisfying property Π?* Notice that a sandwich problem for property Π can be seen a generalization of the recognition problem for property Π. In their seminal paper, Golumbic, Shamir and Kaplan [2] studied a wide range of sandwich problems for subfamilies of perfect graphs. Among those families, they considered comparability graphs (graphs that can be transitively oriented). Assuming G_t and G to be transitive acyclic digraphs (*posets* in the following), sandwich problems on subfamilies of comparability or co-comparability graphs become *poset sandwich problems.*

Problem 1: POSET SANDWICH PROBLEM FOR POSET PROPERTY $\overrightarrow{\Pi}$

Instance: Two posets $P_t = (V, E_t)$ and $P = (V, E)$ such that $E_t \subseteq E$

Question: Does there exist a sandwich poset $P_s = (V, E_s)$ for the pair (P_t, P) satisfying $\overrightarrow{\Pi}$?

One can also consider the problem where the digraphs of the instance have an arbitrary orientation, namely the *directed sandwich problem.*

Problem 2: DIRECTED SANDWICH PROBLEM FOR POSET PROPERTY $\overrightarrow{\Pi}$

Instance: Two digraphs D_t and D such that $D_t \subseteq D$

Question: Does there exist a sandwich digraph D_s for the pair (D_t, D) satisfying $\overrightarrow{\Pi}$?

Notice that for a property $\overrightarrow{\Pi}$ on posets, D_t can be chosen transitively oriented since its transitive closure is contained in any transitively oriented sandwich graph. These two

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sandwich problems satisfy a very simple relation of complexity: Poset sandwich problem for $\overrightarrow{\Pi} \leq_K$ directed sandwich problem for Π (where $\leq_K$ is the Karp reduction). Thus, once a directed sandwich problem is polynomial, so is the poset sandwich problem; also, if the poset sandwich problem is NP-complete, so is the directed sandwich problem.

Since most of the graph classes studied in [2] consisted of comparability graphs, it is reasonable to look for a polynomial transformation between undirected and directed (or poset) sandwich instances. However, even if a property makes sense for both posets and graphs, there may be no relation between the corresponding sandwich graphs as shown in Figure 1. In this example, the undirected property is: *a graph G on V satisfies* Π *if and only if $d_G(a) = 2$ and $d_G(b) = d_G(c) = 1$*; the poset property is similar except that the sandwich has to be a poset. (We write $d_G(a)$ for the degree of the vertex a in G.) Even if there exists an undirected sandwich graph G_s for the instance (G_t, G) satisfying Π, the same edges do not give a sandwich poset satisfying Π in the poset case (for the transitively oriented instance) because the transitivity arc bc is missing.

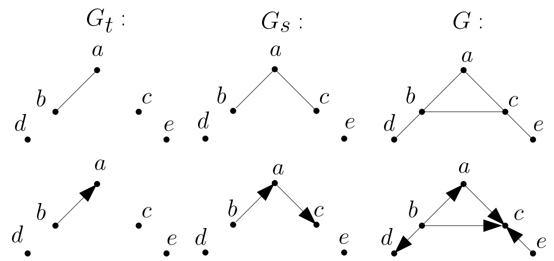


Figure 1: Sandwich graph G_s (undirected case on top and poset case below).

Directed and undirected sandwich problems are not equivalent and the results on undirected sandwich graphs do not give direct answers for poset or directed sandwich problems. In the following, we show that complexity results may be polynomial for both the directed and undirected sandwich problems (series-parallel poset sandwich). Also, an NP-complete undirected sandwich problem can become polynomial in the directed case (interval poset sandwich) or NP-complete in both cases (permutation poset). Thus there seems to be no simple relationship between the undirected and poset sandwiches. The interested reader can find complete proofs of the lemmas in [3].

2 Series-parallel posets

A series-parallel poset is obtained from the single-vertex poset by the application of two composition rules. The *parallel composition* of posets P_1 and P_2 is the poset $P_1 + P_2 = (V_1 \cup V_2, <_+)$ such that $u <_+ v$ if and only if $u, v \in V_1$ and $u <_1 v$ or $u, v \in V_2$ and $u <_2 v$. The *series composition* of posets P_1 and P_2 is the poset $P_1 * P_2 = (V_1 \cup V_2, <_*)$ such that $u <_* v$ if and only if $u, v \in V_1$ and $u <_1 v$ or $u, v \in V_2$ and $u <_2 v$ or $u \in V_1$ and $v \in V_2$. Therefore, series-parallel posets are organized in a tree structure.

The comparability graphs of the series-parallel posets are exactly the cographs [5]. The polynomial time algorithm of Golumbic *et al.* [2] for the cograph sandwich problem can be modified so that it applies to the series-parallel poset sandwich problem. By adding an argument about transitivity to their proof, it follows that the poset sandwich problem is also polynomial. However, this method does not provide an algorithm for the more general *series-parallel directed sandwich problem.* This problem consists of the existence of a series-parallel sandwich poset P_s for the pair (P_t, D) given a poset $P_t = (V, E_t)$ and a digraph $D = (V, E)$ such that $E_t \subseteq E$. We shall provide a polynomial-time algorithm for this problem.

Let S_P be the set of sources of a poset P, $N_P(x)$ be the neighborhood of vertex x in P, $P_{|A}$ be the poset induced by P on a vertex set A. The notation $Succ^P(x)$, denoting the set of out-vertices (successors) of x in P, is extended to sets: $Succ^P(A) = \bigcup_{a\in A} Succ^P(a)$. Without loss of generality, we can assume that P_t is connected.

Lemma 1 *If there is a series-parallel sandwich poset P_s for the instance (P_t, D), then there exists a set $S \subseteq S_{P_t}$ such that $V \setminus S \subseteq Succ^D(S)$.*

Indeed, a series-parallel sandwich poset P_s has a subset S of its set of sources composed in series with the rest of the poset. The algorithm for solving the series-parallel directed sandwich problem recursively reduces the sandwich instance to a set $\tilde{V} \subset V$ by applying the following procedure: Look for a set A among P_t sources satisfying $V \setminus A \subseteq Succ^D(A)$. If none exits, then by Lemma 1, there is no series-parallel poset sandwich. Else remove from V the set A and the set B of isolated vertices in $P_t|_{V\setminus A}$. Then solve the problem on the instance induced by $\tilde{V} = V \setminus (A \cup B)$. The correctness of this procedure is proved by Lemma 2.

Lemma 2 *The following statements are equivalent:*

(i) There is a series-parallel sandwich poset for the instance (P_t, D).

(ii) There is a series-parallel sandwich poset for the instance $(P_t, D)|_{\tilde{V}}$ and $A \neq \emptyset$.

Indeed, the restriction on $\tilde{V}$ of a series-parallel sandwich poset for V is still a series-parallel sandwich poset, and conversely a series-parallel sandwich poset for $\tilde{V}$ can be extended to V by series composition with A and parallel composition with B.

Theorem 1 *The series-parallel directed sandwich problem is polynomial.*

3 Interval posets

This section deals with interval posets, posets that can be represented by assigning a real interval $I_v = [a_v, b_v]$ to each element v in P, such that $v \leq w$ if and only if $b_v \leq a_w$. Given a poset $P_t = (V, E_t)$ and a digraph $D = (V, E)$ such that $E_t \subseteq E$, the existence of an interval sandwich poset P_s for the pair (P_t, D) is to be decided.

We prove the interval directed sandwich problem is polynomial, whereas the interval graph sandwich problem is NP-complete [2]. Therefore, this sandwich problem shows a complexity gap between the oriented and undirected versions. Thus fixing the transitive orientation of the interval graph drastically simplifies the problem (since many transitive orientations are possible). There exists a linear time recognition algorithm for an interval poset [4] based on the following characterization due to Fishburn [1].

Theorem 2 *[1] A poset P is an interval poset iff the set of successors $\big\{Succ^P(v) = \{u \in V, v \leq u\}\big\}_{v \in V}$ is linearly ordered by inclusion.*

Lemma 3, the basis of our algorithm, can be seen as a generalization of the recognition algorithm of [4].

Lemma 3 *If there is an interval sandwich poset P_s for the instance (P_t, D), then there exists a set $S \subseteq S_{P_t}$ such that $V \setminus S_{P_t} \subseteq Succ^D(S)$.*

Let $A' = \{x \in V / Succ^D(x) \supseteq V \setminus S_{P_t}\}$ and $B' = \{x \in V / N_{P_t|_{V \setminus A'}}(x) = \emptyset\}$. We also let $\tilde{V} = V \setminus (A' \cup B')$.

Lemma 4 *The following statements are equivalent:*

(i) There is an interval sandwich poset for the instance (P_t, D).

(ii) There is an interval sandwich poset for the instance $(P_t, D)|_{\tilde{V}}$ and $A' \neq \emptyset$.

Indeed, hereditariness of the interval poset property and Lemma 3 show $(i) \Rightarrow (ii)$. For the converse, an interval sandwich poset on $\tilde{V}$ can be extended to a sandwich on V by adding the vertices B' and arcs from A' towards $\tilde{V} \cup B'$.

We can now describe an algorithm for the interval directed sandwich problem that is very similar to the series-parallel directed sandwich procedure: Look for a set A' among P_t sources satisfying $Succ^D(A') \supseteq V \setminus S_{P_t}$; if none exits, then by Lemma 3 there is no interval poset sandwich. Else remove from V the sets A' and B' and the set of isolated vertices in $P_t|_{V \setminus A'}$. By Lemma 2 there is an interval poset sandwich for the instance if and only if there is one for this reduced instance.

Theorem 3 *The interval directed sandwich problem is polynomial.*

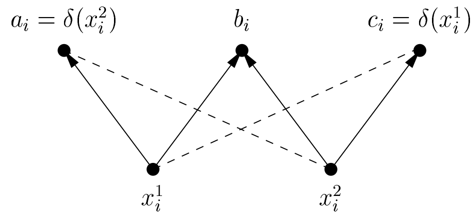
4 2-dimensional posets

Let L and P be, respectively, a total order and a poset on the same vertex set. If $x <_P y$ implies $x <_L y$, then L is a linear extension of P. The dimension of a poset P is the minimum number k of linear extensions such that $x <_P y$ iff $x <_{L_i} y$ for all i, $1 \leqslant i \leqslant k$. The comparability graphs of 2-dimensional posets are the permutation graphs. The permutation graph sandwich problem is known to be NP-complete [2]. For the interval case, we showed that the directed problem is polynomial while the undirected problem is NP-complete. In contrast, we show that the sandwich problem for 2-dimensional posets is also NP-complete. Here, fixing the orientation of the edges does not help.

Given $P_t = (V, E_t)$ and $P = (V, E)$ two posets on the same ground vertex set such that $E_t \subseteq E$, the 2-dimensional poset sandwich problem asks whether a sandwich poset $P_s = (V, E_s)$ of dimension 2 exists.

Theorem 4 *The* 2*-dimensional poset sandwich problem is NP-complete.*

We reduce to the same BETWEENNESS problem that was used in [2] for permutation sandwich graphs: given a ground set S and a set $\mathcal{T} = \{T_1, \ldots T_k\}$ of triples of S, does there exist a linear ordering λ such that for any triple $T_i = (a_i, b_i, c_i)$, either $a_i <_\lambda b_i <_\lambda c_i$ or $c_i <_\lambda b_i <_\lambda a_i$?



Golumbic *et al.* associated the above gadget (undirected) with the triple $T_i = (a_i, b_i, c_i)$. We orient their gadget because we are dealing with posets. To any instance of BETWEENNESS we associate a pair of posets $P_t = (V, E_t)$ and $P = (V, E)$ with $E_t \subseteq E$ as follows:

$$V = S \cup X \text{ where } X = \{x_i^1 | 1 \leqslant i \leqslant k\} \cup \{x_i^2 | 1 \leqslant i \leqslant k\}$$

$$E_t = \bigcup_{1 \leqslant i \leqslant k} \{x_i^1 a_i, x_i^1 b_i, x_i^2 b_i, x_i^2 c_i\} \text{ and } E = X \times S \setminus \{uv | v = \delta(u)\}$$

We use planar lattices to complete the reduction.

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Generalizations of interval graphs defined as intersection graphs

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Abstract

We study properties of some classes of intersection graphs lying between interval graphs and chordal graphs, giving both algorithmic and structural results.

1 Motivation and background

The interval graphs and chordal graphs are well-known classes of intersection graphs, arising respectively from connected subpaths of a path, and from connected subtrees of a tree. In this paper we introduce the graph classes k-catval parameterized by a natural number k where k-catval $\subset (k+1)$-catval, with interval graphs being precisely the 0-catval graphs, and the union of k-catval graphs over all k being precisely the chordal graphs.

In a recent paper [9] a class of graphs lying properly between interval and chordal graphs was introduced, the so-called catval graphs (the intersection graphs of subtrees of a caterpillar tree). The class of catval graphs corresponds to the class we call 1-catval in the present paper. The motivation for studying catval graphs came from memory requirement considerations in dynamic programming algorithms on graphs of bounded treewidth. However, the problem of recognizing catval graphs was not addressed since the characterization given in [9] hints only at a brute force algorithm of complexity $\mathcal{O}(n^6)$. We give a linear-time algorithm for this problem. In essence the algorithm follows from the well-known linear-time algorithms for recognition of interval graphs and the following characterization which we give in the next section: 'G is a catval graph iff the graph $G \setminus \{v : v \text{ simplicial in } G\}$ is interval'.

2 The main result

We consider only connected, simple graphs with at least one edge. A tree is an acyclic graph. A leaf is a vertex with one neighbor. A path is a tree with two leaves. A caterpillar

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is a tree consisting of a path (the 'body') with added leaves (the 'hairs'). A simplicial vertex is a vertex whose neighbors are pairwise adjacent.

Definition 1 *For a natural number k, a tree T is a k-cat if T has a path P with every vertex of T at distance $<= k$ from P.*

Thus caterpillars are 1-cat, and paths are 0-cat.

Definition 2 *A graph G is k-catval if G is the intersection graph of connected subgraphs of a k-cat.*

Thus, interval graphs are 0-catval, any k-catval graph is chordal, and for any chordal graph there exists a k such that it is k-catval. Interval graphs have a characterization in terms of chordal graphs, based on the following definition:

Definition 3 *Three non-adjacent vertices x, y, z of a graph G form an asteroidal triple (AT) if between any two of them there exists a path in G that avoids the neighborhood of the third.*

Theorem 1 *[6] G is an interval graph iff it is chordal and AT-free.*

We generalize the notion of an asteroidal triple:

Definition 4 *An asteroidal triple x, y, z of a graph G is a k-extended asteroidal triple (k-XAT) if there exists additional distinct vertices $x_1, x_2, ..., x_k, y_1, y_2, ..., y_k, z_1, z_2, ..., z_k$ such that $N[x_k] \subset N[x_{k-1}] \subset ... \subset N[x_1] \subset N[x]$ and $N[y_k] \subset N[y_{k-1}] \subset ... \subset N[y_1] \subset N[y]$ and $N[z_k] \subset N[z_{k-1}] \subset ... \subset N[z_1] \subset N[z]$. (where $\subset$ denotes strict inclusion and $N[x] = N(x) \cup \{x\}$ is the closed neighorhood)*

In the terminology introduced here Theorem 1 states that G is a 0-catval graph iff it is chordal and 0-XAT-free. The following result is also known:

Theorem 2 *[9] G is a 1-catval graph iff it is chordal and 1-XAT-free.*

We are currently investigating if this generalizes to 'G is a k-catval graph iff it is chordal and k-XAT-free.'

We now show a characterization of catval graphs in terms of interval graphs:

Theorem 3 *G is a catval graph iff $G' = G \setminus \{v : v \text{ simplicial in } G\}$ is interval.*

Proof: For the forward direction, let G have vertices $\{v_1, v_2, ..., v_n\}$ represented, respectively, by the connected subgraphs induced by the subsets of vertices $\{A_1, A_2, ..., A_n\}$ of a caterpillar A with body B. Thus each A_i induces a connected subgraph of A and $v_i v_j$ is an edge of G iff $A_i \cap A_j \neq \emptyset$. We show that the connected subgraphs induced by subsets of vertices $\{B_i : B_i = A_i \cap B \wedge B_i \neq \emptyset\}$ of the path B is a representation of a subgraph G'' of

G that contains G' as induced subgraph. Firstly, it is clear that G'' is a subgraph of G and that $A_i \cap B = \emptyset$ implies that v_i is simplicial in G so that G' is a subgraph of G''. Moreover, G' is an induced subgraph of G'' since if two adjacent vertices v_i and v_j are non-simplicial in G then since a vertex of $A \setminus B$ has only one neighbor and this neighbor belongs to B we must have $A_i \cap A_j \cap B \neq \emptyset$. Since G' is an induced subgraph of an interval graph we can conclude that G' is itself an interval graph.

For the reverse direction, let G' have vertices $V(G') = \{v_1, v_2, ..., v_m\}$ represented, respectively, by the connected subgraphs induced by the subsets of vertices $\{B_1, B_2, ..., B_m\}$ of a path B, and let $S = \{v_n, v_{n-1}, ..., v_{n-m}\}$ be the set of vertices that are simplicial in G. Let vertex u have neighbors $N(u)$ in G. Let the subgraph of G induced by the simplicial vertices S have components $C_1, C_2, ..., C_t$. It is straight forward to see that since C_i each component C_i must contain pairwise adjacent vertices that all have the same closed neighborhood in G.

For $u \in S$ we must have $N(u) \cap V(G') = \{v_{u_1}, v_{u_2}, ..., v_{u_k}\}$ pairwise adjacent and since a set of subtrees of a tree satisfies the Helly property we must have that the subsets $\{B_{u_1}, B_{u_2}, ..., B_{u_k}\}$ have a nonempty intersection, say for example that vertex $u_{base} \in B$ is in the intersection. We can now describe how to find a representation of G by connected subgraphs of a caterpillar A starting from the given representation of G' by subsets $\{B_1, B_2, ..., B_m\}$ of the path B and augmenting it stepwise. Firstly, initialize $A = B$ and then iterate the following procedure for $i = 1$ to t: for u an arbitrary vertex in component C_i update A to include the new leaf l_i, make this leaf adjacent to u_{base}, add $|C_i|$ subsets $\{l_i\}$ of A as the subgraphs representing each vertex in the component C_i, and add l_i to the subsets representing each $v \in N(u) \setminus S$. The resulting collection of connected subgraphs of the caterpillar A is a representation of G and shows that G is catval. □

To decide if an input graph G is a catval graph we can therefore run the following algorithm:

1. Check that G chordal, else output NO
2. Find the set S of all simplicial vertices of G
3. Output YES if $G \setminus S$ is interval, else output NO

Step 1 is done in linear-time for example by the Maximum Cardinality Search (MCS) algorithm [8]. Step 2 is done in linear-time by a modification of the MCS algorithm that finds all maximal cliques [1] while setting S to be those vertices belonging to only one maximal clique. Step 3 is done in linear-time by one of the standard linear-time interval graph recognition algorithms [2, 5, 4].

Corollary 1 *The recognition problem for catval graphs can be solved in linear time.*

We are currently investigating if these results can be generalized to k-catval graphs by 'G is a k-catval graph iff $G(k)$ is interval', where the definition of $G(k)$ is given inductively by $G(0) = G$ and $G(k+1) = G(k) \setminus \{v : v \text{ simplicial in } G(k)\}$. We also expect some properties with respect to the last vertex visited by a LexBFS [7]. Many such results are already known for subfamilies of chordal graphs [5, 3]

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The Bounded Fragmentation Property and Its Applications In Network Reliability

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Abstract

In this paper, we introduce a new property for graphs called *bounded fragmentation*, by which we mean after removing any set of at most k vertices the number of connected components is bounded only by a function of k. We demonstrate how bounded fragmentation can be used to measure the reliability of a network and introduce several classes of bounded fragmentation graphs. Finally, we pose some open problems related to this concept.

keywords: Bounded Fragmentation, Connectivity, Network Reliability.

1 Introduction

This paper is devoted to a new concept called bounded fragmentation. In fact, this property can be considered as a generalization of connectivity and can be applied to measure the reliability and robustness of a network. In addition, this concept has been used implicitly in other areas such as solving the subgraph isomorphism problem for special kinds of graphs [Haj01, Epp99].

This paper is organized as follows. We start with the terminology and the formal definition of bounded fragmentation in Section 2. In Section 3, we explain how this property can be applied in network reliability. We present some classes and properties which guarantee a graph G to be a bounded fragmentation graph in Section 4. In Section 5, we consider the number of edges of a bounded fragmentation graph. Finally in Section 6, we conclude with a list of open problems and potential extensions for future work.

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2 Basic definitions

We assume the reader is familiar with general concepts of graph theory such as trees and planar graphs. The reader is referred to standard references for appropriate background [BM76].

Our graph terminology is as follows. All graphs are finite, simple and undirected, unless indicated otherwise. A graph G is represented by $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We denote an edge e in a graph G between u and v by $\{u, v\}$. The maximum degree of G is denoted by $\Delta(G)$ and the minimum degree of G is denoted by $\delta(G)$. An *n-clique (K_n)* is a graph G with n vertices in which every pair of vertices is connected by an edge. A graph G is represented by $K_{n,m}$ if its vertices can be partitioned into sets V_1 and V_2 such that $|V_1| = n$, $|V_2| = m$ and edge $\{u, v\} \in E(G)$ if and only if $u \in V_1$ and $v \in V_2$ or vice versa.

A graph $G' = (V', E')$ is a *subgraph of* G if $V' \subseteq V$ and $E' \subseteq E$. A graph $G' = (V', E')$ is an *induced subgraph of* G, denoted by $G[V']$, if $V' \subseteq V$ and E' contains all edges of E which have both end vertices in V'.

The set of components of a graph G is represented by $\mathcal{C}(G)$, where each element of $\mathcal{C}(G)$ is a connected graph. The graph resulting from removal of a set S of vertices and all adjacent edges from G is denoted by $G[V-S]$. A set S is called a *separator* if $|\mathcal{C}(G[V-S])| > 1$. For $k > 0$, graph G is called *k-connected* if every separator has size at least k.

Definition 1 *A graph G is a $(k, g(k))$-*bounded fragmentation *graph if $|\mathcal{C}(G[V-S])| \leq |g(k)|$ for every $S \subseteq V(G)$ of size at most k, where g is a function of k. A graph G is a* totally $g(k)$-bounded fragmentation *graph if it is a $(k, g(k))$-bounded fragmentation graph for all $0 \leq k \leq n$.*

Here, we note that by our definition the number of components of $G[V-S]$ is constant when S has at most k vertices for some constant k. We mainly focus on this property in the rest of this paper.

3 Some applications of bounded fragmentation graphs

Connectivity can be considered as a measure of the reliability of a network. We suppose a network $\mathcal{N}$ is represented by an undirected graph G, in which two computers, namely nodes of the network, can communicate if and only if there is a path in G from one to the other. If G is k-connected, after removing at most $k-1$ vertices of G, the rest of G (which has $n-k+1$ vertices) is still connected. This means that if at most $k-1$ nodes of the network $\mathcal{N}$ fail, the rest of the nodes of the network can communicate with each other.

Bounded fragmentation can play a similar role in the reliability of a network. If G is a $(k, g(k))$-bounded fragmentation graph, after removing at most k vertices we have at least one component which has $\Omega(n)$ vertices. The reason is that after removing at most k vertices the rest of the nodes fall into at most a constant number of connected components ($g(k)$) and thus one component has at least $\Omega(n)$ vertices. Thus, after the failure of at most $k-1$ nodes of $\mathcal{N}$, $\Omega(n)$ nodes in the rest of $\mathcal{N}$ (and not necessarily $n-k$) still can communicate with each other (here we assume k is a constant). Using these facts, bounded fragmentation can be considered as a generalization of connectivity.

Bounded fragmentation also can have another application in the reliability of a network. Suppose that we need to repair the network $\mathcal{N}$ temporarily by adding several links between the current nodes of the network (not by adding any new node because of its high cost) when the number of failing nodes in the network is at most constant k . If G is a $(k, g(k))$-bounded fragmentation graph, then we can simply repair the network by adding at most $g(k)-1$ numbers of links, which is constant. Here, after removing the failing nodes, we find the connected components of G in $O(|V(G)|)$ time. Then we can connect these at most $g(k)$ connected components in the form of a tree, by adding at most $g(k)-1$ edges among them. These two simultaneous properties of bounded fragmentation graphs cause their corresponding networks to be more reliable and robust.

4 Bounded fragmentation graphs

In this section, we focus on classes of bounded fragmentation graphs.

Lemma 1 *Connected graphs with constant maximum degree c are totally ck-bounded fragmentation graphs.*

Theorem 1 *If graph G has a maximum independent set of constant size c, then it is a totally c-bounded fragmentation graph.*

In fact, we can generalize the approach used in Theorem 1 to other maximization problems.

Lemma 2 *Let G be a graph with minimum degree $\delta(G) \geq k+h-1$ for two positive integers k and h. Removing any set S of size at most k can not produce any component with size less than h.*

Using the above lemma, we can obtain the following theorem.

Theorem 2 *Let P be a maximization problem which has a non-zero solution on every connected graph of size at least h, where h is a non-negative constant. We also assume P is additive on components. For any non-negative integer k, if P on a graph G has a maximum solution of constant size c and $\delta(G) \geq k+h-1$ then G is a (k, c)-bounded fragmentation graph.*

For example, the maximum matching problem is a problem which has a non-zero solution on every connected graph of at least two vertices.

Corollary 1 *For any non-negative integer k, if connected graph G has a maximum matching of constant size c and minimum degree at least $k+1$, i.e. $\delta(G) \geq k+1$, then it is a (k, c)-bounded fragmentation graph.*

The reader is referred to Garey and Johnson [GJ79] and Yannakakis [Yan78] to see more problems of this kind.

Example 1 *A complete bipartite graph $K_{n-k-1,k+1}$, where $n \geq 2k+2$, has minimum degree $k+1$ and a maximum matching of size $k+1$. Hence it is a $(k, k+1)$-bounded fragmentation graph.*

The result of Theorem 2 can be generalized to other problems which are not necessarily maximization problems.

Definition 2 Covering a graph by at most m vertex-disjoint paths *means the vertices of a graph can be partitioned into m subsets such that for each set S, there exists a path in a graph that contains exactly the vertices in S.*

Theorem 3 *Graphs whose vertices can be covered by at most c vertex-disjoint paths are totally $(k + c)$-bounded fragmentation graphs.*

Example 2 *Consider a Hamiltonian graph F_n which is constructed from a path of length n by connecting one of its vertices to all its non-neighbors. Since vertices of every Hamiltonian graph can be covered by one path, F_n is a totally $(k + 1)$-bounded fragmentation graph.*

We can also relate bounded fragmentation to other properties of graphs.

Theorem 4 *A planar 3-connected graph is a totally $2k$-bounded fragmentation graph.*

Proof: Suppose we removed a set S of k vertices. Without loss of generality, we assume that no edge can be added to H connecting two vertices in S. Then each component of $H - S$ must occupy a distinct face in the planar embedding of S induced by a unique embedding of H. Since the number of faces of S is at most $2k$ by Euler's Formula [BM76], we obtain the desired result. □

Clearly, a complete graph K_n is a totally 1-bounded fragmentation graph. Intuitively, graphs with large minimum degree are bounded fragmentation graphs. In Theorem 5, we derive an exact bound on the minimum degree of a graph that guarantees the graph to be a bounded fragmentation graph.

The proof of the following theorem is a bit involved and we omit it in this extended abstract.

Theorem 5 *For each constant d, graphs with $\delta(G) \geq \frac{n+d(k-1)}{d+1}$ are (k, d)-bounded fragmentation graphs where $0 \leq k \leq n - d - 1$.*

5 Numbers of edges of bounded fragmentation graphs

As discussed before, bounded fragmentation is a measure in reliability of a network. However, in network design, it is beneficial to have a linear number of communication lines. Thus, an interesting question is whether it is possible to have a linear number of edges and still a graph of bounded fragmentation. The answer to this question is affirmative. Clearly, graphs with constant maximum degree and planar graphs have linear numbers of edges. As shown in Examples 1 and 2, graphs with maximum matchings of constant size or graphs coverable by a constant number of vertex-disjoint paths can also have a linear number of edges.

However, the condition stated in Theorem 5 is valid only for graphs with quadratic numbers of edges. Graphs with constant maximum independent sets have quadratic numbers of edges. The proof follows from the fact that if a graph G has a constant maximum independent set c, its complement $\bar{G}$ has a constant maximum clique c. By Turán's theorem [Tur41, Wes96], $\bar{G}$ has at most $(1-1/(c-1))n^2/2$ edges. Thus G has a quadratic number of edges.

6 Conclusions and future work

In this paper, we introduced applications of bounded fragmentation graphs for networking and mentioned several instances of bounded fragmentation graphs. Here, we present some open problems that can be considered as possible extensions of this paper:

A naive algorithm for testing whether a graph G is (k,c)-bounded fragmentation, for constants k and c, is to check all subsets of vertices of size at most k and count the number of connected components. The running time of this algorithm is $O(n^{k+1})$. It might be possible to give an algorithm whose running time is $O(n^d)$, where d is a constant independent of k. A randomized approach might be another way to solve this problem.

In this paper, we introduced some properties which cause a graph to be bounded fragmentation. Finding other properties of this kind, especially those which impose a linear number of edges (if they exist), and finding an exact characterization of bounded fragmentation graphs are interesting questions. The relation between these properties and treewidth is also interesting, in particular when in solving subgraph isomorphism and minor containment, we search for graphs which are bounded fragmentation and have bounded treewidth (see [Haj01]). A path is a bounded fragmentation graph which has bounded treewidth. Graphs coverable with a constant number of vertex-disjoint paths and graphs with maximum constant degree are the only known classes of bounded fragmentation graphs which have bounded treewidth. Finding other classes with these properties is another possible extension of this paper.

Finally, all graphs introduced in this paper are $(k,O(k))$-bounded fragmentation. It would be instructive to determine whether there is any $(k,g(k))$-bounded fragmentation graph where $g(k)$ is not $O(k)$.

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Subsets with small sumset in $\mathbb{Z}/p\mathbb{Z}$

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Abstract

Let A, B be subsets of $\mathbb{Z}/p\mathbb{Z}$ such that

$$|A + B| \leq |A| + |B| + m.$$

It has been conjectured that, if $|A| \geq m + 3$, $|B| \geq m + 4$ and $|A + B| < p - (m + 3)(m + 4)/2$ then A and B are dense subsets of arithmetic progressions with the same difference. The case $m = -1$ is the theorem of Vosper and the case $m = 0$ has been recently settled by Hamidoune and Rodseth. We prove the case $m = 1$ of the conjecture. The method we use can be extended to other small values of m.

1 Introduction

Let A and B be subsets of residue classes modulo a prime p such that $|A|, |B| \geq 2$. The Cauchy-Davenport Theorem states that

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

Vosper's Theorem [9] states that $|A + B| \geq \min(p - 1, |A| + |B|)$ unless both A and B are arithmetic progressions with a common difference. A recent result by Hamidoune and Rodseth [6] states that

$$|A + B| \leq |A| + |B| \leq p - 5,$$

implies that both A and B are arithmetic progressions with the same difference with at most a single hole. When $A = B$, Freimann [2] proved that $|A+A| \le 5|A|/2$ and $|A| \le p/35$ holds only if A is contained in a short arithmetic progression. A conjecture mentioned in [1] states that $|A+A| \le 3|A|-4$ holds only if A is a subset of a a short arithmetic progression.

Similar results hold in $\mathbb{Z}$, where the following inverse theorem was proved by Freimann.

Theorem 1 (Freimann,[3]) *Let $A \subset [0, a]$ and $B \subset [0, b]$ be subsets of $\mathbb{Z}$ with $0 \in A \cap B$, $a \in A$, $b \in B$ and $b \le a$. Then*

$$|A + B| = |A| + |B| + m, \quad |A| \ge m + 3, |B| \ge m + 4,$$

imply that both A and B are subsets of arithmetic progessions with the same difference and length at most $|A| + m + 1$.

The above results and the rectification principle discussed by Bilu, Lev and Ruzsa [1] suggest the following Conjecture. For a subset $X \subset \mathbb{Z}/p\mathbb{Z}$ let $\ell_r(X)$ denote the cardinality of the smaller arithmetic progression with difference r containing X and $\ell(X) = \min\{\ell_r(X),\ r \in (\mathbb{Z}/p\mathbb{Z})^*\}$.

Conjecture 1 *Let A and B be subsets of $\mathbb{Z}/p\mathbb{Z}$ such that*

$$|A + B| \le |A| + |B| + m \le p - (m + 3)(m + 4)/2.$$

If $|A| \ge m + 3$ and $|B| \ge m + 4$ then there is $r \in (\mathbb{Z}/p\mathbb{Z})^*$ such that

$$\ell_r(A) \le |A| + m + 1 \quad \textit{and } \ell_r(B) \le |B| + m + 1.$$

We use in the present paper an isoperimetric approach to this conjecture allowing to settle the case $m = 1$. The method we use allows certainly to settle the conjecture for some larger values of m at the cost of a more detailed analysis. Below we present succintly the isoperimetric approach.

Let G be an abelian group and let $B \subset G$, be a generating subset such that $0 \in B$. The k-th *isoperimetric number* of B is

$$\kappa_k(B) = \min\{|X + B| - |X| \;\Big|\; |X| \ge k \text{ and } |X + B| \le |G| - k\},$$

where $\min \emptyset = |G|$, by convention. A subset X realising the above minimum is called a *k-fragment* of B. A k-fragment with minimal cardinality is called a *k-atom*.

It is proved in [4] that for any abelian group G and any subset $B \subset G$, a 1-atom of B containing 0 is a subgroup. This result implies easily Mann's generalization of the Cauchy-Davenport Theorem. The structure of 2-atoms is more difficult to describe. For $\kappa_2(B) \le |B|$, it is proved in [5] that a 2-atom of B containing 0 with size ≥ 3 is a subgroup. A critical pair theory is deduced from this description. The structure of 2-atoms in $\mathbb{Z}/p\mathbb{Z}$ is obtained by the authors of [8]. As a consequence it is shown that every set $B \subset \mathbb{Z}/p\mathbb{Z}$ is the union of $h = \kappa_2(B) - |B| + 2$ arithmetic progressions of the same difference provided that $B \leq p-(h+2)^2/2$. In order obtain the conjecture mentioned above, one needs to bound the gaps between consecutive arithmetic progressions. We believe that a description of the k-atoms (suspected to have size k when $|B|$ is not too large) could allow this. We prove here Conjecture 1 for $m=1$.

Theorem 2 *Let A, B be subsets of $\mathbb{Z}/p\mathbb{Z}$ with $|A| \geq 4$ and $|B| \geq 5$. If*

$$|A+B| \leq |A|+|B|+1 \leq p-5,$$

then there is $r \in (\mathbb{Z}/p\mathbb{Z})^*$ such that

$$\ell_r(A) \leq |A|+2 \text{ and } \ell_r(B) \leq |B|+2.$$

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The k-Piece Packing Problem

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Abstract

A *k-piece* is a connected graph with maximum degree equal to k. We consider the problem of covering the maximum number of nodes of a graph by node disjoint k-pieces. When $k = 1$ this is the maximum matching problem and when $k = 2$ this is the problem, recently studied by Kaneko [7], of covering the maximum number of nodes by disjoint paths of length greater than 1. We have a polynomial time algorithm for the problem as well as a Tutte-type existence theorem and a Berge-type min-max formula.

1 Introduction

For a graph G and an integer k, a *k-piece packing* of G is a set of pairwise node disjoint k-piece subgraphs of G. We consider the problem of finding a *maximum k-piece packing*, i. e. a k-piece packing covering the maximum number of nodes. A k-piece packing is *perfect* if it covers every node of the graph. The main result [4] is a $O(en^2)$-time algorithm for solving the k-piece packing problem, from which we derive a characterization of those graphs that have a perfect k-piece packing and we prove a min-max result for the number of nodes of a maximum k-piece packing.

The 1-piece packing problem is just the maximum matching problem for graphs because the only 1-piece is the edge-graph. A classical theorem due to Tutte characterizes those graphs that have a perfect 1-piece packing and a result due to Berge [1] gives a min-max formula. A 2-piece is a circuit or a path of length at least 2. The 2-piece packing problem was recently considered by Kaneko [7], who presented a Tutte-type characterization of those graphs that have a spanning subgraph each connected component of which is a *long path*, i. e. a path of length at least 2. A shorter proof of Kaneko's theorem and a min-max formula was subsequently found by Kano, Katona and Király [8]. Our results generalize all

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of these results and, in particular, yield the first polynomial time algorithm for the 2-piece packing problem.

Several authors [2, 3, 5, 6, 7, 8, 9, 10] have studied the following generalization of matchings. Let $\mathcal{F}$ be a family of graphs; an $\mathcal{F}$-packing of G is a set of node disjoint subgraphs of G, each isomorphic to a member of $\mathcal{F}$. Polynomial time algorithms for finding maximum $\mathcal{F}$-packings have been found for various special families $\mathcal{F}$ of graphs (and digraphs). For instance, there is a polynomial time algorithm for any family $\mathcal{F}$ which consists of K_2 and a set of hypomatchable graphs [3]. Clearly, our problem is the $\mathcal{P}_k$-packing problem where $\mathcal{P}_k$ is the family consisting of all k-pieces. Equivalently, our problem is the $\mathcal{T}_k$-packing problem, where $\mathcal{T}_k$ is the family consisting of all trees with maximum degree k. In [6, 10] the authors study the connection between the existence of efficient algorithms and the structure of node sets coverable by $\mathcal{F}$-packings. In all those cases, the coverable sets formed a matroid if and only if the packing problem was polynomial time solvable. Our algorithm is the first example of a polynomial time solvable $\mathcal{F}$-packing problem for which the coverable sets do not form a matroid. (We are grateful to Martin Loebl for alerting us to this fact.) Indeed, if $k = 2$ and G is the tree consisting of three paths of length two, meeting at a central node, then the maximal coverable sets are the node sets of the paths of length four, yet these are not the bases of a matroid.

Another natural generalization of matching, closely related to k-piece packings, is the concept of a *k-matching*. A *k-matching* of a graph G is a subgraph of G all of whose nodes have degree at most k. The *k-matching problem* is to find a k-matching having a maximum number of edges, called *maximum k-matching*. Clearly, the connected components of a k-matching are t-pieces, with $0 \le t \le k$. We define the *k-piece packing associated with a k-matching M* to consist of those components of M that are k-pieces.

Note that the k-matching problem and the k-piece packing problem are the same when $k = 1$, but different when $k > 1$ – since the k-piece packing problem seeks to maximize the number of *nodes*, whereas the k-matching problem seeks to maximize the number of *edges*. Nevertheless, it turns out that there always exists a maximum k-matching for which the associated k-piece packing is also maximum. In fact, our algorithm is simplified by first finding a maximum k-matching and then modifying it to solve the maximum k-piece packing problem.

2 The main results

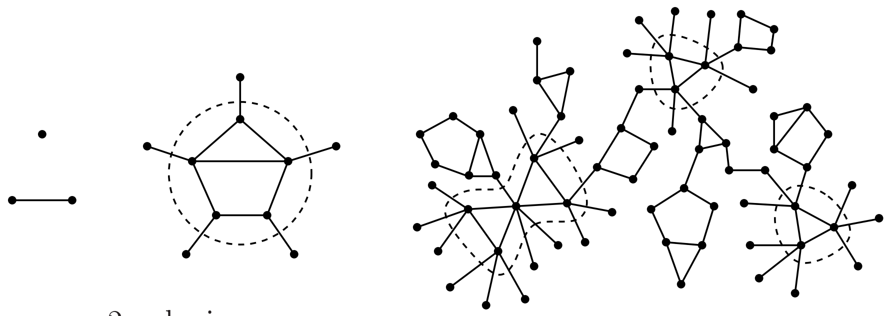
A graph G is *hypomatchable* if it does not have a perfect matching, but each node-removed subgraph $G-v$ has a perfect matching. In our situation, the role analogous to hypomatchable graphs will be played by graphs we call *galaxies*.

Assume k is a fixed positive integer. A graph is a *k-galaxy* if it satisfies the following properties:

1. The subgraph induced by the nodes of degree at least k is a union of hypomatchable graphs.

2. Each node of degree at least k is incident with exactly $k-1$ edges going to nodes of degree at most $k-1$, and all of these $k-1$ edges are cut-edges.

The figure contains examples of galaxies. The nodes of degree at least k are surrounded by dashed lines.



2-galaxies

a 4-galaxy

The 1-galaxies are exactly the hypomatchable graphs. The 2-galaxies were introduced by Kaneko [7] and called ‘suns’. Let k-gal(H) denote the number of connected components of a graph H that are k-galaxies. From the algorithm the following Tutte-type existence theorem can be proved.

Theorem 1 [4] *A graph G has a perfect k-piece packing if and only if*

$$k\text{-gal}(G-S) \le k|S|$$

for every set of nodes $S \subseteq V(G)$.

For the min-max version the following result can be proved, generalizing [1, 8] and Theorem 1. Let $d_k(G)$ denote the number of nodes which are not covered by a maximum k-piece packing of G.

Theorem 2 [4]

$$d_k(G) = \sum_{i=1}^{n} \left(k\text{-gal}_i(G-S_i) - k|S_i|\right),$$

taken over all towers of node sets $V(G) \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n$. Here k-gal$_i(G-S_i)$ is the number of the k-galaxy components H of $G-S_i$ with $d_k(H) \ge i$.

The case $k=1$ is Berge’s theorem on maximum matchings [1] and the case $k=2$ was proved by Kano, Katona and Király [8]. It is interesting to note that this min-max formula refers to a *sequence* of node sets, instead of just one set as in Berge’s theorem or in [2, 3, 5, 9, 10]. This is a consequence of the fact that a maximum k-piece packing of a k-galaxy may leave uncovered an arbitrary number of nodes.

3 Other generalizations

Our algorithm actually solves the node weighted k-piece packing problem; the adaptation requires little additional effort.

Another way of generalization is the following. Consider a graph G and non-negative integers $l(v)$ and $u(v)$ (with $l(v) \leq u(v)$) on every node v of G. An *(l, u)-piece* is a connected subgraph P of G, in which the degree of every node v is at most $u(v)$, and there exists a node w in P with degree in P at least $l(w)$. An *(l, u)-piece packing* of G is a set of pairwise node-disjoint (l, u)-pieces of G. The *(l, u)-piece packing problem* is to find an (l, u)-piece packing covering the maximum number of nodes.

When $l(v) = u(v) = k$ for every node v, we have the k-piece packing problem. On the other hand, one can reduce the (l, u)-piece packing problem to the k-piece packing problem in the following way. Let $k = 1 + \max\{u(x) : x \in V(G)\}$ and A_x, B_x be disjoint sets of new nodes for each $x \in V(G)$ such that $|A_x| = u(x) - l(x) + 1$ and $|B_x| = k - u(x) - 1$. Let the nodes in $A_x \cup B_x$ be adjacent to x and let the nodes in A_x be pairwise adjacent. It is easy to see that the new graph has a perfect k-piece packing if and only if G has a perfect (l, u)-piece packing. This reduction results in a polynomial time algorithm and in a Tutte-type theorem characterizing the graphs having a perfect (l, u)-piece packing. Alternately, one can give an algorithm which solves the (l, u)-piece packing problem directly (not using k-matchings), yielding a Berge-type min-max formula.

We mention two corollaries of the above general Tutte-type theorem. If l and u are constant functions we get the following theorem.

Theorem 3 [4] *Suppose $l < u$ are positive integers. A graph G has a spanning subgraph F such that the maximum degree of each component of F is between l and u if and only if for all node sets S the number of components of $G - S$ with maximum degree at most $l - 1$ does not exceed $u|S|$.*

If $l \equiv 1$ we get a theorem of Las Vergnas [9] (also proved in [5]). A *u-star factor* is a spanning forest in which each connected component is a star, i. e. a tree with one node c of degree i and i nodes of degree 1, with $1 \leq i \leq u(c)$.

Theorem 4 *Suppose u is an integer node function on a graph G, with $u(x) \geq 2$ for all nodes x of G. Then G has a u-star factor if and only if for each node set S the graph $G - S$ has at most $\sum\{u(s) \colon s \in S\}$ isolated nodes.*

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Discrepancy of Linear Hyperplanes in $\mathbb{F}_q^r$

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Abstract

In this paper we study the discrepancy of the hypergraph $\mathcal{H}_r = (\mathbb{F}_q^r, \mathcal{E}_{r,1})$ of linear hyperplanes in $\mathbb{F}_q^r$, where $\mathbb{F}_q^r$ is the r-dimensional vector space over $\mathbb{F}_q$ and $\mathcal{E}_{r,1}$ is the set of all its linear hyperplanes, i.e., the subspaces of codimension one. Let $n := q^r$.

By Spencer's six-standard-deviation theorem and its extension to c colors the discrepancy $\mathrm{disc}(\mathcal{H}_r, c)$ is $O(\sqrt{n})$. The main result of this paper is that this bound is asymptotically tight, i.e. $\mathrm{disc}(\mathcal{H}_r, c) = \Theta(\sqrt{n})$. The method for the lower bound proof might be interesting in itself. Often lower bounds for the discrepancy function are proved via the L^2-discrepancy. Here we use the positive discrepancy $\mathrm{disc}^+(.)$, which is the discrepancy function without the absolute value, to determine a lower bound for $\mathrm{disc}(\mathcal{H}_r, c)$. With Fourier analysis on $\mathbb{F}_q^r$, we show $\mathrm{disc}^+(\mathcal{H}_r) = \Theta(\sqrt{n})$. We present also an extension to subspaces with higher codimension than one.

1 Introduction

Basics about Discrepancy. Let V be a finite set and $\mathcal{E}$ a subset of 2^V. Then $\mathcal{H} := (V, \mathcal{E})$ is called a *hypergraph*. A *c-coloring* of $\mathcal{H}$ is a function $\chi : V \to M_c$, where M_c is any set of cardinality c. For convenience we take $M_c = \{1, 2, \ldots, c\} =: [c]$, but in applications a different choice of M_c can be helpful (see Babai et al., [BHK01]).

Let $A_i := \chi^{-1}(i)$ be the color-class of color $i \in [c]$ in V. The *c-color discrepancy* of $\mathcal{H}$ with respect to χ is defined by

$$\mathrm{disc}(\mathcal{H}, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left| |A_i \cap E| - \frac{|E|}{c} \right|,$$

and the *c-color discrepancy* of $\mathcal{H}$ is

$$\mathrm{disc}(\mathcal{H}, c) = \min_{\chi : V \to [c]} \mathrm{disc}(\mathcal{H}, \chi, c).$$

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For $c = 2$ the c-color discrepancy is exactly half of the common two-color discrepancy where the two colors are represented by 1 and -1. For further information on discrepancies, we refer to Beck and Sós [BS95] and Matoušek [Mat99]. For our purposes, a related discrepancy notion will be relevant. The *positive c-color discrepancy* of $\mathcal{H}$ with respect to χ is

$$\operatorname{disc}^+(\mathcal{H}, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left(|A_i \cap E| - \frac{|E|}{c} \right),$$

and the *positive c-color discrepancy* of $\mathcal{H}$ is

$$\operatorname{disc}^+(\mathcal{H}, c) = \min_{\chi : V \to [c]} \operatorname{disc}^+(\mathcal{H}, \chi, c).$$

Trivially we have $\operatorname{disc}^+(\mathcal{H}, c) \leq \operatorname{disc}(\mathcal{H}, c)$, where equality holds for $c = 2$.

Let $\mathbb{F}_q$ be the field of q elements, where $q = p^k$ is a power of a prime p, $V := \mathbb{F}_q^r$ the r-dimensional vector space over $\mathbb{F}_q$, and let $\mathcal{E}_{r,m}$ be the set of all subspaces of V of codimension m. Put $n := |V| = q^r$. For a set $S \subseteq \mathbb{F}_q^r$ define $S^\sharp := S \setminus \{0\}$. We investigate the discrepancy of the hypergraph $\mathcal{H}_{r,m} = (V, \mathcal{E}_{r,m})$. But let us first focus on $\mathcal{H}_r = (V, \mathcal{E}_{r,1})$. $\mathcal{H}_r$ is an $\frac{n}{q}$-uniform hypergraph on n vertices with $|\mathcal{E}_{r,1}| = \frac{n-1}{q-1}$ hyperedges.

Results. The main result of the paper is the lower bound estimate for the c-color discrepancy $\operatorname{disc}(\mathcal{H}_r, c) \geq \frac{\sqrt{z(1-z)}}{4q(q-1)\sqrt{c}} \sqrt{n} - \frac{q-1}{q}$, where $z := \frac{(q-1) \bmod c}{c}$. By the six-standard-deviation theorem of Spencer [Spe85], and its extension to multicolor discrepancies [DS02], we have $\operatorname{disc}(\mathcal{H}_r, c) = O(\sqrt{n})$. Thus $\operatorname{disc}(\mathcal{H}_r, c) = \Theta(\sqrt{n})$.

Methods. To study the c-color discrepancy we use Fourier analysis and the positive discrepancy function $\operatorname{disc}^+(\mathcal{H}_r, c)$. Since $\sum_{i \in [c]} \left(|A_i \cap E| - \frac{|E|}{c} \right) = 0$ for every $E \in \mathcal{E}_{r,1}$, it holds

$$\operatorname{disc}(\mathcal{H}_r, c) \geq (c-1) \operatorname{disc}^+(\mathcal{H}_r, c).$$

Hence the positive c-color discrepancy is nearly the same as the common c-color discrepancy, but easier to handle with respect to Fourier analysis. First of all, we observe that the positive discrepancy $\operatorname{disc}^+(\mathcal{H}_r, \chi, c)$ for a c-coloring χ of V can be written as

$$\operatorname{disc}^+(\mathcal{H}_r, \chi, c) = \max_{i \in [c]} \max_{E \in \mathcal{E}} \left(\frac{|A_i|}{q} + \frac{1}{q} \hat{A}_i(E^\perp) - \frac{n}{qc} \right), \tag{1}$$

where $\hat{A}_i$ is the Fourier transform of the characteristic function $\mathbf{1}_{A_i}$ in $\mathbb{F}_q^r$ and $\hat{A}_i(E^\perp) := \sum_{x \in E^\perp \setminus \{0\}} \hat{A}_i(x)$. We prove that for any set $A \subseteq \mathbb{F}_q^r$ there exists a $E \in \mathcal{E}$ with $\hat{A}(E^\perp) \geq -\frac{q-1}{q}$. Now a certain tradeoff between the size $|A_i|$ and $\hat{A}_i(E^\perp)$ in (1) can be shown. Finally we get

$$\frac{\sqrt{z(1-z)}}{4q(q-1)\sqrt{c}} \sqrt{n} - \frac{q-1}{q} \leq \operatorname{disc}^+(\mathcal{H}_r, c) \leq \operatorname{disc}(\mathcal{H}_r, c) \leq \alpha \sqrt{\frac{n}{c} \log c} \tag{2}$$

The c-color extension of Spencers six-standard-deviation theorem due to Doerr and Srivastav [DS02] is used for the upper bound in (2).

In the special case $q = 2$, the lower bound is easier to determine in the following way. We define a new hypergraph $\mathcal{H}' := (V', \mathcal{E}')$ with $V' := V \setminus \{0\}$ and $\mathcal{E}' := \{E \cap V' \mid E \in \mathcal{E}_{r,1}\}$. This hypergraph has constant pair-degree, i.e., there exists a $\lambda \in \mathbb{N}$ with

$$|\{E \in \mathcal{E} \mid i, j \in E\}| = \lambda$$

for all $i, j \in V$, $i \neq j$. Thus we can apply the following theorem.

Theorem 1 *Let $\mathcal{H} = (V, \mathcal{E})$ be a finite hypergraph, and let M be the incidence matrix of $\mathcal{H}$. If for some diagonal matrix D the matrix $M^T M - D$ is positive semidefinite, then*

$$\mathrm{disc}(\mathcal{H}, c) \geq \left(\frac{1}{c^2 m} \mathrm{Tr}\, D \right)^{\frac{1}{2}}.$$

Consequently, if $\mathcal{H}$ has constant pair-degree λ and d_v denotes the degree of $v \in V$ then

$$\mathrm{disc}(\mathcal{H}, c) \geq \left(\frac{1}{c^2 m} \sum_{v \in V} (d_v - \lambda) \right)^{\frac{1}{2}}.$$

Note that T stands for transpose and Tr *D is the trace of the matrix D.*

Theorem 1 is a generalisation of Theorem 2.8 and Corollary 2.9 from Beck and Sós [BS95] to c colors following the approach of Doerr and Srivastav in [DS02]. Applying Theorem 1 to the hypergraph $\mathcal{H}'$ we get a lower bound for the c-color discrepancy of our hypergraph $\mathcal{H}_r$.

Corollary 1 *If $q = 2$, then* $\mathrm{disc}(\mathcal{H}_r, c) \geq \frac{1}{2c}\sqrt{n} - \frac{c-1}{c}$.

Finally, in the general case of an arbitrary prime power q, our lower bound proof can be extended to subspaces of codimension $m \in [r-2]$:

$$\mathrm{disc}(\mathcal{H}_{r,m}, c) \geq \frac{1}{4(q-1)} \sqrt{\frac{z(1-z)}{q^{m+1} c}} \sqrt{n} - \frac{q-1}{q}.$$

We hope that the positive discrepancy technique can lead to a better understanding of discrepancy problems also beyond finite vector spaces.

2 Positive Discrepancy of Linear Hyperplanes in $\mathbb{F}_q^r$

Main Result. In this section we sketch the proof of the lower bound for the c-color discrepancy via Fourier analysis and the positive c-color discrepancy.

Theorem 2 *Let $z := \frac{(q-1) \bmod c}{c}$ and $q^{r-1} \geq q^{\frac{r}{2}} + 6q^2$. There exists a constant $\alpha > 0$ such that for every $c \geq 2$ we have*

$$\frac{\sqrt{z(1-z)}}{4q(q-1)\sqrt{c}} \sqrt{n} - \frac{q-1}{q} \leq \mathrm{disc}(\mathcal{H}_r, c) \leq \alpha \sqrt{\frac{n}{c} \log c}.$$

In particular, for a fixed number of colors c it holds $\mathrm{disc}(\mathcal{H}_r, c) = \Theta\left(\sqrt{n}\right)$.

For the proof we need some basic facts and some lemmas about the Fourier transform on $\mathbb{F}_q^r$.

Fourier Transforms over $\mathbb{F}_q^r$. For an introduction to Fourier transformations in finite groups we refer to Lidl and Niederreiter [LN94]. The Fourier transform $\hat{f}$ of a function $f : V \to \mathbb{C}$ is defined by

$$\hat{f} : V \to \mathbb{C}, z \mapsto \sum_{x \in V} f(x) e^{\frac{2\pi i}{p}\langle x,z \rangle}.$$

For convenience we set $\hat{A} := \hat{\mathbf{1}}_A$ for the Fourier transform of the characteristic function $\mathbf{1}_A$ with $A \subseteq V$. For every subspace W of V and every function $f : V \to \mathbb{C}$, we set

$$\hat{f}(W) := \sum_{x \in W^\sharp \setminus \{0\}} \hat{f}(x).$$

We know that for all $E \in \mathcal{E}_{r,1}$ there is a unique subspace $E^\perp$ of V of dimension one, which is orthogonal to E. For all $A \subseteq V$ and $E \in \mathcal{E}_{r,1}$ we set

$$d^+(A, E) \;\; := \;\; |A \cap E| - \frac{|A|}{q}.$$

With this notations we get the first link between Fourier transform and the positive c-color discrepancy of $\mathcal{H}_r$.

Lemma 1 *For all $A \subseteq V$ and all $E \in \mathcal{E}_{r,1}$ we have $d^+(A, E) = \frac{1}{q}\hat{A}(E^\perp)$.*

This lemma immediately implies (1) from the introduction. Using the characterisation of the function d^+ we get the following.

Lemma 2 *For every set $A \subseteq V$ there is a hyperedge $E \in \mathcal{E}_{r,1}$ with $d^+(A, E) \geq -\frac{q-1}{q}$.*

By this lemma and (1) we can maximize the term $\frac{|A_i|}{q} - \frac{n}{qc}$ over all color-classes A_i first, and choose a suitable hyperedge E with "large" $d^+(A_i, E)$ afterwards. The following main lemma for the proof of Theorem 2 helps us, finding a good color-class A_i in this sense.

Lemma 3 *Let $k \in \mathbb{R}^+$ with $\frac{1}{q^{r-1} - q^{\frac{r}{2}} - 1} \leq k \leq \frac{1}{3q}$. Either there exists a color $i \in [c]$ and a hyperedge $E \in \mathcal{E}$ with*

$$|A_i \cap E| - \frac{|E|}{c} > k\frac{\sqrt{n}}{c}$$

or with $x := (q-1) \bmod c$ it holds

$$\sum_{i \in [c]} \sum_{E \in \mathcal{E}} |\hat{A}_i(E^\perp)|^2 \geq |V|(|V| - 1)\frac{cx - k(q-1)^2 - x^2}{c(q-1)}.$$

For the proof of this lemma we need the following statement, which is somehow related to the Parseval Formula.

$$\sum_{E\in\mathcal{E}} |\hat{A}(E^{\perp})|^2 = |V|(q-1)\chi_A(0) - (q-1)|A|^2 + |V| \sum_{W\in\mathcal{E}_1} |A\cap W^{\sharp}|^2,$$

where $\mathcal{E}_1$ is the set of all one-dimensional subspaces of V and A is any subset of V. Using Lemma 3 we can assume the second statement, because otherwise we are done. This assertion leeds us to a color-class A_i with "large" $\frac{|A_i|}{q} - \frac{n}{qc}$.

3 Positive Discrepancy of Subspaces of $\mathbb{F}_q^r$

We can extend the result from linear hyperplanes to subspaces of arbitrary codimensions.

Corollary 2 *Let $m \in [r-2]$ with $q^{r-m} \geq q^{\frac{r-m+1}{2}} + q^2$. Then*

$$\operatorname{disc}(\mathcal{H}_{r,m}, c) \geq \frac{1}{4(q-1)}\sqrt{\frac{z(1-z)}{q^{m+1}c}}\sqrt{n} - \frac{q-1}{q}.$$

We have to look for a subspace of V of codimension m with the desired discrepancy. So we can focus on a subspace V_{m-1} of V with codimension $m-1$. Our main result gives us a linear hyperplane E of V_{m-1} with at most discrepancy $\frac{1}{4(q-1)}\sqrt{\frac{z(1-z)}{q^{m+1}c}}\sqrt{n} - \frac{q-1}{q}$. One can easily check that the codimension of E in V is m. Thus the lower bound is proved.

4 Conclusion

We have shown that the c-color discrepancy as well as the positive c-color discrepancy of linear hyperplanes in $\mathbb{F}_q^r$ is tight. The challenging algorithmic problem still is to construct an $O(\sqrt{n})$-discrepancy coloring for the linear hyperplanes in $\mathbb{F}_q^r$. Furthermore, a more detailed study of positive discrepancy is desirable.

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Tree-Width and Matroids *

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Abstract

We show that tree-width (and a tree decomposition) of a graph can be equivalently defined without refering to graph vertices, and hence the notion of tree-width can be naturally extended to matroids. This work follows an original idea of J.F. Geelen [unpublished].

Keywords: matroid, point configuration, tree-width.
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1 Introduction

We assume that the reader is familiar with basic terms of graph theory. In the past decade, the notion of tree-width (and tree-decompositions) of graphs attracted plenty of attention, both from graph-theoretical and computational points of view. This attention followed the pioneer work of Robertson and Seymour on the Graph Minor Project, and results of various researchers concerning tree-width in (parametrized) complexity.

Definition 1 Let G be a graph. A *tree-decomposition* of G is a pair (T, β), where T is a tree and $\beta : V(T) \to 2^{V(G)}$ is a mapping satisfying the following:

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- For each edge $e = uv \in E(G)$, there is $x \in V(T)$ such that $\{u, v\} \subseteq \beta(x)$.
- (IP) If $x \in V(T)$, and if $y, z \in V(T)$ are two vertices in distinct components of $T - x$, then $\beta(y) \cap \beta(z) \subseteq \beta(x)$.

The width of the decomposition (T, β) of G equals the maximal value of $|\beta(x)| - 1$ over all $x \in V(T)$. The *tree-width* of the graph G is the smallest width over all tree-decompositions of G.

The vertex subsets $\beta(x) \subseteq V(G)$ for $x \in V(T)$ are called "bags". The condition (IP) is called an interpolation property.

We refer the reader to [1] for standard concepts in matroid theory. Here we want to mention few things directly related to our paper. The ground set of a matroid M is denoted by $E(M)$, the matroid *rank* by $\mathrm{r}(M)$, and the *rank function* by $\mathrm{r}_M(\cdot)$. If G is a graph, then its cycle matroid (on the ground set $E(G)$) is denoted by $M(G)$. The rank function of $M(G)$ on a subgraph $H \subseteq G$ (i.e. $\mathrm{r}_{M(G)}\left(E(H)\right)$) equals the number of vertices minus the number of connected components of H, which is also the size of a spanning forest in H.

The notion of "branch-width" is closely related to tree-width on graphs [2], and branch-width routinely extends from graphs to matroids since it refers only to edges. However, in the case of tree-width, vertex sets ("bags") play an important role in the definition, which has no analogue for matroids. Neverthless, a possibile way to extend the definition of tree-width to matroids was first suggested by Geelen [unpublished]. We enhance his idea by allowing matroid elements to be mapped to all vertices of the tree (not only to leaves), and removing a reference to the dual matroid.

2 Matroid Tree-Width

Definition 2 Let M be a matroid. A pair (T, τ), where T is a tree and $\tau : E(M) \to V(T)$ is an arbitrary mapping, is called a *tree-decomposition* of M. Let us, for a vertex x of T, denote by $T_1, \dots, T_d$ the connected components of $T - x$. We define the width of x by

$$\sum_{i=1}^{d} \mathrm{r}_M\left(E(M) - \tau^{-1}(V(T_i))\right) - (d-1) \cdot \mathrm{r}(M)\,,$$

and the *width* of the decomposition (T, τ) as the maximal width over all vertices of T. The *tree-width* of M is the smallest width over all tree-decompositions of M.

At first look, it may be really surprising that this definition has anything in common with the above graph definition. (We emphasize that the mapping τ is *not* an analogue of β above. Instead, τ replaces the first condition of graph tree-width, and the second condition (IP) is "hidden" inside the formula for width.) Still, we prove:

Theorem 1 *Let G be a graph with at least one edge, and let $M = M(G)$ be the cycle matroid of G. Then tree-width of G equals tree-width of M.*

Thus we may equivalently define graph tree-width using the matroidal definition, without refering to bags of vertices. The next section presents an important geometric link between the above two definitions of tree-width.

We remark that, despite close similarlity between definitions, it has not been resolved yet whether branch-width of a graph equals branch-width of its cycle matroid. (The main complication comes from different views of connectivity in graphs and in matroids.)

3 Point Configurations

Let q be a prime or a prime power. We consider *point configurations* in the projective geometry over $GF(q)$, and we denote by $\langle X \rangle$ the span of a set X in the projective geometry. Assume that the column vectors of a matrix $\boldsymbol{A}$ over $GF(q)$ represent a rank-r simple matroid M. Then $\boldsymbol{A}$ may be viewed as a subset P of the elements of the projective geometry $PG(r-1,q)$, i.e. as a point configuration over $GF(q)$.

Definition 3 Let $P \subseteq PG(r,q)$ be a point configuration over $GF(q)$. A *tree-decomposition* of the configuration P is a pair (T,Σ), where T is a tree, and Σ is a mapping from $V(T)$ to subspaces of $PG(r,q)$, satisfying the following:

- For each $p \in P$ there is $x \in V(T)$ such that $p \in \Sigma(x)$.
- (IP') If $v \in V(T)$, $e \in E(T)$ is an edge incident with v, and T_1, T_2 are connected components of $T - e$, then $\langle \Phi_1 \rangle \cap \langle \Phi_2 \rangle \subseteq \Sigma(v)$ where $\Phi_i = \bigcup_{x \in V(T_i)} \Sigma(x)$ for $i = 1, 2$.

The width of the decomposition (T,Σ) of P equals the maximal rank of a subspace $\Sigma(x)$ for $x \in V(T)$. The *tree-width* of the point configurations P is the smallest width over all tree-decompositions of P.

The relation of this definition to the definition of graph tree-width is almost straightforward: The points of P are the edges of a graph, and the subspaces $\Sigma(x)$ correspond to the bags of graph vertices. The second condition (IP') is a version of the interpolation property. (Notice that, unlike for graphs, there is no "$|\cdot| - 1$" in the previous definition of tree-width. That is natural since the "rank" of an n-vertex set is actually $n-1$.)

4 Sketch of the Proof

Let $\boldsymbol{J}(G)$ denote the vertex-edge incidence matrix of a simple graph G. Clearly, the rank of $\boldsymbol{J}(G)$ over $GF(2)$ is $|V(G)| - c(G)$ where $c(G)$ is the number of connected components. It is also well known that $\boldsymbol{J}(G)$ over $GF(2)$ represents the matroid $M(G)$.

Lemma 1 *Let G be a simple connected graph on at least 2 vertices, and let P be the point configuration given by the columns of the matrix $\boldsymbol{J}(G)$ over $GF(2)$. Then tree-width of P is not larger than tree-width of G.*

Sketch of proof. Since the rank of $\boldsymbol{J}(G)$ is $n-1$ where $n = |V(G)|$, all the points of the point configuration P belong to a hyperplane Ψ of $PG(n,2)$. Let (T,β) be a tree-decomposition of the graph G, and let $\mathbf{a}_v$ denotes the unit vector corresponding to the row of the vertex $v \in V(G)$ in the matrix $\boldsymbol{J}(G)$. We define $\Sigma(x) = \big\langle \{\mathbf{a}_v : v \in \beta(x)\} \big\rangle \cap \Psi$ for $x \in V(T)$. Then the rank of $\Sigma(x)$, i.e. the width of x, equals $|\beta(x)|-1$. It is straightforward to verify that (T,Σ) is, indeed, a tree-decomposition of the point configuration P. ∎

So far, we have related graph tree-width to tree-width of its associated point configuration in one direction. (We see no reasonable way how to prove a statement of Lemma 1 as an equivalence.) Now we show equality between tree-widths of a representable matroid and of its point configuration. Our proof uses induction and modularity arguments. Since the proof is quite involved, we skip it here.

Lemma 2 *Let M be a simple $GF(q)$-representable matroid, and let P be a point configuration representing M over $GF(q)$. Then tree-width of M is equal to tree-width of P.*

Finally, we want to get from a tree decomposition of a cycle matroid back to a tree decomposition of the underlying graph. The proof is again rather long, with technical connectivity arguments.

Lemma 3 *Let G be a graph with at least one edge, and let $M = M(G)$ be the cycle matroid of G. Then tree-width of G is not larger than tree-width of M.*

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Antisymmetric Flows in Matroids

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Abstract

We present a seemingly new definition of flows and flow numbers for oriented matroids and prove that the flow number and the antisymmetric flow number of an oriented matroid are bounded with its rank. In particular we show that any orientable matroid $\mathcal{O}$ has an antisymmetric $3^{\lfloor \frac{9}{2} \rfloor r(E)+1}$-flow.

Furthermore, we introduce the notion of a semiflow and show that each oriented matroid has an antisymmetric 3-NZ-semiflow

1 Introduction

When considering flows in matroids the main focus has been on the existence of packings of paths under capacity restrictions [7, 8]. Far less attention has been payed to a possible generalization of the theory of nowhere-zero flows.

Goddyn, Tarsi and Zhang [3] introduced a generalization of the circular flow number to regular and to oriented matroids. Goddyn, Hliněný and Hochstättler [2] renamed this parameter into oriented flow number and showed that it is bounded for oriented matroids of bounded rank.

We present a different generalization of the flow number to oriented matroids using the integer lattice generated by the characteristic vectors of oriented circuits. We do not know whether this parameters is related to the oriented flow number of [2]. Similarly, we generalize the notion of an antisymmetric flow, introduced by Nešetřil and Raspaud [5] and show that the corresponding flow number is well defined and bounded by a function of the rank of the matroid.

Finally, we consider the integer lattice generated by oriented circuits and cocircuits, yielding sums of flows and coflows, which we call semiflows. We show that any oriented matroid for any $l \in \mathbb{Z}$ has a semiflow using only the values l and $l+1$.

Our notation is fairly standard. We assume familiarity with basics of oriented matroid theory and of matroid theory, standard references are [1, 6]. We say that a matroid is *cosimple* if every cocircuit has at least three elements. By $r : E \to \mathbb{N}$ we denote the rank function of the matroid in discussion.

2 A Remark on Kirchhoff's Law

Hartmann and Schneider [4] generalized max-balanced flows to oriented matroids by requiring that a flow f satisfies the "max-version of Kirchhoff's law" for all oriented cocircuits $\vec{X}$, i.e. that

$$\max_{e\in X^+} f(e) = \max_{e\in X^-} f(e). \tag{1}$$

In the setting of nowhere-zero flows an attempt to proceed in a similar fashion by requiring

$$\sum_{e\in X^+} f(e) = \sum_{e\in X^-} f(e). \tag{2}$$

has the following drawback.

Let $\mathcal{O}$ be the rank two oriented matroid associated to n points on the real line. Then up to symmetry $\mathcal{O}$ has n cocircuits $X_1, \ldots, X_n$ where $X_i^+ = \{1, \ldots, i-1\}$ and $X_i^- = \{i+1, \ldots, n\}$. The conditions from (2) can be written as $Af = 0$ where A is the square matrix that has -1s under, 0s on and 1s above the diagonal. This matrix has determinant $= 1$ if n is even and $= 0$ if n is odd.

Since a flow f that satisfies (2) for all cocircuits induces such a flow on each contraction minor, the former implies that f must be zero on each set of points P that can be contracted to an even line. To be more precise: For some set $D \subseteq E$: $E \setminus D$ consists of the even line P and possibly some loops in $\mathcal{O}/D$.

3 Nowhere-Zero Flows in Orientable Matroids

Defining flows as integer sums of circuits instead, seems to be more appropriate in our setting.

Definition 1 *Let $\mathcal{O}$ denote an oriented matroid on a finite set E with circuits $\vec{\mathcal{C}}$. We denote by $\chi_{\vec{C}} : E \to \{0, 1, -1\}$ the characteristic vector of $\vec{C} \in \vec{\mathcal{C}}$ and by $\vec{\mathcal{F}}$ the integer lattice (free integer module) generated by the characteristic vectors of circuits.*

$$\vec{\mathcal{F}} := \{\sum_{\vec{C}\in\mathcal{C}} \lambda_{\vec{C}}\chi_{\vec{C}} \mid \lambda_{\vec{C}} \in \mathbb{Z}\}. \tag{3}$$

A flow in $\mathcal{O}$ *is any* $\vec{f} \in \vec{\mathcal{F}}$. *The flow is said to be a* k-flow, *if* $|\vec{f}(e)| \leq k-1$ *for each* $e \in E$, *it is* nowhere zero *or an* NZ-flow *if* $\vec{f}(e) \neq 0$ *for all* $e \in E$.

Remark 1 *The existence of an NZ-k-flow is invariant under reorientation of the oriented matroid. It might vary for different reorientation classes of an orientable matroid, though.*

Since the complement of a maximal set of pairwise disjoint circuits is independent, we immediately get:

Theorem 1 *Let* $\mathcal{O}$ *be an oriented matroid without coloop of rank* r. *Then* $\mathcal{O}$ *has a NZ-*$(r+2)$*-flow.*

4 Antisymmetric Flows in Oriented Matroids

Definition 2 *A flow* $\vec{f}$ *in an oriented matroid is* antisymmetric *or an ASF if for every pair* $e, g \in E$ *holds* $\vec{f}(e) \neq -\vec{f}(g)$.

Along the lines of the proof of Theorem 4 in [5] we can show

Theorem 2 *Let* $\mathcal{O}$ *be an oriented matroid on a finite set* E *without a directed cocircuit of size 1 or 2. Then* $\mathcal{O}$ *has an* $3^{|E|-1}$*-ASF.*

Finally, we show that we can find a 5-flow that is antisymmetric and nowhere zero on all but at most $\lfloor \frac{9}{2} r(E) \rfloor$ elements. Combining this with Theorem 2 yields.

Theorem 3 *Let* $\mathcal{O}$ *be an oriented matroid without a directed cut of size 1 or 2. Then* $\mathcal{O}$ *has a* $3^{\lfloor \frac{9}{2} r(E) \rfloor + 1}$*-ASF.*

5 Semiflows

Since the flow number is unbounded for cographic matroids, it is clear that even the minimum of the flow number of an oriented matroid and its dual is not bounded even for regular matroids. Things get completely different when we allow sums of a flow and a flow in the dual.

Definition 3 *Let* $\mathcal{O}$ *be an oriented matroid* $\mathcal{O}$ *with circuits* $\mathcal{C}$ *and cocircuits* $\mathcal{C}^*$. *A* semiflow *is any vector from*

$$\vec{\mathcal{F}} + \vec{\mathcal{F}}^* := \{\sum_{\vec{C} \in \mathcal{C}} \lambda_{\vec{C}} \chi_{\vec{C}} + \sum_{\vec{D} \in \mathcal{C}^*} \mu_{\vec{D}} \chi_{\vec{D}} \mid \lambda_{\vec{C}}, \mu_{\vec{D}} \in \mathbb{Z}\}. \tag{4}$$

A semiflow $\vec{f}$ *is said to be a* k-semiflow, *if* $|\vec{f}(e)| \leq k-1$ *for each* $e \in E$, *it is* nowhere zero *or an* NZ-semiflow *if* $\vec{f}(e) \neq 0$ *for all* $e \in E$.

It is well-known that each binary matroid can be partitioned into circuits and cocircuits. As a consequence, each regular matroid has a 2-NZ-semiflow. How about the 4-point line consisting of points p_1, p_2, p_3, p_4 in that order? We have the cocircuit $(0, +, +, +)$ and the circuits $(+, -, +, 0)$ and $(0, +, -, +)$. Thus $(1, 1, 1, 2)$ is a 3-NZ-semiflow.

This example generalizes. Given $l \in \mathbb{Z}$ each oriented matroid has a semiflow using only the numbers $\{l, l+1\}$. This is an easy consequence of the following lemma:

Lemma 1 *Let $\mathcal{O}$ be an oriented matroid on a finite set E, $e \in E$ and $k, l \in \mathbb{Z}$. There exists $f \in \vec{\mathcal{F}} + \vec{\mathcal{F}}^*$ such that $f(e) = k$ and $f(g) \in \{l, l+1\}$ for $g \in E \setminus \{e\}$.*

This not only yields a 3-NZ-semiflow but also an antisymmetric one.

Theorem 4 *Let $\mathcal{O}$ be an oriented matroid. Then $\mathcal{O}$ has an antisymmetric* 3*-NZ-semiflow.*

Note, that this result is best possible even for graphs, as is easily seen considering an acyclic orientation of a triangle.

6 Remarks and Open Problems

We are not aware of previous studies on the integer lattice of circuits of an oriented matroid. Thus it seems to be an open field to figure out, which concepts of flow theory generalize to this setting.

There are also several open problems that are related directly to the work presented here. As usual we refer to the minimum k such that a NZ$-k$-flow, resp. a k-ASF exists, as *(asymmetric) flow number.*

- Is the oriented flow number of [2] related to our flow number? Note, that it is no longer a lower bound. As an example consider the 6 point line which has a 2-NZ-flow in our setting but its oriented flow number is 2.5.

- There are orientations of U_3^6, six points in general position, that give give rise to non-isomorphic lattices. We conjecture that our flow numbers are not matroid invariants as well. So far we are not aware of an example verifying this.

- Give lower bounds on the flow number and the asymmetric flow number for the class of oriented matroids of bounded rank. Can it be much worse than for graphic and cographic matroids, i.e. can the flow number become significantly larger than $\sqrt{r(M)}$?

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An algorithm to find a linkless embedding of a graph

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Abstract

An embedding of a graph G is linkless if any two disjoint circuits of G have zero linking number. Graphs that have a linkless embedding were characterized by Robertson, Seymour, and Thomas as those with no minor in the Petersen family, a set of seven graphs, among them K_6 and the Petersen graph. Open remained the question of finding, in polynomial time, a linkless embedding of a graph if it is known to have a linkless embedding,

We show how to obtain a linkless embedding of a graph G if it is known to have a linkless embedding. This is done by finding in polynomial time a set of generators for the lattice of all disjoint circuits pairs of a graph.

About a new class of matroid-inducing packing families

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Abstract

Let $\mathcal{T}$ be a family of graphs. A $\mathcal{T}$-packing of a graph G is a set of disjoint subgraphs of G, each isomorphic to one of the members of $\mathcal{T}$. We are concerned with families $\mathcal{T}$, such that in every graph G, the subsets of vertices that can be saturated by some $\mathcal{T}$-packing form a collection of independent sets of a matroid. The main purpose of this paper is to introduce a new class of families with this property.

1 Introduction

A matching of a graph G can be viewed as a set of disjoint subgraphs of G, each isomorphic to K_2. A natural generalization, called a *$\mathcal{T}$-packing* of G is a set of disjoint subgraphs of G, each isomorphic to a member of a fixed family $\mathcal{T}$ of graphs. We consider only connected graphs G and connected members of $\mathcal{T}$.

If G is a graph then we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. Analogously to matching which we are generalizing, let us introduce the following terminology: A $\mathcal{T}$-packing *covers* a vertex $v \in V(G)$ if one of the subgraphs included in the packing contains v. A $\mathcal{T}$-packing *saturates* a set of vertices $X \subseteq V(G)$ if it covers every member of X. A $\mathcal{T}$-packing Q is *maximal* if there is no packing Q' with $V(Q') \supsetneq V(Q)$ and is *perfect* if it covers all vertices of G. A $\mathcal{T}$-packing problem consists of finding a $\mathcal{T}$-packing saturating a set of a maximum cardinality.

We are concerned with families $\mathcal{T}$ of graphs such that in every graph G, the subsets of $V(G)$ that can be saturated by some $\mathcal{T}$-packing form a matroid.

Definition 1 *Let X be a set and let $\mathcal{M}$ be a nonempty hereditary system of subsets of X (if $B \subseteq A \subseteq X$ and $A \in \mathcal{M}$ then $B \in \mathcal{M}$). The maximal sets of $\mathcal{M}$ (under the set inclusion) are called* bases. *The system $\mathcal{M}$ is called a* matroid *if the set $\mathcal{B}$ of its bases satisfies the exchange axiom:*

(EA) $\forall B, B' \in \mathcal{B}; \forall x \in B \setminus B'; \exists y \in B' \setminus B : (B' \setminus \{y\}) \cup \{x\} \in \mathcal{B}$.

The exchange axiom implies the fact that all bases of a matroid have the same cardinality. We will often shortly write only $B \setminus y \cup x$ instead of $(B \setminus \{y\}) \cup \{x\}$.

We call a family $\mathcal{T}$ of graphs *matroid-inducing*, if for every graph G, the system of all subsets of $V(G)$ that can be saturated by some $\mathcal{T}$-packing is a matroid.

1.1 Notation and basic notions

If $x \in V(H)$ then we denote by $H \setminus x$ the graph obtained from H by deleting the vertex x.

If Q is a $\mathcal{T}$-packing of a graph G then we denote by $V(Q), E(Q)$ the sets of all vertices and edges of G belonging to some graph of Q. Thus, $G_Q = (V(Q), E(Q))$ is a subgraph of G whose components are graphs isomorphic to members of $\mathcal{T}$, and conversely, this graph uniquely determines Q. The degree of a vertex $x \in V(G)$ in the graph G_Q will be denoted by $deg_Q(x)$.

If G is a graph, Q its $\mathcal{T}$-packing and P a path in G then $Q \triangle P$ will denote the packing (not necessarily a $\mathcal{T}$-packing) constructed by swapping edges and non-edges of Q on P ($\triangle$ is the familliar symmetric difference).

A graph H is called *hypomatchable* if it has no perfect matching but for every vertex x, $H \setminus x$ has a perfect matching. A single vertex is considered hypomatchable.

A *star* S_k is a complete bipartite graph $K_{1,k}$, i.e. the graph with $k+1$ vertices $c, v_1, ..., v_k$ and k edges $cv_1, ..., cv_k$. The number k is called the *power* of the star, the vertex c is called the *center* of the star and the vertices $v_1, ..., v_k$ are called *tips* of the star.

A *two-star* $S_{k,l}$ is a graph constructed from two stars S_k, S_l by connecting their centers by a new edge called a *spine*. The vertices of the two stars used are still called centers and tips. A center of a two-star with m neighboring tips will be sometimes called an *m-center* or a *center of power m*. A tip of a two-star adjacent to a center c of power m will be sometimes called a *c-tip* or a *tip of power m*.

A *k-propeller* ($k \geq 1$) is a graph obtained from a star S_{k+1} by replacing k of its tips by hypomatchable graphs (arbitrarily conected to the center).

1.2 History and results

We are concerned with families $\mathcal{T}$ containing K_2. Considering such families, the matroid-inducing property was proved for all of the basic polynomially solvable cases of the $\mathcal{T}$-packing problem:

(i) Matching ($\{K_2\}$-packing) - Edmonds and Fulkerson, [2]

(ii) Packing by K_2 and hypomatchable graphs - Cornuejols and Hartvigsen, [1]

(iii) Packing by a sequential set of stars $\{S_1, ..., S_r\}$ - Las Vergnas, [5]

A case generalizing all of the previous cases is Packing by an EHP-family of graphs, studied by Loebl and Poljak [6, 7, 8] and by the author [3, 4]. An *EHP-family* is a family consisting of K_2, hypomatchable graphs and propellers. Not all EHP-families are matroid-inducing. For a full characterization of EHP-families with respect to the matroid-inducing property see [3, 4]. The matroid-inducing EHP-families turn out to have a structure generalizing the sequential set of stars. A polynomial algorithm for solving the $\mathcal{T}$-packing problem for matroid-inducing EHP-families was introduced in [7, 8]. The following result about matroid-inducing EHP-families was introduced in [4]:

Theorem 1 *[4] Let $\mathcal{T}$ be a matroid-inducing EHP-family and let H be a graph. If $\mathcal{T}_1 = \mathcal{T} \cup \{H\}$ is matroid-inducing then $\mathcal{T}_1$ is a matroid-inducing EHP-family.*

In other words, if we want to enlarge a matroid-inducing EHP-family $\mathcal{T}$ by one graph H to get another matroid-inducing family, we have to use a graph of the same type as the graphs already included in $\mathcal{T}$ (hypomatchable or a propeller such that $\mathcal{T} \cup \{H\}$ is a matroid-inducing EHP family).

It might be expected that a similar situation occurs when we try to enlarge a matroid-inducing EHP-family by a family of more graphs. We will show that this expectation is false by introducing a rather simple class of non-EHP matroid-inducing families.

2 The Matroid-Inducing Families

Theorem 2 *The family $\mathcal{F}_n = \{S_1, ..., S_n, S_{n+2}, S_{n+1,1}, ..., S_{n+1,n+1}\}$ is matroid-inducing for all $n \geq 1$.*

In other words: In a sequential set of stars $S_1, ..., S_{n+2}$ which is known to be a matroid-inducing family [5], the replacement of the second largest star S_{n+1} by a sequence $S_{n+1,1}, ...,$ $S_{n+1,n+1}$ of two-stars results in a matroid-inducing family. Figure 1 shows the family $\mathcal{F}_1$.

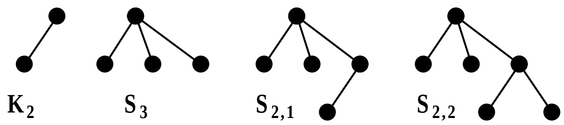

Figure 1: Family $\mathcal{F}_1$

If L is an $\mathcal{F}_n$-packing of a graph G, then every vertex of $V(G)$ covered by a copy of a star $S_k, k < n$, by a tip of S_n (for $n > 1$), by a tip of a two-star or by an m-center of a two-star with $1 \leq m < n+1$ in L, will be called *empty with respect to L*. Let us make an easy observation and then start the proof of Theorem 2:

Observation 1 *Let G be a graph and L an $\mathcal{F}_n$-packing of G. If a vertex uncovered by L is a neighbor (in G) of an empty or uncovered vertex with respect to L then L is not maximal.*

Proof. (Theorem 2) For a contradiction, assume that for some n, $\mathcal{F}_n$ is not matroid-inducing. Hence there exists a graph G and two bases $B, B' \subseteq V(G)$, for which the exchange axiom (EA) does not hold. Thus there is a vertex $x \in B \setminus B'$, such that $\forall y \in B' \setminus B : B' \setminus y \cup x$ is not a base.

Let us denote by Q, Q' the maximal $\mathcal{F}_n$-packings corresponding to B, B', respectively. Without loss of generality we may suppose that G is a graph with minimum $|V(G)|$ and that B, B' and Q, Q' are selected such that the distance $dist(Q, Q') = |E(Q) \setminus E(Q')| + |E(Q') \setminus E(Q)|$ is a minimum. Let us observe that G has no perfect $\mathcal{F}_n$-packing.

Let N, N' be two $\mathcal{F}_n$-packings of G. We say that N' is an *exchange-product* of N, if $V(N') = V(N) \setminus a \cup b$ for some $a \in V(N)$ and $b \notin V(N)$. In the minimum graph G, the following Claim holds:

Claim 1 *Let N be a maximal $\mathcal{F}_n$-packing of G. If N' is an exchange-product of N then N' is maximal.*

The Claim may be proved by inspecting the structure of the two $\mathcal{F}_n$-packings N, N'. Due to space restriction, the proof is omitted here.

Let us continue the proof of Theorem 2 and focus on the two bases B, B' contradicting the (EA) and on the two corresponding $\mathcal{F}_n$-packings Q, Q'. Let $x \in B \setminus B'$ be a vertex contradicting (EA). Let $p \in B$ be a neighbor of x (in Q) with minimum $deg_Q(p)$. Note that if the $deg_Q(p) > 1$ then $deg_Q(x) = 1$. There are several cases:

Case 0: There exists an alternating path P with an even number of edges starting at x with an edge of $E(Q) \setminus E(Q')$, ending with an edge $yq \in E(Q') \setminus E(Q)$, and containing alternately edges of $E(Q)$ and edges of $E(Q')$ that lead to vertex of degree 1 in Q', such that $Q'' = Q' \triangle P$ is an $\mathcal{F}_n$-packing with distance $dist(Q, Q'') < dist(Q, Q')$.

Let $B'' = V(Q'')$. The packing Q'' is an exchange-product of Q' and therefore it is maximal. Hence B'' is a base. If $q \notin Q$ then $B'' = B' \setminus q \cup x$ contradicts the choice of x. Thus $q \in Q$. Because $dist(Q, Q'') < dist(Q, Q')$, (EA) holds for B, B''. Thus, for $q \in B \setminus B''$, there exists $z \in B'' \setminus B$ (thus $z \in B' \setminus B$) such that $B'' \setminus z \cup q = B' \setminus z \cup x$ is a base, which is a contradiction.

Case 1: $1 \leq deg_Q(p) \leq n$. Because Q' is maximal, due to Observation 1 we know that p is not empty with respect to Q'. Thus Q' covers p by a center of S_n, by a center of S_{n+2} or by an $(n+1)$-center of a two-star. In all of the cases we may find a neighbor q of p with $deg_{Q'}(q) = 1$, such that the edge $pq \notin E(Q)$. Considering the path xpq, we get Case 0.

Case 2: $deg_Q(p) = n + 1$. Thus p is an n-center of a copy H of a two-star $S_{n+1,n}$ in Q and x is a p-tip of H. Let pr be the spine of H and let a be an arbitrary r-tip of H.

We know that either $deg_{Q'}(p) = n + 2$ or p is covered by a center of S_n in Q'. If $deg_{Q'}(p) = n + 2$ then there is a neighbor q of p, such that $deg_{Q'}(q) = 1$ and $pq \notin E(Q)$. Hence, considering path xpq, we have Case 0.

If p is covered by a center of a copy T of S_n in Q' and Case 0 does not occur then T covers all p-tips of H different from x and the edge pr. Let us observe that the edge $ra \notin E(Q')$. If a is empty with respect to Q' then we may construct an $\mathcal{F}_n$-packing Q'' with $Q'' \supseteq Q' \cup \{x\}$ contradicting the maximality of Q'. Thus a has a neigbour b in Q', such that $deg_{Q'}(b) = 1$. Obviously the edge $ab \notin E(Q)$ and so, considering the alternating path $xprab$, we are in Case 0.

Case 3: $deg_Q(p) = n + 2$. Again, we know that either p is a center of S_n in Q' or $deg_{Q'}(p) = n+2$. If $deg_{Q'}(p) = n+2$ and Case 0 does not occur then p is an $(n+1)$-center of a copy H of a two-star in Q' and the only edge of $E(Q') \setminus E(Q)$ neighboring with p is the spine pr of H. If H is not a copy of $S_{n+1,n+1}$ then $Q'\triangle\{xp, pr\}$ contradicts the maximality of Q'. If H is a copy of $S_{n+1,n+1}$ then let b be an arbitrary r-tip of H. Let us construct a new packing $Q'' = Q'\triangle\{xp, pr, rb\}$. Define $B'' = V(Q'')$. Q'' is an exchange-product of Q' and so Q'' is a maximal $\mathcal{F}_n$-packing. If $b \notin Q$ then $B'' = B' \setminus b \cup x$ contradicts the choice of x. If $b \in Q$ then note that $pr \notin E(Q)$ and so $dist(Q, Q'') < dist(Q, Q')$. Because Q'' is a maximal $\mathcal{F}_n$-packing of G with $dist(Q, Q'') < dist(Q, Q')$, we get a contradiction as in Case 0.

If p is covered by a center of S_n in Q' and Case 0 does not occur then there is an edge $pq \in E(Q) \setminus E(Q'), x \neq q$. Let us focus on the vertex q and distinguish the following situations:

If q is uncovered by Q', covered by a star $S_j, 1 \leq j \leq n$ in Q' or covered by a tip of a two-star in Q' then it is easy to construct a packing saturating $V(Q') \cup \{x\}$ which contradicts the maximality of Q'.

If q is covered by a center of a copy T of S_{n+2} in Q' then certainly there exists a tip a of T, such that the edge $qa \notin E(Q)$. Let us define a new $\mathcal{F}_n$-packing Q'' by $E(Q'') = E(Q')\triangle\{xp, pq, qa\}$. Q'' is an $\mathcal{F}_n$-packing of G with $dist(Q, Q'') < dist(Q, Q')$. Obviously Q'' is an exchange-product of Q' and so it is maximal. We get a contradiction as in Case 0.

If q is a $(n+1)$-center of a two-star $S_{n+1,l}, 1 \leq l < n+1$ with spine qw in Q' then the packing $Q'\triangle\{xp, pq, qw\}$ contradicts the maximality of Q'.

The last possibility is that q has a neighbor w of degree $n+2$ in Q' (thus q is a tip of S_{n+2} with center w or q is a m-center of a copy T of a two-star $S_{n+1,m}, 1 \leq m \leq n+1$ in Q' and qw is the spine of T). Let $z \neq q$ be a neighbor of w and let us construct a new packing $Q'' = Q'\triangle\{xp, pq, qw, wz\}$. If $z \notin Q$ then $B'' = B' \setminus z \cup x$ is a base which is a contradiction. Thus $z \in Q'$. Note that Q'' is an exchange-product of Q' and that at least one of the edges qw, wz is not in $E(Q)$. Thus Q'' is maximal and $dist(Q, Q'') < dist(Q, Q')$ and again we get a contradiction as in Case 0.

We have inspected all the possible cases and subcases and always got a contradiction. This proves that each $\mathcal{F}_n$ is a matroid-inducing family. $\square$

3 Conclusion

We have introduced a new class of families $\mathcal{F}_n$ of graphs, such that in every graph G, the sets of vertices saturated by some $\mathcal{F}_n$-packing form a matroid.

In the future work, it would be interesting to seek for a polynomial algorithm for finding a maximum $\mathcal{F}_n$-packing. In many of the known algorithms for various cases of the $\mathcal{T}$-packing problem, the polynomial recognition of critical graphs is one of crucial steps. Concerning our class of families $\mathcal{F}_n$, we already know that the only $\mathcal{F}_n$-critical graphs are odd cycles (in case of $\mathcal{F}_1$) and a single vertex.

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Rainbow Arithmetic Progressions and Anti-Ramsey Results

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1 Extended abstract

In 1916, Schur [17] proved that for every k there exists n with the property that every k-coloring of $[n] := \{1, \ldots, n\}$ contains a monochromatic solution of the equation $x + y = z$. More than seven decades later, Alekseev and Savchev [1] considered what Bill Sands calls an *un-Schur* problem [10]. They proved that for every equinumerous 3-coloring of $[3n]$ (i.e., a coloring in which different color classes have the same cardinality), equation $x+y = z$ has a solution with x, y and z belonging to different color classes. Such solutions will be called *rainbow* solutions. E. and G. Szekeres asked whether the condition of equal cardinalities for three color classes can be weakened [19]. Indeed, Schönheim [16] proved that for every 3-coloring of $[n]$, such that every color class has cardinality greater than $n/4$, equation $x + y = z$ has rainbow solutions. Moreover, he showed that $n/4$ is optimal.

Inspired by the problem above, Radoičić posed the following conjecture at an open problem session of the MIT Combinatorics Seminar in 2000.

Conjecture 1 *For every equinumerous* 3*-coloring of* $[3n]$ *there exists a rainbow* $AP(3)$*, i.e., a solution to the equation* $x + y = 2z$ *in which* x*,* y*, and* z *are colored with three different colors.*

This conjecture can be considered as the counterpart of van der Waerden's theorem in Ramsey theory. Van der Waerden's theorem states that for every k and t there exists N with the property that every k-coloring of $[N]$ contains a monochromatic t-term arithmetic progression.

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Backed by the computer evidence ($n \leq 56$), we pose the following stronger form of Conjecture 1.

Conjecture 2 *For every $n \geq 3$, every partition of $[n]$ into three color classes R, G, and B with $\min(|R|, |G|, |B|) > r(n)$, where*

$$r(n) := \begin{cases} \lfloor (n+2)/6 \rfloor & \text{if } n \not\equiv 2 \pmod 6 \\ (n+4)/6 & \text{if } n \equiv 2 \pmod 6 \end{cases} \tag{1}$$

contains a rainbow $AP(3)$.

Unable to settle the above conjectures, we prove the following *infinite* version of Conjecture 2.

Let $c : \mathbb{N} \mapsto \{R, G, B\}$ be a 3-coloring of the set of natural numbers with colors Red, Green, and Blue. We can also think of c as an infinite sequence of the elements of $\{R, G, B\}$. Let $R_c(n)$ be the number of integers less than or equal to n that are colored red. In other words, $R_c(n) := |[n] \cap \{i : c(i) = R\}|$. $G_c(n)$ and $B_c(n)$ are defined similarly. A rainbow $AP(3)$ is a sequence a_1, a_2, a_3 such that $a_1 + a_3 = 2a_2$ and $c(a_i) \neq c(a_j)$ for every $i \neq j$.

Theorem 1 *Let c be a 3-coloring of $\mathbb{N}$ such that*

$$\limsup_{n \to \infty} (\min(R_c(n), G_c(n), B_c(n)) - n/6) = +\infty. \tag{2}$$

Then c contains a rainbow $AP(3)$.

For the proof of this theorem, as well as the remaining results mentioned in this abstract, please refer to our paper [12].

We also show that there exist a 3-coloring of $[n]$ with $\min(|R|, |G|, |B|) = r(n)$, where r is the function defined in (1), that contains no rainbow $AP(3)$. This shows that Conjecture 2, if true, is the best possible.

Previous work regarding the existence of rainbow structures in a colored universe has been done in the context of *canonical* Ramsey theory (see [6, 5, 14, 13, 12] and references therein). However, the canonical theorems prove the existence of *either* a monochromatic structure *or* a rainbow structure. Our results are not "either-or" statements and, thus, are the first results in literature guaranteeing the sole existence of rainbow arithmetic progressions. In a sense, the conjectures and theorems above can be thought of as the first rainbow counterparts of classical theorems in Ramsey theory, such as van der Waerden's, Rado's and Szemerédi's theorems [9]. It is curious to note that anti-Ramsey problems have received great attention in the context of graph theory as well (see [7, 4, 2, 15, 3, 12] and references therein).

An interesting corollary of Theorem 1 is a *modular* version of Conjecture 2, which states that if $\mathbb{Z}_n$ is colored with 3 colors such that the size of every color class is greater than $n/6$, then there exist x, y and z, each of a different color with $x + y \equiv 2z \pmod n$. It turns out that in this case $n/6$ is not the best possible.

Theorem 2 *Let n be an odd number and q be the smallest prime factor of n. Then every 3-coloring c of $\mathbb{Z}_n$ with $\min(|R_c|, |G_c|, |B_c|) > n/q$ contains a rainbow $AP(3)$.*

For any integer n, we define $m(n)$ as the largest integer m for which there is a rainbow-free 3-coloring c of $\mathbb{Z}_n$ such that $|R_c|, |G_c|, |B_c| \geq m$. Theorem 2 shows that for every integer n, $m(n) \leq \min(n/6, n/q)$, where q is the smallest prime factor of n. Computing the exact value of $m(n)$ for every n remains a challenge. The following theorem gives a general lower bound for the value of $m(n)$.

Theorem 3 *For every integer n that is not a power of 2, if q denotes the smallest odd prime factor of n, then $m(n) \geq \lfloor \frac{n}{2q} \rfloor$.*

In the following theorem we characterize the set of natural numbers n for which $m(n) = 0$.

Theorem 4 *For every integer n, there is a rainbow-free 3-coloring of $\mathbb{Z}_n$ with non-empty color classes if and only if n does* not *satisfy any of the following conditions:*

(a) n is a power of 2.

(b) n is a prime and $\mathrm{ord}_n(2) = n - 1$ (i.e., 2 is a generator of $\mathbb{Z}_n$).

(c) n is a prime, $\mathrm{ord}_n(2) = (n-1)/2$, and $(n-1)/2$ is an odd number.

Strong inverse theorems from additive number theory have proved to be useful tools in Ramsey theory. For example, Gowers' proof of Szemerédi theorem relies on the theorem of Freiman [8]. Likewise, we will use a recent theorem of Hamidoune and Rødseth [11], generalizing the classical Vosper's theorem, to prove that *almost* every coloring of $\mathbb{Z}_p$ with three colors has rainbow solutions for *almost* all linear equivalence relations in three variables in $\mathbb{Z}_p$, where p denotes a prime number. Moreover, we classify all the exceptions.

Theorem 5 *Let $a, b, c, e \in \mathbb{Z}_p$, with $abc \not\equiv 0 \pmod p$. Then every coloring of $\mathbb{Z}_p = R \cup B \cup G$ with $|R|, |B|, |G| \geq 4$, contains a rainbow solution of $ax + by + cz \equiv e \pmod p$ with the only exception being the case when $a = b = c$ and every color class is an arithmetic progression with the same common difference d, so that $d^{-1}R = \{i\}_{i=a_1}^{a_2-1}$, $d^{-1}B = \{i\}_{i=a_2}^{a_3-1}$ and $d^{-1}G = \{i\}_{i=a_3}^{a_1-1}$, where $(a_1 + a_2 + a_3) \equiv e + 1$ or $e + 2 \pmod p$.*

Finally, there are many more directions and generalization one might consider.

One natural direction is generalizing the problems above for rainbow solutions of any homogeneous equation, imitating Rado's theorem about the monochromatic analogue. We have already showed an example of this in Theorem 5.

Search for a rainbow counterpart of the Hales-Jewett theorem, though an exciting possibility, led us to some negative results. First, recall some notation from [9]. Define C_t^n, the n-cube over t elements by $C_t^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, t-1\}\}$. A *geometric line* in C_t^n is a set of (suitably ordered) points $\mathbf{x_0}, \ldots, \mathbf{x_{t-1}}$, $\mathbf{x_i} = (x_{i,1}, \ldots, x_{i,n})$ so that in each coordinate j, $1 \leq j \leq n$, either $x_{0,j} = x_{1,j} = \ldots = x_{t-1,j}$, or $x_{s,j} = s$

for every $0 \le s < t$, or $x_{s,j} = n - s$ for every $0 \le s < t$. The Hales-Jewett theorem states that for every t and k, if n is sufficiently large, every k-coloring of C_t^n contains a monochromatic geometric line. This motivates the following question: Is it true that for every equinumerous t-coloring of C_t^n there exists a rainbow geometric line? The following coloring shows that the answer is negative even for small values of t and n. A 3-coloring of C_3^3 defined by $C_1 = \{000, 002, 020, 200, 220, 022, 202, 222, 001\}$, $C_2 = \{011, 021, 101, 201, 111, 221, 010, 210, 012\}$, and $C_3 = \{100, 110, 120, 121, 211, 102, 112, 122, 212\}$ (parentheses and commas being removed for clarity), has no rainbow geometric lines.

Another generic direction we considered is increasing the number of colors and the length of a rainbow AP.

Proposition 1 *For every n and $k > 3$, there exists a k-coloring of $[n]$ with no rainbow $AP(k)$ and with each color of size at least $\lfloor \frac{n+2}{3\lfloor (k+4)/3 \rfloor} \rfloor$.*

One is tempted to also generalize Theorem 1 and conjecture that any partition $\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_k$ into k color classes, with every color class having density greater than $\frac{1}{3\lfloor (k+4)/3 \rfloor}$, contains a rainbow $AP(k)$. However, it is easy to verify that the following equinumerous colorings of $\mathbb{N}$ do not contain any rainbow $AP(5)$, and hence the generalization of Radoičić's conjecture is not true for $k = 5, 6$.

$$c_5(i) := \begin{cases} 1 & \text{if } i \equiv 1, 3 \pmod{10} \\ 2 & \text{if } i \equiv 2, 5 \pmod{10} \\ 3 & \text{if } i \equiv 4, 8 \pmod{10} \\ 4 & \text{if } i \equiv 6, 7 \pmod{10} \\ 5 & \text{if } i \equiv 9, 0 \pmod{10} \end{cases} \qquad c_6(i) := \begin{cases} 1 & \text{if } i \equiv 1, 3 \pmod{12} \\ 2 & \text{if } i \equiv 2, 4 \pmod{12} \\ 3 & \text{if } i \equiv 5, 7 \pmod{12} \\ 4 & \text{if } i \equiv 6, 8 \pmod{12} \\ 5 & \text{if } i \equiv 9, 11 \pmod{12} \\ 6 & \text{if } i \equiv 10, 0 \pmod{12} \end{cases}$$

We still do not know whether there is a similar example when the number of colors is $k = 4$ or $k > 6$. If the number of colors is infinite, the following proposition shows that one cannot guarantee even the existence of a rainbow $AP(3)$ with the assumption that each color has a positive density.

Proposition 2 *There is an infinite coloring of $\mathbb{N}$ with each color having positive density such that there is no rainbow $AP(3)$.*

Yet another direction is limiting our attention to equinumerous colorings and letting the number of colors be different from the desired length of a rainbow AP. Let T_k denote the minimal number $t \in \mathbb{N}$ such that there is a rainbow $AP(k)$ in every equinumerous t-coloring of $[tn]$ for every $n \in \mathbb{N}$. We have the following lower and upper bounds on T_k.

Proposition 3 *For every $k \ge 3$, $\lfloor \frac{k^2}{4} \rfloor < T_k \le \frac{k(k-1)^2}{2}$.*

Note that Proposition 3 gives $3 \le T_3 \le 6$, while Conjecture 1 claims that $T_3 = 3$.

Conjecture 3 *For all $k \ge 3$, $T_k = \Theta(k^2)$.*

It is easy to show that the maximal number of rainbow $AP(3)$'s over all equinumerous 3-colorings of $[3n]$ is $\lfloor 3n^2/2 \rfloor$, this being achieved for the unique 3-coloring with color classes $R = \{n|n \equiv 0 \pmod 3\}$, $B = \{n|n \equiv 1 \pmod 3\}$ and $G = \{n|n \equiv 2 \pmod 3\}$. It seems very difficult to characterize those equinumerous 3-colorings (in general, k-colorings) that minimize the number of rainbow $AP(3)$'s. Letting $f_k(n)$ denote the minimal number of rainbow $AP(k)$'s, over all equinumerous k-colorings of $[kn]$, we pose the following conjecture.

Conjecture 4 *$f_3(n) = \Omega(n)$.*

If we define $g_k(n)$ as the minimal number of rainbow $AP(k)$'s, over all equinumerous k-colorings of $\mathbb{Z}_{kn}$, then a straightforward counting argument shows that $g_3(n) \geq n$, when n is odd.

Finally, the further generalization of Vosper's theorem, due to Serra and Zémor [18], may lead to a generalization of Theorem 5 for more than 3 color classes.

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The Erdős–Heilbronn problem in the additive theory and the polynomial method

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1 The results

Let $G \neq 0$ denote any Abelian group. Define $p(G)$ as the smallest positive integer p for which there exists a nonzero element g of G with $pg = 0$. If no such integer exists, we write $p(G) = \infty$. For nonempty subsets $A, B \subseteq G$ with $|A| = k$ and $|B| = \ell$, we will consider the sets

$$A + B = \{a + b \mid a \in A, b \in B\}$$

and

$$A \dot{+} B = \{a + b \mid a \in A, b \in B, a \neq b\}.$$

The following extension of the Cauchy–Davenport theorem can be obtained easily from Kneser's theorem or can be proved directly with a combinatorial argument, see [4].

Theorem 1 *If A and B are nonempty subsets of an Abelian group G such that $p(G) \geq |A| + |B| - 1$, then $|A + B| \geq |A| + |B| - 1$.*

It follows from Vosper's inverse theorem, that if $A \subset \mathbb{Z}/p\mathbb{Z}$ where p is a prime larger than $2|A| - 1$, then $|A + A| = 2|A| - 1$ if and only if the elements of A form an arithmetic progression modulo p.

Much less is known in the case of restricted addition. In 1994 Dias da Silva and Hamidoune proved that for $A \subset \mathbb{Z}/p\mathbb{Z}$, p a prime,

$$|A \dot{+} A| \geq \min\{p, 2|A| - 3\},$$

thus settling a problem of Erdős and Heilbronn. Later Alon, Nathanson and Ruzsa applying the so-called 'polynomial method' gave a simpler proof that also yields $|A \dot{+} B| \geq \min\{p, |A| + |B| - 2\}$ if $|A| \neq |B|$. Here we prove the following extension of the Dias da Silva–Hamidoune theorem:

Theorem 2 *If A is a k-element subset of an Abelian group G, then*

$$|A\dot{+}A| \geq \min\{p(G), 2k-3\}.$$

We begin with a more general result for cyclic groups of prime power order.

Theorem 3 *Let $A, B \subseteq \mathbb{Z}/q\mathbb{Z}$, where $q = p^\alpha$ is a power of a prime p. Then*

$$|A\dot{+}B| \geq \min\{p, |A| + |B| - 3\}.$$

This result we prove in Section 3. Now Theorem 2 can be derived as follows. First of all, since we are dealing with a finite problem, we may assume that G is finitely generated. It is easy to see that the result is valid if $G = \mathbb{Z}$. Thus, in view of Theorem 3 and the structure theorem for finitely generated Abelian groups, it only remains to prove that if the statement of Theorem 2 is true for two Abelian groups G^1 and G^2, then it is also valid for their direct sum $G^1 \oplus G^2$. The long and technical proof can be found in [3].

In Section 4 we prove the following inverse theorem.

Theorem 4 *Let A be a set of $k \geq 5$ residue classes modulo a prime $p > 2k-3$. Then $|A\dot{+}A| = 2k-3$ if and only if A is an arithmetic progression.*

Finally we note that this result can also be transferred to direct sums to obtain that if G does not contain any element whose order is a square of a prime, then $A \subset G$, $7 \leq |A\dot{+}A| = 2|A| - 3 < p(G)$ implies that A is an arithmetic progression.

2 Zeroes of multivariate polynomials

The so-called polynomial method exploits the fact that, over any field, the number of roots of a nonzero polynomial cannot exceed the degree of the polynomial. The proof of our results depend on multivariate analogues of the above fact, proved by Alon [1].

One is the so-called 'Combinatorial Nullstellensatz'.

Lemma 1 *Let F be an arbitrary field and let $f = f(x_1, \ldots, x_k)$ be a polynomial in $F[x_1, \ldots, x_k]$. Let $S_1, \ldots, S_k$ be nonempty subsets of F and define $g_i(x) = \prod_{s \in S_i}(x_i - s)$. If $f(s_1, s_2, \ldots, s_k) = 0$ for all $s_i \in S_i$, then there exist polynomials $h_1, h_2, \ldots, h_k \in F[x_1, \ldots, x_k]$ satisfying $\deg(h_i) \leq \deg(f) - \deg(g_i)$ such that $f = \sum_{i=1}^k h_i g_i$.*

This result in turn implies what we will call the 'polynomial lemma'.

Lemma 2 *Let F be an arbitrary field and let $f = f(x_1, \ldots, x_k)$ be a polynomial in $F[x_1, \ldots, x_k]$. Suppose that there is a monomial $\prod_{i=1}^k x_i^{t_i}$ such that $\sum_{i=1}^k t_i$ equals the degree of f and whose coefficient in f is nonzero. Then, if $S_1, \ldots, S_k$ are subsets of F with $|S_i| > t_i$ then there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_k \in S_k$ such that $f(s_1, \ldots, s_k) \neq 0$.*

3 The proof of Theorem 3

We may clearly assume that $|A| = k \geq 2$ and $|B| = \ell \geq 2$. Since $A' \supseteq A$ and $B' \supseteq B$ implies $|A' \dot{+} B'| \geq |A \dot{+} B|$, we also may assume that $k + \ell - 3 \leq p$.

Our proof will depend on the polynomial lemma, used in a multiplicative setting. Let $\varepsilon = e^{2\pi i/q}$ and consider the unique embedding $\varphi : G \hookrightarrow \mathbb{C}^\times$ of G into the multiplicative group of the field of complex numbers with the property $\varphi(1) = \varepsilon$. Write $C = A \dot{+} B$ and define

$$\tilde{A} = \{\varphi(a) \mid a \in A\}, \ \tilde{B} = \{\varphi(b)^{-1} \mid b \in B\}, \ \tilde{C} = \{\varphi(c) \mid c \in C\}.$$

Observe that for $a \in A$ and $b \in B$,

$$a = b \Longleftrightarrow \varphi(a)\varphi(b)^{-1} - 1 = 0$$

and

$$a + b = c \Longleftrightarrow \varphi(a) - \varphi(c)\varphi(b)^{-1} = 0.$$

Thus, if $x \in \tilde{A}$ and $y \in \tilde{B}$, then either $xy - 1 = 0$, or there exists a $c \in \tilde{C}$ such that $x - cy = 0$.

We wish to prove that $|C| \geq k+\ell-3$. Assume that on the contrary, $|C| = |\tilde{C}| \leq k+\ell-4$, and choose any set $\tilde{C}' \subseteq G$, of cardinality $k+\ell-4$, that contains $\tilde{C}$. Consider the polynomial $P \in \mathbb{C}[x, y]$ defined as

$$P(x, y) = (xy - 1) \prod_{c \in \tilde{C}'} (x - cy),$$

then $P(x, y) = 0$ for every $x \in \tilde{A}$, $y \in \tilde{B}$. Since the degree of P is clearly not greater than $k + \ell - 2$, in view of Lemma 2, the desired contradiction comes from the fact that the coefficient of the monomial $x^{k-1}y^{\ell-1}$ in P is different from 0.

To verify this fact, observe that writing $\tilde{C}' = \{c_1, c_2, \ldots, c_{k+\ell-4}\}$, this coefficient is

$$\mathrm{coeff}_P(x^{k-1}y^{\ell-1}) = (-1)^{\ell-2} Q(c_1, c_2, \ldots, c_{k+\ell-4}),$$

where $Q(x_1, x_2, \ldots, x_{k+\ell-4})$ is the $(\ell - 2)^{nd}$ elementary symmetrical polynomial in the variables $x_1, \ldots, x_{k+\ell-4}$. In particular, $Q(c_1, c_2, \ldots, c_{k+\ell-4})$ is the sum of $\binom{k+\ell-4}{\ell-2}$ numbers, each of which is a product of $\ell - 2$ terms. These terms, each being equal to some c_i, are all elements of $\varphi(G)$. Consequently, each of the $\binom{k+\ell-4}{\ell-2}$ summands is an element of $\varphi(G)$, hence equals some q^{th} root of unity.

Lemma 3 ([2]) *If $\varepsilon_1, \ldots, \varepsilon_m$ are q^{th} roots of unity such that $\sum_{i=1}^m \varepsilon_i = 0$, then m is divisible by p.*

As $p > k + \ell - 4$, the binomial coefficient $\binom{k+\ell-4}{\ell-2}$ is not divisible by p. Thus, it follows from Lemma 3 that $Q(c_1, c_2, \ldots, c_{k+\ell-4})$ cannot be zero. Accordingly, $\mathrm{coeff}_P(x^{k-1}y^{\ell-1}) \neq 0$, which completes the proof of Theorem 3.

4 Sketch of the proof of Theorem 4

To prove this theorem we first embed the group $\mathbb{Z}_p$ that contains A in an algebraically closed field $\bar{\mathbb{F}}_p$ of characteristic p. We assume that $C = A\dot{+}A$ has a cardinality $2k-3$ and the elements of A are $a_1, a_2, \ldots, a_k$. We define the polynomial

$$f(x,y) = (x-y)\prod_{c\in C}(x+y-c)$$

and also an auxiliary polynomial $g(z) = \prod_{i=1}^{k}(z-a_i)$. Notice that $f(x,y) = 0$ for arbitrary $x, y \in A$. According to the Combinatorial Nullstellensatz (Lemma 1) there exist polynomials $h', h'' \in \bar{\mathbb{F}}_p[x,y]$ of degree at most $k-2$ such that

$$f(x,y) = h'(x,y)g(x) + h''(x,y)g(y).$$

Since the polynomial f alternates we can write

$$f(x,y) = -f(y,x) = -h'(y,x)g(y) - h''(y,x)g(x)$$

to obtain that

$$f(x,y) = h(x,y)g(x) - h(y,x)g(y), \tag{1}$$

where $h(x,y) = (1/2)(h'(x,y) - h''(y,x))$ is a polynomial of degree at most $k-2$. Thus we can write

$$h(x,y) = \sum_{i=0}^{k-2} h_i(x,y) = \sum_{i=0}^{k-2}\sum_{j=0}^{i} A_{ij}x^j y^{i-j}.$$

We can also rewrite $f(x,y)$ in the form

$$f(x,y) = \sum_{i=0}^{2k-3}(-1)^i s_i p_{2k-2-i}(x,y).$$

Here $s_0 = 1$ and, for $1 \le i \le 2k-3$, s_i is the i^{th} elementary symmetrical polynomial of the elements of C, while

$$p_i(x,y) = (x-y)(x+y)^{i-1} = \sum_{j=0}^{i} B_{ij}x^j y^{i-j},$$

where $B_{ii} = 1$, $B_{i,0} = -1$, and otherwise $B_{ij} = \binom{i-1}{j-1} - \binom{i-1}{j} = \frac{2j-i}{j}\binom{i-1}{i-j}$. If we also denote, for $1 \le i \le k$, by σ_i the i^{th} elementary symmetrical polynomial in $a_1, a_2, \ldots, a_k$, after comparing coefficients we arrive at certain relations between the numbers s_i, the numbers σ_i and the coefficients A_{ij}.

In particular, after a lengthy argument (see [5]) one can conclude that the values of σ_1 and σ_2 uniquely determine those of $\sigma_3, \sigma_4, \ldots, \sigma_k$, and in turn also the elements of A, since they are the k solutions of the equation

$$g(z) = z^k - \sigma_1 z^{k-1} + \sigma_2 z^{k-2} - \ldots + (-1)^k\sigma_k = 0.$$

Since $\bar{\mathbb{F}}_p$ is an algebraically closed field of characteristic $p > 2k - 3$, there is a unique set $A_0 \subset \bar{\mathbb{F}}_p$ whose elements form a k-term arithmetical progression, and the first two elementary symmetrical polynomials of the elements of A_0 are exactly σ_1 and σ_2. If this progression is not constant, then $|A_0 \dot{+} A_0| = 2k - 3$, thus the above argument shows that the remaining $k - 2$ elementary symmetrical polynomials of the elements of A_0 must be equal to $\sigma_3, \sigma_4, \ldots, \sigma_k$. It is not difficult to arrive at the same conclusion in the hypothetical singular case when $|A_0| = 1$. In any case, this implies that $A = A_0$ and thus the elements of A indeed form an arithmetical progression.

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A coding problem for pairs of subsets

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Abstract

Let X be an n–element finite set, $0 < k < n/2$ an integer. Suppose that $\{A_1, B_1\}$ and $\{A_2, B_2\}$ are unordered pairs of disjoint k-element subsets of X (that is, $|A_1| = |B_1| = |A_2| = |B_2| = k, A_1 \cap B_1 = \emptyset, A_2 \cap B_2 = \emptyset$). Define the distance of these pairs by $d(\{A_1, B_1\}, \{A_2, B_2\}) = \min\{|A_1 - A_2| + |B_1 - B_2|, |A_1 - B_2| + |B_1 - A_2|\}$. It is known ([4]) that this is really a distance on the space of such pairs. Estimates are given on the maximum number of such pairs with pairwise distance at least d. Another variant is studied, too, when the additional condition forbiding the usage of one k-element set in two pairs.

Keywords: families of subsets, pairs of disjoint subsets, codes, hamiltonian cycle

1 Introduction

Motivated by database theory, the following extremal problem was posed in [3] for matrices. Let the entries of the $n \times m$ matrix M be 0, 1 and 2. Each column contains exactly three 1s and three 2s, the other entries are 0s. Suppose that the triplets defined by the row indecies of the theree 1s and three 2s, resp. for all columns are all different. A further condition is that if the rows of the 1s coincide in two places for any two given columns then this cannot happen with the 2s (for the same two columns). This condition must hold if the 1s are replaced by 2s and the 2s are replaced by 1s in one of the two columns. Given n, maximize m under the above conditions.

It is obvious that the problem can be formulated in terms of (unordered) pairs $\{A_i, B_i\}$ of subsets of an n-element set X. Namely, let X be the set of rows of the matrix and let A_i (B_i) be the set of rows where the ith column contains a 1 (2). The conditions on the matrix can be reformulated in the following way. $|A_i| = |B_i| = 3, A_i \cap B_i = \emptyset$ all the three element sets A_i, B_j $(1 \le i, j \le m)$ are distinct, if $|A_i \cap A_j| = 2$ then $|B_i \cap B_j| < 2$, if $|A_i \cap B_j| = 2$ then $|B_i \cap A_j| < 2$. Given the size n of X maximize the number m of pairs.

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[3] has solved this problem in a more general context, namely for pairs of disjoint k-element sets. Here $A_i, B_i \subset X, |X| = n, A_i \cap B_i = \emptyset, |A_i| = |B_i| = k$, all the k-element sets $A_i, B_j (1 \le i, j \le m)$ are distinct, $|A_i \cap A_j| \ge \lceil \frac{k+1}{2} \rceil$ implies $|B_i \cap B_j| < \lceil \frac{k+1}{2} \rceil$ and $|A_i \cap B_j| \ge \lceil \frac{k+1}{2} \rceil$ implies $|B_i \cap A_j| \ge \lceil \frac{k+1}{2} \rceil$, that is, the pairs are considered unordered. Again, the maximum of m is determined for a given n. It is obvious that one cannot choose more than $\lfloor \frac{1}{2} \binom{n}{k} \rfloor$ such pairs. It has been proved in [3] that this many pairs can really be chosen, that is, this is the desired maximum. The method of the proof uses a Hamiltonian type theorem, formulated later in the paper.

It was realised in [4] that the problem is, in fact, a coding type problem. Introduce the following distance for pairs of disjoint k-element sets.

$$d(\{A_1, B_1\}, \{A_2, B_2\}) = \min\{|A_1 - A_2| + |B_1 - B_2|, |A_1 - B_2| + |B_1 - A_2|\} \qquad (1)$$

It was shown in [4] that one choose $\lfloor \frac{1}{2} \binom{n}{k} \rfloor$ such pairs with pairwise distance at least k. It is easy to see that this theorem implies the previous one.

[2] started to investigate the problem as a coding type one.

We say that a set $\mathcal{C}$ of such pairs of subsets is a $2-(n,k,d)$-code if the distance of any two elements of $\mathcal{C}$ is at least d. Let $C(n,k,d)$ be the maximum size of a $2-(n,k,d)$-code. $C'(n,k,d)$ denotes the same under the additional condition that a

$$k\text{–element subset may occur only once in the pairs } \{A, B\} \in \mathcal{C} \text{ as } A \text{ or } B. \qquad (2)$$

We give estimates on these numbers.

2 The starting problem

Theorem 1 [3] *Let $|X| = n, 2 \le k, n_0(k) \le n$ be given integers. The family of all k-element subsets of X can be partitioned into unordered pairs (except possibly one if $\binom{n}{k}$ is odd), so that paired k-element subsets are disjoint and if $\{A_1, B_1\}$ and $\{A_2, B_2\}$ are two such pairs then $|A_1 \cap A_2| \ge \lceil \frac{k+1}{2} \rceil$ and $|B_1 \cap B_2| \ge \lceil \frac{k+1}{2} \rceil$ cannot both hold.*

Sketch of the proof. The following theorem (a generalization of Dirac's theorem) is used.

Theorem 2 [3] *Let $G_0 = (V, E_0)$ and $G_1 = (V, E_1)$ be simple graphs on the same vertex set $|V| = N$, such that $E_0 \cap E_1 = \emptyset$. The 4-tuple (x, y, z, v) is called an* alternating cycle *if (x, y) and (z, v) are in E_0 and (y, z) and (x, v) are in E_1. Let r be the minimum degree of G_0 and let s be the maximum degree of G_1. Suppose, that*

$$2\,r - 8\,s^2 - s - 1 > N,$$

then there is a Hamiltonian cycle in G_0 such that if (a, b) and (c, d) are both edges of the cycle, then (a, b, c, d) is not an alternating cycle.

We construct graphs $G_0 = (V, E_0)$ and $G_1 = (V, E_1)$ that satisfy the requirements of Theorem 2. The vertex set V consists of the q-subsets of X, $|V| = \binom{n}{k} = N$. Two k-element subsets are adjacent in G_0 if their intersection is empty, while two k-element subsets are adjacent in G_1 if they intersect in at least $\lceil \frac{k+1}{2} \rceil$ elements. The minimum degree of G_0 is $r = \binom{n-k}{k}$ and the maximum degree of G_1 is $s = \sum_{j=k}^{k} \binom{k}{j} \binom{n-k}{k-j}$. It is easy to check that $2\,r - 8\,s^2 - s - 1 > N = \binom{n}{k}$, provided $n \geq n_0(k)$.

According to Theorem 2, there is a Hamiltonian cycle H in G_0 that does not contain two disjoint edges that span an alternating cycle. Now the required partition of the k-subsets into disjoint pairs can be obtained by going around H, every other edge will form a good pair. $\square$

3 The coding problem without repeating sets

It is proved in [4] that (1) is really a distance, that is, it satisfies the triangle inequality.

Theorem 3 *Let $|X| = n \geq n_0(k)$. The family of all k-element subsets of X can be partitioned into disjoint pairs (except possibly one if $\binom{n}{k}$ is odd), so that $d(\{A_1, B_1\}, \{A_2, B_2\}) \geq k$ holds for any two such pairs $\{A_1, B_1\}$ and $\{A_2, B_2\}$. In other words, $C'(n, k, d) = \lfloor \frac{1}{2}\binom{n}{k} \rfloor$.*

Sketch of the proof. The proof is similar to the proof of Theorem 1 but we have to use here weighted graphs. G_0 and r are defined as before. The edges of G_1 are weighted by positive integers. The weight of $e \in E_1$ is denoted by $l(e)$. Denote the number of edges of weight i starting from the vertex v by

$$d(v, i) = |\{e \in E_1 : v \in e, l(e) = i\}|.$$

Let s be the maximum degree in G_1, that is,

$$s = \max_{v \in V}\{\sum_i d(v, i)\} \qquad (3).$$

Another parameter t is defined by

$$t = t(k) = \max_{v \in V}\{\sum_i d(v, i) \max_{w \in V}\{ \sum_{k+1-i \leq j} d(w, j)\}\} \qquad (4).$$

A 4-tuple (x, y, z, v) of vertices is called *heavy* C_4 iff $(x, y), (z, v) \in E_0$, $(y, z), (x, v) \in E_1, l((y, z)) + l((x, v)) \geq k + 1$. After these definitions we are able to formulate our theorem.

Theorem 4 *Suppose, that*

$$2\,r - 4\,t - s - 1 > N.$$

Then there is a Hamiltonian cycle in G_0 such that if (a, b) and (c, d) are both edges of the cycle, then (a, b, c, d) is not a heavy C_4.

In order to prove Theorem 3, we have to apply Theorem 4 with the weight $l((A,B)) = |A \cap B|$ on the edge $\{A,B\}$ of G_1. G_0 (as before) is regular with degree $r = \binom{n-k}{k} = \frac{1}{k!}n^k + O(n^{k-1})$. In G_1 we have

$$d(v,i) = d(i) = \binom{k}{i}\binom{n-k}{k-i} \; (1 \leq i < k). \tag{5}$$

By (3) and (5) we have

$$s = \sum_{i=1}^{k-1} \binom{k}{i}\binom{n-k}{k-i} = \frac{k}{(k-1)!}n^{k-1} + O(n^{k-2}).$$

On the other hand (4) and (5) imply

$$t = \sum_{k+1 \leq i+j} d(i)d(j) = \sum_{k+1 \leq i+j} \binom{k}{i}\binom{n-k}{k-i}\binom{k}{j}\binom{n-k}{k-j} =$$

$$= n^{k-1}\sum_{i=2}^{k-1} \binom{k}{i}\binom{k}{k+1-i}\frac{1}{(k-i)!}\frac{1}{(i-1)!} + O(n^{k-2}).$$

It is easy to check that $2r - 4t - s - 1 > N = \binom{n}{k} = \frac{1}{k!}n^k + O(n^{k-1})$, provided $n \geq n_0(k)$.

According to Theorem 4, there is a Hamiltonian cycle in G_0 which ensures the existence of the set of pairs desired in Theorem 3.

4 The "real" coding problem

It is easy to see that Theorem 3 implies

$$C'(n,k,d) = C'(n,k,k) = \left\lfloor \frac{1}{2}\binom{n}{k} \right\rfloor$$

if $d \leq k$. On the other hand,

$$C'(n,k,d) = C(n,k,d)$$

for $d > k$, since the coincidence of two k-element sets implies that the distance of the pairs is at most k. Therefore the only remaining problem is to determine $C(n,k.d)$ for $d > k$. [2] gave an upper estimate for this case.

Theorem 5 *Let $d \leq 2k \leq n$ be integers. Then*

$$C(n,k,d) \leq \frac{1}{2}\frac{n(n-1)\cdots(n-2k+d)}{k(k-1)\cdots\lceil\frac{d+1}{2}\rceil \cdot k(k-1)\cdots\lfloor\frac{d+1}{2}\rfloor}$$

holds.

It is be proved in [1] that this upper estimate is asymptotically sharp.

Theorem 6 *Suppose that $k < d \leq 2k \leq n$. Then*

$$\lim_{n\to\infty} \frac{C(n,k,d)}{n^{2k-d+1}} = \frac{1}{2} \frac{1}{k(k-1)\cdots\lceil\frac{d+1}{2}\rceil \cdot k(k-1)\cdots\lfloor\frac{d+1}{2}\rfloor}.$$

The proof heavily uses the Rödl method ([7]), namely the theorem of Frankl and Rödl ([5]).

We conjecture that the upper estimate of Theorem 5 is sharp for any given k and d for infinitely many n. Paper [1] shows a method which might work for the case when $d = 2k-1$, but we were actually be able to prove the conjecture on for the cases $C(n,2,3)$ and $C(n,3,5)$.

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Descrying induced interval graphs

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Abstract

In this paper we address the problem of finding an induced interval graph with a maximum number of vertices in a given graph G. We show that the problem is NP-complete for bipartite planar graphs and split graphs. We can show that there exist efficient algorithms for graphs with bounded treewidth, (tree-)cographs, and distance hereditary graphs. We show that there exists an efficient algorithm for graphs which are intersection graphs of intervals on lines intersecting in one point.

1 Introduction

Given a graph $G = (V, E)$ the *maximum interval graph problem* (MIGP for short) is that of finding a maximum induced interval subgraph of G by deleting the minimum number of vertices of G.

MIGP has an application to physical mapping of a target DNA molecule [1, 3]. Suppose that we are given a set of fragments (i.e., clones). Each fragment corresponds to a substring of the target DNA. The aim of the physical mapping problem is to reconstruct the relative position of these fragments along the target DNA based on the information of their pairwise intersections. We can build a graph $G = (V, E)$ according to the fragments and their pairwise intersections as follows. Each fragment is represented by a vertex and two vertices are adjacent if and only if their corresponding fragments intersect. If the fragments are accurate and cover the whole target DNA then the physical mapping problem becomes the one of recognizing whether G is an interval graph and constructing its interval model if so. In laboratories however, some biological techniques may occasionally produce some

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new and bad fragments, such as chimeric fragments, in a way such that G is no longer an interval graph. Therefore we want to remove the minimum number of bad fragments to obtain a maximum induced interval graph.

The MIGP is NP-complete according to the results of [5]. We show that MIGP is NP-complete on split graphs and planar bipartite graphs with maximum degree 3. We propose polynomial time algorithms for MIGP on graphs with bounded treewidth, tree-cographs, and intersection graphs of intervals on lines intersecting in one point.

2 NP-completeness results

Yannakakis showed that finding the maximum chain graph in a bipartite graph is NP-complete. This leads to the following result:

Theorem 1 *The* MAXIMUM INTERVAL GRAPH *problem is NP-complete on split graphs.*

In [5], Lewis and Yannakakis showed that the maximum interval graph problem is NP-complete on planar graphs. Using a different reduction, Yannakakis shows that the problem remains NP-complete on bipartite graphs. We obtain the following result:

Theorem 2 *The* MAXIMUM INTERVAL GRAPH *problem is NP-complete on planar bipartite graphs with maximum degree 3.*

Proof: MIGP is in NP since the problem of determining whether a graph is an interval graph can be done in linear time. In the following, we will show that MIGP on planar bipartite graphs is NP-hard by a reduction from the Hamiltonian path problem on 3-regular planar graphs.

Given a 3-regular graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, we transform it to a new graph $G' = (V', E')$ as follows. We first replace each edge $(v_i, v_j) \in E$ by a path $P_{i,j} = (v_i, u_{i,j}, v_j)$ of length 2 and then replace each vertex $v_i \in V$ by a tree $K_{1,3}$, denoted by T_i, with an internal vertex w_i and three leaves v_i^1, v_i^2 and v_i^3. Clearly, G' is planar and its vertices can be partitioned into two independent sets X and $Y \cup Z$, where $X = \{v_i^1, v_i^2, v_i^3 | 1 \leq i \leq n\}$, $Y = \{w_i | 1 \leq i \leq n\}$ and $Z = \{u_{i,j} | (v_i, v_j) \in E\}$. Hence, G' is a planar bipartite graph. Moreover, G' contains $4n + \frac{3n}{2}$ vertices and the degrees of its vertices are either 2 or 3 (i.e., G' is a planar bipartite graph with maximum degree 3).

To complete the proof, we show that G has a Hamiltonian path if and only if G' has a connected interval subgraph with at least $5n - 1$ vertices. We omit the details of the proof. $\square$

3 Graphs with bounded treewidth

A rooted tree decomposition is called *nice*[4] if every node of T has at most two children and the following conditions are satisfied. If a node i has two children j and k then

$S_i = S_j = S_k$ and in this case we call the node i a *join node*. If a node i has only one child j then either $|S_i| = |S_j| + 1$ and $S_j \subseteq S_i$ (in which case the node i is called an *introduce node*) or $|S_i| = |S_j| - 1$ and $S_j \subseteq S_i$ (and in this case the node i is a *forget node*). Finally, if a node i has no children at all and is not the root then i is called a *start node*. We may assume that a nice tree decomposition $(T, \mathcal{S})$ of G of width k is part of the input. For a node i in the tree decomposition let T_i be the subtree of T rooted at node i. We let $G_i = (V_i, E_i)$ stand for the subgraph induced by the vertices occurring in bags $S_j \in \mathcal{S}$ for $j \in V(T_i)$.

Our algorithm works from the leaves in T up to the root computing partial solutions for $G_i = (V_i, E_i)$ on the way. Consider an interval graph contained in G_i. Let $\mathbb{C} = [C_1, \ldots, C_t]$ be the consecutive clique arrangement of such an interval graph. The *characteristic* of this consecutive clique arrangement consists first of all of $[C_1 \cap S_i, \ldots, C_t \cap S_i]$ in which duplicates are removed. If some consecutive elements are empty, we keep one *i-long* empty set in the representation. We write $\mathbb{C}_r = [c_1, \ldots, c_r]$ for this representative arrangement.

For each non-empty clique in $\mathbb{C}_r$, we indicate if it represents a clique which also contains vertices outside S_i. If this is the case we call it an *i-long clique*. If the original clique is contained in S_i then we call it *i-short*.

Assume that the interval graph with clique arrangement $\mathbb{C}$ is disconnected. For ease of exposition, we put an *i-short* empty set between the components. All possible permutations of the components are stored.

The second part of the characteristic is the *maximum number of vertices* that *any* interval graph in G_i can have that gives rise to the same consecutive clique arrangement $\mathbb{C}_r$ in S_i. We need to describe for each type of node how a table with characteristics is updated. In this abstract we only give the details for the join node.

Let **i** be a join node and let j and k be the two children of i. In this case we have $S_i = S_j = S_k$. First, all table entries at node j and all table entries at node k are stored at node i.

Now consider entries at node j and node k which are similar except for empty sets, cardinalities, and long/short indications. The following compatibility requirements must be met. For each non-empty clique in the representative clique arrangement, it can be only either j-long or k-long, but not both. Consider two non-empty j-long cliques which are consecutive except for possibly a j-long empty set between them. In that case there can be no k-long empty set between them in the representative arrangement at node k. [1] Of course, a similar requirement holds for two non-empty k-long cliques which are consecutive except for possibly a k-long empty set.

If the table entries at nodes j and k are compatible, the *union* is taken. That is, if a clique was j-long or k-long then it is made i-long. If there is a j-long empty set or a k-long empty set between two cliques (or both) then one i-long empty set is put at the same position. The same for possibly empty sets at the beginning or end. There is an i-short empty set only if there is a j-short and a k-short empty set exactly at the same position, except

[1] Notice that if there is a clique completely contained in $V_k - S_i$ between the two j-long cliques, then it can be moved forward or backward in the representation. Hence the requirement is not a restriction.

possibly for j-long or k-long empty sets.
The cardinality of the entry is the sum of the two cardinalities at j and k minus the number of vertices of S_i contained in the representative clique arrangement.
Finally duplicate elements in the table at node i, except for the cardinalities, are replaced by a single entry with only the maximum cardinality of the copies.

Theorem 3 *There exists a linear time algorithm to compute the maximum cardinality of an induced interval graph contained in a graph of bounded treewidth.*

4 Intervals on lines intersecting in one point

Consider a set of lines that have a common intersection point. We study the MAXIMUM INTERVAL GRAPH problem for the intersection graphs of intervals on these lines. These graphs were studied in [2]. [2] Notice that the class is properly contained in the class of *path graphs* (i.e., the intersection graphs of paths in a tree).
Let G be the intersection graph of n intervals contained in a collection of lines intersecting in one point. We refer to this common point as ω. The *line segments* with one end point ω are called *half-lines*. Notice that *all* intervals containing ω form a clique. We refer to this clique as Ω. We can obtain the following facts:

1. Every connected component of an interval graph contained in G has intervals not in Ω on at most two half-lines of *different* lines.

2. Let x and y be in a connected component C of the interval subgraph. Assume furthermore that x and y are on the two different half-lines of one line ℓ, and x and y have *private neighbors* in Ω. Then the C contains no other half-line with a vertex not in Ω.

We describe a polynomial time algorithm that solves MIGP for this class of graphs. We maximize over all pairs of half-lines containing vertices x and y such that these have private neighbors in Ω. First we consider the case that x and y are on the same line.

Lemma 1 *If x and y have private neighbors and are on the same line, then all intervals on this line are in the maximum interval graph.*

For every other line, we try all pairs of intervals α and β on different sides of ω. Remove all intervals fully contained in the same half-line as α (β) with one endpoint closer to ω than both endpoints of α (β) and we remove also all neighbors of α and β in Ω. Maximize over α and β the number of remaining intervals on this line.
We now assume that the maximum interval graph has no two vertices x and y with private neighbors on the same line. Assume that x and y are vertices with private neighbors on

[2] It was shown in [2] that the DOMINATION problem can be solved in polynomial time for this class of graphs. For path graphs in general the DOMINATION problem remains NP-complete.

different lines. We take x and y such that one endpoint is as close as possible to ω (we maximize over all possible choices of pairs x and y). Hence the interval graph contains no interval p fully contained in the same half-line as x with one endpoint closer to ω than both endpoints of x.
Consider the line that contains x. Remove all lines fully contained in the half-line that contains x with one endpoint closer to ω than both endpoints of x. We try all vertices q on the other side of ω. Remove all intervals not containing ω on the same half-line as q with at least one endpoint closer to ω than both endpoints of q. Also remove those elements of Ω which are private neighbors of q with respect to x if x has a private neighbor with respect to q. Maximize over q the remaining number of vertices on this line. The same procedure is applied to the line that contains y.
For lines not containing x or y we take intervals α and β on different sides of ω. Remove all intervals not containing ω with one endpoint closer to ω than both endpoints of α (β). Remove also all neighbors of α and of β in Ω. Maximize over α and β the number of remaining vertices. (Notice that there may be vertices of Ω which are not removed.) This way we obtain:

Theorem 4 *There exists a polynomial time algorithm to solve the* MAXIMUM INTERVAL GRAPH *problem on the intersection graph of intervals on lines intersecting in one point.*

5 Conclusion

We have shown that finding an induced interval graph with a maximum number of vertices in a given graph G is NP-complete even for bipartite planar graphs and split graphs. We have proved that there exist efficient algorithms for graphs with bounded treewidth and for the intersection graphs of intervals on lines intersecting in one point. Also for tree-cographs and distance hereditary graphs there are efficient algorithms. Because of space limitations we have to omit the details. We conjecture that the problem can be solved efficiently for directed path graphs.

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A Reduced Non-blocking Routing Network

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Abstract

We will present a general construction scheme for a reduced routing network connecting a set L of vertices of order n to a set R of order N, where N is large compared to n. We require that n arbitrary connections can be routed concurrently. In terms of graph theory, if $L = \{l(1), ..., l(n)\}$, we require that for any n distinct elements $r(1), ..., r(n)$ of R there are vertex-disjoint paths $p(1), ..., p(n)$ such that $p(i)$ is a path from $l(i)$ to $r(i)$ for $i = 1, ..., n$. We will construct networks that are considerably smaller than the complete bipartite graph at the expense of only a single additional intermediate level: the size of this network is proportional to $\sqrt{N}$. Moreover, we construct networks that are particularly appropriate for VLSI implementations of shared-memory switches and we present an efficient construction of the control logic.

Application

Shared-memory switch architectures are well established as building blocks for high-speed routers and switches. We study a 3232 switch with 1024 packet locations in the shared memory. The CMOS implementation of the data section for such a switch consists of three major parts.

1. The shared memory is typically constructed with many 2-port memories, each storing a fixed-size packet;
2. the input router, and
3. the output router.

These routers connect any input to any memory, and any memory to any output. The input and output routers can take up to 70a good candidate for optimization. This is one example of applications in which a small set of n components is connected to a large set of

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N components in such a way that n arbitrary concurrent electrical connections can be built in each cycle. Other examples are multiprocessor designs that share common resources. These applications require a small data transfer latency, which can be translated into the requirement of a small number of intermediate levels between components.

The full paper can be found at: `www.meditrans.com/paper/koehl_luijten.pdf`

Discrepancy and Eigenvalues of Cayley Graphs (extended abstract)

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Abstract

We consider *quasirandom* properties for *Cayley graphs* of finite abelian groups. In particular, we show that having uniform edge-distribution (i.e., small discrepancy) and having large eigenvalue gap are equivalent properties for Cayley graphs, even if they are *sparse*. This positively answers a question of Chung and Graham ["Sparse quasi-random graphs", Combinatorica **22** (2002), no. 2, 217–244] for the particular case of Cayley graphs, while in general the answer is negative.

1 Introduction

Our aim here is to investigate certain aspects of a well known connection between the eigenvalue gap property and quasirandomness of graphs.

Let an n-vertex graph G be given. Recall that the eigenvalues of G are simply the eigenvalues of the n by n, 0–1 adjacency matrix of G, with 1 indicating edges. As usual, let $\lambda_k = \lambda_k(G)$ be the kth largest eigenvalue of G, in absolute value. Recall that G is said to be "quasirandom" if the edges of G are "uniformly distributed" (we postpone the precise definition, see Definition 1).

Thanks to the work of Tanner [14], Alon and Milman [3] and Alon [1] (see also Alon and Spencer [4, Chapter 9]) it is well known that a *gap* between the largest and the second largest eigenvalue of a graph G is related to the quasirandomness of G. Here, the concept of "quasirandomness" will be that of Chung, Graham, and Wilson [7].

Recall that [7] presents a "theory of quasirandomness" for *dense* graphs, exhibiting several, quite disparate almost sure properties of random graphs that are, quite surprisingly,

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equivalent in a deterministic sense. (Earlier work in this direction is due to Thomason [15] (see also [16]), and several other authors [1, 2, 8, 13].) One of the so-called "quasirandom properties" that is presented in [7] is the eigenvalue gap between λ_1 and λ_k ($k \geq 2$).

More recently, Chung and Graham [6] set out to investigate the extension of the results in [7] to *sparse graphs*, that is, graphs with vanishing edge-density. As it turns out, a naïve approach to such a project fails, as the results in [7] *do not* fully generalise to the "sparse case" in the expected manner (for a thorough discussion on this point, see [6] and also to [9, 11]).

On the other hand, some positive results have been established. In particular it was shown in [6] that a large eigenvalue gap implies uniform edge-distribution. Chung and Graham asked whether the converse also holds (see [6, p. 230]). An affirmative answer to this question would fully generalise the relationship between these two concepts to the sparse case.

However, Krivelevich and Sudakov [12] discovered that, unfortunately, the answer to the question posed by Chung and Graham is negative, by constructing a suitable family of counterexamples. Here, our aim is to show that *the answer is positive if one considers Cayley graphs of finite abelian groups, regardless of the density of the graph.* We leave the non-abelian case as an open question. It is worth noting that several explicit constructions of quasirandom graphs are indeed Cayley graphs (see, e.g., [16] and [12, Section 3]).

Before we proceed to state our result precisely, we mention that our proof method also sheds some light on the investigation of quasirandom subsets of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, in the spirit of Chung and Graham [5], in the *sparse case* (and for general abelian groups, as suggested in [5, p. 85]). We shall come back to this topic in the near future.

2 Statement of the main result

We use the following notation. If $G = (V, E)$ is a graph, we write $e(G)$ for the number of edges $|E|$ in G. If $U \subset V$ is a set of vertices of G, then $G[U]$ denotes the subgraph of G induced by U. Furthermore, if $W \subset V$ is disjoint from U, then we write $G[U, W]$ for the bipartite subgraph of G naturally induced by the pair (U, W). We also sometimes write $E(U, W) = E_G(U, W)$ for the edge set of $G[U, W]$. If $\delta > 0$, we write $x \sim_\delta y$ to mean that

$$(1 - \delta)y \leq x \leq (1 + \delta)y.$$

Definition 1 (DISC(δ)) *Let $0 < \delta \leq 1$ be given. We say that an n-vertex graph G ($n \geq 2$) satisfies property* DISC(δ) *if the following assertion holds: for all $U \subset V(G)$ with $|U| \geq \delta n$, we have*

$$e(G[U]) \sim_\delta e(G)\binom{|U|}{2} \Big/ \binom{n}{2}.$$

Given a graph G, let $\mathbf{A} = \mathbf{A}(G) = (a_{vv'})_{v,v' \in V(G)}$ be the 0–1 adjacency matrix of G, with 1 denoting edges. The *eigenvalues* of G are simply the eigenvalues of $\mathbf{A}$. Since $\mathbf{A}$

is symmetric, its eigenvalues are real. As usual, we adjust the notation so that these eigenvalues are such that

$$\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|. \tag{1}$$

Definition 2 (EIG(ε)) *Let $0 < \varepsilon \leq 1$ be given. We say that an n-vertex graph G satisfies property* EIG(ε) *if the following holds. Let $\bar{d} = \bar{d}(G) = 2e(G)/n$ be the average degree of G, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G, with the notation adjusted in such a way that* (1) *holds. Then*

(i) $\lambda_1 \sim_\varepsilon \bar{d}$,

(ii) $|\lambda_i| \leq \varepsilon\bar{d}$ *for all* $1 < i \leq n$.

Finally, we define Cayley graphs.

Definition 3 (Cayley graph $G = G(\Gamma, A)$) *Let Γ be an abelian group, and suppose $A \subset \Gamma \setminus \{0\}$ is symmetric, that is, $A = -A$. The Cayley graph $G = G(\Gamma, A)$ is defined to be the graph on Γ, with two vertices γ and $\gamma' \in \Gamma$ adjacent in G if and only if $\gamma' - \gamma \in A$.*

We only consider finite graphs and finite abelian groups.

The following theorem answers a question of Chung and Graham from [6] in the positive for the interesting class of Cayley graphs.

Theorem 1 *For any $\varepsilon > 0$, there are constants $\delta > 0$ and $n_0 \geq 1$ for which the following holds. If $G = G(\Gamma, A)$ is a Cayley graph for some abelian group Γ and symmetric set $A = -A \subseteq \Gamma \setminus \{0\}$, the number of vertices $n = |\Gamma|$ of G satisfies $n \geq n_0$, and G satisfies property* DISC(δ), *then G satisfies* EIG(ε).

The somewhat technical proof of Theorem 1 is given in the full version of this paper [10].

3 Remarks on the result

Let us discuss some points concerning Theorem 1 (for the proofs see [10]). We first observe that Theorem 1 together with the results of Chung and Graham [6] imply that properties DISC and EIG are equivalent for Cayley graphs. We say that DISC implies EIG for Cayley graphs if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that, for any sequence of n-vertex d_n-regular Cayley graphs G_n with d_n tending to infinity as $n \to \infty$, the following holds: *if all but finitely many graphs G_n satisfy* DISC(δ), *then all but finitely many G_n satisfy* EIG(ε). Theorem 1 tells us that DISC implies EIG for sequences of Cayley graphs. In [6] it is proved that EIG implies DISC in the same sense for sequences of arbitrary graphs with average degree tending to infinity.

Secondly, we note that in general it is not true that DISC implies EIG for arbitrary sequences of graphs. This was already pointed out by Krivelevich and Sudakov in [12]. For every $\varepsilon > 0$ and every $\delta > 0$, they constructed an infinite sequence of graphs that satisfy

DISC(δ) but fail to satisfy (i) in the definition of EIG(ε) (see Definition 2). A different construction to be presented in the full version of this paper gives additional control over a constant number of the largest eigenvalues.

At last, we wish to compare our main result, Theorem 1, with the earlier work of Chung and Graham [6]. Let us consider the following property.

Definition 4 (CIRCUIT$_t(\xi)$) *Let $0 < \xi \leq 1$ and an integer $t \geq 3$ be given. We say that an n-vertex graph G with average degree $\bar{d}(G)$ satisfies property* CIRCUIT$_t(\xi)$ *if the number of t-circuits C_t^* in G, i.e., closed walks of length t, satisfies*

$$\#\{C_t^* \hookrightarrow G\} \sim_\xi \bar{d}(G)^t\,.$$

Similarly to our Theorem 1, Theorem 6 in [6] establishes the implication DISC $\Rightarrow$ EIG. This implication is proved in [6] for arbitrary graphs under some additional conditions. These additional conditions, combined with DISC, also imply CIRCUIT$_{2\ell}$ for some $\ell > 1$. The following fact shows that Theorem 6 in [6] does not imply our main result, as it says that there are sequences of Cayley graphs that satisfy both DISC and EIG, but fail to meet CIRCUIT$_{2\ell}$ for every $\ell > 1$.

Fact 1 *There is an infinite sequence G_N of N-vertex Cayley graphs ($N \to \infty$) for which the following holds:*

(i) for every $\delta > 0$ all but finitely many graphs G_N satisfy DISC(δ)*,*

(ii) for every $\varepsilon > 0$ all but finitely many graphs G_N satisfy EIG(ε)*, and*

(iii) for every integer $\ell > 1$ and every $\xi > 0$ only finitely many graphs G_N satisfy CIRCUIT$_{2\ell}(\xi)$.

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Generalized covers and their approximations

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Set cover is one of the oldest and most studied NP-hard problems [9, 8, 10, 4, 4]. Given a ground set U of m elements, the goal is to cover U with the smallest possible number of sets. One of the best polynomial time algorithms for approximating *set cover* is the greedy algorithm: at each step choose the unused set which covers the largest number of remaining elements. Lovasz [10] and Johnson [8] showed that the performance ratio of the greedy algorithm is no worse than $H(m)$, where $H(m) = 1 + 1/2 + \ldots + 1/m$ is the mth harmonic number, a value which is clearly between $\ln m$ and $1 + \ln m$. Similar results were obtained in [11, 14]. Slavik [16] showed that the performance ratio of the greedy algorithm is exactly $\ln m - \ln \ln m + \Theta(1)$. Feige [5] proved that for any $\varepsilon > 0$ no polynomial time algorithm can approximate *set cover* within $(1-\varepsilon)\ln m$ unless $NP \subset DTIME[n^{O(\log\log n)}]$.

It is well-known that *set cover* forms an important class of integer programs

$$\{\min \mathbf{cx} |\ A\mathbf{x} \geq \mathbf{b}, \quad \mathbf{x} \in \{0,1\}^n\}, \tag{1}$$

where $\mathbf{c} = (1, ..., 1)$, $\mathbf{b} = (1, ..., 1)^T$ and $A = (a_{ij})$ is an arbitrary $m \times n$ $(0,1)$-matrix.

In fact, covering problems with restrictions on the number of 1's in each row and column in $(0,1)$-matrices appeared often [6]. It is known that for any (0,1)-matrix of size $m \times n$ with at least k 1's in each row the size of the minimum cover $C(A)$ satisfies

$$C(A) \leq \frac{n}{k}(1 + \ln \frac{mk}{n}). \tag{2}$$

Better bounds were obtained under more restrictive conditions [13, 7, 2, 12]. On the other side, it is interesting to explore how far general bounds like [10, 8] or (2) can be extended to integer programs (IPs). Chvatal [4] extended the results [10, 8] to the weighted version of set cover, e.g. for IPs (1) with $\mathbf{c} = (c_1, ..., c_n)$. Then general bounds were obtained for so-called covering IPs, e.g. programs where all elements of A, $\mathbf{b}$, and $\mathbf{c}$ are nonnegative [15].

We address a generalization of *set cover* where the matrix $A = (a_{ij})$ in (1) is allowed to be $(0, \pm 1)$-matrix, e.g. $a_{ij} \in \{-1, 0, 1\}$ and $\mathbf{c} = (1, ..., 1)$, $\mathbf{b} = (1, ..., 1)^T$. Let us denote the

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optimum value of the objective function in (1) with this condition by $C(A)$. Let m, n, k, t be nonnegative integers, $k > t$, and let $R(m, n, k, t)$ denote the class of all matrices of size $m \times n$ with elements $-1, 0, +1$ where each matrix satisfies the following conditions: the number of $+1$'s in each row is at least k and the number of -1's in each row is at most t. Our main results are the following.

Theorem 1. For any matrix $A \in R(m, n, k, t)$

$$C(A) \leq c\frac{n}{k-t} \cdot \frac{k+t}{k-t} \cdot \ln(m+1), \tag{3}$$

where $c > 0$ is some absolute constant.

Note, that Theorem 1 gives non-trivial upper bound for the problem when

$$c\frac{k+t}{(k-t)^2} \cdot \ln(m+1) < 1, \tag{4}$$

because otherwise trivial bound $C(A) \leq n$ is better. Next theorem shows that bound (3) is tight for some range of parameters.

Theorem 2. Let $n = c'm$ for some constant c', and let k and t satisfy the conditions $(n \to \infty)$:
1) $k = o(n)$, 2) $k - t = o(k)$, 3) $\frac{k+t}{(k-t)^2} = o(m^{-\delta})$,
for some constant $\delta > 0$. Then there is an absolute constant $c_1 > 0$ such that for sufficiently large n there is a matrix $A \in R(m, n, k, t)$ with

$$C(A) \geq c_1\frac{n}{k-t} \cdot \frac{k}{k-t} \cdot \ln m. \tag{5}$$

Example. Let $n = m$, $k = m^{1/3}$, $t = m^{1/3} - m^{1/4}$. The conditions 1 and 2 of Theorem 2 obviously hold. The left hand side of the condition 3 is: $\frac{k+t}{(k-t)^2} < \frac{2m^{1/3}}{m^{1/2}} = O(m^{-1/6})$, all the conditions of Theorem 2 hold and

$$C(A) \geq c\frac{m}{m^{1/4}} \cdot \frac{m^{1/3}}{m^{1/4}} \cdot \ln m = cm^{5/6} \ln m.$$

We give only the sketch of the proof of Theorem 1. It is based on upper bounds for probabilities of large deviations of sums of independent random variables. Consider fractional feasible solution to (1) $\mathbf{x} = (\frac{1}{k-t}, \ldots, \frac{1}{k-t})$ and multiply it by some factor K. Clearly, a new vector $(\frac{K}{k-t}, \ldots, \frac{K}{k-t})$ is a feasible solution of the same problem. In fact, we choose $K = 4(e-2)(k+t)/(k-t)\ln(m+1)$ and assume that the condition (4) holds (with $c = 4(e-2)$). This implies $K \leq k - t$.

Then define a random vector $\mathbf{z}$ by considering each $K/(k-t)$ as the probability p rounding $K/(k-t)$ to 1 with probability p, and to 0 with probability $1-p$. To show that the random vector $\mathbf{z}$ is a feasible solution to (1) we use the following lemma.

Lemma 1. Let x_i, $i = 1, \ldots, n$, be independent random variables taking values a_i, $|a_i| \leq 1$, and 0 with probabilities p_i and $1 - p_i$, respectively. Let $X = \sum_{i=1}^n x_i$, $EX = \sum_{i=1}^n a_i p_i > 0$, and $d > 0$ be such that $EX^2 \leq dEX$. Then for any $0 < \delta < 2d(e-2)$

$$P\{X - EX < -\delta EX\} \leq \exp\{-\frac{\delta^2 EX}{4d(e-2)}\}.$$

We have, $p = \frac{K}{k-t}$, and the event, that a constraint is violated by $\mathbf{z}$ can be written as

$$P\{X \le 0\} = P\{X - EX < -\delta EX\},$$

where $\delta = 1$, $X = \sum_{j=1}^{n} a_{ij} z_j$. Clearly, $EX = p(k-t) = K$. To apply Lemma 1 we need to estimate d. We have $\sum_{i=1}^{n} a_i^2 p_i = p(k+t)$. The inequality $p(k+t) \le dp(k-t)$ implies $d \ge \frac{k+t}{k-t}$, so we can choose $d = \frac{k+t}{k-t}$. By Lemma 1 (with $\delta = 1$)

$$P\{X \le 0\} \le \exp\{-\frac{\delta^2 EX}{4d(e-2)}\} = \exp\{-\frac{(k-t)K}{4(e-2)(k+t)}\} =$$

$$= \exp\{-\ln(m+1)\} = \frac{1}{m+1}.$$

It is easy to see now that the random vector $\mathbf{z}$ is feasible and the value of the objective function is at most

$$(1+o(1))4(e-2)\frac{n}{k-t} \cdot \frac{k+t}{k-t} \cdot \ln(m+1).$$

To prove Theorem 2 we prove that under the conditions of Theorem 2 for almost all $(0, \pm 1)$-matrices having exactly k 1's in each row and exactly t elements equal to -1 in each row the inequality (4) holds. Picking such matrices at random with equal probabilities we can consider elements a_{ij} as random variables. It is not difficult to check that $P\{a_{ij} = 1\} = k/n$, $P\{a_{ij} = -1\} = t/n$, $P\{a_{ij} = 0\} = 1 - (k+t)/n$. Moreover, the random variables a_{ij}, $i = 1, \ldots, m$ are independent (for each j fixed).

Fixing a subset consisting of l columns and a row, let Y be the random variable equal to the number of $+1$'s in the intersection of the row with these l columns, and let Z be the random variable equal to the number of -1's in the same intersection. Let $X(l)$ be the random variable equal to the number of covers of size l (the number of feasible solutions with exactly l components equal to 1). Our aim is to estimate the expectation of $X(l)$ and to show that it tends to 0 when $l = c_1 \frac{n}{k-t} \cdot \frac{k}{k-t} \cdot \ln m$ and $n \to \infty$. We have

$$EX(l) = \binom{n}{l} (1 - P\{Z \ge Y\})^m .$$

To complete the proof of Theorem 2 it is sufficient to find an appropriate lower bound for $P\{Z \ge Y\}$ which is the main technical point of the proof.

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Many Isomorphism Types of 6- and 7-Designs

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A simple t-(v, k, λ) design consists of a point set V of size v and a set of blocks $\mathcal{B}$ where each block is a subset of k points and each subset of t points lies in exactly λ blocks.

Until the discovery of 6-$(33, 8, 36)$ designs by Magliveras and Leavitt in the early 80'th [15], it was suspected that simple t-designs might exist only for $t \leq 5$. The big surprise then was the famous Theorem by Teirlinck in 1987 [17] showing that simple t-designs exist for all t. Teirlinck's method is recursive, called large set recursion, and produces point sets of astronomical size.

Since then, successful efforts have been made to directly construct t-designs with "large" t for small point sets and to refine the recursive methods to also reduce the size of the point sets, notably by Ajoodani-Namini [1]. Directly constructed large sets of t-designs are still needed as starting points for recursions. Several t-designs have been constructed by A. Betten, A. Wassermann and the author with their software package DISCRETA for $6 \leq t \leq 9$[3, 4, 2, 6, 13] and further ones are presented here.

We solve the isomorphism problem for directly constructed t-designs with the following theorem.

Theorem 1[16, 8, 13] *Let $G = S_V$ be the full symmetric group on the point set V. Let $A \leq G$ be a group of automorphisms of each t-(v, k, λ) design in a set Ω.*

a) If A is the full automorphism group of each of the designs in Ω then two of these designs are isomorphic only if some permutation in the normalizer $N_G(A)$ of A maps one design onto the other. If Ω then consists of all t-(v, k, λ) designs with full automorphism group A then the number of different isomorphism types in Ω is $\frac{|\Omega|}{|N_G(A)|/|A|}$.

b) If A contains a Sylow subgroup P of the full automorphism group of each of the designs in Ω then two of these designs are isomorphic only if some permutation in $N_G(P)$ maps one design onto the other.

van Leijenhorst [14] and Tran van Trung [18] construct a new t-design (TvT-construction) from any pair of t-designs whose parameter sets are those of a derived and a residual design from the same parameter set. One can replace one of the two designs by isomorphic copies to obtain further designs with the same parameters. Using double cosets in the full symmetric group on the point set we obtain bounds on the number of isomorphism types from this construction, see [13].

Theorem 2 *If there are n_1 isomorphism types of t-$(v,k-1,\lambda)$ designs and n_2 isomorphism types of t-$(v,k,\lambda(v-k+1)/(k-t))$ designs with automorphism groups of order at most a then there exist at least*

$$\frac{n_1 \cdot n_2 \cdot v!}{(v+1)a^2}$$

isomorphism types of t-$(v+1,k,\lambda(v-t+1)/(k-t))$ designs with automorphism group of order at most $a(v+1)$.

We have constructed new 6-designs and 7-designs and applied Theorems 1 and 2, see the table which also reports the known 6-designs up to 20 points. A $G+$ denotes a group with an added fixed point. A $G-$ denotes a point stabilizer. It should be noted that up to now no 6-design has been found where λ is less than 4. The only 6-designs on less than 19 points known are those listed on 14 points. A 6-$(19,7,5)$ design and any 6-$(19,8,\lambda)$ design are still missing. The TvT-construction is applied iteratively to the 7-designs on 24 points, the previously known 7-$(24,8,6)$, 7-$(24,9,48)$, and 7-$(25,9,54)$ designs [13], and their complementary designs.

The latter parameter sets belong to a *family* of parameter sets that are all obtainable from a unique formally admissible parameter set by computing the parameters of derived designs, residual designs, and regarding a t-design as a s-design for some $s \leq t$. This source of parameter sets itself is neither residual nor derived nor a t-design for some larger t and is called the *ancestor* of the family.

Theorem 3 *Each parameter set of a non-trivial design belongs to a finite family with a unique ancestor.*

For all parameter sets of 7-designs in the family of 15-$(32,16,6)$ we could establish the existence of such designs. We present these parameter sets in a table where a lower bound for the number of isomorphism types is given below each parameter set. No 8-design is known with parameters from this family. For further families see [5].

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Table of 6-Designs and some 7-Designs

Parameters	*Origin*	*No of Isomorphism Types*	*Reference*
$6-(14,7,4)$	$C_{13}+$	2	[11]
	A_4	1	[12]
	C_3	4	[9]
	C_7	2	[9]
	C_2	4	[9]
$6-(19,7,4)$	$Hol(C_{17})++$	1	[3]
$6-(19,7,6)$	$Hol(C_{19})$	3	[3]
$6-(19,9,126)$	$(\mathbf{PGL(2,8)\times C_2})+$	4	
$6-(19,9,134)$	$(\mathbf{PSL(2,9)\times C_2})-$	1	
$6-(20,9,112)$	$PGL(2,19)$	2	[10]
	$\mathbf{PGL(2,9)\times C_2}$	98	
	$\mathbf{PSL(3,4)}-$	19892	
$6-(20,10,336)$	$\mathbf{PSL(3,4)}-$	≥ 683616	
$6-(20,10,406)$	$7-(20,10,116)$		[3]
	$PSL(2,19)$	7	
	TvT construction with complementary design from 6-(19,9,116)	≥ 13199346316	
$6-(20,10,434)$	$7-(20,10,124)$		[3]
	$PSL(2,19)$	1	[3]
	TvT construction with complementary design from 6-(19,9,124)	≥ 13199346316	
$6-(20,10,441)$	$7-(20,10,126)$		
	$\mathbf{PGL(2,8)\times C_2}++$	4	
	TvT construction with complementary design from 6-(19,9,126)	≥ 13199346316	
$6-(20,10,469)$	$7-(20,10,134)$		
	$PSL(2,19)$	56	[3]
	TvT construction with complementary design from 6-(19,9,134)	≥ 13199346316	
$6-(20,10,476)$	$\mathbf{PSL(3,4)}-$	≥ 20000000	
$7-(20,10,126)$	$\mathbf{PGL(2,8)\times C_2}++$	4	
$7-(20,10,134)$	$\mathbf{PGL(2,9)\times C_2}$	1	
$7-(24,11,840)$	$\mathbf{PGL(2,23)}$	≥ 155	
$7-(24,12,2184)$	$\mathbf{PGL(2,23)}$	≥ 28500	

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Parameter Sets of 7-Designs in the Family of 15-(32, 16, 6)

7-(32,16,721050)
>= 1.5E4165

7-(31,15,259578) 7-(31,16,461472)
>= 4.0E2080 >= 4.0E2080

7-(30,14,86526) 7-(30,15,173052) 7-(30,16,288420)
>= 9.7E1035 >= 1.3E1039 >= 9.7E1035

7-(29,13,26334) 7-(29,14,60192) 7-(29,15,112860) 7-(29,16,175560)
>= 3.0E511 >= 3.8E514 >= 3.8E514 >= 3.0E511

7-(28,12,7182) 7-(28,13,19152) 7-(28,14,41040) 7-(28,15,71820) 7-(28,16,103740)
>= 3.9E252 >= 2.8E250 >= 5.0E255 >= 2.8E250 >= 3.9E252

7-(27,11,1710) 7-(27,12,5472) 7-(27,13,13680) 7-(27,14,27360) 7-(27,15,44460) 7-(27,16,59280)
>= 1.5E125 >= 6.2E117 >= 1.1E123 >= 1.1E123 >= 6.2E117 >= 1.5E125

7-(26,10,342) 7-(26,11,1368) 7-(26,12,4104) 7-(26,13,9576) 7-(26,14,17784) 7-(26,15,26676) 7-(26,16,32604)
>= 1E62 >= 6.6E51 >= 4.0E54 >= 1.2E57 >= 4.0E54 >= 6.6E51 >= 1E62

7-(25,9,54) 7-(25,10,288) 7-(25,11,1080) 7-(25,12,3024) 7-(25,13,6552) 7-(25,14,11232) 7-(25,15,15444) 7-(25,16,17160)
>= 1.2E20 >= 2.3E19 >= 4.5E19 >= 1.4E22 >= 1.4E22 >= 4.5E19 >= 2.3E19 >= 1.2E20

7-(24,8,6) 7-(24,9,48) 7-(24,10,240) 7-(24,11,840) 7-(24,12,2184) 7-(24,13,4368) 7-(24,14,6864) 7-(24,15,8580)
>= 132 >= 2827 >= 91 >= 155 >= 28500 >= 155 >= 91 >= 2827

7-(24,16,8580)
>= 132

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Minimum tree decompositions in simple graphs

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Let $G = (V, E)$ be a connected simple graph. An *edge decomposition* of G is a family of subgraphs $G_1, \dots, G_k$ whose sets of edges partition E. We write

$$G = G_1 \oplus \cdots \oplus G_k.$$

When each subgraph G_i is acyclic we have a *forest decomposition*. The *arboricity* $a(G)$ is the minimum number of forests in a forest decomposition of G. When each forest is connected, we have a *tree decomposition*. The minimum number of trees in a tree decomposition of G is denoted by $\tau(G)$. Since each forest on n vertices has at most $n-1$ edges, $a_0(G) = \lceil |E|/(|V|-1) \rceil$ is a trivial lower bound for both the arboricity and $\tau(G)$.

There are several classes of graphs for which $\tau(G)$ attains its minimum value $a_0(G)$. Kampen [3] proved that maximal planar (mp) graphs can be decomposed into 3 edge-disjoint trees, therefore we have $\tau(G) = a_0(G) = 3$. Ringel [11] proved that maximal planar bipartite (mpb) graphs verify, $\tau(G) = a_0(G) = 2$. Shi, Li and Tian introduce in [8] a class of graphs with uniform edge-density: $G = (V, E)$ is called a $\mathcal{P}_k$-*graph*, if $|V| \geq 3$ and $|E| = k(|V| - 2)$ and for any subgraph of G, $H = (V', E')$ with $|V'| \geq 3$, $|E'| = k(|V'| - 2)$. In particular $\mathcal{P}_2$ contains the mpb graphs and the mp graphs are contained in $\mathcal{P}_3$. These authors proved that the equality $a_0(G) = \tau(G)$ also holds for a graph G in $\mathcal{P}_2 \cup \mathcal{P}_3$.

Chung [1] obtained the nontrivial upper bound, $\tau(G) \leq \lceil |V|/2 \rceil$, for connected graphs with no multiple edges. Thus, for complete graphs, $\tau(K_n) = a_0(K_n) = \lceil n/2 \rceil$. Truszczyński [12] showed that the equality $a_0(G) = \tau(G)$ also holds for complete bipartite graphs and hypercubes.

For regular graphs of even degree and maximum edge-connectivity it is shown in [5] that the above equality holds. When the degree is odd, the use of higher measures of edge-connectivity gives also the result, [6]. In [7] it is proved that any graph G of order n and minimum degree $\delta \geq (n-1)/2$ has a decomposition in $a_0(G)$ trees.

Here we are interested in the structure of the trees involved in a tree decomposition. We say that a tree decomposition $G = T_1 \oplus \cdots \oplus T_k$ is of type $(a_1, \dots, a_k)$ if T_i has $n - a_i$ vertices, $i = 1, \dots, k$, where n is the order of G.

Ringel [11] conjectured that in mpb graphs the two possible kinds, $(1, 1)$ and $(0, 2)$, of decompositions in two trees exist. The conjecture was proved by Ouyang and Liu [10]. A decomposition into three trees of a mp graph can be of type $(1,1,1)$, $(0,1,2)$ or $(0,0,3)$. Shi, Li and Tian [8] proved that all these types of decompositions exist for any graph in $\mathcal{P}_3$. For regular graphs, the minimum tree decompositions obtained in [5, 6] are of type $(0,\ldots,0,l)$.

We show that arbitrary trees may appear in different kinds of minimum tree decompositions (*MTD*). We consider mpb graphs, mp graphs and regular graphs.

We denote by $P_m^{r,s}$ the tree obtained from a path P_m with m edges by adding $r \geq 1$ leaves to one end vertex and $s \geq 1$ leaves to the other one. When $r = s$ we write P_m^r.

Theorem 1 *Let T be an arbitrary tree. There is a maximal planar bipartite graph G which admits T as a factor in a MTD of type (0,2) or (1,1). Moreover, T can be chosen to be the spanning tree in a decomposition of type $(0,2)$ if and only if T is neither a star S_m, $m \geq 2$, nor a $P_2^{r,s}$-tree, $r, s \geq 1$.*

Proof. The proof is by induction on the order n of the given tree T. Let T be a given tree of order $n > 4$. Let us denote by $\mathcal{F}$ the graphs which are either stars S_m or $P_2^{r,s}$-trees, $r, s \geq 1$.

Let l be an end vertex in $T \neq P_5$ such that $T' = T - l \notin \mathcal{F}$, or any end vertex if $T \in \mathcal{F}$, and let vl be the edge of T incident to l. By the induction hypothesis, there is a mpb graph G' which admits T' as a factor in a minimum tree decomposition of both types, unless $T' \in \mathcal{F}$.

Let R be the set of faces incident to a vertex v in a planar embedding of G', and let U be the set of vertices incident to the faces in R in the same chromatic class as v in the bipartition of G'.

Let $G' = T_1' \oplus T_2'$ be a tree decomposition. Suppose that there is a vertex $u \in U \cap V(T_2')$. Then, we obtain a mpb graph G which admits a decomposition of the same type as the one of G'. Suppose now that $U \cap V(T_2') = \emptyset$. This implies that $T = P_2^{1,t-1}$ and we can obtain a mpb graph G which admits a decomposition of type (1,1).

If $T = P_5$, Figure 1 shows mpb graphs admitting a path of five edges P_5, in any type of *MTD*. $\square$

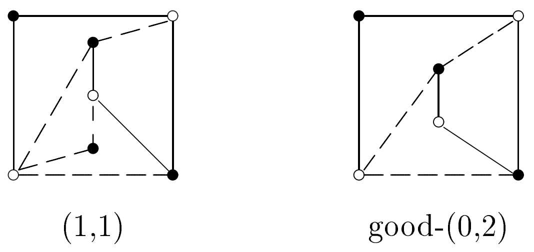

Figure 1: mpb graphs admiting P_5 as a factor.

By induction on the order of T we prove that every tree is a factor of a mp graph in a decomposition of a chosen type.

Theorem 2 *Let T be an arbitrary tree. There is a maximal planar graph G which admits T as a factor in a MTD of type (a,b,c) for any choice of $0 \leq a \leq b \leq c$ with $a+b+c=3$. Moreover, T can be chosen to be a spanning tree in a decomposition of type $(0,0,3)$ if T is not a star S_m, $m \geq 2$.* □

Finally we consider the problem for regular graphs. By using the Erdös-Gallai characterization of graphical sequences and a result by Kleitman and Wang [4] about the existence of graphs which admit a certain number of edge-disjoint spanning trees, we show that every tree can appear in a *MTD* of regular graphs, provided that the natural necessary conditions hold.

Theorem 3 *Let $d \geq 2$ be an integer and let T be a tree of order $n \geq d+1$ with nd even and maximum degree $\Delta(T) \leq \lceil \frac{d}{2} \rceil + 1$. Then, there is a regular graph G of order n and degree d such that*
$$G = T \oplus T_2 \oplus \cdots \oplus T_k \oplus T_{k+1},\ k = \lfloor d/2 \rfloor$$
is a MTD of type $(0,\ldots,0,l)$, $l = \frac{(2-\epsilon)}{2}n - k - 1$, $\epsilon \equiv d \pmod 2$, except if d is odd and either $d = n-3$ and $T = P_1^{k+1}$ or $d = n-1$ with $\Delta(T) = k+2$.

Proof. The strategy of the proof is to show that the tree T together with an appropriate T_{k+1} can be packed in a d-regular graph G of order n together with another graph of order n, G_1, which has $k-1$ edge disjoint trees. The existence of graph G_1 is guaranteed by the two theorems above-named, while as for the existence of T_{k+1} we introduce the following lemma.

Lemma 1 *Let T be a tree of order n and $k \geq 2$ an integer.*

(i) If $n \geq 2k+1$ and $\Delta(T) \leq k+1$ then there is a packing of T with a path P of order $k+1$ such that $\Delta(T \oplus P) \leq k+1$.

(ii) For even n, if either $n \geq 2k+4$, $\Delta(T) \leq k+2$ and $T \neq P_1^{k+1}$ or $n = 2k+2$ and $\Delta(T) \leq k+1$ there exists a packing of T with a path P of order $n/2+k+1$ such that $\Delta(T \oplus P) \leq k+2$. □

Note that, as a consequence of Theorem 3, we have the following corollary.

Corollary 1 *Every spanning tree of order n with maximim degree $\Delta(T) \leq \lfloor \frac{n-1}{2} \rfloor + 1$ can be extended to a MTD in a complete graph K_n.* □

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A polyhedral approach to even factors

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Abstract

Generalizing path-matchings, W.H. Cunningham and J.F. Geelen introduced the even factors. In weakly symmetric directed graphs they proved a min-max formula for the maximum even factor by algebraic method and also dealt with the weighted case. Later, Gy. Pap and L. Szegő proved a simplified formula by purely combinatorial method. We consider polyhedra related to even factors, and prove integrality and total dual integrality, without using earlier unweighted results.

1 Introduction

Cunningham and Geelen [1] introduced the notion of *even factor* in directed graphs (*digraphs*) as the edge set of vertex disjoint union of directed paths (*dipaths*) and even directed cycles (*dicycles*). The maximum cardinality even factor problem is NP-hard in general but there are special classes of digraphs where it can be solved.

We consider digraphs with no loops and with no parallel edges, but oppositely directed edges are permitted. An edge of a digraph is called *symmetric*, if the reserved edge is in the edge set of the digraph, too. A digraph is symmetric if all its edges are symmetric, while a digraph is called *weakly symmetric* if its strongly connected components are symmetric. Then weak symmetry means that the edges contained in dicycle are symmetric. Pap and Szegő [5] introduced the more general class of *hardly symmetric* digraphs where the problem remains tractable. A digraph is said to be hardly symmetric if the edges contained in odd dicycle are symmetric.

Using an algebraic approach, Cunningham and Geelen [1] proved a min-max formula for the maximum cardinality even factor problem in weakly symmetric digraphs. Later, Pap and Szegő [5] proved a simpler formula for the same problem with purely combinatorial methods and they also observed that the same proof and formula works for the hardly symmetric case.

As the unweighted problem is tractable only in special graphs, the weight function also has to possess certain symmetries. If $G' = (V', A')$ is a weakly symmetric graph and

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$c' : A' \to \mathbb{R}$ is a weight function s.t. $c'(uv) = c'(vu)$ if $uv, vu \in A'$, then the *pair* (G', c') is called *weakly symmetric*. Cunningham and Geelen [1] considered weighted generalization for weakly symmetric pairs.

Here we deal with a more general class of digraphs and weight functions which turn out to be general enough to contain the mentioned cardinality and weighted cases. We prove integrality of polyhedra related to even factors and total dual integrality, without using earlier unweighted results, which rather follow as consequences. This approach enables us to derive weighted min-max formulas, nevertheless we omit to present them explicitly, since these are direct consequences of the polyhedral theorems.

2 Preliminaries and results

In a graph, let uv and $\{u, v\}$ denote respectively the directed edge with tail u and head v, and the undirected edge with endvertices u and v. If $G = (V, E = E_\mathrm{d} \cup E_\mathrm{u})$ is a *mixed graph*, E_d and E_u denote respectively the set of its directed edges and the set of its undirected edges and $y \in \mathbb{R}^E$, then for $u \in V$ and $S \subseteq V$ we will use the notations
$d_y(u) = \sum_{\{u,v\}\in E_\mathrm{u}} y(\{u, v\}) + \sum_{uv\in E_\mathrm{d}} y(uv) + \sum_{vu\in E_\mathrm{d}} y(vu)$, $i_y(S) =$
$\sum \{y(\{u, v\}) : \{u, v\} \in E_\mathrm{u},\ u, v \in S\} + \sum \{y(uv) : uv \in E_\mathrm{d},\ u, v \in S\}$,
$\varrho_y(S) = \sum\{y(uv) : uv \in E_\mathrm{d}, u \notin S, v \in S\}$ and $\delta_y(S) = \varrho_y(V - S)$.

For any digraph $G = (V, A)$, we can consider the following system

$$x \in \mathbb{R}^A,\ x \geq 0 \tag{1}$$
$$\varrho_x(v) \leq 1 \quad \text{for every } v \in V \tag{2}$$
$$\delta_x(v) \leq 1 \quad \text{for every } v \in V \tag{3}$$
$$i_x(S) \leq |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{C}(G) \tag{4}$$

whose solution set is denoted by $\mathrm{EF}(G)$ and $\mathcal{C}(G)$ denotes the set of vertex sets of the strongly connected components of G. We can observe that the integer solutions of $\mathrm{EF}(G)$ are the even factors of G. However, this polyhedron is not integer. Next, we define a class of graphs and weight functions where we can state that $\max\{cx : x \in \mathrm{EF}(G)\}$ is attained on an integer vector.

For a digraph G', $\overline{G'}$ denotes its underlying undirected graph. If $U \in \mathcal{C}(G)$ and $|U| \geq 2$, then let $\mathcal{D}(U)$ denote the collection of maximal sets $S \subseteq U$ s.t. $G[S]$ is strongly connected and $\overline{G[S]}$ is 2-vertex-connected, where $G[S]$ denotes the subgraph of G induced by S. If $U \in \mathcal{C}(G)$ and $|U| = 1$, then let $\mathcal{D}(U) = U$. Then for $S, T \in \mathcal{D}(U)$, $|S \cap T| \leq 1$, and the hypergraph $(U, \mathcal{D}(U))$ does not contain undirected cycle. A digraph G' is called bipartite if $\overline{G'}$ is bipartite. Throughout the paper, we deal with digraphs s.t. for any $U \in \mathcal{C}(G)$, $S \in \mathcal{D}(U)$, $G[S]$ is symmetric or bipartite. We consider families $\mathcal{S}, \mathcal{B} \subseteq 2^V$ s.t. $\mathcal{S} \cap \mathcal{B} = \emptyset$, $\mathcal{S} \cup \mathcal{B} = \cup_{U\in\mathcal{C}(G)}\mathcal{D}(U)$, for any $S \in \mathcal{S}$, $G[S]$ is symmetric and for any $S \in \mathcal{B}$, $G[S]$ is bipartite. A digraph is called *evenly symmetric* if there exist families satisfying the above criteria. If $G = (V, A)$ is an evenly symmetric graph and $c : A \to \mathbb{R}$ is a weight function s.t. $c(uv) = c(vu)$ if $uv, vu \in A$ and $\exists S \in \mathcal{S}$ s.t. $u, v \in S$, then the *pair* (G, c) is said to be *evenly symmetric*. Let $\mathcal{S}^\star$ be the family of maximal members of $\{T : \exists \mathcal{T} \subseteq \mathcal{S}, T = \cup\mathcal{T}$ s.t. the hypergraph $(T, \mathcal{T})$ is connected$\}$. It is easy to see that in the system defining $\mathrm{EF}(G)$, we can change (4) to

$$i_x(S) \le |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}^\star. \tag{5}$$

The solution set of (1), (2), (3), (4) and the solution set of (1), (2), (3), (5) equals. Moreover, with respect to some objective function, the two systems have integer optimal dual solutions at the same time. A family $\mathcal{F}$ of subsets of V is said to be *laminar* if for any $X, Y \in \mathcal{F}$, at least one of $X \subseteq Y$, $Y \subseteq X$ or $X \cap Y = \emptyset$ holds. Now we are ready to state a result on dual integrality.

Theorem 1 *If (G, c) is evenly symmetric and c is an integer vector, then the dual of the linear program* $\max\{cx : (1), (2), (3), (5)\}$ *has integer optimal solution $(\lambda_v^\varrho, \lambda_v^\delta, z_S)$. Moreover, there is one s.t. $\{S : z_S > 0\}$ is laminar.*

To deal with primal integrality and to exploit the theory of TDI systems, the symmetric edges spanned by the members of $\mathcal{S}$ are considered as undirected edges and we take the *mixed graph* with vertex set V and edge set $E = E_\mathrm{d} \cup E_\mathrm{u}$, $E_\mathrm{u} = \{\{u, v\} : uv, vu \in A,\ u, v \in U \text{ for some } U \in \mathcal{S}\}$, $E_\mathrm{d} = \{uv \in A : \{u, v\} \notin E_\mathrm{u}\}$. We need the polyhedral description of $\mathrm{SEF}(G)$ defined by

$$\left\{ y \in \mathbb{R}^E : \exists x \in \mathrm{EF}(G),\ y(e) = \left\{ \begin{array}{rl} x(uv) + x(vu) & \text{if } e = \{u, v\} \in E_\mathrm{u} \\ x(uv) & \text{if } e = uv \in E_\mathrm{d} \end{array} \right\} \right\}.$$

Theorem 2 *For the evenly symmetric digraph $G = (V, A)$,* $\mathrm{SEF}(G)$ *is the solution set of*

$$y \in \mathbb{R}^E,\ y \ge 0 \tag{6}$$
$$d_y(v) \le 2 \quad \text{for every } v \in V \tag{7}$$
$$i_y(S) \le |S| - 1 \quad \text{for every } S \subseteq U,\ |S| \text{ odd},\ U \in \mathcal{S}^\star \tag{8}$$
$$i_y(S) + \varrho_y(S) \le |S| \quad \text{for every } S \in \mathcal{U} \tag{9}$$
$$i_y(S) + \delta_y(S) \le |S| \quad \text{for every } S \in \mathcal{U} \tag{10}$$

where $\mathcal{U} = \{S : \exists U \in \mathcal{S}^\star \text{ s.t. } S \subseteq U\} \cup \{v \in V : v \notin \cup\{S : S \in \mathcal{S}^\star\}\}$.

Observe that, if $|\mathcal{C}(G)| = 1$, then this specializes to $2\mathrm{M}(G)$ where $\mathrm{M}(G)$ is the *matching polyhedron* defined by

$$\left\{ y \in \mathbb{R}^E : y \ge 0, \quad d_y(v) \le 1\ \forall v \in V, \quad i_y(S) \le \tfrac{|S|-1}{2}\ \forall S \subseteq V,\ |S| \text{ odd} \right\}.$$

This system was proved to be a TDI system by Cunningham and Marsh [3].

Another well-known special case is when the strongly connected components are bipartite. Then $\left\{ y \in \mathbb{R}^E : y \ge 0, \quad \varrho_y(v) \le 1,\ \delta_y(v) \le 1\ \forall v \in V \right\}$ is obtained, which is TDI. Also, we can see easily that any integer solution of this system is a vertex disjoint union of dipaths and even dicycles.

A more general special case is the *path-matching polyhedron* which was introduced by Cunningham and Geelen. For a detailed description, the interested reader is referred to the paper of Cunningham and Geelen [2] and Frank and Szegő [4]. Here, V is the disjoint union of T_1, R and T_2, T_1 and T_2 are stable sets, $G[R]$ is symmetric, no edge enters T_1 and no edge leaves T_2. System (6), (7), (8), (9), (10) defined on this graph specializes to

$$\begin{aligned} y \in \mathbb{R}^E,\ y \geq 0 & \\ d_y(v) \leq 2 & \quad \text{for every } v \in R \\ i_y(S) \leq |S| - 1 & \quad \text{for every } S \subseteq R,\ |S| \text{ odd} \\ i_y(S) + \varrho_y(S) \leq |S|,\ i_y(S) + \delta_y(S) \leq |S| & \quad \text{for every } S \subseteq R \\ \varrho_y(v) \leq 1,\ \delta_y(v) \leq 1 & \quad \text{for every } v \in T_2. \end{aligned}$$

which is called path-matching polyhedron. This system was proved to be TDI by Cunningham and Geelen [2]. Our main theorem is the following.

Theorem 3 *If $G = (V, A)$ is evenly symmetric, then (6), (7), (8), (9), (10) is TDI. In addition, if $c \in \mathbb{Z}^E$, then there is an integer optimal solution $(\mu_v^d, \mu_S^i, \mu_S^{i,\varrho}, \mu_S^{i,\delta})$ of the dual of* $\max\{cy : (6), (7), (8), (9), (10)\}$ *s.t.*

$\mathcal{G} = \{S : \mu_S^i > 0\} \cup \{S : \mu_S^{i,\varrho} > 0\} \cup \{S : \mu_S^{i,\delta} > 0\}$ is a laminar family,
if $\mu_S^i > 0$, $\mu_T^{i,\varrho} > 0$ or $\mu_T^{i,\delta} > 0$, $S \cap T \neq \emptyset$, then $S \subseteq T$,
$\mu_S^{i,\delta} > 0, \mu_T^{i,\varrho} > 0$ implies $S \cap T = \emptyset$.

This contains the weakly symmetric case. Due to the characterization of hardly symmetric graphs, the hardly symmetric case is also included. Z. Király observed that if $G = (V, A)$ is a strongly connected hardly symmetric digraph and $\overline{G}$ is 2-vertex-connected, then G is symmetric or bipartite.

Cunningham and Geelen [1] proved polyhedral results using their unweighted min-max formula. The opposite direction also can be followed, the unweighted formula of Pap and Szegő [5] can be obtained as a consequence of Theorem 3. In a digraph $G = (V, A)$, we let $N_G^+(X) = \{v \in V - X : \exists u \in X, uv \in A\}$ and $\mathrm{odd}(G)$ denotes the *number of strongly connected components of G having an odd number of vertices with no entering arc.*

Theorem 4 (Pap and Szegő) *If $G = (V, A)$ is a hardly symmetric digraph, then the maximum cardinality of an even factor is*

$$\min_{X \subseteq V} |V| + |N_G^+(X)| - \mathrm{odd}(G[X]).$$

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On Restricted Min-Wise Independence of Permutations

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Abstract

A family of permutations $\mathcal{F} \subseteq S_n$ with a probability distribution on it is called *k-restricted min-wise independent* if we have $\Pr[\min \pi(X) = \pi(x)] = \frac{1}{|X|}$ for every subset $X \subseteq [n]$ with $|X| \leq k$, every $x \in X$, and $\pi \in \mathcal{F}$ chosen at random. We present a simple proof of a result of Norin: every such family has size at least $\binom{n-1}{\lfloor \frac{k-1}{2} \rfloor}$. Some features of our method might be of independent interest.

The best available upper bound for the size of such family is $1 + \sum_{j=2}^{k}(j-1)\binom{n}{j}$. We show that this bound is tight if the goal is to imitate not the uniform distribution on S_n, but a distribution given by assigning suitable priorities to the elements of $[n]$ (the stationary distribution of the *Tsetlin library*, or self-organizing lists). This is analogous to a result of Karloff and Mansour for k-wise independent random variables.

1 Introduction

Let S_n denote the set of all permutations of $[n] = \{1, 2, \ldots, n\}$. A family $\mathcal{F} \subseteq S_n$ of permutations with a probability distribution on it is called *min-wise independent* if, for any set $X \subseteq [n]$, the following condition holds: For every $x \in X$ and for π chosen at random from $\mathcal{F}$, we have $\Pr[\min \pi(X) = \pi(x)] = \frac{1}{|X|}$. That is, we require that all elements of X have equal chance of becoming the minimum element of the image of X under π. We call this condition the *min-uniform condition for X*.

As for the probability distribution on $\mathcal{F}$, two basic cases can be distinguished: Either the distribution is required to be uniform (then we call $\mathcal{F}$ a *uniform* family), or it can be arbitrary (then we speak of a *biased* family).

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Min-wise independent families of permutations were introduced and thoroughly investigated by Broder, Charikar, Frieze, and Mitzenmacher [2]. They are essential to algorithms for detecting near-duplicate documents, such as used in practice by the AltaVista Web indexing software. Broder et al. [2] showed that min-wise independent families are too large and impractical for some applications.

Restricted min-wise independence. In this paper we study a relaxation of min-wise independence, where the min-uniform condition is required to hold exactly, but only for some sets X. Namely, we call a family $\mathcal{F}$ *k-restricted min-wise independent* if the min-uniform condition holds for all $X \subseteq [n]$ with $|X| \leq k$.

This perhaps most natural concept of restricted min-wise independence was also introduced by Broder et al. [2]. By a sophisticated method using graph entropy, they proved a lower bound of $\Omega\big(k2^{k/2}\log(n/k)\big)$ for the size of any (possibly biased) k-restricted min-wise independent family. They also noted that for uniform k-restricted min-wise independent families, one obtains a lower bound of $e^{k-o(k)}$ by the method used for uniform min-wise families, and that the upper bound of $\sum_{j=2}^{k} j\binom{n}{j}$ $(= \Theta(n^k)$ for k fixed) for *biased* k-restricted min-wise independent families follows by the linear programming approach ([7]). The problem was further investigated by Itoh et al. [4], who constructed an explicit uniform k-restricted min-wise independent family of size $n^{(1+\frac{2}{\lg n})k}$, and proved a lower bound of $n{-}1$ (for any $k \geq 3$). Recently Norin [8] proved the following stronger lower bound:

Theorem 1 ([8]) *Let $\mathcal{F}$ be a k-restricted min-wise independent family (with an arbitrary, possibly biased, probability distribution), and let $n \geq 2k$. Then*

$$|\mathcal{F}| \geq \binom{n-1}{\lfloor \frac{k-1}{2} \rfloor}.$$

In Section 2 we present an alternative proof of this fact. Although we found it without knowing of Norin's work, it has some similarities to his proof, but it is considerably simpler and shorter. Some of the features of our proof, such as the use of complex random variables, appear to be new in this context and could be useful in other problems as well.

Recently we have learned that Itoh et al. [5] independently proved a lower bound similar to Theorem 1 but slightly stronger, namely, $\sum_{i=0}^{(k-1)/2}\binom{n-1}{i}$ for k odd and $\sum_{i=0}^{k/2-1}\binom{n-1}{i} + \binom{n-2}{k/2-1}$ for k even. Their method differs from both Norin's and ours in several ways.

Non-uniform distributions on S_n. While we do not know whether the lower bound in Theorem 1 can be improved, we show that in a more general setting, the upper bound of order n^k obtained by the linear programming approach is tight.

In that more general setting, we do not want our family $\mathcal{F}$ of permutations to imitate, as far as the minima of at most k-element sets are concerned, the uniform distribution on S_n, but rather distributions where a random permutation is selected according to some given *priorities* of the elements, positive real numbers $w_1, w_2, \ldots, w_n$. Random permutation π is selected in n steps as follows: We maintain a current set C of elements, which is initialized to $[n]$ before the first step. In the ith step, we pick the element that becomes the ith element in the ordering given by π (that is, we pick $\pi^{-1}(i)$). We pick it at random from the current set C, where each $a \in C$ is chosen with probability $w_a / \sum_{b \in C} w_b$. The chosen element is deleted from C and we continue with the next step.

The probability distribution $\mu(w_1, \ldots, w_n)$ on S_n defined in this manner is known as the stationary distribution of the *Tsetlin library* Markov chain [9], and it arises naturally in computer science (e.g., for self-organizing lists), as well as in many applied contexts.

It should be clear what is meant by saying that a family $\mathcal{F} \subseteq S_n$ (with a probability distribution on it) is k-restricted min-wise independent *with respect to a given probability distribution* μ *on* S_n: For any at most k-element $X \subseteq [n]$ and any $x \in X$, the probability of $\min \pi(X) = \pi(x)$ is the same for π selected at random from $\mathcal{F}$ and for π selected at random from μ. Now we can state our result.

Theorem 2

(i) *Let μ be an arbitrary probability distribution on S_n. Then there exists a (biased) family $\mathcal{F} \subset S_n$ of size at most $1 + \sum_{j=2}^{k} (j-1)\binom{n}{j}$ that is k-restricted min-wise independent with respect to μ.*

(ii) *There exist positive priorities $w_1, \ldots, w_n$ such that the upper bound in* (i) *cannot be improved for the distribution $\mu = \mu(w_1, \ldots, w_n)$. In fact, the set of positive vectors $(w_1, \ldots, w_n) \in \mathbb{R}^n$ that do not have this property is contained in the zero set of a non-zero polynomial in $w_1, \ldots, w_n$, and thus it is nowhere dense and of measure zero.*

In particular, for $k = n$ we obtain the bound $(n-2)2^{n-1}$. The proof of (i) is standard by the LP-approach, introduced in [7].

Part (ii) is proven in several steps. If $w = (w_1, \ldots, w_n)$ is a vector of priorities and $\mathcal{F} \subset S_n$ is k-restricted min-wise independent (with respect to the distribution $\mu(w)$) family of size less then the bound from (i), then it can be proven that w is a zero of the appropriately defined rational function $F(x_1, \ldots, x_n)$. On the other hand, we can show that $F(x_1, \ldots, x_n)$ is a linear combination of linearly independent rational functions and thus it is a non-zero function.

2 The lower bound for k-restricted min-wise independent families

All known proofs of Theorem 1 use the following somewhat surprising property of k-restricted min-wise independent families.

Proposition 1 ([8]) *Let $\mathcal{F}$ be a k-restricted min-wise independent family, let X be a k-element subset of $[n]$, and let $X = A \cup \{x\} \cup B$ be a partition of X. Then, for π chosen from $\mathcal{F}$ at random,*

$$\Pr[\pi(A) < \pi(x) < \pi(B)] = \frac{1}{(a+1)\binom{k}{a+1}}, \tag{1}$$

where $a = |A|$ (this probability is the same as for π chosen uniformly at random from S_n).

Let $L_n(i,j)$, $1 \le j \le i \le n$, be the 0/1 matrix with rows indexed by all i-element subsets $S \subseteq [n]$, columns indexed by all j-element subsets $T \subseteq [n]$, and with the entry at position (S,T) equal to 1 if $T \subseteq S$ and equal to 0 otherwise. This matrix is a representative of a wide class of so-called inclusion matrices. We need the following statement.

Theorem 3 (Gottlieb [3]) *For every i,j, $1 \le j \le i \le n$, the matrix $L_n(i,j)$ has full rank; that is* $\operatorname{rank} L_n(i,j) = \min\left\{\binom{n}{i}, \binom{n}{j}\right\}$.

Proof of Theorem 1. Our basic strategy is inspired by Alon, Babai, and Itai [1]. We suppose that a k-restricted min-wise independent family $\mathcal{F}$ with its probability distribution is given, and we define a suitable family $(X_i : i \in I)$ of random variables on $\mathcal{F}$. We consider the $|I| \times |I|$ matrix $M_I := \Big(\mathbf{E}[X_i X_j]\Big)_{i,j\in I}$ (where $\mathbf{E}[X]$ denotes the expectation of X) and prove that it is non-singular. Then it follows that $|\mathcal{F}| \ge |I|$. Indeed, let us consider each random variable X_i as a vector with $|\mathcal{F}|$ components. We verify that these $|I|$ vectors are linearly independent and, consequently, the vector space of all $|\mathcal{F}|$-component vectors has dimension at least $|I|$.

To check the linear independence of the vectors X_i, let us suppose that $(\alpha_i : i \in I)$ are scalars such that $\sum_{i\in I} \alpha_i X_i = 0$. Then, for every $j \in I$, $X_j \sum_{i\in I} \alpha_i X_i = \sum_{i\in I} \alpha_i X_i X_j = 0$, and therefore,

$$\mathbf{E}\Big[\sum_{i\in I} \alpha_i X_i X_j\Big] = \sum_{i\in I} \alpha_i \mathbf{E}[X_i X_j] = 0.$$

Since M_I is non-singular, it follows that $\alpha_i = 0$ for all $i \in I$, and so the X_i are linearly independent.

We let $\ell := \lfloor \frac{k-1}{2} \rfloor$. Our random variables will be indexed by ℓ-element subsets $S \subset [n-1]$; that is, $I = \binom{[n-1]}{\ell}$. They are defined as follows:

$$X_S(\pi) = \begin{cases} 1 & \text{if } \pi(n) < \pi(S), \\ \mathrm{i} & \text{if } \pi(S) < \pi(n), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathrm{i} = \sqrt{-1}$ is the imaginary unit.

Using Proposition 1, we can calculate the expectations $\mathbf{E}[X_S X_T]$, i.e., the matrix M_I:

$$\mathbf{E}[X_S X_T] = \begin{cases} 0 & \text{if } S \cap T \ne \emptyset, \text{ and} \\ \mathrm{i}\frac{2}{(\ell+1)\binom{2\ell+1}{\ell+1}} & \text{if } S \cap T = \emptyset. \end{cases}$$

Thus, M_I is a constant multiple of the "intersection matrix" with the entry (S,T) being 1 for $S \cap T = \emptyset$ and 0 otherwise. This, in turn, is the same as the matrix $L_n(n-\ell-1, \ell)$, which is non-singular by Theorem 3. Theorem 1 is proved. □

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Repeat structure in regular graphs and digraphs

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1 Introduction

The problem of finding digraphs of largest order $n_{d,k}$ given maximum out-degree d and diameter at most k is well known as the *degree/diameter problem* and has been the subject of investigation for many years. A natural upper bound for $n_{d,k}$ can be found by using a standard spanning tree argument; the bound is called the *Moore bound*, $M_{d,k}$,

$$n_{d,k} \leq M_{d,k} = 1 + d + d^2 + \ldots + d^k.$$

A result by Plesník and Znám [15], and later independently by Bridges and Toueg [7], shows that *Moore digraphs*, i.e., digraphs of order $M_{d,k}$, only exist when $d = 1$ or $k = 1$; they are respectively the directed cycles C_{k+1} and the complete digraphs K_{d+1}.

Since there are no Moore digraphs for $d \geq 2$ or $k \geq 2$, the study of the existence of large digraphs focuses on digraphs of order close to the Moore bound, that is, digraphs of order $n = M_{d,k} - \Delta$, with the smallest possible Δ. Throughout the paper, for Δ a nonnegative integer we denote by *digraph with defect* Δ or (d, k, Δ)*-digraph* the digraph of order $M_{d,k} - \Delta$. Note that the $(d, k, 0)$-digraphs are the well-known Moore digraphs and the $(d, k, 1)$-digraphs are the so-called *almost Moore digraphs*.

In the case of $k = 2$, it is known that line digraphs of complete digraphs are examples of $(d, 2, 1)$-digraphs. For $d = 2$, there exist two additional digraphs. For $d \geq 3$, it has been proved by Gimbert [9] that line digraphs of K_{d+1} are the only $(d, 2, 1)$-digraphs. In the remaining case of $k \geq 3$, improvements for the upper bounds of $n_{d,k}$ are only known for out-degrees $d = 2$ and 3. For $d = 2$, $n_{2,k} \leq M_{2,k} - 3$ by the results of Miller and Fris [11] and Miller and Širáň [13]. For $d = 3$, $n_{3,k} \leq M_{3,k} - 2$ by the result of Baskoro, Miller, Širáň and Sutton [6]. The best-known lower bound on the order of $(2, k, \Delta)$-digraphs for $k \geq 4$ is obtained from *Alegre digraph*, $n_{d,k} \geq 25 \times 2^{k-4}$. For the remaining cases of out-degree and diameter, a general lower bound is $n_{d,k} = d^k + d^{k-1}$, obtained from *Kautz digraphs* (i.e., the line digraphs of complete digraphs without loops). For the current state of the degree/diameter problem, we refer to the survey article of Miller and Širáň [14].

To deal with (d, k, Δ)-digraphs, for $\Delta = 1$, Miller and Fris [10] introduced a notion called "repeat". The concept of a repeat has been used to prove several structural and nonexistence results for the $(d, k, 1)$-digraphs. A *repeat* of a vertex u in a $(d, k, 1)$-digraph is defined as the unique vertex $r(u)$ having exactly two $v \to r(v)$ directed walks of lengths at most k. In general, for a subset S of the vertex set of a $(d, k, 1)$-digraph, the *repeat of* S, $r(S)$ is simply the set af all the repeats of the vertices in S.

One important property of a repeat, presented here as Lemma 1, was proved by Baskoro, Miller, Plesník, and Znám [5]. Here by $N^+(u)$ we denote the set of all out-neighbours of the vertex u and, for a subset S of the vertex set of the digraph, and by $N^+(S)$ we mean the set of all out-neighbours of the vertices in S.

Lemma 1 *[5]* **(Neighbourhood Lemma)**
If u is a vertex in a diregular $(d, k, 1)$-digraph then $N^+(r(u)) = r(N^+(u))$ and $N^-(r(u)) = r(N^-(u))$.

This lemma asserts that if a $(d, k, 1)$-digraph G is diregular then the map r is an automorphism in G. Since all $(d, k, 1)$-digraphs are diregular [12], it follows that the map r is an automorphism in every $(d, k, 1)$-digraph.

The Neighbourhood Lemma has been very useful to prove results concerning the characterisations of $(d, k, 1)$-digraphs (see [1, 2, 3, 4, 5]) and also to prove the nonexistence of $(2, k, 1)$-digraphs [11] and $(3, k, 1)$-digraphs [6] for $k \geq 3$.

Simanjuntak and Miller [16] generalised the Neighbourhood Lemma to apply also to undirected graphs with defect $\Delta = 1$ or 2. In this paper we generalise the Neighbourhood Lemma much further so that it can be used for both directed and undirected graphs and for any defect $\Delta \geq 1$. The new generalised Neighbourhood Lemma is referred to as the Neighbourhood Theorem (Theorem 1). This is the main result of this paper.

2 The result

In studying (d, k, Δ)-digraphs, sometimes it is useful to depict the structure of the digraph by means of a so-called layered diagram of the digraph. For a digraph G, a vertex u in $V(G)$, and an integer l, the *layered diagram* of *depth l rooted* at u, $\mathcal{L}^+(G, u, l)$, contains all the directed walks of length $\leq l$ from u. That is, the first level contains the single vertex u, the second level contains all the neighbours of u, and, in general, the ith level contains the endpoints of all walks of distance $i - 1$ from u, $i = 1, \ldots, l$.

Similarly, the *layered diagram* of *height l terminated* at u, $\mathcal{L}^-(G, u, l)$, contains all the directed walks of length $\leq l$ to u. In the layered diagrams, every vertex occurs at least once. However, some vertices may occur more than once. Note that for a fixed graph G, a vertex u, and a nonnegative integer l, both $\mathcal{L}^+(G,u,l)$ and $\mathcal{L}^-(G,u,l)$ are uniquely determined. As an example, we present the layered diagram $\mathcal{L}^+(G,0,3)$ of the unique diregular $(3,2,2)$-digraph G on 11 vertices in Figure 1.

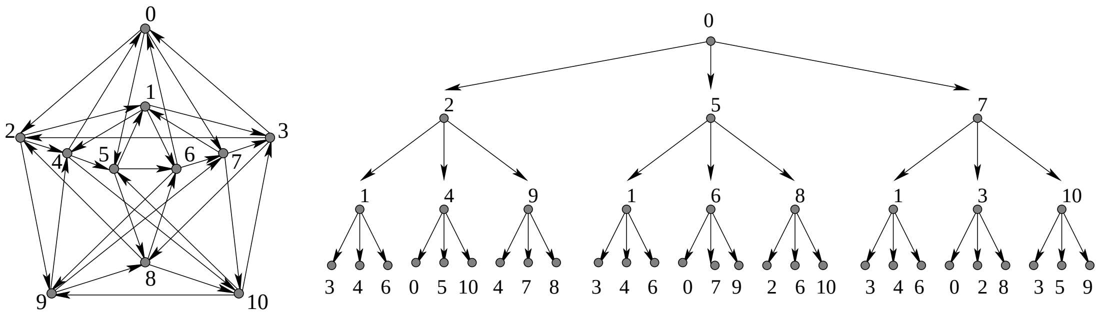

Figure 1: The unique diregular $(3,2,2)$-digraph G and its layered diagram of depth 3 rooted at 0, $\mathcal{L}^+(G,0,3)$.

Let G be an out-diregular (d,k,Δ)-digraph. For each u in G we define the following sets:

$$\begin{aligned} S_1^+(u) &= \{v \mid \exists!\ u \to v \text{ walk of length} \leq k\}, \\ S_2^+(u) &= \{v \mid \exists \text{ exactly } \textit{two}\ u \to v \text{ walks of length} \leq k\}, \\ &\vdots \\ S_\Delta^+(u) &= \{v \mid \exists \text{ exactly } \Delta\ u \to v \text{ walks of length} \leq k\}. \end{aligned}$$

Note that for every $i = 1, \dots, \Delta$, the set $S_i^+(u)$ contains all vertices that occur i times in the layered diagram $\mathcal{L}^+(G,u,k)$.

Since G is out-diregular and of order $M_{d,k} - \Delta$, for each u in G the following properties hold.

Observation 1 $\sum_i |S_i^+(u)| = M_{d,k} - \Delta$.

Observation 2 *For* $\Delta \geq 2$, $\sum_{i=2}^{\Delta}(i-1)|S_i^+(u)| = \Delta$.

For $i \geq 1$ and for each vertex u in G, we call $v \in S_i^+(u)$ a *repeat* of u with *multiplicity* $m_u(v) = (i-1)$. Thus for every u in G, the equation in Observation 2 can be rewritten as $\sum_v m_u(v) = \Delta$. The multiplicity $m_u(v)$ of a vertex v can also be seen as the number of replicate occurences of v in the layered diagram $\mathcal{L}^+(G,u,k)$. We denote by $R(u)$ the multiset of all repeats of u and, in general, we denote by $R(M)$ the multiset of all the repeats of the elements of a multiset M.

Recall that Lemma 1 applies to $(d,k,1)$-digraphs. We shall next generalise this lemma by considering (d,k,Δ)-digraphs with $\Delta > 1$. Note that by $T_j^+(u)$, we mean the multiset of the terminating vertices of all the directed walks of lengths $\leq j$ starting from u. For a multiset M, by $T_j^+(M)$ we mean the multiset containing all the vertices in $T_j^+(u), \forall u \in G$, and by $N^+(M) = N_1^+(M)$ we mean the multiset of all out-neighbours of vertices in M.

Theorem 1 (Neighbourhood Theorem)
If G is a diregular (d,k,Δ)-digraph then for every vertex u in G, $N^+(R(u)) = R(N^+(u))$.

Proof. Consider an arbitrary vertex u in G and denote by $u_1, \ldots, u_d$ the neighbours of u.

To show that $N^+(R(u)) = R(N^+(u))$, we shall express the multiset $T_{k+1}^+(u)$ in two different ways:

(1) $T_{k+1}^+(u) = \{u\} \cup T_k^+(N^+(u)) = \{u\} \cup T_k^+(u_1) \cup \ldots \cup T_k^+(u_d)$ and $T_k^+(u_i) = V(G) \cup \{R(u_i)\}$ for every $i = 1, \ldots, d$. Thus

$$T_{k+1}^+(u) = \{u\} \cup dV(G) \cup R(u_1) \cup \ldots R(u_d) = \{u\} \cup dV(G) \cup R(N^+(u))$$

and

$$|T_{k+1}^+(u)| = 1 + dn + |R(N^+(u))|.$$

(2) Since for $i = 1, \ldots, d$, each vertex u_i has to reach every other vertex in directed walks of lengths at most k and since the in-degree of every vertex is d, it follows that each vertex in G must occur at least d times in $T_{k+1}^+(u)$. Moreover, $T_{k+1}^+(u)$ also contains the out-neighbours of all the repeats of u. Thus we have

$$T_{k+1}^+(u) \supseteq \{u\} \cup dV(G) \cup N^+(R(u))$$

and so

$$|T_{k+1}^+(u)| \geq 1 + dn + |N^+(R(u))|.$$

Summarising, we have $R(N^+(u)) \supseteq N^+(R(u))$. On the other hand, $|R(N^+(u))| = |N^+(R(u))| = \Delta d$, which results in $R(N^+(u)) = N^+(R(u))$.

Corollary 1 *If G is a diregular (d,k,Δ)-digraph then for every vertex u in G, $N^-(R(u)) = R(N^-(u))$.*

At this stage we have proved that the Neighbourhood Theorem holds for all digraphs which are diregular. Unfortunately, little is known about the diregularity of (d,k,Δ)-digraphs in general. For small values of defect we have: For $\Delta = 1$, all $(d,k,1)$-digraphs are diregular [12]. For $\Delta = 2$, the diregularity has been proved only for $d = 2$ and $k \geq 3$ [17].

Open Problem 1 *Is the diregularity of a (d,k,Δ)-digraph a sufficient condition for the Neigbourhood Theorem to hold?*

Note that the Neighbourhood Theorem does not hold only for digraphs. Using a somewhat more complicated counting argument we can prove the undirected version of Theorem 1.

Theorem 2 (Neighbourhood Theorem for undirected graphs)
For every vertex u in a regular (d, k, Δ)*-graph,* $N(R(u)) = R(N(u))$.

Proof. The proof is too long to include here. □

The Neighbourhood Theorem is very useful for exploiting structural properties of graphs and digraphs. For example, we have used the undirected version of the theorem to prove that there is no graph of 35 vertices, degree 6 and diameter 2, which improves the upper bound for $n_{6,2}$ so that we now have $32 \leq n_{6,2} \leq 34$.

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Observability of odd graphs, bi-odd graphs and recursive clique-trees

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Abstract

An edge-coloring is *proper* if no two adjacent edges are assigned the same color; it is *vertex-distinguishing* if for any two distinct vertices u and v, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. The observability of a graph G is the minimum number of colors required for a vertex-distinguishing proper edge-coloring of G. We determine the observability of three types of graphs : two regular graphs, *odd* and *bi-odd graphs* and a new category of small-world graphs, *the recursive clique-trees*, introduced in [CFR02].

Let $G = (V, E)$ be a simple graph with no more than one isolated vertex and no isolated edges. An edge coloring is called **proper** if no two adjacent edges are assigned the same color and an edge coloring is called **vertex-distinguishing** if for any two distinct vertices u and v, the set of colors assigned to the edges incident to u differs from the set of colors assigned to the edges incident to v. The minimum number of colors required for a vertex-distinguishing proper edge-coloring of G is called **observability** and is noted by $Obs(G)$ [CHS96] or $\chi'_s(G)$ [BS97]. A graph G is *vdp*-colorable, if there exists a vertex-distinguishing proper edge-coloring of G.

The concept of vertex-distinguishing colorings has been very studied this last years in severals papers [BS97, CHS96] and give many interesting challenging problems.

A first observation is that the number of colors must be sufficiently large so that there are at least as many color sets of size d as there are vertices of degree d, then:

$$Obs(G) \geq min\left\{j : 1 \leq d \leq \Delta(G), \left(\begin{array}{c} j \\ d \end{array}\right) \geq n_d\right\}$$

where n_d is the number of vertices of degree d. In particular, if G contains more than one vertex of degree d, then $Obs(G) \geq d + 1$.

In [BS97], Burris and Schelp made the following conjecture:

Conjecture 1 *Let G be a vdp-colorable graph and let k be the minimum integer such that $\binom{k}{d} \geq n_d$ for all d with $1 \leq d \leq \Delta(G)$. Then, $Obs(G) = k$ or $k+1$.*

Some graphs, including complete graphs, complete bipartite graphs, paths, cycles, wheels, many trees, verify the conjecture 1 [CHS96, BS97].

In this paper, we determine the observability of the odd graphs, the bi-odd graphs, which are regular graphs, and of a certain class of k-trees [Bod88]. This class is the class of **recursive clique-trees** introduced in [CFR02], which provide scale-free small-world networks with adjustable clustering and values of the parameter of the power law associated with the degree distribution. For further details, see for examples [CFR02]. These graphs verify the conjecture 1.

Odd graphs. The odd graph $\mathcal{O}_k$ is the graph (V, E) defined as follow :

- V is the k-subsets of $\{0, 1, 2, \ldots, 2k-2\}$;
- the vertices u and v are linked if their corresponding k-subsets have exactly one element in common (see figure 1).

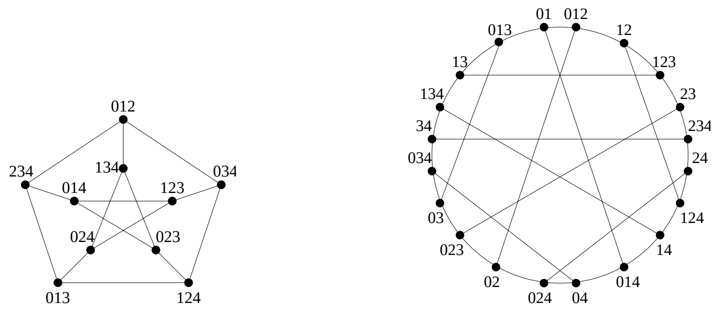

Figure 1: Odd graph $\mathcal{O}_3$ and bi-odd graph $2\mathcal{O}_3$.

The odd graph $\mathcal{O}_k$ is k-regular and has $\binom{2k-1}{k}$ vertices.

Theorem 1 *Let $\mathcal{O}_k$ be an odd graph, with $k \geq 2$, then $Obs(\mathcal{O}_k) = 2k-1$.*

Usually, color a regular graph of degree d and $\binom{c}{d}$ vertices with c colors for a given topology is not evident. However, the topology of the odd-graph fits well on the notion of observability and it is easy to obtain the previous result. Then, we can spread the result to the bi-odd graphs [Bre00].

Bi-odd graphs. The bi-odd graph $2\mathcal{O}_k$ is the bipartite graph (V, E) defined as follow :

- V is the k-subsets and the $(k-1)$-subsets of $\{0, 1, 2, \ldots, 2k-2\}$;
- the vertices u and v are linked if $(k-1)$-subsets of one is included in the k-subset of the other (see figure 1).

The bi-odd graph $2\mathcal{O}_k$ is k-regular and has $2\binom{2k-1}{k}$ vertices.

Theorem 2 *Let $2\mathcal{O}_k$ be a bi-odd graph, with $k \geq 2$.*

- *if* $\binom{2k-1}{k} \equiv 0 \mod 2$, *then* $Obs(2\mathcal{O}_k) = 2k$,
- *if* $\binom{2k-1}{k} \equiv 1 \mod 2$, *then* $Obs(2\mathcal{O}_k) = 2k+1$.

Until now, we just consider regular graphs, but what happens when some vertices have high degree. We can think that these vertices need lot of colors and these colors will be enough to color the edges of the vertices of small degree. We will now consider a model of small-world graphs : the recursive clique-trees.

Recursive clique-tree. Recursive clique-tree $K(q, t)$, $q \geq 2$, is the graph constructed as follow:

- for $t = 0$, the recursive clique-tree is a complete graph K_q (q-clique);
- for $t \geq 1$, the recursive clique-tree $K(q, t)$ is constructed from $K(q, t-1)$ in which, for all subgraphs Ω isomorphic to a q-clique, a new vertex is added and joined to all vertices of Ω (see figure 2).

There is another very interesting way to construct $K(q, t)$, which shows the recursive structure of such networks. We call **native clique** of $K(q, t)$ its q-clique at step 0.

Recursive construction of $K(q, t)$.

- for $t = 0$, $K(q, 0)$ is a q-clique.
- for $t \geq 1$, $K(q, t)$ is constructed as follow : let K_{q+1} be a $(q+1)$-clique. Every subgraph isomorphic to a q-clique in K_{q+1} is a native clique of a $K(q, t-1)$ (see figure 2).

The maximum degree at step t is $\Delta_t = \frac{q^t-1}{q-1} + q - 1$ [CFR02].

Theorem 3 *Let $K(q, t)$ be a recursive clique-tree, $q \geq 2$ then :*

- $Obs(K(q, 0)) = q$ *if q is odd, and $q+1$ otherwise.*

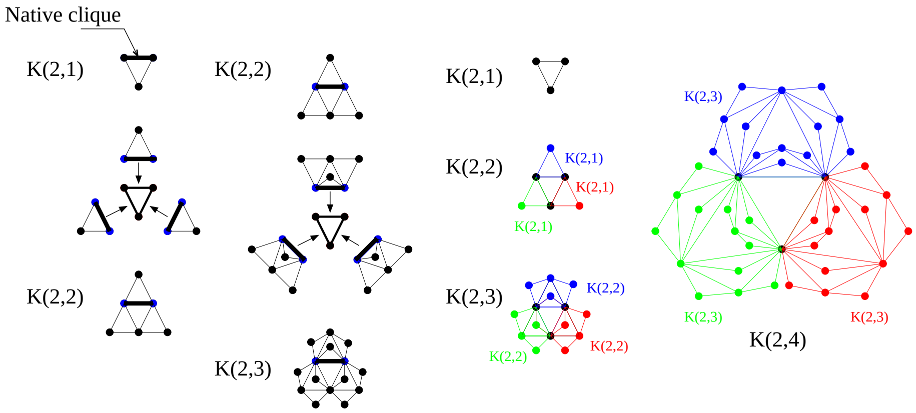

Figure 2: Example of recursive clique-tree and recursive construction. To construct $K(q,t)$, on each q-clique of a $(q+1)$-clique, we *glue* a native clique of a $K(q,t-1)$.

- *$Obs(K(q,1)) = q+1$ if q is even, and $q+2$ otherwise.*
- *for $t \geq 2$,*

$$Obs(K(q,t)) = \frac{q^t - 1}{q-1} + q$$

The proof is based on an induction on the step t and on a decomposition of the problem into two sub-problems of coloring. First, we construct a new graph, $\overline{K}(q,t)$, which represents the transition between the step t and the step $t+1$. A particular coloring of this object, s-coloring, helps us for the proof. The first sub-problem is the vertex-distinguishing proper edge-coloring of $K(q,2)$. This proof uses the adjacency matrix of the graph and theorems of coloring on the complete graph (Vizing for *proper coloring*, Hornak for *vertex-distinguishing proper edge-coloring* [CHS96]). The second sub-problem is the s-coloring of $\overline{K}(q,t)$. The proof is based on the adjacency matrix and on some properties of this object (bipartite, recursively constructed ...).

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K_5-free Bound for the Class of Planar Graphs

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1 Introduction

Let G and H be graphs. A homomorphism f of G to H is an edge preserving mapping of $V(G)$ to $V(H)$. Homomorphism defines a quasiorder (a reflexive and transitive binary relation) on the class of graphs, by $G \preccurlyeq H$ if and only if there is a homomorphism from G to H. Given a class $\mathcal{C}$ of graphs we say that $\mathcal{C}$ is bounded by H if for every $G \in \mathcal{C}$ we have $G \preccurlyeq H$. In this setting the four-color theorem simply states that the class P of planar graphs is bounded by K_4. Similarly Grötsch theorem (every triangle free planar graph is 3 colorable) states that the class of triangle free planar graphs is bounded by K_3.

The problem of the existence of a bound with some special properties, for a given class of graphs has been a subject of study in graph homomorphism. Some particular cases of this problem has been studied in [3, 4, 5, 6]. In [6] authors have proved the existence of K_k-free bound for any minor closed class of K_k-free graphs. This indeed provides a K_5-free graph H which bounds the class of planar graphs, therefore answering a question of authors posed in [1]. Note that if we take the categorical product of H with K_5 we will find a bound for the set of planar graphs which is smaller than K_5 in the sense of homomorphism, this is a stronger result than the five-color theorem whose proof does not use the four-color theorem.

In this note we define a new type of vertex coloring which we call it diverse coloring. We give an algorithm to find a k-diverse coloring for any planar graph using a fixed number of colors. Using the coloring obtained from this algorithm we construct a K_5 bound for the class of planar graphs. This will be a completely different method for proving the existence of a K_5-free bound for the class of planar graphs.

2 Diverse coloring

Definition 1 *For the given integers k and l we say that an l-coloring c, of a given graph*

G is a k-diverse coloring if at least $min\{d(x), k\}$ different colors appear on the neighbors of x. A k-diverse coloring which uses at most l colors will be denoted by (k, l)-coloring.

Theorem 1 *Given an integer $k \geq 11$, every planar graph admits a $(k, 5k + 8)$-coloring.*

Sketch of the proof. This will be proven by induction on the number of vertices of G. For graphs on at most $5k + 8$ vertices we can color all the vertices with different colors. Suppose we have found a $(k, 5k + 8)$-coloring for every planar graph on at most n vertices and let G be a planar graph on $n + 1$ vertices. If G has a vertex x with $d(x) = 1$ then any $(k, 5k + 8)$-coloring of $G \backslash x$ can be extended to a $(k, 5k + 8)$-coloring of G.

If G does not have a vertex of degree 1 but it has a vertex x of degree 2 then identify x with one of its neighbors, remove the loop and the possible multiedge. Call the new graph G_x. By induction G_x admits a $(k, 5k+8)$-coloring c_x. By counting the number of colors we using them to color x would violating the diversity condition we see that c_x is extendable to a $(k, 5k + 8)$-coloring of G.

If neither of the previous two cases happen then by a theorem of Kotzig we will find an edge $e = uv$ in G such that $d(u)+d(v) \leq 13$. Without loss of generality assume $d(u) \leq d(v)$. Therefore $d(u) \leq 6$. Identify u and v, remove loops and possible multiple edges and call the new graph G_e. Let v' to be the new vertex in G_e (obtained from identifying u and v) then $d(v') \leq 11$. By induction G_e admits a $(k, 5k + 8)$-coloring, we denote this coloring by c_e. Use c_e to color all vertices of but u, where v has taken the color of v'. Again a simple counting will show us the existence of a possible color for u without violating the diversity condition. □

Notice that in the proof of this theorem we introduced an inductive algorithm to find a $(k, 5k + 8)$-coloring of any planar graph. We will call this algorithm *k-diverse coloring algorithm.* In the next section we will show that the coloring obtained from 11-diverse coloring algorithm satisfies the condition of the Theorem 7 of [3] and hence provides us with a K_5-free bound for the class of planar graphs.

3 K_5-free bound

Theorem 2 *Let G be a planar graph and c a $(11, 63)$-coloring of G obtained from 11-diverse coloring algorithm. Then c has the property that every 5-chromatic subgraph of G takes at least 6 different colors.*

Sketch of the proof. By contradiction suppose this is not true and algorithm fails. We consider G to be the smallest graph for which the 11-diverse algorithm fails. Then G must contain a 5-critical subgraph which takes only 5 different colors. By analyzing the algorithm we will see that this is impossible. We will use the fact that a 5-critical planar graph must have minimum degree at least 5. □

Using this theorem and the results of [3] one can construct a K_5-free bound for the class of planar graphs. For the sake of completeness we will repeat the construction here.

Constructing $H_{(11,63)}$: The vertices of this graph are the pairs (i, φ) where $i \in [63]$ and φ is any function from $\{S | S \subset [63], i \in S \textit{ and } \quad |S| = 5\}$ to $\{1, 2, 3, 4\}$. For the edge set; (i, φ) is adjacent to (j, ψ) if $i \neq j$ and $\varphi(S) \neq \psi(S)$ whenever both $\varphi(S)$ and $\varphi(S)$ are well defined, i.e. $\{i, j\} \subset S$.

Lemma 1 *(c. [3]) $H_{(11,63)}$ is K_5-free.*

Theorem 3 *$H_{(11,63)}$ bounds class of planar graphs.*

Proof: Let G be a planar graph and c a $(11, 63)$-coloring of G obtained from the algorithm. By Theorem 2, for any set S of 5 colors, the subgraph G_S induced on these colors is 4-colorable. Let ρ_S to be a 4-coloring of G_S. The homomorphism of G to $H_{(11,63)}$ is defined by:

$$\text{for } x \in V(G) \text{ let } f(x) = (c(x), \varphi_x) \text{ where } \varphi_x(S) = \rho_S(x).$$

□

4 Remarks

The K_5-free bound for the class of planar graphs we provided here, has a large number of vertices. The following problem, if it is answered without using the four-color theorem, will provide us with a better K_5-free bound which has fractional chromatic number $\frac{24}{25}$. Notice that the best known bound for the fractional chromatic number of planar graphs without using the four-color theorem is that it is strictly smaller than 5.

Problem 1 *Show that every planar graph can be colored using 6 colors in such a way that every 5-chromatic subgraph of it receives all 6 different colors.*

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The Interval Function of a Connected Graph and Road Systems

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In this paper we assume that a finite nonempty set V is given. We denote by V^* the set of all finite sequences over V including the empty sequence denoted by ι. The elements of V^* will be considered as formal words. If $u_0, \ldots, u_n \in V$, where $n \geq 0$, and if $\alpha = u_0 \ldots u_n$, then we put $F\alpha = u_0$, $L\alpha = u_n$, and $\bar{\alpha} = u_n \ldots u_0$. We denote by V^+ the set of all sequences $\alpha \in V^* \setminus \{\iota\}$ such that if $\alpha = u_0 \ldots u_n$, where $n \geq 0$ and $u_0, \ldots, u_n \in V$, then $u_0, \ldots, u_n$ are pairwise distinct.

By a *road system* on V we mean a subset R of V^+ satisfying the following properties for all $u, v, x, y \in V$ and all $\alpha, \beta, \gamma, \delta \in V^*$:

there exists $\pi \in R$ such that $F\pi = u$ and $L\pi = v$;

if $\alpha uv\beta \in R$, then $\alpha uv, uv\beta \in R$;

if $u\alpha v\beta w \in R$, then $uw \notin R$;

if $u\alpha v, \beta u\gamma v\delta \in R$, then $\beta u\alpha v\gamma \in R$.

Let R be a road system on V. By the graph determined by R we mean the graph G with $V(G) = V$ and

$$E(G) = \{\{u, v\}; u, v \in V, \text{ and } uv \in R \text{ or } vu \in R\}.$$

It is clear that G is connected and every $\alpha \in R$ is a path in G.

Let G be a connected graph such that $V(G) = V$, and let R denote the set of all geodesics (i.e. shortest paths) in G. We can see that R is a road system on V.

Let R be a road system on V. It is obvious that if R is the set of all geodesics in a connected graph G, then G is the graph determined by R.

The characterization of the set of all geodesics in a connected graph given in [2] can be reformulated as follows:

Theorem A *Let R be a road system on V, and let G be the graph determined by R. Then R is the set of all geodesics in G if and only if R satisfies the following axioms:*

(**r**0) *if $\alpha \in R$, then $\bar{\alpha} \in R$ for every $\alpha \in V^*$;*

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(**r**1) *if $uv\alpha x, vu\beta y, v\alpha xy \in R$, then $u\beta yx \in R$ for all $u, v, x, y \in V$ and all $\alpha, \beta \in V^*$.*

(**r**2) *if $uv\alpha x, xy \in R$, then there exists $\pi \in V^*$ such that $u\pi yx \in R$ or $uv\pi y \in R$ or $v\pi xy \in R$ for all $u, v, x, y \in V$ and all $\alpha, \beta \in V^*$.*

Proofs of Theorem A can be found in [2], [5], and [6].

By an *intervaloid function* on V we mean a mapping I of $V \times V$ into 2^V such that I satisfies the following properties for all $u, v, x, y \in V$:

$u, v \in I(u, v)$;

$I(u, u) = \{u\}$;

$x \in I(u, v)$ and $y \in I(x, v)$ if and only if $y \in I(u, v)$ and $x \in I(u, y)$.

Let I be an intervaloid function on V. By the graph determined by I we mean the graph G with $V(G) = V$ and

$$E(G) = \{\{u, v\}; u, v \in V, u \neq v, \text{ and } I(u, v) = \{u, v\} \text{ or } I(v, u) = \{u, v\}\}.$$

Let G be a connected graph with $V(G) = V$. Following by H.M. Mulder [1], by the *interval function* of G we mean the mapping I of $V \times V$ into 2^V such that

$$I(u, v) = \{x \in V;\ x \text{ lies on an } u - v \text{ geodesic in } G\}$$

for all $u, v \in V$.

Let G be a connected graph G with $V = V(G)$, and let I denote the interval function of G. We can see that I is an intervaloid function on V and G is the graph determined by I.

Let I be an intervaloid function on V. It is obvious that if I is the interval function of a connected graph G, then G is determined by I.

The characterization of the interval function of a connected graph given in [3] can be reformulated as follows:

Theorem B *Let I be an intervaloid function on V, and let G be the graph determined by I. Then G is connected and I is the interval function of G if and only if I satisfies the following axioms:*

(**i**0) *$I(v, u) = I(u, v)$ for all $u, v \in V$;*

(**i**1) *if $|I(u, v)| = I(x, y)| = 2$, $v \in I(u, x)$ and $u, x \in I(v, y)$, then $y \in I(u, x)$ for all $u, v, x, y \in V$.*

(**i**2) *if $|I(u, v)| = I(x, y)| = 2$ and $v \in I(u, x)$, then $y \in I(u, x)$ or $v \in I(u, y)$ or $x \in I(v, y)$ for all $u, v, x, y \in V$.*

Proofs of Theorem B can be found in [3], [4], and [7]; one more proof of Theorem B can be easily derived from the proof of Theorem 1 in [8]. Theorem B is deeper than Theorem A and each of the mentioned proofs of Theorem B is considerably complicated than any proof of Theorem A. None of those proofs of Theorem B depends on Theorem A.

In the present paper Theorem B will be deduced from Theorem A. A very natural bijection of the set of all road systems on V onto the set of all intervaloid functions on V will be presented in our Theorem 1. Theorem B will be a consequence of Theorems A and 1.

For every road system R on V, we denote by $\mathcal{I}(R)$ the mapping I of $V \times V$ into 2^V such that $I(u,v) =$

$$\{x \in V; \text{there exists } \alpha \in V^* \text{ such that } u = F\alpha, v = L\alpha, \text{ and } x \text{ lies on } \alpha\}$$

for all $u, v \in V$.

For every intervaloid function I on V, we denote by $\mathcal{R}(I)$ the set of all $\alpha \in V^*$ such that there exist $v_0, \dots, v_n \in V$, where $n \geq 0$, $\alpha = v_0 \dots v_n$,

$$v_j \in I(v_i, v_k) \text{ for all } 0 \leq i < j < k \leq n,$$

and

$$v_m \neq v_{m+1} \text{ and } I(v_m, v_{m+1}) = \{v_m, v_{m+1}\} \text{ for all } m, 0 \leq m < n.$$

The next theorem is the main result of this paper:

Theorem 1 *Let* $\mathbf{I}$ *denote the set of all intervaloid functions on* V *and let* $\mathbf{R}$ *denote the set of all road systems on* V*. Then* $\mathcal{R}$ *is a bijection of* $\mathbf{I}$ *onto* $\mathbf{R}$*,* $\mathcal{I} = \mathcal{R}^{-1}$*, and the following statements hold for each* $I \in \mathbf{I}$*:*

(A) *the graph determined by* $\mathcal{R}(I)$ *is identical to the graph determined by* I*;*

(B) $\mathcal{R}(I)$ *satisfies the axiom* ($\mathbf{r}n$) *if and only if* I *satisfies the axiom* ($\mathbf{i}n$) *for each* $n = 0, 1, 2$.

It is easy to see that Theorem B is a consequence of Theorems A and 1.

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Algorithm for Computing Structural Descriptions and Obstructions of Tree Ideals

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Abstract

A *Tree ideal* is a set of finite acyclic graphs that is closed under either minor or topological minor relation. Robertson, Seymour and Thomas have given a finite structural description for rooted, directed finite tree ideals closed under topological inclusion. A recent result extends their description to undirected and unrooted tree ideals closed under either minor or topological minor inclusion. Non-constructive proofs have been given about existence of an algorithm that computes a finite structural description given the obstruction set of $\mathcal{I}$ and vice versa. We present an explicit algorithm that works in both directions.

The free complex of a two-dimensional generalized convex shelling

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Abstract

In this article, we study two open problems posed by Edelman & Reiner about topology of certain simplicial complexes associated with abstract convex geometries. As a special case, we concentrate on generalized convex shellings, which were introduced by Kashiwabara, Nakamura & Okamoto for their representation theorem of abstract convex geometries. We answer one of the problems affirmatively for 2-dimensional separable generalized convex shellings, and the other problem negatively already for this case.

1 Introduction

An (abstract) convex geometry was introduced by Edelman & Jamison [2] as a combinatorial abstraction of "convexity" appearing in a lot of objects. Recently, a representation theorem for convex geometries has been established by Kashiwabara, Nakamura & Okamoto [6], which states that any convex geometry is isomorphic to some "separable generalized convex shelling." A generalized convex shelling is defined via two finite point sets in a certain dimension. Therefore, their representation theorem gives a stratification of the convex geometries by the minimum dimension in which a convex geometry can be realized as a generalized convex shelling. We will study the topology of the free complex of a 2-dimensional generalized convex shelling.

The motivation of this work stems from Edelman & Reiner [3]. An Euler-Poincaré type formula for the number of interior points in a d-dimensional point configuration was proved by Ahrens, Gordon & McMahon [1] when $d = 2$, and proved by Edelman & Reiner [3] and Klain [7] independently for any d. The approach by Klain [7] used a more general theorem on valuation, while that by Edelman & Reiner [3] was topological. (Their topological proof was later simplified with use of oriented matroids by Edelman, Reiner & Welker [4].) In the paper by Edelman & Reiner [3], they studied the topology of deletions in the free complex of a convex shelling (which arises from a point configuration), and also asked a possible generalization to any convex geometry, i.e., the following problems.

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Open Problems 1 (Edelman & Reiner [3]) *Let $\mathcal{L}$ be a convex geometry on E and denote the free complex of $\mathcal{L}$ by* $\mathrm{Free}(\mathcal{L})$.

1. *Is* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *of $x \in E$ contractible if* $\mathrm{Dep}_{\mathcal{L}}(x) \neq E$*?*
2. *Is* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *homotopy equivalent to a bouquet of some equidimensional spheres*[1] *if* $\mathrm{Dep}_{\mathcal{L}}(x) = E$*? Does* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *have the same homology type as a bouquet of some equidimensional spheres for such a case?*

Edelman & Reiner showed, in the same paper [3], that Open Problems 1 are valid for some special cases and conjectured that the first part of the problems is valid for all convex geometries. Later, Edelman, Reiner & Welker [4] showed that Open Problems 1 are also valid for another special case.

Open Problems 1 ask for a generalization to all convex geometries, and thanks to Kashiwabara, Nakamura & Okamoto [6] every convex geometry is isomorphic to some separable generalized convex shelling. That is why it is worth looking at 2-dimensional separable generalized convex shellings as a simpler case. Our result states that Open Problem 1.1 is valid for 2-dimensional separable generalized convex shellings, but Open Problem 1.2 is not valid. Namely, to be more precise for the first statement, we will prove the following.

Theorem 1 *Let P and Q be nonempty finite point sets in $\mathbb{R}^2$ such that* $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$. *In addition, let $\mathcal{L}$ be the generalized convex shelling on P with respect to Q. Consider the free complex* $\mathrm{Free}(\mathcal{L})$ *of $\mathcal{L}$. Then the deletion* $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ *of an element $x \in P$ is contractible if* $\mathrm{Dep}_{\mathcal{L}}(x) \neq P$.

2 Preliminaries

Due to the space limitation, we omit the necessary definitions of graph theory and topology.

A *convex geometry* on a nonempty finite set E is a family $\mathcal{L}$ of subsets of E satisfying the following three conditions: (1) $\emptyset, E \in \mathcal{L}$, (2) if $X, Y \in \mathcal{L}$ then $X \cap Y \in \mathcal{L}$, and (3) if $X \in \mathcal{L} \setminus \{E\}$ then there exists $e \in E \setminus X$ such that $X \cup \{e\} \in \mathcal{L}$. For a convex geometry $\mathcal{L}$ on E, we define an operator $\tau_{\mathcal{L}} : 2^E \to 2^E$ as $\tau_{\mathcal{L}}(A) = \bigcap\{X \in \mathcal{L} : A \subseteq X\}$. The operator $\tau_{\mathcal{L}}$ is called the *closure operator* of $\mathcal{L}$.

For a set $A \subseteq E$, an element $e \in A$ is called an *extreme point* if $e \notin \tau_{\mathcal{L}}(A \setminus \{e\})$. We denote the set of extreme points of A by $\mathrm{ex}_{\mathcal{L}}(A)$. Namely, define the operator $\mathrm{ex}_{\mathcal{L}} : 2^E \to 2^E$ as $\mathrm{ex}_{\mathcal{L}}(A) = \{e \in A : e \text{ is an extreme point of } A\}$. We call $\mathrm{ex}_{\mathcal{L}}$ the *extreme point operator*.

A set $A \subseteq E$ is called *independent* if $\mathrm{ex}_{\mathcal{L}}(A) = A$. We say that e *depends on* f if there exists an independent set A such that $f \in A$, $e \in \tau_{\mathcal{L}}(A)$ and $e \notin \tau_{\mathcal{L}}(A \setminus \{f\})$. We denote the set of all elements f on which e depends by $\mathrm{Dep}_{\mathcal{L}}(e)$ and call it the *dependency set* of e. A set $X \subseteq E$ is called *free* if $X \in \mathcal{L}$ and $\mathrm{ex}_{\mathcal{L}}(X) = X$. We denote the family of free sets of a convex geometry $\mathcal{L}$ by $\mathrm{Free}(\mathcal{L})$. Note that $\mathrm{Free}(\mathcal{L})$ forms a simplicial complex for any

[1] Note that a contractible complex is not considered as a bouquet of some equidimensional spheres.

convex geometry $\mathcal{L}$. Thus, it is natural that we call Free($\mathcal{L}$) the *free complex* of a convex geometry $\mathcal{L}$.

Now we will define a generalized convex shelling. Let P and Q be finite point sets in $\mathbb{R}^d$ (where d is a positive integer) such that $P \cap \mathrm{conv}(Q) = \emptyset$. Then the *generalized convex shelling* on P with respect to Q is a convex geometry $\mathcal{L}$ defined as follows: $\mathcal{L} = \{X \subseteq P : P \cap \mathrm{conv}(X \cup Q) = X\}$.[2] We also call a convex geometry $\mathcal{L}$ a *d-dimensional generalized convex shelling* if there exist finite point sets P and Q in $\mathbb{R}^d$ such that $P \cap \mathrm{conv}(Q) = \emptyset$ and $\mathcal{L}$ is isomorphic to the generalized convex shelling on P with respect to Q. A generalized convex shelling on P with respect to Q is called *separable* if $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$; otherwise it is *non-separable*. The next lemma tells us the closure operator and the extreme point operator of a generalized convex shelling.

Lemma 1 *Let $\mathcal{L}$ be a generalized convex shelling on P with respect to Q. Then, for a set $X \subseteq P$ we have $\tau_{\mathcal{L}}(X) = P \cap \mathrm{conv}(X \cup Q)$ and $\mathrm{ex}_{\mathcal{L}}(X) = \{x \in X : x$ is an extreme point of $\mathrm{conv}(X \cup Q)\}$.[3] In particular, $X \subseteq P$ is free if and only if $P \cap \mathrm{conv}(X \cup Q) = X$ and every element of X is an extreme point of $\mathrm{conv}(X \cup Q)$.*

In this paper, we study the free complex of a 2-dimensional separable generalized convex shelling. Since Open Problems 1 are valid if $Q = \emptyset$ [3], we want to concentrate on the case in which $Q \neq \emptyset$. Let us state that as an assumption.

Assumption 1 *When we talk about the generalized convex shelling on P with respect to Q in the rest of this paper, Q is always nonempty unless stated otherwise.*

Here, we define the clique complex of a graph. Let G be a graph. A *clique* of G is a vertex subset of G which induces a complete subgraph. The *clique complex* of G is the family of cliques of G. We treat the empty set and the single vertices as cliques, so the clique complex is actually a simplicial complex. In the literature, a clique complex is also called a *flag complex*.

3 On Open Problem 1.1 (proof of Theorem 1)

Now we concentrate on 2-dimensional separable generalized convex shellings. Namely, P and Q are nonempty finite point sets in $\mathbb{R}^2$ satisfying that $\mathrm{conv}(P) \cap \mathrm{conv}(Q) = \emptyset$ and $\mathcal{L}$ is the generalized convex shelling on P with respect to Q.

To prove Theorem 1, we will use the following fact, which is a consequence of a proposition by Hachimori & Nakamura [5].

Lemma 2 *A minimal nonface of the free complex* Free($\mathcal{L}$) *of a d-dimensional generalized convex shelling is of size at most d.*

[2] Actually, Q does not have to be a finite point set. It can be any subset of $\mathbb{R}^d$ satisfying that $P \cap \mathrm{conv}(Q) = \emptyset$. But for the simplicity, we will require Q to be a finite set.

[3] Here, you would notice that we are using the words "extreme point" in two different ways. One for an extreme point of a convex geometry, one for an extreme point of the convex hull. But they should be clear from the context.

It is well known that a simplicial complex whose minimal nonfaces are of size 2 is a clique complex of some graph. Therefore, so is the free complex of a 2-dimensional generalized convex shelling $\mathcal{L}$ by Lemma 2 (and this graph is actually the 1-dimensional skeleton of $\mathrm{Free}(\mathcal{L})$). Denote by $G(\mathcal{L})$ a unique graph whose clique complex is $\mathrm{Free}(\mathcal{L})$. The next is a key lemma.

Lemma 3 *$G(\mathcal{L})$ is chordal and connected.*

Note that Lemma 3 fails for a non-separable case. Since the clique complex of a connected chordal graph is contractible (not difficult to show), we can immediately find that the free complex of a 2-dimensional separable generalized convex shelling is contractible. Note that this holds for all d-dimensional (possibly non-separable) generalized convex shellings even if $Q = \emptyset$. A proof of this fact has already been given by Edelman & Reiner [3] (based on a theorem in Edelman & Jamison [2]). Notice that our approach is discrete-geometric while they used tools from combinatorial topology. However, we are not aware of a discrete-geometric proof for the higher-dimensional case.

Since an induced subgraph of a chordal graph is also chordal, we can immediately see that if x be a vertex of $G(\mathcal{L})$ and c_x be the number of connected components of $G(\mathcal{L}) - x$ then $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(x)$ is homotopy equivalent to c_x distinct points. Therefore, to prove Theorem 1, we only have to show the following lemma.

Lemma 4 *A vertex x of $G(\mathcal{L})$ is not a cut-vertex of $G(\mathcal{L})$ if $\mathrm{Dep}_{\mathcal{L}}(x) \neq P$.*

Thus, we are able to finish the proof of Theorem 1. Note that the converse of Lemma 4 does not hold in general. That is exactly the reason why Open Problem 1.2 is invalid, as we will see in the next section.

4 On Open Problem 1.2 (a negative answer)

Here we will give an example which answers Open Problem 1.2 negatively. Look at the figure. In this example, $P = \{1, 2, 3, 4, 5\}$ and $Q = \{q_1, q_2\}$. Let $\mathcal{L}$ be the generalized convex shelling on P with respect to Q. The solid lines show the edges of $G(\mathcal{L})$. We can observe that $\mathrm{Dep}_{\mathcal{L}}(4) = P$. However, the deletion of 4 from $G(\mathcal{L})$ results in a connected graph, therefore $\mathrm{del}_{\mathrm{Free}(\mathcal{L})}(4)$ is contractible, which implies that this is not homotopy equivalent to (or does not have the same homology type as) a bouquet of equidimensional spheres.

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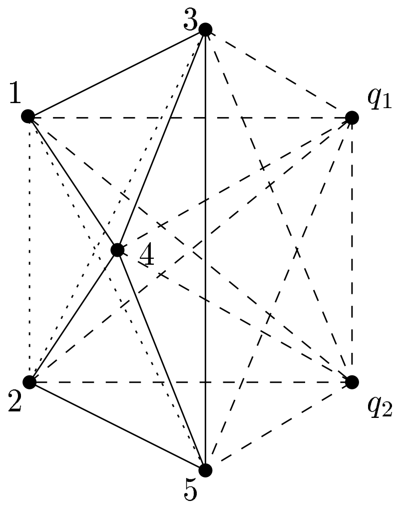

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The polytope of non-crossing graphs on a planar point set

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1 Introduction

The set of (straight-line, or geometric) non-crossing graphs with a given set of vertices $\mathcal{A}$ in the plane is of interest in Computational Geometry, Geometric Combinatorics, and related areas. In particular, much effort has been directed towards enumeration, counting and optimization on the set of maximal such graphs, that is to say, triangulations of $\mathcal{A}$. But little is known about the poset structure of the set of all non-crossing subgraphs under inclusion. In this paper we associate to $\mathcal{A}$ a polytope whose face poset contains the poset of non-crossing graphs on $\mathcal{A}$ embedded in a very nice way.

The construction is based on [5], where a polyhedron $\overline{X_f}(\mathcal{A})$ of dimension $2n-3$ with face poset (opposite to) that of pointed non-crossing graphs on $\mathcal{A}$ is constructed. A straight-line graph embedded in the plane is called *pointed* if the edges incident to every vertex span an angle smaller than 180 degrees. Let n denote the size of $\mathcal{A}$, n_b and n_i the number of points in the boundary and the interior of its convex hull, respectively. The polyhedron $\overline{X_f}(\mathcal{A})$ has a unique maximal bounded face $X_f(\mathcal{A})$, of dimension $2n_i+n_b-3$, there called the *polytope of pointed pseudo-triangulations* of $\mathcal{A}$.

Our main new ingredient is that we consider "marked" non-crossing graphs, meaning non-crossing graphs together with the specification of a subset of their pointed vertices. With similar ideas but with n extra coordinates for the n possible marks, we get a polyhedron $\overline{Y_f}(\mathcal{A})$ of dimension $3n-3$ with a unique maximal bounded face $Y_f(\mathcal{A})$ of dimension $3n_i+n_b-3$.

The technical tools both in our construction and in [5] are *pseudo-triangulations* of planar point sets and their relation to structural rigidity of non-crossing graphs. Pseudo-triangulations, first introduced by Pocchiola and Vegter around 1995 (see [3]), have by

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now been used in many Computational Geometry applications, among them visibility [4, 3], ray shooting [1], and kinetic data structures [2]. Streinu [6] introduced the *minimum* or *pointed pseudo-triangulations*, and used them to prove the Carpenter's Rule Theorem. Pointed pseudo-triangulations turn out to coincide with the maximal non-crossing and pointed graphs; that is to say, with the vertices of the polytope $X_f(\mathcal{A})$. Our method extends that construction and $Y_f(\mathcal{A})$ is the *polytope of pseudo-triangulations*, since its vertices correspond to all pseudo-triangulations of $\mathcal{A}$.

2 Overview of the method and results

In the sequel we assume our point set $\mathcal{A}$ to be in general position, although the results are proved in the full paper for point sets in degenerate position as well. Let n_i and n_b be the number of points of $\mathcal{A}$ in the interior and boundary of $\mathrm{conv}(\mathcal{A})$, respectively, and let $n = n_i + n_b$. We define:

Definition 1 (Flips in pseudo-triangulations) Let T be a pseudo-triangulation of $\mathcal{A}$. We call *flips* in T the following three types of operations, all producing pseudo-triangulations.

- (*Deletion flip*). The removal of an edge $e \in T$, if $T \setminus e$ is a pseudo-triangulation.
- (*Insertion flip*). The insertion of an edge $e \notin T$, if $T \cup e$ is a pseudo-triangulation.
- (*Diagonal flip*). The exchange of an edge $e \in T$, if $T \setminus e$ is not a pseudo-triangulation, for the unique edge e' such that $(T \setminus e) \cup e'$ is a pseudo-triangulation.

The *graph of pseudo-triangulations* of $\mathcal{A}$ has as vertices all the pseudo-triangulations of $\mathcal{A}$ and as edges all flips of any of the types.

Proposition 1 *The graph of pseudo-triangulations of $\mathcal{A}$ is connected and regular of degree $3n_i + n_b - 3 = 3n - 2n_b - 3$.*

As happened with pointed pseudo-triangulations, Proposition 1 suggests that the graph of pseudo-triangulations of $\mathcal{A}$ may be the skeleton of a simple polytope of dimension $3n_i + n_b - 3$. As a step towards this result we look at what the face poset of such a polytope should be. The polytope being simple means that we want to regard each pseudo-triangulation T as the upper bound element in a Boolean poset of order $3n - 3 - 2n_b$. This number equals the number of interior edges plus interior pointed vertices in T:

Definition 2 A *marked graph* on $\mathcal{A}$ is a geometric graph with vertex set $\mathcal{A}$ together with a subset of its vertices, that we call "marked". We call a marked graph *non-crossing* if it is non-crossing as a graph and marks arise only in pointed vertices.

We call a non-crossing marked graph *fully-marked* if it is marked at all pointed vertices. If, in addition, it is a pseudo-triangulation, then we call it a *fully-marked pseudo-triangulation*, abbreviated as *f.m.p.t.*

We start defining a linear cone $\overline{Y_0}(\mathcal{A})$ by one inequality for each possible edge and each point of $\mathcal{A}$. Its $\binom{n+1}{2}$ facets are then translated using the entries of a vector f in $\mathbb{R}^{\binom{n+1}{2}}$ to produce a polyhedron $\overline{Y_f}(\mathcal{A})$ which has as unique maximal bounded face a polytope $Y_f(\mathcal{A})$. Our proof goes by analyzing the necessary and sufficient conditions for f to produce a polytope with the desired properties. We get the next result, where flips in fully-marked pseudo-triangulations are defined in the natural way from those in pseudo-triangulations:

Theorem 1 (The polyhedron of marked non-crossing graphs) *If f is a valid choice of parameters, then there is a simple polyhedron $\overline{Y_f}(\mathcal{A})$ of dimension $3n-3$ whose face poset equals (the opposite of) the poset of non-crossing marked graphs on $\mathcal{A}$. In particular:*

(a) *Vertices of the polyhedron are in 1-to-1 correspondence with fully-marked pseudo-triangulations of $\mathcal{A}$.*

(b) *Bounded edges correspond to flips of interior edges or marks in fully-marked pseudo-triangulations, i.e., to fully-marked pseudo-triangulations with one interior edge or mark removed.*

(c) *Extreme rays correspond to fully-marked pseudo-triangulations with one convex hull edge or mark removed.*

We prove valid choices of f to be the interior of a convex polyhedron defined by $\binom{n}{4}$ strict inequalities and give an explicit choice. Then, from the existence of a valid f and the theorem above, the following result is concluded:

Theorem 2 (The polytope of all pseudo-triangulations) *Let $Y_f(\mathcal{A})$ be the face of $\overline{Y_f}(\mathcal{A})$ defined turning into equalities the equations which correspond to convex hull edges or convex hull points of $\mathcal{A}$, and assume f to be a valid choice. Then:*

1. *$Y_f(\mathcal{A})$ is a simple polytope of dimension $3n-2n_b-3$ whose 1-skeleton is the graph of pseudo-triangulations of $\mathcal{A}$. (In particular, $Y_f(\mathcal{A})$ is the unique maximal bounded face of $\overline{Y_f}(\mathcal{A})$).*

2. *Let F be the face of $Y_f(\mathcal{A})$ defined by turning into equalities the equations corresponding to interior points. Then, the complement of the star of F in the proper face-poset of $Y_f(\mathcal{A})$ equals (the opposite of) the poset of non-crossing graphs on $\mathcal{A}$ which use all the convex hull edges.*

This result, which proves the claims advanced in the introduction, deserves some words of explanation:

- Since convex hull edges are irrelevant to crossingness, the poset of *all* non-crossing graphs on $\mathcal{A}$ is the direct product of the poset in the statement and a Boolean poset of rank n_b.

- By "proper" face poset of a polytope we mean that the polytope itself is not considered a face.

- We give a fully explicit facet description of $Y_f(\mathcal{A})$. It lives in $\mathbb{R}^{3n}$ and is defined by 3 linear equalities and $\binom{n}{2} + n$ linear inequalities.

It is worth mentioning that our results have some rigid-theoretic consequences. Namely:

Theorem 3 *Let T be a pseudo-triangulation of a planar point set A. Let G be its underlying graph. Then:*

1. *G is infinitesimally rigid, hence rigid and generically rigid.*

2. *There are at least $2k + 3l$ edges of T incident to any subset of k pointed plus l non-pointed vertices of T.*

If the pseudo-triangulation is pointed, then it has $2n - 3$ edges and part (2) is just the Laman characterization of isostatic graphs in the plane as graphs with $2n - 3$ edges with every k vertices incident to at least $2k$ edges. In particular, Theorem 3 generalizes Ileana Streinu's result [6] that pointed pseudo-triangulations are isostatic graphs.

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The $\mathcal{H}$-factor problem in directed graphs

Gyula Pap*

1 Introduction

In this extended abstract we discuss a new attempt to handle factorization problems, where the question is, whether a graph $G = (V, E)$ – directed or undirected – can be covered by a disjoint union of allowed cycles. The set of allowed cycles is a set $\mathcal{H}$ of some cycles in G. In [3] Cornuéjols and Hartvigsen gave a min-max relation for the maximum size of a triangle-free 2-matching, which could be extended to exclusion of higher poligons [3, 2]. In [6] L. Szegő and the author gave a formula for the so-called maximum even factor problem in weakly symmetric graphs. These two results can be derived from the main result of the paper.

The main result, Theorem 1. solves a factorization problem when the directed graph G is $\mathcal{H}$-symmetric, defined later. In fact, we get a min-max formula for a deficit version of the problem. The definition of $\mathcal{H}$-symmetric graphs was motivated by weakly symmetry (as for the even factor problem). The formula has an aspect similar to those in [3, 2]: the right hand side is defined implicitly, that means we do not have a good characterization in terms of co-$\mathcal{NP}$. A special feature of the notions is, that factorcriticality can be captured as in the Gallai-Edmonds decomposition (see [5]).

2 Main result

Let $G = (V, E)$ be a directed graph. A cycle is the edgeset of a closed directed walk without repetition of edges or vertices. A cycle is *even*, if it has even number of edges (a cycle of length two is regarded as even). A cycle is *asymmetric*, whenever it has at least one edge uv such that $vu \notin E$. Let $\mathcal{H}$ be a collection of some cycles in G such that $\mathcal{H}$ contains all even cycles and all asymmetric cycles (and probably some others). If we have a pair $(G, \mathcal{H})$ as above, we say G is *$\mathcal{H}$-symmetric*. An *$\mathcal{H}$-factor* is a subset M of edges, such that the components of (V, M) are directed paths and $\mathcal{H}$-cycles. Let $\nu^{\mathcal{H}}(G)$ denote the maximum cardinality of an $\mathcal{H}$-factor.

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We consider the problem of determining $\nu^{\mathcal{H}}(G)$. The set $\mathcal{H}$ can be chosen in many ways, thus we neglect computational complexity.

Theorem 1 *If $G=(V,E)$ is an $\mathcal{H}$-symmetric directed graph, then*

$$\nu^{\mathcal{H}}(G)=\min_{X\subseteq V}|V|+|N_G^+(X)|-sc^{\mathcal{H}}G[X] \tag{1}$$

where $N_G^+(X):=\{x\in V-X:\exists y\in X, yx\in E\}$, and $sc^{\mathcal{H}}G[X]$ denotes the following number. Consider the strongly connected components in $G[X]$, we take the source components: $sc^{\mathcal{H}}G[X]$ is the number of source components in $G[X]$, that can not be covered by $\mathcal{H}$-cycles.

Theorem 1. generalizes two problems: even factors and $\mathcal{H}'$-factors in undirected graphs.

The notion of even factors was introduced by W.H. Cunningham in [1]. An *even factor* is a subset M of edges, such that the components of (V,M) are directed paths and even cycles. $\nu(G)$ denotes the maximum cardinality of an even factor. Determining $\nu(G)$ is $\mathcal{NP}$-complete in general. Thus we consider weakly symmetric graphs, that means: Let $G=(V,E)$ be a directed graph such that if an edge $uv\in E$ is contained in an odd cycle, then $vu\in E$. W.H. Cunningham and J.F. Geelen discovered an algebraic method to compute the maximum cardinality of an even factor in a weakly symmetric graph, later L. Szegő and the author [6] proved a compact formula:

Theorem 2 *If $G=(V,E)$ is a weakly symmetric directed graph, then*

$$\nu(G)=\min_{X\subseteq V}|V|+|N_G^+(X)|-odd\,G[X] \tag{2}$$

where $odd\,G[X]$ denotes the number of source components in $G[X]$ having an odd number of nodes.

Theorem 1. generalizes formula (2), since if G is weakly symmetric and $\mathcal{H}$ consists only of the even cycles, then G is $\mathcal{H}$-symmetric, as well.

Cornuéjols and Pulleyblank [4] derived a min-max formula for the maximum size of a triangle-free 2-matching in an undirected graph. Cornuéjols and Hartvigsen [3] discovered an extension from triangles to a collection of factorcritical subgraphs. More generally, in [2] J. Szabó and Z. Király considered general factorization problems in undirected graphs, here we only look at the relevant special case. Let $G=(V,E)$ be an undirected graph, and let $\mathcal{H}'$ be a set of allowed subgraphs: For each edge $uv\in E$ let $\{uv\}\in\mathcal{H}'$, and the other members in $\mathcal{H}'$ are some odd cycles. An *$\mathcal{H}'$-factor* is a vertex-disjoint union of some members of $\mathcal{H}'$.

Theorem 3 *The minimum number of nodes exposed by an $\mathcal{H}'$-factor is equal to*

$$\max_{Y\subseteq V}c^{\mathcal{H}'}(Y)-|Y|,$$

where $c^{\mathcal{H}'}(Y)$ denotes the number of components C of $G-Y$ which can not be covered by disjoint members H of $\mathcal{H}'$ such that $V(H)\subseteq C$.

This theorem can be derived from Theorem 1. by the following reduction principle: For an undirected graph G we construct a directed graph on the same vertex set by replacing the edges by a pair of oppositely oriented edges. The set $\mathcal{H}$ is constructed in the obvious way: we take all even cycles and the projections of the odd cycles in $\mathcal{H}'$.

We have the following strengthening of Theorem 1. Under the assumptions of Theorem 1, we have a minimizing set X in the formula (1) such that the source components of $G[X]$ are *factorcritical* (i.e. they arise from a factorcritical graph by the reduction principle above). This gives us the statement mentioned in [3, 2]: $\mathcal{H}'$-criticality implies factorcriticality.

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A construction for non-rank facets of stable set polytopes of webs

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Abstract

Graphs with circular symmetry, called webs, are relevant w.r.t. describing the stable set polytopes of two larger graph classes, quasi-line graphs [6, 10] and claw-free graphs [5, 6]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [7]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [3, 13] while there are examples with clique number ≥ 4 having non-rank facets [8, 9, 10, 12].

In this paper, we provide a construction for non-rank facets of stable set polytopes of webs. We use this construction to prove, for several fixed values of ω including all odd values at least 5, that there are only finitely many webs with clique number ω whose stable set polytopes admit rank facets only.

Kywords: web, rank-perfect graph, stable set polytope, (non-)rank facet

A natural generalization of odd holes and odd antiholes are graphs with circular symmetry of their maximum cliques and stable sets, called *webs*: a web W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (mod n) and $i \neq j$.

The webs W_9^k on nine nodes are depicted in Figure 1. Notice that the clique number (maximum size of a set of pairwise adjacent vertices) of a web W_n^k is $k+1$ and that webs are also called circulant graphs C_n^k [2]. Furthermore, graphs $W(n,k)$ with $n \geq 2$, $1 \leq k \leq \frac{1}{2}n$ and $W(n,k) = \overline{W}_n^{k-1}$ were introduced in [13].

Webs and line graphs belong to the classes of *quasi-line graphs* (the neighborhood of any node can be partitioned into two cliques) and *claw-free graphs* (the neighborhood of

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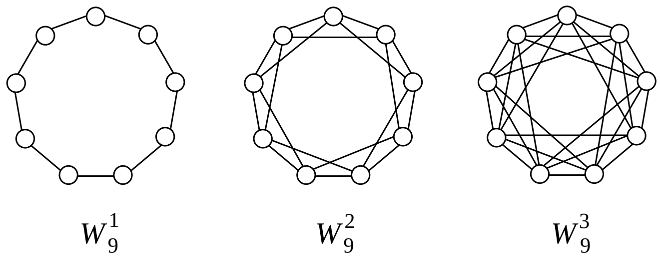

Figure 1

any node does not contain a stable set of size 3). The *line graph* $L(H)$ of a graph H is obtained by taking the edges of H as nodes of $L(H)$ and connecting two nodes in $L(H)$ iff the corresponding edges of H are incident.

Webs and line graphs are relevant w.r.t. describing the stable set polytopes of those larger graph classes [5, 6, 10]. The *stable set polytope* STAB(G) of G is defined as the convex hull of the incidence vectors of all stable sets of the graph $G = (V, E)$. A *rank constraint* associated with an induced subgraph $G' \subseteq G$ is the 0/1-constraint of the form

$$\sum_{i \in G'} x_i \leq \alpha(G')$$

(where $\alpha(G')$ denotes the cardinality of a maximum stable set in G'). A graph is *rank-perfect* if all non-trivial facets of its stable set polytope are rank constraints. The class of rank-perfect graphs contains all *perfect graphs* due to [1] since all non-trivial facets of their stable set polytopes are rank constraints

$$\sum_{i \in Q} x_i \leq 1$$

associated with (inclusion-wise maximal) cliques Q (note $\alpha(G') = 1$ holds iff G' is a clique).

All facets of the stable set polytope of line graphs are known from matching theory [4]. In particular, line graphs are rank-perfect. A characterization of the *rank* facets in stable set polytope of claw-free graphs was given by Galluccio & Sassano [5]. They showed that all rank facets can be constructed by means of standard operations from rank constraints associated with cliques, partitionable webs, or line graphs of 2-connected, critical hypomatchable graphs.

However, we are still far from having a complete description for the stable set polytopes of webs and, therefore, of quasi-line and claw-free graphs, too. Finding a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [7]. Claw-free graphs are not rank-perfect: Giles & Trotter [6], Oriolo [10], and Liebling et al [9] found non-rank facets which occur even in the stable set polytopes of quasi-line graphs. These non-rank facets rely on combinations of joined webs.

Several further authors studied the stable set polytopes of webs. Obviously, webs W_n^1 with clique number 2 are holes, hence they are perfect if n is even and minimally imperfect if n is odd. Thus, all webs with clique number 2 are rank-perfect due to [1, 11]. Dahl [3] studied webs W_n^2 with clique number 3 and showed that they are rank-perfect. On the other hand, Kind [8] found (by means of the PORTA software[1]) examples of webs with clique number > 4 the stable set polytopes of which have *non*-rank facets, e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9. Oriolo [10], Liebling et al [9], and Pêcher and Wagler [12] presented further examples of such webs.

In this paper, we prove with the help of a construction for non-rank facets that there are, for several fixed values of k, only finitely many rank-perfect webs W_n^k: all webs W_n^4 with $n > 45$ are not rank-perfect and for every odd $\omega \geq 7$, all webs $W_n^{\omega-1}$ with $n \geq 2\omega^3$ are not rank-perfect. This indicates how difficult finding a complete linear description of the stable set polytopes of webs is.

Outline of the proof

Our main result is Theorem 1 which states informally that in a weak non-rank facet (without going into details, weak non-rank facets are a subclass of non-rank facets, which is not very restrictive) of a web W_n^k, one can insert a maximum clique and get a weak non-rank facet of W_{n+k+1}^k.

Theorem 1 *If* $\mathrm{STAB}(W_n^k)$ *has a weak non-rank facet then* $\mathrm{STAB}(W_{n+k+1}^k)$ *has a weak non-rank facet.*

Therefore, if W_n^k has a weak non-rank facet then all webs $W_{n+\lambda(k+1)}^k$ $(\lambda \geq 0)$ are not rank-perfect, too. Hence Theorem 1 implies the following corollary:

Corollary 1 *If there are* $k+1$ *webs* $W_{n_0}^k, \dots, W_{n_k}^k$ *such that*

- $\mathrm{STAB}(W_{n_i}^k)$ *has a weak non-rank facet*
- $n_i = i \pmod{k+1}$

then all webs W_n^k *with* $n \geq \max\{n_0, \dots, n_k\} - k$ *are not rank-perfect.*

We provide such a set $W_{n_0}^k, \dots, W_{n_k}^k$ of not rank-perfect webs for $k = 3$ and each even $k \geq 4$.

Theorem 2 *If* $l = 2 \pmod 3$ *and* $l \geq 11$ *then* $\mathrm{STAB}(W_{3l}^3)$ *has a weak non-rank facet.*

Due to Theorem 2 from [12], the stable set polytopes of the webs W_{33}^3, W_{42}^3, W_{51}^3 and W_{60}^3 have a weak non-rank facet. Hence by Corollary 1 all webs W_n^3 with $n > 56$ are not rank-perfect.

Theorem 3 *If* $l = 1 \pmod 3$ *and* $l \geq 13$ *then* $\mathrm{STAB}(W_{2l}^4)$ *has a weak non-rank facet.*

[1]By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

Due to Theorem 3, the stable set polytopes of the webs W^4_{26}, W^4_{32}, W^4_{38}, W^4_{44} and W^4_{50} have a weak non-rank facet. Hence Corollary 1 implies again that all webs W^4_n with $n > 45$ are not rank-perfect.

At last, the fact that all webs $W^{\omega-1}_n$ with $n \geq 2\omega^3$ are not rank-perfect for each odd $\omega \geq 7$ follows from:

Theorem 4 *Let $\omega \geq 7$ be an odd integer. Then for every integer $j \geq 1$ holds*

- $\mathrm{STAB}\left(W^{\omega-1}_{\left(\frac{\omega+5}{2}+(\omega+1)j\right)(\omega-1)}\right)$ *has a weak non-rank facet if $\omega = 1 \pmod 4$;*
- $\mathrm{STAB}\left(W^{\omega-1}_{\left(\frac{\omega+3}{2}+(\omega-1)j\right)(\omega-1)}\right)$ *has a weak non-rank facet if $\omega = 3 \pmod 4$.*

This finally shows that, for $\omega = 4$ and for each odd $\omega \geq 5$, there are only finitely many rank-perfect webs with clique number ω, hence, almost all of them are not rank-perfect.

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A combinatorial measure of information

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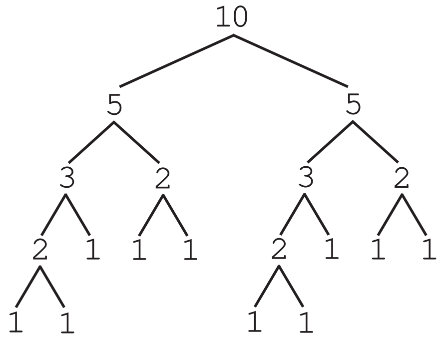
1 Introduction

Shannon information was introduced in 1948 and came to play an important role in pure and applied mathematics. Its success contributed to obscuring the fact that the concept of information admits alternative formulations, equally sound and perhaps more strongly motivated. The purpose of this paper is to explore one such formulation, based on ideas of combinatorics. The significance of this formulation is that it is derived in an elementary way from first principles, and intuitively it is very natural and powerfully motivated. Moreover, the measure of information introduced here — though discrete and not involving logarithms — has properties which are surprisingly similar to those of Shannon information. We begin with a review of partition trees.

2 Partition Trees

Partition trees have been used in the context of the following classical problem of combinatorics: There is a set A and an unknown element x in A whose identity we wish to discover. In order to do so, we are given a set $\pi_1, \pi_2, \ldots, \pi_n$ of bipartitions of A, and are allowed to ask a sequence of questions of the form "which of the two classes of bipartition π_i contains x?" (We may frame the queries as yes/no questions which ask whether x is, or is not, in a designated class of the partition.) The objective of the inquiry is to determine the least number of yes/no questions which suffice to identify any unknown element x. The connection with binary trees is obvious: For example, we ask: What is the least number of yes/no questions required, on average, to determine the identity of an unknown x in a set A of 10 elements? By inspection of the search tree shown below, it is evident that the average number of queries is 3.4. (It is the average depth of the leaves of the tree, because each level of the tree represents one question.) Note especially that the number of queries is not equal to $\log_2 10$ as occasionally suggested in expository discussions.



When speaking of a *tree* in this Section, we shall mean a rooted, directed binary tree. If T is a tree and v is any vertex of T, the *depth* of v in T is the length of the path from the root to v. The set of vertices of T is naturally partitioned into *levels*, where level k contains all those vertices whose depth is k.

A *partition tree* is a tree T together with a mapping ϵ from T to the set of non-negative integers, where ϵ has the following properties:
1. If w is a vertex and u and v are the children of w, then $\epsilon(w) = \epsilon(u) + \epsilon(v)$.
2. If v is a vertex of T, then v is a leaf iff $\epsilon(v) = 1$.

Let $d(v)$ denote the depth of the vertex v. If $\{f_1, f_2, \ldots, f_\ell\}$ is the set of nonzero leaves of T, then the *average depth* of T, denoted by $d(T)$, is

$$d(T) = \frac{1}{\ell}\sum_{i=1}^{\ell} d(f_i)$$

The *cumulative depth* of T is $D(T) = \sum_{i=1}^{\ell} d(f_i)$. If T is a partition tree whose root is r, and if $\epsilon(r) = m$, we say that T is a partition tree for m.

Definition 1 *A partition tree T for m is called* compact *if the cumulative depth of T is minimal among partition trees for m.*

Let $\Delta(m) = min\{D(T) : T$ is a partition tree for $m\}$. Then $\delta(m) = \Delta(m)/m$ is the minimum value of the average depth of a binary tree for m. From Definition 1, $\Delta(m)$ is the cumulative depth of any compact tree for m.

A *bipartition* of an integer m is a pair of nonzero integers (a, b) whose sum is m. We come now to the definition of the core concept of this paper. With every bipartition (a, b), we associate a rational number called the *information* of (a, b), defined by:

Definition 2 *The* information *of (a, b) is the quantity $I(a, b)$ given by:*

$$I(a, b) = \delta(a + b) - \frac{1}{a + b}[a\delta(a) + b\delta(b)] \tag{1}$$

The motivation behind this definition is the following: Recall that in our paradigmatic example, $\delta(m)$ is the least value of the mean number of queries needed to identify a target element in an m- element set. Now (a,b) partitions this m-element set into two classes, one with a elements, the other with b elements. Then the term $\frac{1}{a+b}[a\delta(a)+b\delta(b)]$ represents the minimum number of queries needed, on average, to pick the target element from either the a-element class or the b-element class, whichever the target is in. Thus, $I(a,b)$ is meant to be viewed as the amount of a resource called "information" contributed by the bipartition (a,b) toward the goal of partitioning an m-element set into singletons (thereby identifying any target element).

If vertices u and v are children of a common vertex w, then u and v are called *siblings*. The definition of *balanced tree* given next departs slightly from the more common definition of this term.

Definition 3 *A partition tree T is said to be* balanced *if, for any pair of sibling vertices u and v, there is an integer k such that*

$$2^k \leq \epsilon(u) \leq 2^{k+1} \ \ \textit{and} \ \ 2^k \leq \epsilon(v) \leq 2^{k+1}$$

The theorem which follows is a known result of the theory of combinatorial search. A recent proof may be found in [2].

Theorem 1 *(a) A partition tree T is compact iff T is balanced.*
(b) If T is a compact tree for m and has L levels, then $D(T) = (L+1)m - 2^L$.

From this we deduce that for any positive integer m, if $2^{L-1} \leq m \leq 2^L$ then $\delta(m) = (L+1) - 2^L/m$. Note that $\delta(2^k) = k = \log_2(2^k)$. As explained in the Introduction, $\delta(m)$ (and not $\log_2 m$) is the number of queries needed, on average, to single out one target element from a set of m. In certain contexts, the function δ may be viewed as a discrete counterpart of the function $\log_2$. It is known that on every interval $[2^k, 2^{k+1}]$, δ is a very close approximation of $\log_2$: The maximum difference $\log_2(m) - \delta(m)$, over all values of m, is equal to .086. (See Knuth [1], p.194.)

The *entropy* of a random variable $\mathbf{x}$ with sample space $\{x_1, x_2\}$ and probabilities p_1, p_2 is given by $H(\mathbf{x}) = -p_1\log(p_1) - p_2\log(p_2)$. We assume here that p_1 and p_2 are rational numbers, so there are integers a, b and m such that $a+b=m$, $p_1 = a/m$ and $p_2 = b/m$. Then the entropy, to be denoted here by $H(a,b)$, is given by

$$\begin{aligned} H(a,b) &= \frac{a}{m}\log\frac{m}{a} + \frac{b}{m}\log\frac{m}{b} \\ &= (\frac{a}{m} + \frac{b}{m})\log m - [\frac{a}{m}\log a + \frac{b}{m}\log b] \\ &= \log(a+b) - \frac{1}{a+b}[a\log a + b\log b] \end{aligned} \tag{2}$$

which is to be compared with (1). It is therefore clear that if we associate δ with $\log_2$, we are likewise associating $I(a,b)$ with $H(a,b)$.

The definition of $I(x_1, x_2)$ is easily generalized to n variables and is sometimes referred to as the *discrete entropy* function. It is shown in [3] that Shannon's first and second theorems hold for discrete entropy.

3 Defects in Partition Trees

If T is a partition tree for m and T is not balanced, then $D(T)$, the cumulative depth of T, exceeds $\Delta(m)$. If $E(T) = D(T) - \Delta(m)$, then $E(T)$ is the excess of the depth of T over the *minimum* depth of a tree for m. It is shown next that this excess depends on the number of powers of 2 which lie between a and b. Let $a, b \neq 0$:

Definition 4 *Let (a, b) be a bipartition of m, and let k be an integer such that exactly k successive powers of 2 lie between a and b. That is, let us assume that there are positive integers n and k such that*

$$2^n \leq a < 2^{n+1} < \ldots < 2^{n+k} < b \leq 2^{n+k+1} \tag{3}$$

For each j, $(1 \leq j \leq k)$, let t_j be the lesser of $|a - 2^{n+j}|$ and $|b - 2^{n+j}|$, and let

$$df(a, b) = \sum_{j=1}^{k} t_j$$

Then $df(a, b)$ is called the **defect** *of the bipartition (a, b).*

If T is a partition tree, w is an internal node of T, and u, v are the children of w, then the integer $\epsilon(w)$ splits into the two terms $\epsilon(u)$ and $\epsilon(v)$. We shall refer to this bipartition as an *internal bipartition* of T. Clearly, every internal node of T corresponds to an internal bipartition. Define the *defect of T* to be the sum of the defects of the internal bipartitions of T, and denote it by $df(T)$. If w is an internal node with children u and v, we let the symbol $df(w)$ mean $df(\epsilon(u), \epsilon(v))$.

Theorem 2 *(Pinter [2].) Let $w_1, w_2, \ldots, w_r$ be the internal nodes of a partition tree T for m, and let $df(T) = \sum_{i=1}^{r} df(w_i)$. Then the cumulative depth of T has the value*

$$D(T) = \Delta(m) + df(T) \tag{4}$$

Definition 5 *The quantity $\theta(a, b) = df(a, b)/(a + b)$ is called the* unit defect *of (a, b).*

Theorem 3 *(Pinter [2]). $I(a, b) = 1 - \theta(a, b)$*

We conclude that the amount of information associated with a bipartition (a, b) is maximal (that is, equal to 1) iff a and b lie between a common pair of consecutive powers of 2. If that fails to be the case —if one or more powers of 2 lie strictly between a and b — then $1 - I(a, b)$ is equal to the unit defect of (a, b). So it is of interest to note, for example, that $I(32, 64) = 1$, while $I(7, 9) = .94$.

4 Mutual Information

A partition tree represents a succession of partitions acting on a set A. If a first partition α divides A into two classes, then the succeeding partition β divides each of the classes of α into two subclasses. We shall now describe the relationship between the information of partitions which act upon one another. The same situation arises in Shannon information theory when two random variables $\mathbf{x}$ and $\mathbf{y}$ are not independent, and this has led to the notions of conditional and mutual information. Here we shall briefly discuss the discrete versions of these concepts.

Let us recall from Shannon information theory that if $\mathbf{x}$ and $\mathbf{y}$ are random variables with joint probability distribution $p(x_i, y_j)$, $1 \leq x, y \leq 2$, then the conditional entropy of $\mathbf{x}$ given $\mathbf{y}$ is defined as $H(\mathbf{x}|\mathbf{y}) = -\sum\sum p(x_i, y_j)\log p(x_i|y_j)$. If it is assumed that the probabilities are rational numbers, then there are integers a_c, a_d, b_c, b_d and n such that the probability distribution is $\{a_c/n, a_d/n, b_c/n, b_d/n\}$. In that case,

$$H(\mathbf{x}|\mathbf{y}) = [aH(a_c, a_d) + bH(b_c, b_d)]/(a+b) \tag{5}$$

where H(,) is defined in (2).

If π_1 is a bipartition of a set S into $\{A, B\}$ and π_2 is a bipartition of S into $\{C, D\}$, then π_2 induces a bipartition of A into $\{A \cap C, A \cap D\}$ and of B into $\{B \cap C, B \cap D\}$. Let $a = |A|$, $b = |B|$, $a_c = |A \cap C|$, $a_d = |A \cap D|$, $b_c = |B \cap C|$ and $b_d = |B \cap D|$. By analogy with (5), the *conditional information* of π_2, given π_1, is defined as follows:

$$I(\pi_2|\pi_1) = [aI(a_c, a_d) + bI(b_c, b_d)]/(a+b)$$

Thus, $I(\pi_2|\pi_1)$ is equal to the average of the information of the bipartitions which π_2 induces on the two classes of π_1. The *mutual information* of π_1 and π_2 is defined by:

$$M(\pi_1, \pi_2) = I(\pi_2) - I(\pi_2|\pi_1) \tag{6}$$

Theorem 4 *As above, let π_1 and π_2 be the partitions $\pi_1 = (a, b)$ and $\pi_2 = (c, d)$.*

$$M(\pi_1, \pi_2) = \theta(a_c, a_d) + \theta(b_c, b_d) - \theta(c, d) \tag{7}$$

The proof requires nothing more than making the required substitutions.

Formula (7) is quite remarkable as an "explanation" of mutual information in terms of defect. It is worth commenting further on this: Let S be the disjoint union of A and B and let π be a bipartition of S into $\{C, D\}$. Then π induces bipartitions of A into $\{A\cap C, A\cap D\}$ and of B into $\{B \cap C, B \cap D\}$, these bipartitions to be denoted by $\pi \restriction A$ and $\pi \restriction B$. Then the expression for the mutual information in (7) may be written as follows:

$$\theta(\pi \restriction A) + \theta(\pi \restriction B) - \theta(\pi)$$

Formula (7) suggests an observation worth commenting on: Mutual information is conceptually different from conditional and joint information. The latter two are, indeed, measures of information, but "mutual information" is a measure of defect.

Finally, while mutual information does not admit negative values in Shannon's theory, the discrete version may have negative values, as illustrated next:
Example. Let $\alpha = (24, 12)$ be the bipartition of a set A into classes C and D of cardinalities $|C| = 24$ and $|D| = 12$. Let $\beta = (24, 12)$ be another bipartition of A such that $\beta \restriction C = (16, 8)$ and $\beta \restriction D = (8, 4)$. From Formula 7, $M(\alpha, \beta) = 0 + 0 - 4$.

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A proof of two Erdős' conjectures on restricted addition and further results

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Paul Erdős was interested in restricted addition, an operation which coincides with the classical addition except that the addition of two identical elements is not allowed. He asked many questions on the subject. More than twenty years ago, the two following conjectures were again stated in the important book of problems and questions *Old and new problems and results in combinatorial number theory* (Monographies de L'Enseignement Mathématique **28**, 1980) by Erdős and Graham : A set $\mathcal{A}$ of non-negative integers being given,

- if there exists an integer h such that the upper density of $h\mathcal{A}$, where $h\mathcal{A}$ is defined as
$$h\mathcal{A} = \{a_1 + \cdots + a_h, \text{ with } a_1, \ldots, a_h \in \mathcal{A}\},$$
verifies $\overline{\mathrm{d}}(h\mathcal{A}) > 0$, then the upper density of $h \times \mathcal{A}$, where $h \times \mathcal{A}$ is defined as
$$h \times \mathcal{A} = \{a_1 + \cdots + a_h, \text{ with } a_1, \ldots, a_h \in \mathcal{A}, a_i \neq a_j \text{ (if } i \neq j)\}$$
satisfies also $\overline{\mathrm{d}}(h \times \mathcal{A}) > 0$;
- if $\mathcal{A}$ is an additive basis of order r (which means that every large enough integer can be expressed as the sum of r elements taken from $\mathcal{A}$, with possible repetitions), then the lower density $\underline{\mathrm{d}}(r \times \mathcal{A})$ is strictly positive.

In this talk, we introduce a new method which allows us to settle both of these conjectures and even to be a little more precise. In addition, we raise some further questions which are partially solved.

This is a joint work with Norbert Hegyvári (Budapest) and François Hennecart (Bordeaux and Saint-Étienne) which is about to appear in *Crelle's*.

A New Relaxation Method for the Generalized Minimum Spanning Tree Problem

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Abstract

We consider the Generalized Minimum Spanning Tree Problem denoted by GMST. It is known that the GMST problem is NP-hard. We present several mixed integer programming formulations of the problem. Based on a new formulation of the GMST problem we give a solution procedure, and discuss the advantages of our approach in comparison with an earlier method.

1 Introduction

We are concerned with the generalized version of the minimum spanning tree problem (MST) called the generalized minimum spanning tree problem (GMST), introduced by Myung, Lee and Tcha in [2]. Given an undirected graph whose nodes are partitioned into a number of subsets (clusters), the GMST problem is then to find a minimum-cost tree which includes *exactly* one node from each cluster. Therefore, the MST is a special case of the GMST problem where each cluster consists of exactly one node.
The model fits various problems: of determining the location of regional service centers (e.g. public facilities, branches, distribution centers) which should be connected by building links (e.g. highways, communication links), of designing metropolitan area networks and regional area networks, where we are to interconnect a number of local area networks.

2 Definition and complexity of the problem

The GMST problem is defined on an undirected graph $G = (V, E)$ with nodes partitioned into m clusters: $V_1, V_2, ..., V_m$. Let $|V| = n$ and $K = \{1, 2, \dots, m\}$ be the index set of the node sets (clusters). We assume that edges are defined only between nodes belonging to different clusters and each edge $e = \{i, j\} \in E$ has a nonnegative cost c_{ij}.
The GMST problem is the problem of finding a minimum-cost tree spanning a subset of nodes which includes exactly one node from each cluster. We will call a tree containing one node from each cluster a generalized spanning tree.

Theorem 1 *The GMST problem is NP-hard.*

In [3], we proved that even defined on trees the GMST problem is NP-hard.

3 Integer Programming Formulations

The GMST problem can be formulated as an integer program in many different ways, cf. [1], [2], and [3]. For example, introducing the variables $x_e \in \{0,1\}$, $e \in E$ and $z_i \in \{0,1\}$, $i \in V$, to indicate whether an edge e respectively a node i is contained in the spanning tree, we obtain a valid formulation (so-called *generalized cutset formulation*) as follows:

$$
\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
s.t. \quad & z(V_k) = 1, \qquad \forall\, k \in K = \{1, ..., m\} && (1)\\
& x(E) = m - 1 && (2)\\
& x(\delta(S)) \geq z_i + z_j - 1, \ \forall\, i \in S \subset V, j \notin S && (3)\\
& x_e, z_i \in \{0,1\}, \qquad \forall\, e \in E,\ i \in V. && (4)
\end{aligned}
$$

Here we use the standard shorthand notations: $x(F) = \sum_{e \in F} x_e$, $F \subseteq E$ and $z(S) = \sum_{i \in S} z_i$, $S \subseteq V$ and for $S \subseteq V$, the *cutset*, denoted by $\delta(S)$, is defined as usually: $\delta(S) = \{\ e = (i,j) \in E \mid i \in S,\ j \notin S\}$.

The constraints in the generalized cutset formulation imply that a feasible solution defines a connected subgraph (constraints (3)), with $m-1$ edges (constraint (2)) and exactly one node from each cluster (constraints (1)) , i.e., a generalized spanning tree.

Our new model arises from distinguishing between *global*, i.e. inter-cluster connections, and *local* ones, i.e. connections between nodes from different clusters. We introduce variables y_{ij} $(i, j \in \{1, ..., m\})$ to describe the global connections. So $y_{ij} = 1$ if cluster V_i is connected to cluster V_j and $y_{ij} = 0$ otherwise and we assume that y represents a spanning tree. The convex hull of all these y-vectors is generally known as the spanning tree polytope (on the contracted graph with vertex set $\{V_1, ..., V_m\}$ which we assume to be complete).

Following Yannakakis [4] this polytope, denoted by P_{MST}, can be represented by the following polynomial number of constraints:

$$
\begin{aligned}
\sum_{\{i,j\}} y_{ij} &= m - 1 \\
y_{ij} &= \lambda_{kij} + \lambda_{kji}, \text{ for } 1 \leq k, i, j \leq m \text{ and } i \neq j && (5)\\
\sum_{j} \lambda_{kij} &= 1, \qquad \text{for } 1 \leq k, i, j \leq m \text{ and } i \neq k && (6)\\
\lambda_{kkj} &= 0, \qquad \text{for } 1 \leq k, j \leq m && (7)\\
y_{ij}, \lambda_{kij} &\geq 0, \qquad \text{for } 1 \leq k, i, j \leq m.
\end{aligned}
$$

where the variables λ_{kij} are defined for every triple of nodes k, i, j, with $i \neq j \neq k$ and their value for a spanning tree is: $\lambda_{kij} = 1$ if j is the parent of i when we root the tree at k and 0 otherwise.
If the vector y describes a spanning tree on the contracted graph, the corresponding best (w.r.t. minimization of the costs) *local solution* $x \in \{0,1\}^{|E|}$ can be obtained either by using dynamic programming, see [3], or by solving the following 0-1 programming problem:

$$
\begin{array}{lll}
\min & \displaystyle\sum_{e \in E} c_e x_e & \\
s.t. & z(V_k) = 1, & \forall\, k \in K = \{1, ..., m\} \\
 & x(V_l, V_r) = y_{lr}, & \forall\, l, r \in K = \{1, ..., m\},\ l \neq r \\
 & x(i, V_r) \leq z_i, & \forall\, r \in K, \forall\, i \in V \setminus V_r \\
 & x_e, z_i \in \{0,1\}, & \forall\, e = (i,j) \in E,\ \forall\, i \in V,
\end{array}
$$

where $x(V_l, V_r) = \displaystyle\sum_{i \in V_l, j \in V_r} x_{ij}$ and $x(i, V_r) = \displaystyle\sum_{j \in V_r} x_{ij}$.
For given y, we denote the feasible set of the linear programming relaxation of this program by $P_{local}(y)$. The following result holds:

Proposition 1 *If y is the 0-1 incidence vector of a spanning tree of the contracted graph then the polyhedron $P_{local}(y)$ is integral.*

In order to prove this result we showed that every solution of the linear programming relaxation can be written as a convex combination of solutions corresponding to spanning trees.
The observations presented so far lead to our final formulation, called *local-global formulation* of the GMST problem as an 0-1 mixed integer programming problem, where only the global variables y are forced to be integral:

$$
\begin{array}{lll}
(P_0) & \min & \displaystyle\sum_{e \in E} c_e x_e \\
 & s.t. & y \in P_{MST} \\
 & & (x, z) \in P_{local}(y) \\
 & & y_{lr} \in \{0,1\}, \quad \forall\, 1 \leq l, r \leq m.
\end{array}
$$

This new formulation of the GMST problem was obtained by incorporating the constraints characterizing P_{MST}, with $y \in \{0,1\}$, into $P_{local}(y)$.
In the next section we present a solution procedure for solving the GMST problem based on the local-global formulation and we report on our computational results for many instances of the problem.

4 A New Solution Procedure

There are different ways to solve the GMST problem with the help of formulation (P_0). The first possibility is to consider the mixed integer program (P_0) and solve it directly (for example with CPLEX).
Secondly we considered the relaxation of (P_0) obtained by choosing randomly one cluster V_k and rooting the global tree only at the root k.
If the optimal solution of this relaxation produces a generalized spanning tree, then we have given the optimal solution of the GMST problem. Otherwise we choose another root or add a second root and proceed in this way till we get the optimal solution of the GMST problem. We call this procedure the *rooting procedure*.
We compare the computational results for solving the problem using our rooting procedure with the computational results given by Myung *et al.* in [2]. According to the method of generating the edge costs, the problems generated are classified into three types: the structured Euclidean case, the unstructured Euclidean case and the non-Euclidean case.
Our algorithms have been coded in C and compiled with a HP-UX cc compiler. For solving the linear and mixed integer programming problems we used CPLEX. The computational experiments were performed on a HP 9000/735 computer with a 125 Mhz processor and 144 Mb memory.
In all the problems that we considered, the optimal solution of the GMST problem has been found by using our rooting procedure. These numerical experiences with the new formulation of the GMST problem are very promising.

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An Algorithm For Finding A Hamiltonian Cycle In Middle Two Ranks of Poset Bn

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Abstract

In this paper, we study middle two ranks of poset Bn (partially ordered set of subsets of [n], when n is odd. The middle two ranks of poset Bn form a (n+1/2)- regular graph which we call G(Bn). We'll show the relation of G(Bn) to a subgraph of Qn graph. Further, by using Hamiltonian cycles in Qn graphs, we'll find a Hamiltonian cycle in G(Bn).

Flag-Transitive Symmetric Designs

Eugenia O'Reilly Regueiro*

1 Definitions

This work presents some results on symmetric designs, and in particular on biplanes, which are classified according to certain properties of their automorphism groups.

A (v, k, λ)-*symmetric design* is an incidence structure of v points and v blocks such that any point is incident with exactly k blocks, and every pair of blocks is incident with exactly λ points. In symmetric designs the roles of points and blocks are dual. The *complement* D' of a (v, k, λ)-symmetric design D is a $(v, v-k, v-2k+\lambda)$-symmetric design whose points are the same points of D, and whose blocks are the complements of the blocks of D, that is, take the same set of points and replace incidence by non-incidence and vice-versa.

If D is a (v, k, λ)- symmetric design, we call $n = k - \lambda$ the *order* of D, (note that the order of D is the same as that of D'). A *flag* in a symmetric design D is an ordered pair (p, B) such that p is a point of D, B is a block of D, and they are incident.

An *automorphism* of a symmetric design D is a permutation of the points of D that also permutes the blocks, and preserves the incidence relation. The set of automorphisms of a symmetric design D, together with the composition of functions form a group, the *automorphism group* G of D. Therefore if D is a symmetric design, and G is its automorphism group, then G is *flag-transitive* if it is transitive on the flags of D. Finally, a $(v, k, 1)$-symmetric design is a *projective plane*, and a $(v, k, 2)$-symmetric design is a *biplane*.

2 Background

Flag-transitivity is just one of many conditions that can be imposed on the automorphism group of a symmetric design. In the case $\lambda = 1$, Kantor [3] proved that either D is Desarguesian and $G \rhd PSL(3, n)$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, and $n^2 + n + 1$ is a prime.

Here we will consider symmetric designs with $\lambda > 1$. There is a considerable difference between these two cases in that for $\lambda = 1$ there are infinitely many symmetric designs,

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whereas for any given $\lambda > 1$ there are finitely many known examples, and indeed it is conjectured that for any given $\lambda > 1$ only finitely many exist.

The only values of k for which examples of biplanes are known are $k = 3, 4, 5, 6, 9, 11$, and 13. Due to arithmetical restrictions on the parameters, there are no examples with $k =$ 7, 8, 10, or 12. In the case of flag-transitivity, in [2] it is shown that there are no linear spaces admitting a flag-transitive imprimitive automorphism group, and Davies [1] proved that for any given λ, there are finitely many (v, k, λ)-designs (not necessarily symmetric) admitting an imprimitive, flag-transitive automorphism group, by showing that if the automorphism group is imprimitive, then k is bounded, thus allowing only a finite number of possibilities. Although it is shown in [1] that k is bounded, an explicit bound is not given. In [6] we calculate for symmetric designs a bound for k in terms of λ and give some conditions that the parameters must satisfy. We analyse the case of biplanes, showing that the only admissible parameters for a biplane with a flag-transitive, imprimitive automorphism group are (16,6,2). The reduction to primitive groups allows us to use the O'Nan-Scott Theorem, which classifies primitive groups into five types, and so we may analyse each case separately. We prove a reduction theorem for small values of λ, and use this reduction to classify flag-transitive biplanes, showing that their automorphism group can only be of affine, or almost simple type.

The list of flag-transitive biplanes along with their full automorphism groups and point-stabilisers is as follows:

1. (4,3,2), S_4, S_3.
2. (7,4,2), $PSL_2(7)$, S_4.
3. (11,5,2), $PSL_2(11)$, A_5.
4. (16,6,2), 2^4S_6, S_6.
5. (16,6,2), $(\mathbb{Z}_2 \times \mathbb{Z}_8)\,(S_4.2)$, $(S_4.2)$.
6. (37,9,2), $\mathbb{Z}_{37} \cdot \mathbb{Z}_9$, $\mathbb{Z}_9$.

The biplanes 1, 4, and 5 have automorphism groups of affine type with dimension greater than one. The biplanes 2 and 3 have automorphism groups of almost simple type. Also, biplanes 3 and 6 are both examples of the one-dimensional affine case.

3 Results

Theorem 1 *If D is a (v, k, λ)-symmetric design admitting a flag-transitive, imprimitive automorphism group G, then one of the following holds:*

1. *(v, k, λ) is one of the following:*
   (a) (35,18,9)

   (b) (70,24,8)
   (c) (247,42,7)
   (d) $(\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$, *or*

2. $k \leq \lambda(\lambda-2)$.

Corollary 1 *If G is a flag-transitive automorphism group of a (v,k,λ)-symmetric design D with $\lambda \leq 3$, then either G is primitive, or D has parameters (16,6,2), (45,12,3).*

Theorem 2 *If D is a (v,k,λ)-symmetric design admitting a flag-transitive, primitive automorphism group G with $\lambda \leq 3$, then G is of affine, or almost simple type.*

Theorem 3 *If D is a biplane with a primitive, flag-transitive automorphism group G of affine type, then one of the following holds:*

1. *D has parameters (4,3,2).*
2. *D has parameters (16,6,2).*
3. *$G \leq A\Gamma L_1(q)$, for some odd prime power q.*

Corollary 2 *If D is a biplane with a flag-transitive automorphism group G, then one of the following holds:*

1. *D has parameters (4,3,2).*
2. *D has parameters (16,6,2).*
3. *$G \leq A\Gamma L_1(q)$, for some odd prime power q.*
4. *G is of almost simple type.*

Theorem 4 *If D is a biplane with a primitive, flag-transitive automorphism group of almost simple type, then D has parameters either (7,4,2), or (11,5,2), and is unique up to isomorphism.*

Corollary 3 *If D is a biplane with a flag-transitive automorphism group G, then one of the following holds:*

1. *D has parameters (4,3,2),*
2. *D has parameters (7,4,2),*
3. *D has parameters (11,5,2),*
4. *D has parameters (16,6,2), or*
5. *$G \leq A\Gamma L_1(q)$, for some odd prime power q.*

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Graphs as topological objects

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The following proposition is equivalent to the graph minors theorem of Seymour and Robertson:

Proposition: *Given any infinite collection $\mathcal{F}$ of compact connected topological spaces such that, for all members of $\mathcal{F}$:*

- *all singletons are either open or closed, and their boundaries consist of at most two points;*
- *the set of closed sets is itself a topology.*

Then there exist A, B in $\mathcal{F}$, a subspace Y of A and a continuous map from Y onto B such that the inverse images of connected sets are connected.

Since the above statement sounds more topological than combinatorial, one question which arises is: is there a "topological" proof of the graph minors theorem? If yes, one would expect it to be relatively short. The next question is: can any of the topological assumptions on the members of $\mathcal{F}$ be relaxed to obtain a class of objects which covers more than just finite graphs?

In this presentation I shall not tackle the above questions, other than to explain the equivalence and illustrate how the topological framework upon which my work with Bruce Richter is based gives a structure which allows us to interpolate between finite graphs and (possibly infinite) topological spaces and provides the appropriate concepts to phrase interesting questions, in particular with regards to infinite graphs. But I will attempt to show how in our work, which deals with generalizations of well known results concerning the cycle space of a finite graph to a larger class of objects, this setup has indeed been fruitful. I will also relate our work to the recent results of Diestel and Kühn concerning the cycle space of infinite graphs.

The traditional way of viewing a graph as a topological object is to give it the structure of a one-dimensional cell-complex; the resulting topological space can be thought of as an embedding of the graph in three-dimensional Euclidean space with the subspace topology, or more abstractly as a topological space obtained by "gluing together" homeomorphs of $[0, 1]$ for edges and points for vertices while respecting the edge-vertex incidences.

We adopt a different approach by imposing a topological structure on the graph itself, so that vertices *and* edges are *points* in the ground set. The motivation is to reduce the notion of connectedness in graph theory, based on finite paths, to the standard notion of connectedness of point-set topology, in a direct fashion, without recourse to the continuum, which appears extraneous to the combinatorial concept of an abstract graph.

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Definition: *A* topologized graph *is a topological space equipped with a bipartition* (V, E) *of its ground set such that:*
- $\forall v \in V$, $\{v\}$ *is closed;*
- $\forall e \in E$, $\{e\}$ *is open;*
- $\forall e \in E$, $|\partial(\{e\})| = 2$.

Given a finite graph as a purely combinatorial object, with V as the set of vertices, E as the set of edges and a prescribed edge-vertex incidence relation, there is a unique way of assigning to $V \cup E$ the structure of a topologized graph with bipartition (V, E). One way of describing this topology is to say that a set is closed iff it constitutes a subgraph, i.e. whenever it contains an edge it also contains all incident vertices. This construction is well behaved, in the sense that the inherited subspace topology on a subgraph agrees with the topology constructed directly from the subgraph. Moreover, it satisfies the requirement we set ourselves, i.e. the topological space is (topologically) connected if and only if the original graph is (graph-theoretically) connected. The boundary of an edge-singleton consists precisely of its two endvertices.

In the infinite case, the structure of a topologized graph is no longer uniquely determined by the edge-vertex incidences. Consider for example the graph consisting of a one-way-infinite path and a vertex v of degree 0; then a possibility for the topology would be as above, i.e. to take a set U to be open iff whenever it contains a vertex, it also contains all incident edges. This results in a disconnected topologized graph with bipartition (V, E), but if we modify it slightly by taking exception at the one "special" vertex v, requiring that $v \in U$ implies also that U is the complement of a finite subset, we obtain a connected topologized graph, also with bipartition (V, E). This second possibility corresponds closely to the standard compactification of the one-way-infinite path using the concept of end, which I will discuss shortly.

The topology becomes reconstructible from the combinatorial information if we further assume that the following equivalent properties hold:

1. the arbitrary intersection of open sets is open;
2. closed sets form a topology,i.e. the arbitrary union of closed sets is closed;
3. $\forall v \in V$, the set consisting of v and all incident edges is open;
4. subgraphs are closed.

These properties necessarily hold in the finite case; in general, we call a topologized graph with these properties *classical*. In the context of classical topologized graphs, our construction amounts to a change of language, from combinatorial to topological; although this in itself might still turn out to be enlightening, it is not only unnecessarily restrictive for our purposes but also unnatural in the light of the recent work of Diestel and Kühn from which emerges a fruitful trend to consider ends at par with vertices, given that assumption (3) fails for ends.

Although without the above assumptions the topology is not reconstructible from the graph, of course we can still carry out the inverse process: starting from a topologized graph, classical or not, simply consider the graph having vertex set V and edge set E, with e incident with v if and only if $v \in \partial(\{e\})$. Hence we focus on the class of topologized graphs rather than graphs; the choice of the term "topologized graph" is dictated by our philosophy to sift the combinatorial from the topological, and regard our objects as graphs, carrying combinatorial information, equipped with more structure, namely the topology, which defines what it means to be "connected", and consequently what is a path, a cycle, a spanning tree, etc.

In classical topologized graphs these concepts, which can be defined in the traditional way, can be characterized in topological language in various ways. For example, a path from a vertex u to a vertex v in a classical topologized graph is characterizable as a minimal connected set containing u and v. Also, a classical topologized graph is a (possibly infinite) path iff it is connected and any triple of points contains one which separates the other two, in the sense that deleting it leaves the other two in different connected components. It turns out that topological spaces with this property, which we call *prepaths*, are necessarily topologized graphs. They can be quite pathological, but locally-connected prepaths, which we call *paths*, are well behaved and include graph-theoretic paths, the seemingly disparate unit interval of real numbers and all connected orderable topological spaces. In order to generalize from finite graphs to arbitrary topologized graphs, we use the appropriate characterization (in topological language) of the combinatorial entity in question.

Definition: *A* spanning tree *of a topologized graph is a minimal connected set containing the vertex set.*

Of course, this definition is equivalent to the traditional one in the case of classical topologized graphs; moreover, it is also essentially the same as that of *topological spanning trees* as defined by Diestel and Kühn [2] in the special case of a topological construction which they obtain from infinite graphs using *ends*. Ends of graphs were introduced by Halin in 1964 [5]; they are equivalence classes of rays (one-way-infinite paths), two rays being equivalent if they are joined by infinitely many disjoint paths, or equivalently if no finite set of vertices separates them.

The topological spaces considered by Diestel and Kühn are obtained by adding the ends as "points at infinity" to the standard cell-complex associated with an infinite graph. As they point out in [3], in the case of locally finite graphs, the graph-theoretic notion of end coincides with the topological notion of end introduced by Freudenthal in 1931 [4] to construct a compactification of rim-compact Tychonoff spaces; in fact, local finiteness translates into rim-compactness and the topological spaces that Diestel and Kühn work on in [1] are precisely the Freudenthal compactifications of the 1-complexes associated with infinite graphs. When the graphs are not locally finite, the topological space can still be constructed, but it is no longer compact; I will refer to it as the *direction extension* of a graph. In [2] Diestel and Kühn assume that the original graph satisfies a certain property and modify the direction extension by identifying ends with certain vertices. In both cases, the resulting space can be reduced to the corresponding topologized graph by thinking of ends as vertices and replacing the edges (which for them correspond to arcs) with open singletons. After this reduction, topological spanning trees are seen to be spanning trees of (special) topologized graphs.

The question of the existence of spanning trees seems to be hard; it is rather pleasing that a good old-fashioned topological property, compactness, makes life easier.

Theorem: *A compact feebly Hausdorff topologized graph must have a spanning tree.*

Most of the interesting results in classical topology hold for spaces which are at least T_1, typically Hausdorff (T_2). Topologized graphs are necessarily T_0 but they are only interesting when they are *not* T_1. Thus it is not surprising that we need some separation axiom to exclude pathologies. A space is *weakly Hausdorff* if for any two points x, y there exist open sets U_x, U_y containing x, y respectively such that $U_x \cap U_y$ is finite. The notion of *feebly Hausdorff* is a relaxation of this, which is also quite natural but requires some concepts which I omit here in the interests of brevity. It is interesting that Freudenthal compactifications of locally finite graphs correspond to weakly Hausdorff topologized graphs, while if the graph has infinite degrees the direction extension need not correspond to a feebly Hausdorff space, but it does after the identification operation of Diestel and Kühn. Moreover, under their assumptions, if the graph is two-connected, then their construction is compact, and the existence of a spanning tree for their spaces (which they also prove) follows from the above theorem.

A central point motivating our work, and also that of Diestel and Kühn, was to extend the following well known result from finite to infinite graphs: given a spanning tree, the fundamental cycles generate the cycle space. Clearly, one must know what a "cycle" is in the new context. Again, we proceed by picking the appropriate topological characterization from the context of classical topologized graphs:

Definition: *A* cycle *is a connected topologized graph with no cutpoints equipped with a cyclic order on the set of edges such that if (a, b, c), (b, c, d) and (c, d, a) all belong to the cyclical order, then $\{a, c\}$ separates b and d.*

With this definition, we have that adding an edge to spanning tree introduces a cycle, and the set of its edges is unique. Thus, it makes sense to speak of fundamental edgesets. Among other results we obtain the following:

Theorem: *If G is a compact weakly Hausdorff topologized graph, and T a spanning tree of G, then the fundamental edgesets with respect to T generate the cycle space of G.*

Again, the analogous results obtained by Diestel and Kühn are a special case of the above for locally finite graphs, and follow easily in the general case.

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The Cameron-Erdös Conjecture

A. A. Sapozhenko*

Keywords: sum-free sets, independent sets

A subset A of the integers is called *sum-free* (abbreviated, SF), if $a + b \notin A$ for all $a, b \in A$. Let $s(n)$ denote the number of sum-free sets in the interval $[1, n]$. Cameron and Erdös [1] conjectured that $s(n) = O(2^{n/2})$. Here we prove this conjecture. Namely, we prove the following

Theorem 1.
$s(n) \sim c_0 2^{\lceil n/2 \rceil}$, where $5.036 \leq c_0 \leq 5.079$ for even n.
$s(n) \sim c_1 2^{\lceil n/2 \rceil}$, where $4.086 \leq c_1 \leq 4.095$ for odd n.

Put $N = [1, n]$. Let N^0 (N^1) be the subset of even (odd) integer from N. We say that a family $\mathcal{B} \subseteq 2^N$ *covers* $\mathcal{A} \subseteq 2^N$ if for any $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ s.t. $A \subseteq B$. The elements $B \in \mathcal{B}$ will be called *containers* and $\mathcal{B}$ will be called the *system* of containers for $\mathcal{A}$. If $A \subseteq N$ and X is an arbitrary segment of N, than the subset $D \cap X$ is called the *X-fragment* (or, sometimes, *$|X|$-fragment*) of A. Set $\hat{q} = n^{3/4} \log_2 n$, $\tilde{q} = \hat{q} \log_2 n$. Let $S(n)$ be the family of SF sets of N. A system $\mathcal{B}$ of containers for $S(n)$ is called *correct* if the following properties hold.

(1) For any $B \in \mathcal{B}$

$$|B| \leq n/2 + O(\hat{q});$$

(2) $|\mathcal{B}| \leq 2^{\hat{q}/\sqrt{\log_n}}$;

(3) For $B \in \mathcal{B}$ and any segment X of N with $|X| \geq \tilde{q}$

$$\Big| |B \cap X| - |X|/2 \Big| = O(\hat{q});$$

(4) For $B \in \mathcal{B}$ and any segment X of N with $|X| \geq \tilde{q}$

$$\Big| |B \cap X \cap N^0| - |X|/4 \Big| = O(\hat{q}).$$

A system $\mathcal{B}$ of containers for $S(n)$ is called *almost correct* if it is correct for some subfamily $S'(n) \subseteq S(n)$ s.t. $|S(n)| - |S'(n)| = o(2^{n/2})$.

First we prove

Theorem 2. *Let us denote $S(t, n) = \{A \in S(n) : A \subseteq [t, n]\}$, $S^1(n) = 2^{N^1}$, and $\tilde{S}(n) = S(n) \setminus (S(n/3, n) \cup) S^1(n)$. Then $\tilde{S}(n)$ has almost correct sistem of containers.*

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Then using the Freiman's result from [2], we obtain

Theorem 3. $|\tilde{S}(n)| = o(2^{n/2})$.

Now Theorem 1 is proved.

When proving Thorem 2, we use the estimates for the number of independent sets in graphs. A graph G on n vertices with minimal degree l and maximal degree m will be called $(n, l, k, m, \delta, \Delta, \theta)$-*graph* if the fraction of the vertices with degree less than $k - \theta$ equals δ and the fraction of the vertices with degree bigger than $k - \theta$ equals Δ. We based on the following (see [3])

Theorem 4. ([3]) *Let G be a $(n, l, k, m, \delta, \Delta, \theta)$-graph. Then*

$$I(G) \leq 2^{\frac{n}{2}\left(1+\delta\left(1-\frac{l}{k}\right)+\Delta\left(\frac{m}{k}-1\right)+O\left(\frac{\theta}{k}+\sqrt{\frac{\log k}{k}}\right)\right)}.$$

We also use the result on the structure of SF sets from [4].

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Towards a characterization of uni-stochastic matrices

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1 Introduction

We denote by Ω_n the polytope of all doubly-stochastic matrices of size n. A doubly-stochastic matrix M is *ortho-stochastic* (for short, o.s.) if $M_{i,j} = O_{i,j}^2$, where O is a real-orthogonal matrix. We denote by Ω_n^O the set of all ortho-stochastic matrices of size n. A doubly-stochastic matrix M is *uni-stochastic* (for short, u.s.) if $M_{i,j} = |U_{i,j}|^2$, where U is a unitary matrix. We denote by Ω_n^U the set of all uni-stochastic matrices of size n. When it is not necessary to specify the size of the matrices, we write Ω (resp. Ω^O and Ω^U). We denote by $D(M)$ the digraph of a matrix M. With an abuse of notation, if a digraph D is the digraph of an o.s. (resp. u.s.) matrix, we then write $D \in \Omega^O$ (resp. $D \in \Omega^U$).

Fifteen years ago, Fiedler [3] proposed to characterize o.s. and u.s. matrices. The problem was reproposed by Louck [4], in relation to foundational issues of Quantum Mechanics. Recently, the problem appeared again, this time in the study of quantum chaos and its links to Random Matrix Theory (see, *e.g.*, Życzkowski *et al.* [10]), and in the context of Quantum Information Theory (see, *e.g.*, Nielsen [6]).

It is well-known that $\Omega_n^U \subset \Omega_n$, for $n \geq 3$. Au-Yeung and Poon [1], and Nakazato [5], characterized Ω_3^U. Beasley *et al.* [2] observed that if $M \in \Omega^O$ then M is combinatorially orthogonal (for short, c-orthogonal), and the converse is not generally true.

Our aim is to study combinatorial properties of o.s. and u.s. matrices. We adopt a graph-theoretic approach. We prove some general results and some results about Cayley (di)graphs. We point out that the best-known topologies of point-to-point interconnection networks are based on (di)graphs of u.s. matrices. Finally, we list some open problems.

2 General results

We first introduce the notions of c-orthogonal and sc-orthogonal matrix. We then prove:

Proposition 1 *If a matrix M is inseparable and c-orthogonal then $D(M)$ has no cut-vertex, directed bridge, bridge.*

Proposition 2 *If a matrix M if sc-orthogonal then M is p-equivalent to a direct sum of primitive matrices.*

Proposition 3 *If a matrix $M \in \Omega^U$ then M is sc-orthogonal. The converse is not generally true.*

Proposition 4 *If a graph $G \in \Omega_n^U$ is bipartite then G has a perfect matching.*

Proposition 5 ([8]) *Let D be a line digraph. Then $D \in \Omega^U$ if and only if every connected component of D is Eulerian.*

We state some corollaries of these observations and propose the following conjecture:

Conjecture 1 *Let $D \in \Omega_n^U$ be a connected (di)graph. Then D is Hamiltonian.*

3 Results about Cayley (di)graphs

Let Γ a finite group with identity element e. We denote by $Cay\,(\Gamma, S)$ the Cayley digraph of Γ with respect to $S \subset \Gamma$ ($e \notin S$). A *Cayley graph* is a Cayley digraph $Cay\,(\Gamma, S)$ such that $S = S^{-1}$.

Proposition 6 *Let Γ be 2-generated. Then there exists some $S \subset \Gamma$, such that $\Gamma = \langle S \rangle$ and $Cay\,(\Gamma, S) \in \Omega^U$.*

Proposition 7 *Let S_n be the symmetric group on an $[n]$-set. Let $S = \{s_1, s_2\}$, where $s_1 = (1\ 2\ \ldots\ n)$ and $s_2 = (2\ 3\ \ldots\ n)$. Then $Cay\,(S_n, S) \in \Omega_{n!}^O$.*

Proposition 8 *Let $Cay\,(\Gamma, S) \in \Omega_n^U$ be a connected Cayley (di)graph, and let $|S| = 2$. If S contains a transposition then $Cay\,(\Gamma, S)$ is Hamiltonian.*

Proposition 9 *Let $D = Cay\,(\Gamma, S) \in \Omega_n^U$ be a connected Cayley (di)graph. Let D have (di)cycles C_{m_1} and C_{m_2}, with $1 \leq m_1, m_2 \leq n$ and $m_1 + m_2 = n$. Then D is Hamiltonian.*

4 Interconnection networks

Many models of point-to-point interconnection networks are line digraphs of regular digraphs (*e.g.* de Bruijn, Reddy-Pradhan-Kuhl, butterflies, *etc.*). By Proposition 5, these digraphs are then in Ω^U. We denote by Q_n the n-dimensional hypercube. The graph Q_n models a widely-used network. The graph Q_n is not a line digraph. We prove:

Proposition 10 *The graph $Q_n \in \Omega_{2^n}^O$. Moreover Q_n is the graph of a weighing matrix $W(n, 2^n)$.*

We conjecture the following:

Conjecture 2 *The Cayley graph $Cay\,(\mathbb{Z}_2^n, S) \in \Omega_{2^n}^O$, for any S. Moreover $Cay\,(\mathbb{Z}_2^n, S)$ is the graph of a weighing matrix $W\,(|S|\,, 2^n)$, for any S.*

We then consider double-loop networks and we prove the following elementary results:

Proposition 11 *Let $Cay\,(\Gamma, \{s_1, s_2\}) \in \Omega_n^U$, with $s_1 < s_2$. Then:*

1. *For every $1 \leq h, i, j, k \leq n$, if $g_j = s_1 g_i, g_k = s_1 g_h$ and $g_k = s_2 g_i$ then $g_j = s_2 g_h$.*
2. *For every $s_1, s_2 \in S$, $s_1 s_2^{-1} = s_2 s_1^{-1}$.*
3. *The graph $Cay\,(\Gamma, \{s_1, s_2\})$ is on a even number of vertices.*
4. *If Γ is abelian then $2s_1 = 2s_2$.*

Proposition 12 *Let $D = Cay\,(\mathbb{Z}_n; \{s_1, s_2\}) \in \Omega_n^U$. Then:*

1. *$s_2 = s_1 + \frac{n}{2} \,(\mathrm{mod}\, n)$.*
2. *$\mathbb{Z}_n = \langle s_1, s_2 \rangle$ if and only if: s_1 is odd; or s_1 is even and there exists m such that $n = 4m + 2$ (that is, $\frac{n}{2}$ is odd).*
3. *If $\mathbb{Z}_n = \langle s_1, s_2 \rangle$ then D is a line digraph and its diameter is $\frac{n}{2}$.*
4. *D is a graph if and only if $s_1 = \frac{n}{4}$.*
5. *D is a generalized cycle.*

Finally, we ask if a graph in Ω^U can have long induced cycles. Let C_n be an n-cycle. We show that the graph of the circulant weighing matrix $W\,(n; 4)$, defined by the first row

$$((1, -)\,, (\frac{n}{2} - 1, +), (\frac{n}{2} + 1, +), (n - 1, +)),$$

has $C_{2/n}$ as an induced subgraph.

5 Open problems

Problem 1 *Delorme and Poljak [7] stated the following conjecture, which Steger confirmed for $d = 3$: for $d \geq 3$, every d-regular graph G without bridges admits a collection of even Eulerian subgraphs such that every edge of G belongs to the same number of subgraphs from the collection. Is the conjecture confirmed in the case $G \in \Omega^U$?*

Problem 2 *The evolution of a quantum mechanical system isolated from the environment is induced by unitary matrices. Then classical processes induced by matrices whose (di)graphs are in Ω^U share some of their properties with quantum evolution. Can this type of "combinatorially quantum" behaviour be satisfactorially characterized? Characterizing the eigenvalues of graphs in Ω^U can be of help in this task?*

Problem 3 *Subsets of $\mathbb{Z}_2^n$ are naturally associated to Boolean functions. Suppose Conjecture 2 false. Let $\mathcal{S} = \left\{S : Cay(\mathbb{Z}_2^n, S) \in \Omega_{2^n}^U\right\}$. What is the complexity (maybe determined via Fourier analysis on $\mathbb{Z}_2^n$) of the characteristic functions of these sets? Suppose Conjecture 2 true. The same question may be then asked for families of subsets from which arise unitary matrices with specific properties (for example, unitary matrices with a tensor product decomposition).*

To conclude, it has to be pointed out that the problem of characterizing o.s. and u.s. matrices remains open.

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On Matching and Colouring of Graphs

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Abstract

We define an *r-bounded cover* of a graph G to be a spanning subgraph $T \subseteq G$ such that each component of T is a complete subgraph of G of order at most r. A 2-bounded cover of a graph G corresponds to a matching of G, and an $\omega(G)$-bounded cover of G corresponds to a colouring of the complement $\overline{G}$. A number of results on matching and colouring can be generalised to r-bounded covers. We present a generalisation of the classical decomposition theorem of Gallai and Edmonds, and some of its corollaries.

1 Introduction

Matching and colouring theory represent two of the most developed branches of graph theory, yet there seems to be little overlap between them. The purpose of this talk is to show that, in fact, some theorems on matching and colouring have a common generalisation in terms of what we call *r-bounded covers*. An r-bounded cover of a graph G is a spanning subgraph $T \subseteq G$ such that each component of T is a complete subgraph of G of order at most r. (We shall always assume that $r \geqslant 2$.) Thus a 2-bounded cover of a graph G corresponds to a matching of G, while an $\omega(G)$-bounded cover of G corresponds to a colouring of the complement $\overline{G}$, where $\omega(G)$ denotes the clique number of G. An r-bounded cover T of G is *minimal* if it has the minimal number of components. The number of components of a minimal r-bounded cover of G is the *r-bounded cover number* $\overline{\chi}_r(G)$ of G. Thus $\overline{\chi}_{\omega(G)}(G)$ equals the chromatic number $\chi(\overline{G})$ of the complement, and it is not hard to check that $\overline{\chi}_2(G) = |G| - \nu(G)$, where $\nu(G)$ is the matching number of G.

The idea of r-bounded covers is not new: Gallai mentioned it as a remark in [5], but he did not elaborate on the idea and concentrated only on $\omega(G)$-bounded covers.

If T_1 and T_2 are minimal 2-bounded covers of G, the components of $T_1 \cup T_2$ have a very simple structure: they are paths of even (possibly zero) length, even cycles, or paths of length one. Moreover, a component containing an isolated vertex of T_1 or T_2 must be a path of even (possibly zero) length, and it contains exactly one isolated vertex of T_1 and

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one isolated vertex of T_2. This last observation can serve as the basis for a proof of a number of results in matching theory, including the decomposition theorem of Gallai and Edmonds [3, 5, 6].

If T_1 and T_2 are minimal r-bounded covers of G and $r \geqslant 3$, the structure of the components $T_1 \cup T_2$ is not as simple, and in particular a component of $T_1 \cup T_2$ containing an isolated vertex of T_1 or T_2 may contain any number of other isolated vertices of T_1 and T_2. However, we can concentrate on a special class of minimal r-bounded covers, which we call *extreme*. An r-bounded cover is extreme if it is minimal and contains the minimal number of isolated vertices. We can prove the following crucial lemma:

Lemma 1 *If T_1 and T_2 are extreme r-bounded covers of a graph G, then every component of $T_1 \cup T_2$ either has no isolated vertex of T_1 and T_2, or has exactly one isolated vertex of T_1 and one isolated vertex of T_2.*

Note that a 2-bounded cover is minimal if and only if it is extreme. Thus an extreme r-bounded cover can be viewed as a generalisation of a maximal matching. By considering the complement, it can also be viewed as a special class of colouring.

2 The Gallai-Edmonds Decomposition

Given a graph G, let us partition the vertex set into three subsets as follows. Let $D_r(G)$ be the set of all vertices of G which are isolated in some extreme r-bounded cover of G, let $A_r(G)$ be the set of all vertices in $V(G) \setminus D_r(G)$ which are adjacent to at least one vertex in $D_r(G)$, and finally put $C_r(G) = V(G) \setminus (A_r(G) \cup D_r(G))$.

The following fundamental theorem was proved for 2-bounded covers by Gallai [5, 6] and Edmonds [3]. Gallai [5] also proved parts of Theorem 1 for the case $r = \omega(G)$, and he knew that the results also hold for any $r \geqslant 2$. His proof relies heavily on matching theory, whereas our proof [9] is entirely self-contained and uses no matching theory.

Theorem 1 *The following assertions hold for any graph G.*

1. *The components G_i of the subgraph induced by $D_r(G)$ are r-bounded cover-critical graphs such that $|G_i| = 2\overline{\chi}_r(G) - 1$.*
2. *The subgraph induced by $C_r(G)$ has a minimal r-bounded cover with no isolated vertices.*
3. *Every subset $X \subseteq A_r(G)$ has neighbours in at least $|X| + 1$ components of $G[D_r(G)]$.*
4. $|C_r(G)|/r \leqslant \overline{\chi}_r(G) - |D_r(G)|/2 - c(G[D_r(G)])/2 \leqslant |C_r(G)|/2$.
5. $\mathrm{def}_r(G) = c(G[D_r(G)]) - |A_r(G)|$.

Note that a graph G is *r-bounded cover-critical* if $\overline{\chi}_r(G - x) = \overline{\chi}_r(G) - 1$ for every vertex x of G, $c(G)$ denotes the number of components of G, and $\mathrm{def}_r(G)$ denotes the number of isolated vertices of an extreme r-bounded cover of G.

3 Some Applications of Theorem 1

Theorem 1 can be used to prove the following common generalisation of Gallai's classical result [5] on colour-critical graphs, and Gallai's lemma [4] on factor-critical graphs.

Theorem 2 *Every connected r-bounded cover-critical graph has at least $2\overline{\chi}_r(G) - 1$ vertices.*

Let $\beta_r(G)$ denote the number of components G_i of a graph G that are r-bounded cover-critical graphs and $|G_i| = 2\overline{\chi}_r(G) - 1$. Theorem 1 can be used to deduce the following result, which in turn implies Tutte's [11] characterisation of graphs with perfect matchings.

Theorem 3 *A graph G has an extreme r-bounded cover with no isolated vertices if and only if $\beta_r(G - X) \leqslant |X|$ for all $X \subseteq V(G)$.*

4 Conclusion

A number of other results on matching and colouring can be generalised to r-bounded covers, including theorems of Berge [1], Dirac [2], Gallai [5], Stehlík [8] and Toft [10]. On the whole, generalising the matching results to r-bounded covers has been more difficult, and therefore more interesting, than generalising the colouring results. Those interested are referred to [7, 9].

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The complexity of assigning genotypes to people in a pedigree consistently

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Abstract

We discuss the complexity of a combinatorial problem in the field of genetics. The problem, which we call Genotype ASsignability problem and abbreviate as GAS, is defined as follows:

"A pedigree is given and, for a certain position (locus) on a certain pair of chromosomes, the genotypes at the locus of some people in the pedigree are given. Is it possible to assign all other people (i.e., all of the people of which genotypes are not given) genotypes for the locus so as not to cause contradiction in the transmission of genes at the locus in the whole of the pedigree?" GAS can be used to compute, from the genotypes of some people in a pedigree, the genotypes that each person in the pedigree can possess. Although many combinatorial problems have been studied so far, GAS seems to have not been done yet. Let m be the number of different genes in a pedigree and n that of all people in the pedigree. We prove that GAS is NP-complete when $m \geq 3$ and that it can be solved in linear time $O(n)$ when $m = 2$.
Keywords bio-informatics, pedigree, genotype, algorithm, complexity, NP-complete

1 Problem

In this paper, we consider a combinatorial problem in the field of genetics. The problem, which we call Genotype ASsignability problem and abbreviate as GAS, is genetically defined as follows:

> A pedigree is given and, for a certain locus on a certain pair of chromosomes, the genotypes at the locus of some people in the pedigree are given. Is it possible to assign all other people (i.e., all of the people of which genotypes are not given) genotypes for the locus so as not to cause contradiction in the transmission of genes at the locus in the whole of the pedigree?

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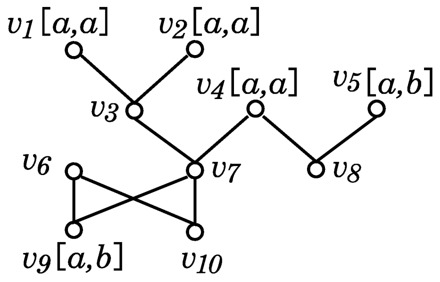

Figure 1: A problem instance of GAS.

We will define GAS formally. (1) A pedigree is a directed acyclic graph $G = (V, E)$ such that the number of parents of each vertex is two or zero, where a vertex and an edge in G indicate a person and a parent-child relation in the pedigree, respectively. (2) A genotype is a multiset that consists of two genes at a position (locus) on a pair of chromosomes [1]. The genotype (multiset) of genes a and b is denoted by $[a, b]$. (3) Note that GAS deals with only a fixed locus. Let M be the set of different genes that can appear at the locus. An assignment of genotypes at the locus to some people $W \subset V$ that is given as an input data can be expressed as a mapping $g : W \to \{[x, y] \mid x, y \in M\}$. (4) An assignment of genotypes at the locus to all people V can also be expressed as a mapping $f : V \to \{[x, y] \mid x, y \in M\}$. (5) In the world of nature, a human inherits one gene in a genotype from his/her father and another from his/her mother [1]. Express it in terms of genotypes, in the world of nature, among the genotype $f(v)$ at a locus of a human v, the genotype $f(v')$ at the locus of the father v' of v and the genotype $f(v'')$ at the locus of the mother v'' of v, the following relation holds:

$$R1 : \qquad f(v) \in \{[x, y] \mid x \in f(v'), y \in f(v'')\},$$

which we call "genotype heredity rule" in this paper. The phrase "not to cause contradiction in the transmission of genes at the locus in the whole of the pedigree" in the above definition of GAS means "to satisfy the genotype heredity rule ($R1$) for the genotypes at the locus of all people in the pedigree." As the result of (1) to (5), GAS is formally defined as follows:

Genotype ASsignability problem (GAS) (i) A directed acyclic graph $G = (V, E)$ such that the number of parents of each vertex is two or zero, (ii) a set $M = \{a_1, \ldots, a_m\}$, and (iii) a mapping $g : W \to \{[x, y] \mid x, y \in M\}$ for a set of vertexes $W \subset V$ are given. Is there a mapping $f : V \to \{[x, y] \mid x, y \in M\}$ for V such that (iv) $f(v^*) = g(v^*)$ for all $v^* \in W$ and $f(v) \in \{[x, y] \mid x \in f(v'),\ y \in f(v''),$ where $v'(v'')$ is the father(mother) of $v\}$ for all $v \in V$? $\Box$

We call a mapping from a set of vertexes (people) to a set of genotypes GenoType Assignment, abbreviated as GTA, and write such as the GTA for W, the GTA g and a GTA f. We also write "a GTA f is consistent" when a GTA f for V satisfies the condition (iv), i.e., "$f(v^*) = g(v^*)$ for $\cdots$ for all $v \in V$", in the above formal definition of GAS.

Example 1 Consider the problem instance in Figure 1, where $V = \{v_1, \ldots, v_{10}\}, M = \{a, b\}, W = \{v_1, v_2, v_4, v_5, v_9\}, g(v_1) = g(v_2) = g(v_4) = [a, a]$ and $g(v_5) = g(v_9) = [a, b]$. We omitted the arrow heads of all edges in the graph, promising that all edges in graphs point below through this paper. For all $v^* \in W$, $f(v^*) = g(v^*)$ must hold. Since $f(v_1) = f(v_2) = [a, a]$, $f(v_3)$ must be $f(v_3) = [a, a]$ in order to satisfy the genotype heredity rule $(R1)$ for v_3. Similarly, since $f(v_3) = f(v_4) = [a, a]$, $f(v_7)$ must be $f(v_7) = [a, a]$. For example, if we assume $f(v_6) = f(v_8) = f(v_{10}) = [a, b]$ for $V - W - \{v_3, v_7\}$, the f becomes consistent. Therefore, there exists a consistent GTA. We shall see this example again in Section 3. □

Example 2 Consider the problem instance of Example 1 where the genotype of v_9 has been changed to $[b, b]$ $(g(v_9) = [b, b])$. As in Example 1, $f(v_7)$ must be $f(v_7) = [a, a]$. Since $f(v_7) = [a, a]$ and $f(v_9) = [b, b]$, the genotype heredity rule does not hold for v_9 in spite of the value of $f(v_6)$. Therefore, there exists no consistent GTA. □

Application 1 Consider computing the genotypes that v_9 can possess, assuming that the genotype of v_9 is not given in the problem instance of Example 1. Since $M = \{a, b\}$, the genotypes that v_9 can possess are some of $[a, a], [a, b]$ and $[b, b]$. We can think that v_9 can possess a genotype $[x, y]$ $(x, y \in M)$ if and only if the problem instance in Example 1 where the genotype of v_9 is set to $[x, y]$ has a consistent GTA. For $[x, y] = [a, b]$, as was shown in Example 1, there exists a consistent GTA. For $[x, y] = [b, b]$, as in Example 2, there exists no consistent GTA. Finally, for $[x, y] = [a, a]$, as is easily understood, there exists a consistent GTA. Therefore, v_9 can possess the genotypes $[a, a]$ and $[a, b]$. If we consider v_7, it can possess only one genotype $[a, a]$, as was shown in Example 1. □

In the world of nature, there are some loci such that the value of it (i.e., the genotype at it) determines the value of a certain observable attribute (phenotype) of a human [1]. For example, the locus for the ABO blood type is its example, and his/her blood type becomes O (A, B, AB) when his/her genotype at the locus is $[o, o]$ ($[a, a]$ or $[a, o]$, $[b, b]$ or $[b, o]$, $[a, b]$), where a, b and o are genes. Therefore, knowing the genotypes of people at such a locus is important in genetics. We hope that, when carrying out medical inspections to know the genotypes of people is expensive or impossible because they are already dead or not born yet, computing their genotypes by solving GAS repeatedly as Application 1 will be useful for researches and applications of genetics.

Although many combinatorial problems have been studied so far, GAS seems to have not been done yet [2, 3]. In this paper, we discuss the complexity of GAS.

2 NP-completeness

We call a restricted version of GAS such that the number of different genes is three ($|M| = m = 3$) 3GAS.

Theorem 1 3GAS is NP-complete.

Proof We prove that the 3-satisfiability problem (3SAT) is polynomially transformable to 3GAS. We call an expression $I = (G(V, E), M, W, g)$ that gives an instance of GAS a GAS expression. First, we give the procedure for constructing, from the CNF expression C in an arbitrary instance of 3SAT, the GAS expression $I = (G(V, E), M, W, g)$ with $m = 3$

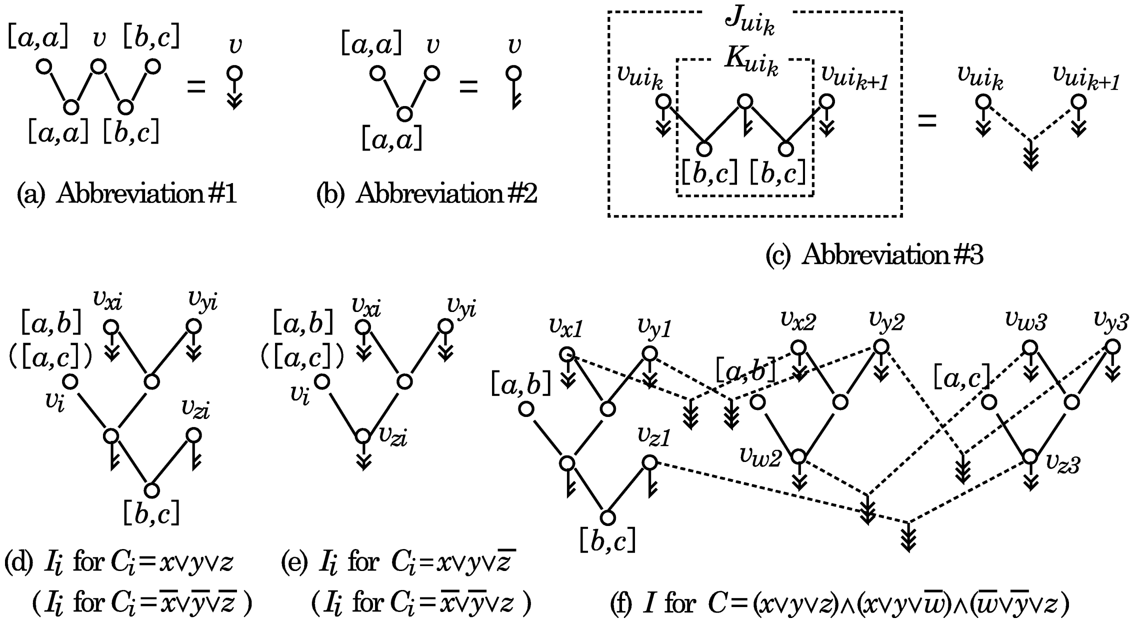

Figure 2: Abbreviations and GAS expressions to give 3GAS's.

((1), (2) and (3) below). Second, we show that the procedure is executed in time linear in the size of C ((i) below) and that C is satisfiable if and only if I has a consistent GTA f ((ii) and (iii) below).

We prepare to give the procedure. Let the CNF expression C be $C = C_1 \wedge C_2 \wedge \cdots \wedge C_p$, where each $C_i, i = 1, \ldots, p$, is a logical sum of three literals out of the literals $u_1, \overline{u_1}, u_2, \overline{u_2}, \ldots, u_q, \overline{u_q}$ of the logical variables $u_1, u_2, \ldots, u_q$. Let a, b and c be different genes, and the one-to-one mapping that makes the logical constant 0 (1) correspond to the genotype $[a, b]$ ($[a, c]$) be denoted by t (i.e., $t(0) = [a, b]$ and $t(1) = [a, c]$). Let some parts of I be abbreviated as (a), (b) and (c) in Figure 2. Then, the procedure to construct I with $m = 3$ from $C = C_1 \wedge \cdots \wedge C_p$ is given by the following steps (1), (2) and (3).

(1) For each $C_i, i = 1, \ldots, p$, construct the GAS expression I_i as follows. Let three variables in C_i be $x, y, z \in \{u_1, \ldots, u_q\}$. When C_i has three positive literals ($C_i = x \vee y \vee z$), construct, as I_i, (d) in Figure 2 in which the genotype of the vertex v_i is $[a, b]$. When C_i has two positive literals ($C_i = x \vee y \vee \overline{z}$), construct, as I_i, (e) in which the genotype of v_i is $[a, b]$. When C_i has one positive literal ($C_i = \overline{x} \vee \overline{y} \vee z$), construct, as I_i, (e) in which the genotype of v_i is $[a, c]$. Finally, when C_i has no positive literal ($C_i = \overline{x} \vee \overline{y} \vee \overline{z}$), construct, as I_i, (d) in which the genotype of v_i is $[a, c]$.

Every GAS expression I_i has the following properties (1a), (1b), (1c) and (1d). (1a) The different genes in I_i are a, b and c. (1b) I_i contains the vertexes v_{xi}, v_{yi} and v_{zi} that correspond to the variables x, y and z respectively. (1c) If an assignment $(x, y, z) = (d_x, d_y, d_z)$, where d_x, d_y and d_z are 0 or 1, makes the value of C_i 1, there exists a consistent GTA f_i for I_i such that the genotypes $f_i(v_{xi}), f_i(v_{yi})$ and $f_i(v_{zi})$ of v_{xi}, v_{yi} and v_{zi} are eqaul to

$t(d_x), t(d_y)$ and $t(d_z)$ respectively. Conversely, (1d) if there exists a consistent GTA f_i for I_i, the genotypes $f_i(v_{xi}), f_i(v_{yi})$ and $f_i(v_{zi})$ of v_{xi}, v_{yi} and v_{zi} are $[a, b]$ or $[a, c]$, and the inverse of the mapping t of those genotypes, $(x, y, z) = (t^{-1}(f_i(v_{xi})), t^{-1}(f_i(v_{yi})), t^{-1}(f_i(v_{zi})))$, makes the value of C_i 1.

(2) For each variable, denoted by u, in $C = C_1 \wedge \cdots \wedge C_p$, let all clauses containing u be $C_{i_1}, C_{i_2}, \ldots, C_{i_r}$. For each clause $C_{i_k}, k = 1, \ldots, r$, let v_{ui_k} be the vertex corresponding to u in I_{i_k} constructed for C_{i_k} in (1). For each pair of vertexes $(v_{ui_k}, v_{ui_{k+1}}), k = 1, \ldots, r-1$, construct K_{ui_k} in (c) in Figure 2, where K_{ui_k} contains the two edges intersecting the dotted lines.

Let J_{ui_k} ((c) in Figure 2) be the GAS expression obtained by adding, to K_{ui_k}, the two vertexes $v_{ui_k}, v_{ui_{k+1}}$ and the two segments with two arrow heads each that are hanging down from those vertexes. Every GAS expression J_{ui_k} has the following properties (2a) and (2b). (2a) The different genes in J_{ui_k} are a, b and c. (2b) For J_{ui_k}, there exist a consistent GTA f' satisfying $f'(v_{ui_k}) = f'(v_{ui_{k+1}}) = [a, b]$ and a consistent GTA f'' satisfying $f''(v_{ui_k}) = f''(v_{ui_{k+1}}) = [a, c]$, and there exists no consistent GTA other than f' and f''.

(3) Construct I as the combination of all I_i constructed in (1) and all K_{ui_k} constructed in (2), where I becomes a GAS expression with $m = 3$ from the properties (1a) and (2a).

For example, when C is $C = (x \vee y \vee z) \wedge (x \vee y \vee \overline{w}) \wedge (\overline{w} \vee \overline{y} \vee z)$, I becomes (f) in Figure 2.

(i) The number of I_i constructed in (1) is p, and that of the vertexes (edges) in each I_i is 17 or 19 (16 or 18). The number of K_{ui_k} constructed in (2) is no more than $(p-1) \times 3$, and that of the vertexes (edges) in each K_{ui_k} is 5 (6). Therefore, the above procedure ((1), (2) and (3)) can be executed in time $O(p)$ linear in the size of C.

(ii) Assuming that $C = C_1 \wedge \cdots \wedge C_p$ is satisfiable, let an arbitrary satisfying assignment be denoted by $(u_1, \ldots, u_q) = (d_1, \ldots, d_q)$, where each d_j is 0 or 1. Since $(u_1, \ldots, u_q) = (d_1, \ldots, d_q)$ makes the value of each $C_i, i = 1, \ldots, p$, 1, there exists, from the property (1c), a consistent GTA f_i for each I_i such that $f_i(v_{u_j i}) = t(d_j)$ for all variables u_j in C_i. Since, for such GTA's f_i, the genotypes of the vertexes $v_{u_j i_k}$ and $v_{u_j i_{k+1}}$ at both ends of $J_{u_j i_k}$ coincide to become $[a, b]$ or $[a, c]$ $(f_{i_k}(v_{u_j i_k}) = f_{i_{k+1}}(v_{u_j i_{k+1}}) = t(d_j) = [a, b]$ or $[a, c])$, there exists, from the property (2b), a consistent GTA f_h^* such that $f_h^*(v_{u_j i_k}) = f_h^*(v_{u_j i_{k+1}}) = t(d_j)$ for each $J_{u_j i_k}$. Consequently, if we consider the GTA f obtained by combining such GTA's f_i and such GTA's f_h^*, the f becomes consistent for the whole of I.

(iii) Assuming that there exists a consistent GTA for I, let an arbitrary consistent GTA be denoted by f. (a) Since f is also consistent for each I_i, the assignment $(u_{j_1}, u_{j_2}, u_{j_3}) = (t^{-1}(f(v_{u_{j_1} i})), t^{-1}(f(v_{u_{j_2} i})), t^{-1}(f(v_{u_{j_3} i})))$ makes the value of C_i 1 from the property (1d), where u_{j_1}, u_{j_2} and u_{j_3} are all variables in C_i. (b) Since f is also consistent for all $J_{u_j i_k}$, the genotypes $f(v_{u_j i})$ of all vertexes $v_{u_j i}$ corresponding to each variable u_j in C must, from the property (2b), coincide to become a certain value, which is denoted by e_j. ¿From (a) and (b), since, for each C_i, the assignment $(u_{j_1}, u_{j_2}, u_{j_3}) = (t^{-1}(e_{j_1}), t^{-1}(e_{j_2}), t^{-1}(e_{j_3}))$ makes the value of C_i 1, the assignment $(u_1, \ldots, u_q) = (t^{-1}(e_1), \ldots, t^{-1}(e_q))$ makes the value of C 1.

From (i), (ii) and (iii), 3SAT is polynomially transformable to 3GAS. 3GAS is also in the class $\mathcal{NP}$. Consequently, 3GAS is NP-complete. □

3 Linear time algorithm for the case of $m = 2$

Let n be the number of all vertexes in GAS. We give an algorithm that can solve GAS for $m = 2$ in linear time $O(n)$ in the following. The two genes are denoted by a and b.

step1: Assign every vertex $v \in V$ a number $ord(v)$ so that $ord(\text{parent}) < ord(\text{child})$ holds for every child and each of its parents.

step2: $f(v^*) = g(v^*)$ for every $v^* \in W$. Carry out the following processing for every $v \in V - W$ in ascending order of $ord(v)$. If both of the genotypes of the parents of v are $[a, a]([b, b])$, $f(v) = [a, a]$ $([b, b])$. Otherwise $f(v) = [a, b]$.

step3: If f satisfies the genotype heredity rule $(R1)$ for all $v \in V$, output "yes" (i.e., there exists a consistent GTA). Otherwise output "no." □

Consider applying the above algorithm to GAS of Figure 1 in Section 1. Step1 computes as the order given by ord's, for example, the order of $v_1, v_2, v_4, v_5, v_6, v_3, v_8, v_7, v_9, v_{10}$. Step2 computes $f(v_1) = f(v_2) = f(v_4) = [a, a]$ and $f(v_5) = f(v_9) = [a, b]$ for W, and $f(v_3) = f(v_7) = [a, a]$ and $f(v_6) = f(v_8) = f(v_{10}) = [a, b]$ for $V - W$. Since f satisfies the genotype heredity rule $(R1)$ for V, step3 outputs "yes." The GTA f in Example 1 in Section 1 was computed in this way.

When the particular GTA f computed by the algorithm is consistent, there certainly exists a consistent GTA. On the other hand, it can be proved that there exists no consistent GTA when the f is inconsistent. Therefore, the algorithm can output an answer correctly.

Since all of the worst case time complexities of step1, step2 and step3 are $O(n)$, that of the algorithm is also $O(n)$.

In this paper, we discussed the complexity of GAS. Algorithms to solve GAS for $m \geq 3$ and the complexities of various problems relevant to GAS are the subjects for future studies.

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On constructive characterizations of (k, l)-sparse graphs

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Abstract

In this paper we concern constructive characterizations of graphs satisfying tree-connectivity requirements.

1 Constructive characterizations

By a constructive characterization of a graph property, we mean a building procedure consisting of some simple steps so that the graphs obtained from a specified initial graph are precisely those having the property. For example, a graph is connected if and only if it can be obtained from a node by the operation: add a new edge connecting an existing node with either an existing node or a new one. A slightly less trivial known result is the so called ear-decomposition of 2-connected graphs.

A graph is said to be *k-edge-connected* if the deletion of at most $k-1$ edges leaves a connected graph. From now on, adding an edge means adding a new edge connecting two existing nodes. This new edge can be parallel to existing ones, but it cannot be a loop unless otherwise stated. In 1976 Lovász [4] proved the following result.

Theorem 1 *An undirected graph $G = (V, E)$ is $2k$-edge-connected if and only if G can be obtained from a single node by the following two operations:*

(i) *add a new edge (possibly a loop),*

(ii) *add a new node z, subdivide k existing edges by new nodes, and identify the k subdividing nodes with z.*

Similar constructive characterizations for $2k+1$-edge-connectivity was given by Mader. A directed counterpart of these results is also due to Mader [5]. This kind of characterizations can be very useful. For example, Lovász used his result to derive Nash-Williams' theorem

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on k-edge-connected orientations of graphs, while Mader used his result to derive Edmonds' theorem on disjoint arborescences.

k-edge-connectivity is the usual way to formulate one's intuitive feeling for high 'edge-connection' of an undirected graph but there may be other possibilities, as well. We call an undirected graph *k-tree-connected* if it contains k edge-disjoint spanning trees. The constructive characterization of k-tree-connected graphs was given by Frank in [1] by observing that a combination of a theorem of Mader and a theorem of Tutte gives rise to the following.

Theorem 2 *An undirected graph $G = (V, E)$ is k-tree-connected if and only if G can be built from a single node by the following two operations:*

(i) *add a new edge,*

(ii) *add a new node z and k new edges ending at z,*

(iii) *pinch i $(1 \leq i \leq k-1)$ existing edges with a new node z, and add $k-i$ new edges connecting z with existing nodes.*

In [2] two variants of the notion of k-tree-connectivity were considered. A graph G (with at least 2 nodes) is called *nearly k-tree-connected* if G is not k-tree-connected but adding any new edge to G results in a k-tree-connected graph. Let K_2^{k-1} denote the graph on two nodes with $k-1$ parallel edges.

Theorem 3 *An undirected graph $G = (V, E)$ is nearly k-tree-connected if and only if G can be built from K_2^{k-1} by applying the following operations:*

(O1') *add a new node z and k new edges ending at z so that no k parallel edges can arise,*

(O2') *choose a subset F of i existing edges $(1 \leq i \leq k-1)$, pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other nodes so that there are no k parallel edges in the resulting graph.*

Actually, we proved this result in a slightly more general form. We proved the following conjecture in case $l = 1$. A graph $G = (V, E)$ is said to be *(k,l)-sparse* if $\gamma_G(X) \leq k|X| - (k+l)$ for all $X \subseteq V, |X| \geq 2$, where $\gamma_G(X)$ denotes the number of the edges of G with both end-nodes in $X \subseteq V$. (By convention the graph with one single node is (k,l)-sparse.)

Conjecture 1 *Let $1 \leq l \leq \frac{k+2}{3}$. An undirected graph $G = (V, E)$ is (k,l)-sparse if and only if G can be built from a single node by applying the following operations:*

(P1) *add a new node z and at most k new edges ending at z so that no $k-l+1$ parallel edges can arise.*

(P2) *Choose a subset F of i existing edges $(1 \leq i \leq k-1)$, pinch the elements of F with a new node z, and add $k-i$ new edges connecting z with other nodes so that there are no $k-l+1$ parallel edges in the resulting graph.*

Note that there is no (k,l)-sparse graph on 3 nodes for $\frac{k}{2} < l$ for which $|E| = 3k-(k+l) = 2k-l$. Hence we concern only the case of $l \leq \frac{k}{2}$. (By a theorem of Nash-Williams, a graph $G=(V,E)$ with $|E| = k|V|-(k+1)$ is nearly k-tree-connected if and only if G is $(k,1)$-sparse.)

We call a graph *highly k-tree-connected* if the deletion of any existing edge leaves a k-tree-connected graph. Frank and Király [3] gave a constructive characterization (among others) for highly 2-tree-connected graphs. In [2] this was extended for arbitrary $k \geq 2$.

2 Construction of (k,l)-sparse graphs

We have the following theorem, which follows easily from the definition of (k,l)-sparse graphs.

Theorem 4 *If an undirected graph $G=(V,E)$ can be built up from a single node by applying the operations* (P1) *and* (P2)*, then G is (k,l)-sparse.*

Inspiring by the previous constructive characterizations we would conjecture that the reverse of the above theorem is also true for all k and l satisfying $\frac{k}{2} \geq l$. But as we will show in Section 4, this is not true if $l \geq \frac{k+2}{3}$. We believe that Conjecture 1 will be proved soon.

3 Splittings for (k,l)-sparse graphs

The main ingredient of the proof of Conjecture 1 in case $l=1$ is a necessary and sufficient condition for a node which can be the last node of a construction. In this chapter we prove the corresponding condition for (k,l)-sparse graphs.

A splitting off in a (k,l)-sparse graph G is *admissible* if the resulting graph on node set $V-z$ is (k,l)-sparse.

Definition 1 *Let b_G denote the following function on the subsets of V with cardinality at least 2:*

$$b_G(X) := k|X| - (k+l) - \gamma_G(X).$$

By this definition a graph $G=(V,E)$ is (k,l)-sparse if and only if $b_G(X) \geq 0$ for all subsets $X \subseteq V, |X| \geq 2$. If $b_G(X)=0$ and $X \neq V$, then X is said to be a *G-tight* set. Furthermore G is a union of k spanning trees after adding arbitrary l edges if and only if G is (k,l)-sparse and $b_G(V)=0$. We will abbreviate b_G by b.

Observation 1 *Splitting off zu and zv at node z is non-admissible (that is, adding the edge uv to the induced subgraph of G on $V-z$ does not result in a (k,l)-sparse graph) if and only if there exists a tight subset in $V-z$ containing u and v.*

We say that splitting off j disjoint pairs of edges ($1 \le j \le k-1$) at node z is *admissible* if it consists of admissible splittings. Obviously the order of the pairs in a *splitting sequence* is irrelevant. The *length* of a splitting sequence $\mathcal{S}$ is the number of its pairs and it is denoted by $|\mathcal{S}|$.

An admissible splitting sequence at node z of length $d_G(z) - k =: i$ is called a *full splitting* for $d_G(z) \ge k+1$. For the sake of convenience, at a node z with degree at most k the inverse of operation (P1) (that is, the deletion of z and all of its adjacent edges) is also called a *full splitting*. Note that $b_G(X)$ is an upper bound for the number of split edges induced by $X \subseteq V - z$ provided by an admissible sequence of splittings at some node z.

Now we prove that if $d_G(z)$ is at most $k+l$, then a full splitting always exists at z.

Proposition 1 *Let G be a (k,l)-sparse graph. If $z \in V$ has degree at most $k+l$, then there exists a full splitting at z.*

Let $i := d_G(z) - k$ (here $2 \le i \le k-1$). Call a node z *small* if $k+l+1 \le d_G(z) \le 2k-1$.

Theorem 5 *A small node z of G does not admit a full splitting if and only if z has a neighbour t and there is a family $\mathcal{P}_z$ of subsets of $V-z$ with at least two elements such that:*

$$X \cap Y = \{t\} \quad \textit{for} \quad X, Y \in \mathcal{P}_z, \tag{$*$}$$

$$\sum_{X \in \mathcal{P}_z} b(X) < d_G(z,t) - (k-i) - d_G(z, V - z - \cup\mathcal{P}_z). \tag{$**$}$$

4 Counterexamples

In this section we give a (k,l)-sparse graph for any $k \ge 2$, $\frac{k+2}{3} \le l \le \frac{k}{2}$ which cannot be obtained by the operations of Theorem 4. This is surprising because we managed to prove almost all the ingredients of the proof of the constructive characterization of $(k,1)$-sparse graphs also for these graphs. We remark that, for the given graph $G_{(k,l)} = (V_{(k,l)}, E_{(k,l)})$, $|V_{(k,l)}| = 15k - 5l + 10$, which is 60 in the smallest case $(4,2)$ and 85 in case $(6,3)$.

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Tamari Lattices for Types B and D

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Abstract

The usual, or Type A, Tamari lattice is a partial order on T_A^n, the triangulations of an $n+3$-gon. We define a partial order on T_B^n, the set of centrally symmetric triangulations of a $2n+2$-gon, and show that it is a lattice, and that it shares certain other nice properties of the Type A Tamari lattice, and therefore that it deserves to be considered the Type B Tamari lattice.

We define a bijection between T_B^n and non-crossing partitions of Type B. Reiner has defined the noncrossing partitions of Type D^n as a subset of those of Type B^n. We show that the elements of T_B^n which correspond to the noncrossing partitions of Type D^n form a lattice under the order induced from their inclusion in T_B^n, which therefore deserves to be called the Type D^n Tamari lattice.

1 Introduction

Let T_A^n denote the set of triangulations of an $n+3$-gon. Conventionally, we will number the vertices of our $n+3$-gon counterclockwise from 0 to $n+2$, with a long top edge connecting vertices 0 and $n+2$.

Let $S \in T_A^n$. We colour the chords of S red and green, as follows. A chord C of S is the diagonal of a quadrilateral $Q(C)$ in S. If C is the diagonal in $Q(C)$ which is connected to the vertex with the largest label, we colour it green; otherwise we colour it red.

We partially order T_A^n by giving covering relations: T covers S if they coincide except that some green chord in S has been replaced by the other diagonal of $Q(C)$ (which is red). This is one way to construct the the Tamari lattice, which was introduced in [8] and which has been since been studied by several authors (see [5, 4, 1]).

Although this is not clear from the elementary description given here, the Tamari lattice should be thought of as belonging to Type A. One indication of why can be found in [1], where it is shown that it is a quotient of the weak order on the symmetric group. Another is that the elements of T_A^n index clusters in the A^n root system (see [2]). Once one has this idea, it is natural to ask whether there exist Tamari lattices in other types.

For reasons which we shall go into further below, the Type B triangulations, denoted T_B^n, are the triangulations of a $2n+2$-gon which are fixed under a half-turn rotation.

These triangulations have already appeared in the work of Simion [7], and of Fomin and Zelevinsky [2], where they index the clusters in the B^n root system. One goal of our paper is to define a partial order on T_B^n and to prove that it is a lattice. The definition is analogous to that already given for the Type A Tamari lattice: it is given in terms of covering relations, and S covers T in T_B^n if S is obtained from T by replacing a symmetric pair of chords $C, \bar{C}$ by the other diagonals of $Q(C), Q(\bar{C})$. The details of the definition are a trifle complicated, so we defer them for the main body of the paper. This definition was arrived at independently and more or less simultaneously by Reading [5]. He has also proved that T_B^n is a lattice, using a rather different approach.

What objects should be considered as Type D triangulations is not as settled as in Type B, although certain information is known, such as the desired cardinality. One candidate is provided in [2], and used there to index the clusters in the Type D^n root system. We follow a different approach. First, we find a bijection between Type B triangulations and Type B non-crossing partitions, which were introduced by Reiner [6]. Motivated by Reiner's definition of non-crossing partitions for Type D as a subset of those for Type B (which has the desired cardinality), we take our Type D triangulations T_D^n to be the corresponding subset of the T_B^n. (It is not clear whether there is any natural bijection between our T_D^n and the Type D triangulations of [2].) Our second chief result is to show that the order induced on T_D^n from its inclusion in T_B^n gives it a lattice structure also. In fact, our approach to type D works for any of the interpolating pseudo-types between B^n and D^n (for the definitions of which, see [6]), though here we shall restrict our discussion to Type D.

From the results in this paper one could proceed in two directions. One direction is to consider the existence of Tamari lattices in all Coxeter types. The other direction is to investigate further the lattices defined here, to see how many of the properties of the usual Tamari lattice carry over. We have already had some success along these lines: for instance, we can show that T_B^n and T_D^n are EL-shellable.

2 Type B Triangulations

Recall that the Type B^n Weyl group consists of signed permutations of n. We can think of these as permutations of $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ fixed under interchanging i and $\bar{i}$ for all $1 \leq i \leq n$. By analogy, Type B^n triangulations, T_B^n, are defined to be Type A triangulations of a $2n+2$-gon fixed under a half-turn. There is general consensus that this is the correct choice of Type B triangulation: see [7, 2].

We number the vertices of our standard $2n+2$-gon counterclockwise from $n+1$ to 1 and then from $\overline{n+1}$ to $\bar{1}$.

For $S \in T_B^n$, consider a chord C of S. A chord is said to be *pure* if it connects two barred vertices or two unbarred vertices; otherwise it is said to be *mixed*. The chord C is the diagonal of a quadrilateral, which we denote $Q(C)$. If C is pure, then we colour it red if $Q(C)$ contains another vertex of the same type as those of C whose label is higher, and green otherwise. If it is mixed, we colour it red if $Q(C)$ contains an unbarred vertex whose label is higher than the unbarred vertex of the chord, or a barred vertex whose label is higher than the label of the barred vertex of the chord. Otherwise we colour it green. We write $\bar{C}$ for the symmetric pair of a chord C (that is to say, the image of C under a half turn). Observe that C and $\bar{C}$ are assigned the same colour.

Lemma 1 *Consider a chord C in a triangulation S. Let S' be the triangulation obtained by replacing C by C', the other diagonal of $Q(C)$, and also replacing $\bar{C}$ by $\bar{C}'$. Then the colours of C in S and C' in S' are opposite.*

We can now state the first main theorem of this paper (which, as was already remarked, was arrived at and proved independently and more or less simultaneously by Reading [5]).

Theorem 1 *There is a lattice structure on T_B^n whose covering relations are given by $S \lessdot T$ iff S and T differ in that a green chord C in S is replaced in T by the other diagonal of $Q(C)$ (which will be red), and symmetrically for $\bar{C}$. We call this lattice the Type B Tamari lattice.*

The first ingredient in our proof of Theorem 1 is some further analysis of the red and green edges of triangulations.

Fix a triangulation S. For $1 \leq i \leq n$, consider those chords of S which are attached to i and let C_i be the first of these encountered in searching clockwise starting at $\bar{1}$. If none is encountered before reaching $i+1$, then C_i is not defined. Let $R(S)$ be the set of these chords, together with their symmetric pairs.

Lemma 2 *For any triangulation S, $R(S)$ consists of the red chords; the chords not in $R(S)$ are green. S is the unique triangulation whose set of red chords is exactly $R(S)$.*

3 Bracket Vectors in Types A and B

We briefly recall some well-known facts about the Type A Tamari lattice, which serve as motivation for our work in Type B.

To any triangulation $S \in T_A^n$, there can be associated a bracket vector $l(S) = (l_1(S), \ldots, l_{n+1}(S))$. Let $c_i(S)$ be the least vertex attached to i. Then $l_i(S) = i - 1 - c_i(S)$.

An $n+1$-tuple of positive integers is a bracket vector for some triangulation iff it satisfies the following two properties:

(i) $l_j \leq l_i - (i-j)$ provided $i-j$ and $l_i - (i-j)$ are both non-negative,

(ii) $0 \leq l_i \leq i-1$.

The order relation on triangulations can be given a simple interpretation in terms of bracket vectors: $S \leq T$ iff $l_i(S) \leq l_i(T)$ for all i. Further, $l(S \wedge T)_i = \min(l_i(S), l_i(T))$. Additionally, for x any n-tuple of numbers satisfying only condition (ii), there is a unique triangulation $\uparrow(x)$ with the property that if $S \in T_A^n$, then $l_i(S) \geq x_i$ iff $S \geq \uparrow(x)$. With the aid of this construction we can say $l(S \vee T) = \uparrow(\max(l(S), l(T)))$ (where max is taken coordinatewise).

We would like to do exactly the same thing in Type B, and we will largely be successful.

To a triangulation $S \in T_B^n$ we associate a bracket vector $l(S) = (l_1(S), \ldots, l_n(S))$. For $1 \leq i \leq n$, let $c_i(S)$ denote the first vertex adjacent to i encountered proceeding clockwise starting at $\bar{1}$. If the counter-clockwise distance from $i-1$ to $c_i(S)$ is less than or equal to $n-1$, set $l_i(S)$ to be that distance. Otherwise, set $l_i(S) = *$.

Conventions regarding $*$: $*$ is considered to be greater than any integer. $*$ plus an integer (or $*$) equals $*$.

Lemma 3 *The map from T_B^n to bracket vectors is injective.*

Lemma 4 *Type B bracket vectors are characterized by the following two properties:*
i) For $1 \leq i < j \leq n$, $l_i \leq l_j - (j-i)$ if $l_j - (j-i)$ is well-defined and non-negative.
ii) If $* > l_i \geq i$, then $l_{n+i-l_i} = *$.

We will now define an order on T_B^n. For $S, T \in T_B^n$, let $S \leq T$ iff for all i, $l_i(S) \leq l_i(T)$.

Proposition 1 *The covering relations of the order defined in the previous paragraph are as described in Theorem 1.*

To show that this order is in fact a lattice, we need some preliminary results.

Let M^n denote the n-tuples with entries in $[0, n-1] \cup \{*\}$, with the Cartesian product order. Let $M_{(i)}^n$ denote the elements of M^n which satisfy condition (i) of Lemma 4. Let $M_{(ii)}^n$ denote the elements of M^n which satisfy condition (ii) of Lemma 4.

Proposition 2 *There exist maps $\uparrow : M_{(ii)}^n \to T_B^n$, $\downarrow : M_{(i)}^n \to T_B^n$, which satisfy the following conditions:*

$$f \leq l(S) \text{ iff } \uparrow(f) \leq S$$
$$l(S) \leq f \text{ iff } S \leq \downarrow(f).$$

Using these maps, we can prove that meet and join exist in T_B^n by giving simple descriptions of them.

Proposition 3 *The Tamari order on T_B^n is a lattice. The lattice operations are given as follows: For $S, T \in T_B^n$, $S \vee T = \uparrow(\max(l(S), l(T)))$ and $S \wedge T = \downarrow(\min(l(S), l(T)))$.*

This completes the proof of Theorem 1.

4 Non-crossing partitions

The A^n non-crossing partitions are partitions of $n+1$ into sets such that if $v_1, \ldots, v_{n+1}$ are $n+1$ points on a circle, labelled in cyclic order, and if $B_1, \ldots, B_r$ are the convex hulls of the sets of vertices corresponding to the blocks of the partition, then the B_i are non-intersecting.

There is a bijection from T_A^n to A^n non-crossing partitions as follows. For $S \in T_A^n$, erase all the green edges of S, and move the endpoints of each red edge a little bit, the lower end point a little counterclockwise, the higher endpoint a little clockwise (so i and j are both on the upper side of the edge). These edges now divide the vertices in $[n+1]$ into subsets, which form a non-crossing partition by construction.

Note that the non-crossing partitions are often considered as being ordered by refinement; this order is quite different from the Tamari order.

The Type B non-crossing partitions, NC_B^n, are partitions of the set $1, \ldots, n, \bar{1}, \ldots, \bar{n}$, which have the properties that the partition remains fixed under interchanging barred and unbarred elements, and that if $2n$ points are chosen around a circle and labelled cyclically $v_1, \ldots, v_n, v_{\bar{1}}, \ldots, v_{\bar{n}}$, then the convex hulls of the vertices corresponding to the blocks of the partition do not intersect.

We now define a map ψ from T_B^n to NC_B^n, analogous to that in Type A. Erase all green edges. For every red edge which connects a barred vertex and an unbarred vertex, move both endpoints slightly counterclockwise. For a red edge which connects two vertices of the same type, move the endpoints slightly together (so that the vertices both lie on the side of the edge which includes the larger part of the polygon). Erase the vertices $n+1$ and $\overline{n+1}$. The remaining vertices are now partitioned by the red edges, in what is clearly a Type B non-crossing partition.

Proposition 4 *The map ψ is a bijection from T_B^n to NC_B^n.*

5 Type D

The Type B partitions, Π_B^n, are by definition those partitions of the set $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ which are fixed under the map interchanging i and $\bar{i}$, and such that there is at most one block which contains both i and $\bar{i}$, for any i. These partitions are in bijection with the elements of the intersection lattice of the Type B hyperplane arrangement.

The intersection lattice of the Type D hyperplane arrangement is a subset of that of Type B. This allows a natural definition of Type D partitions as a subset of Π_B^n. By this approach, one obtains that Π_D^n consists of those partitions of Π_B^n which do not contain any block $\{i, \bar{i}\}$. In [6], Reiner defined the non-crossing partitions of Type D, $\mathrm{NC}_D^n = \mathrm{NC}_B^n \cap \Pi_D^n$. (For more details on the material sketched in the two preceding paragraphs, see [6].)

We now define T_D^n to be those triangulations which correspond under ψ to partitions in NC_D^n.

Lemma 5 *T_D^n consists of those triangulations which do not contain the triangle $i, \bar{i}, i+1$ and $i, \bar{i}, \overline{i+1}$ for any $1 \le i \le n$. T_D^n can also be characterized as the set of triangulations S such that $l(S)_i \neq n-1$ for any $i \in [n]$.*

Proposition 5 *If $S, T \in T_D^n$, then $S \wedge T \in T_D^n$.*

This implies the following theorem:

Theorem 2 *The partial order induced on T_D^n from its inclusion in T_B^n is a lattice.*

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Packing tripods: a computational approach *

Alexander Tiskin †

1 Introduction

In [4, 2], S. K. Stein and S. Szabó posed a problem concerning simple three-dimensional polyominoes, called "semicrosses" in [4], and "tripods" in [2]. A *tripod* of order n is formed by a corner and the three adjacent edges of an integer $n \times n \times n$ cube (see Figure 1). How densely can one fill the space with non-overlapping tripods of order n? In [6], the author showed that the fraction of the space occupied by tripods must be infinitely small as $n \to \infty$.

Here we address a more general problem, also introduced in [4, 2]. Now, tripods of order n are to be packed with identical orientation and without overlap, so that their corner cubes coincide with one of the unit cells of a fixed $n \times n \times n$ cube. The number of tripods in the packing is called the packing *size*. Following Stein, we denote by $f(n)$ the maximum size of a tripod packing for a given order n.

The maximum fraction of space that can be filled with arbitrary non-overlapping tripods is proportional to $f(n)/n^2$. Our previous result can now be restated as follows: $f(n) = o(n^2)$ as $n \to \infty$. Our current goal is to study the behaviour of Stein's function $f(n)$ in more detail. In this paper, we approach this goal from two directions:

- computing good packings for small values of n;
- combining such packings recursively to obtain an asymptotic lower bound on $f(n)$.

The best previously known lower bound for Stein's function is $f(n) = \Omega(n^{1.5235})$. We show that this lower bound can be improved to $\Omega(n^{1.5287})$. We also make some conjectures on the structure of good packings, and suggest possible further improvements to our experiments.

2 Packings of small order

For a given tripod packing of order n, the only known method of deciding its optimality is by brute-force search. Since the number of possible tripod packings grows exponentially,

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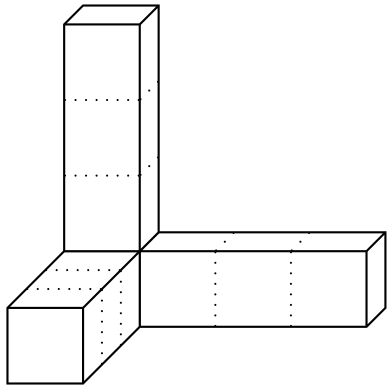

Figure 1: A tripod of order $n = 4$

```
0  0.  02.  02..  02...  02....  02.4...  02....3.
   .1  ..0  ..02  ..02.  ..02..  ..05...  ..0.2..3
       1.2  13..  13.4.  13.4..  1.26...  14..5...
            ..13  ..1.2  ..1.24  ....012  67......
                  ..3.4  ..35..  3...46.  ..1..25.
                         ....35  .3....4  ..4.6.7.
                                 ..3.5.6  ...0.4.5
                                          ...1.6.7
```

Figure 2: Best known tripod packings of order $n \leq 8$

obtaining the precise value of the function $f(n)$ is difficult even for very moderate values of n.

Figure 2 shows the best known tripod packings of order $n \leq 8$. Each packing is represented by a $n \times n$ matrix. A tripod in position (i, j, k) is represented by the number k in row i and column j of the matrix (with all indices running from 0 to $n-1$).

The packings with $n \leq 6$ are the only ones known to be optimal. Note that these optimal packings have a very simple structure: apart from the corner tripods $(0,0,0)$ and $(n-1, n-1, n-1)$, they contain a triple of tripods $(0,1,2)$, $(1,2,0)$, $(2,0,1)$. The remaining tripods can obtained by repeatedly translating this triple along the main diagonal of the cube, with simultaneous rotation by 60° in every step. The total number of tripods obtained by this construction is $3n-4$.

The packing of degree 7 illustrates another, much more powerful method of constructing tripod packings. This packing can be obtained from the packing of degree 3 by "blowing up" the two corner positions to $3 \times 3 \times 3$ subcubes, each with 5 tripods. The edge positions become "blown up" to $3 \times 3 \times 1$ cuboids; three of these cuboids contain 3 tripods each. The total number of tripods is 19. Every packing of order m and size d can be "blown up" to a packing of order m^2 and size d^2. By repeating this process, we can obtain a packing of order m^k and size d^k for any $k \geq 0$. Hence, $f(m^k) \geq d^k$, and therefore $f(n) = \Omega(n^{\log d/\log m})$. The best lower bound given by the packings in Figure 2 is $f(n) = \Omega(n^{\log 19/\log 7}) = \Omega(n^{1.5131})$.

In contrast with the previous packings, the packing of degree 8 lacks any obvious structure (apart from the symmetry that we discuss below). This "sporadic" packing appears to be better than all known systematic packings of the same order. This motivates us in searching exhaustively for good packings of relatively low degrees.

In order to search the space of all packings efficiently, we need to exploit the various symmetries of this space. For instance, a packing remains valid if the cube is rotated by 120° around its main diagonal. In fact, all packings in Figure 2 are invariant under such rotation: the existence of a tripod (i, j, k) implies the existence of tripods (j, k, i) and (k, i, j). We call this property *3-symmetry.* All known optimal packings are either 3-symmetric, or can be transformed into 3-symmetric form without losing any tripods. Therefore, the following conjecture seems plausible.

Conjecture (3-symmetry) *For every n, there exists a 3-symmetric tripod packing of size $f(n)$.*

In other words, we conjecture that $f(n) = g(n)$, where $g(n)$ is the maximum possible size of a 3-symmetric packing. The 3-symmetry conjecture allows us to perform exhaustive search for much higher values of n. Table 1 shows the best known values of $g(n)$ (which are also the best known values of $f(n)$) for $n \leq 15$. These values are known to be optimal up to $g(14)$.

Another transformation preserving the validity of a tripod packing is *mirror reflection*: a tripod (i, j, k) is mapped to (j, i, k). No non-trivial packing can be invariant under this transformation; still, breaking the mirror symmetry of the search space approximately halves the amount of search.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$g(n)$	1	2	5	8	11	14	19	23	28	32	37	43	49	55	61

Table 1: Best known tripod packings of order $n \leq 15$

The last global transformation that preserves the validity of a packing is *duality*: a tripod (i, j, k) is mapped to $(n-i-1, n-j-1, n-k-1)$. All packings in Figure 2 except the last one are self-dual (i.e. invariant under duality). It might be tempting to strengthen our symmetry conjecture by considering packings that are both 3-symmetric and self-dual. However, since the value of $g(8) = 23$ is odd, it cannot correspond to any self-dual packing. Still, exploiting duality allows some partial symmetry breaking and further reduction in the amount of search.

In addition to these global symmetries, it is also possible to reduce the search by considering transformations that affect individual tripods. For example, it is easy to see that a packing only needs be considered if it satisfies the following property: no tripod (i, j, k) can be moved to a position (i, j, k') with $k' < k$. This property, which we call *gravity*, is easy to enforce (even in conjunction with 3-symmetry), and provides significant further savings.

By running a branch-and-bound algorithm with all the above optimisations, we have been able to compute Table 1 (and the corresponding packing matrices) in about 40 hours on a Sun Ultra 5 workstation. In the next section, we describe how these results feed into the search for the best asymptotic lower-bound packing exponent.

3 Packing asymptotics

Previous to this work, the best asymptotic lower bound on the function $f(x)$ has been given by D. Hickerson [3]. Hickerson's construction is based on recursion similar to the one we have seen in the example of $n = 7$. We start with a simple "external" packing, and blow up each position by substituting suitably chosen "internal" packings. Both the external and the internal packings are in general cuboids with unequal dimensions. Hickerson ran a few levels of recursion by hand, using just two external cuboid packings: one of order $2 \times 2 \times 1$ and size 2, another of order $3 \times 3 \times 3$ and size 5. He obtained a packing of order 255 and size 4639, implying $f(n) = \Omega(n^{\log 4639/\log 255}) = \Omega(n^{1.5235})$.

Hickerson's method has been programmed by G. Stoneman as part of his undergraduate project [5]. Additionally, a database of cuboid packings generalising Table 1 was compiled and used for selection of internal packings. One of the resulting recursive packings had order 255 and size 4776, implying $f(n) = \Omega(n^{\log 4776/\log 255}) = \Omega(n^{1.5287})$. This is the best currently known asymptotic lower bound, which corresponds to the densest known infinite family of tripod packings.

4 Conclusion

We have presented a systematic computational approach to Stein's tripod packing problem, which resulted in a new asymptotic lower bound on the maximum size of a packing: $f(n) = \Omega(n^{1.5287})$. There remains a substantial gap between this lower bound and the upper bound $f(n) = o(n^2)$ from [6]. It is likely that the lower bound can be further improved by a more sophisticated application of the methods described here. For instance, the exhaustive search for recursive packings in [5] uses just two external cuboids. Using more external cuboids could yield even better results. Furthermore, it may be possible to collect statistics on cuboid dimensions in order to direct the search to the more promising regions of the cuboid dimension space. A proof or disproof of the 3-symmetry conjecture, which has eluded us so far, could provide new valuable insights into the structure of tripod packing. Finally, our computation techniques could be extended to other challenging combinatorial problems.

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On 2-connected spanning subgraphs of given maximum degree

Michal Tkáč* and Heinz-Jürgen Voss†

1 The Results

Finding hamiltonian paths and cycles in graphs is a well investigated subject in graph theory. There are probably hundreds of papers devoted to results of the form "If G satisfies property P, then G is hamiltonian". If P is replaced by a weaker condition P' then G is perhaps no longer hamiltonian, but has a generalized structure near to hamiltonicity. Such generalizations have been stated in [7]. Here we generalize a hamiltonian cycle in a different way.

Let G be a connected graph. A subset S of the vertex set of G *separates* G if the graph $G \setminus S$ obtained from G by deleting the vertices of S is disconnected. If $|S| = r$, S is said to be an *r-separator of* G. A cycle $\mathcal{C}$ of a graph G is said to be a *separating i-cycle* (SiC) if $|V(\mathcal{C})| = i$ and $V(\mathcal{C})$ separates G.

A graph G is said to be *chordal* if each cycle $\mathcal{C}$ of length at least 4 has a chord; by a chord we mean an edge which is not in $\mathcal{C}$ but its end vertices are on $\mathcal{C}$. A graph G is said to be *sep-chordal* if each *separating* cycle $\mathcal{C}$ of length at least 4 has a chord. Here we will discuss a special class of (sep)-chordal graphs, namely, the (sep)-chordal polyhedral graphs.

By Steinitz's theorem a polyhedral graph is a planar 3-connected graph. Moreover, it is uniquely embeddable on the sphere. Any 3-separator of a chordal polyhedral graph G forms a separating 3-cycle (S3C). This is a cycle Δ of length three and $G \setminus \Delta$ consist of exactly two components. If one of these components is a single vertex we will call it a simplicial vertex. A 3-cycle whose all three vertices are adjacent with a vertex x will be denoted by Δ_x. The vertex x is said to be *a contact vertex* of Δ_x. It is easy to verify that in a chordal polyhedral graph each 3-cycle has a contact vertex and each S3C has exactly two contact vertices.

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A subgraph of G is *a spanning subgraph of* G if it contains all vertices of G. The 2-connected spanning subgraphs in which all vertices have degree at most k, $k \geq 2$, are called *k-trestles*. We will say that a graph G is *k-trestled* provided that G has a k-trestle.

The property P investigated here is the toughness of a graph G. A graph G is said to be *t-tough* if for every separating set $S \subseteq V(G)$ the number $\omega(G \setminus S)$ of components of $G \setminus S$ is at most $\frac{|S|}{t}$. The toughness $\tau(G)$ of a non-complete graph G is defined to be the largest number $t > 0$ such that G is t-tough. For a complete graph K_n let $\tau(K_n) = \infty$. The concept of toughness was introduced by Chvátal [4].

It is easy to see that a 2-trestle in G is a Hamiltonian cycle; so every graph with toughness less than one has no 2-trestles. The following Lemma from [8] shows that every graph has a similar property with respect to k-trestles, $k \geq 2$.

Lemma 1 *Every graph G with toughness $\tau(G) < \frac{2}{k}$ (where the integer k is greater than one) has no k-trestle.*

In [4] Chvátal conjectured :

Conjecture 1 *(Chvátal) There is a real number $t_0 > 0$ such that every t_0-tough graph has a Hamiltonian cycle (i.e., a 2-trestle).*

It seems to be interesting to consider relations between t-tough and k-trestled graphs in general. We posed in [8] the following conjecture.

Conjecture 2 *For every integer k greater than one there is a real number $t_k > 0$ such that every t_k-tough graph has a k-trestle.*

Chen, Jacobson, Kézdy, and Lehel [3] proved the validity of the Conjectures 1 and 2 for chordal graphs.

Theorem 1 *Each chordal graph G of toughness > 18 has a 2-trestle, i. e., a Hamiltonian cycle.*

For chordal polyhedral graphs Böhme, Harant, and Tkáč [2] found a sharp bound for the toughness.

Theorem 2 *Each chordal planar graph G of toughness > 1 has a 2-trestle, i. e., a Hamiltonian cycle. Moreover, there are 1-tough chordal polyhedral graphs without a Hamiltonian cycle.*

Gerlach [6] strengthened this result.

Theorem 3 *Each sep-chordal planar graph G of toughness > 1 has a 2-trestle, i. e., a Hamiltonian cycle. Moreover, there are even 1-tough chordal polyhedral graphs without a Hamiltonian cycle.*

There are several papers which deal with k-trestled polyhedral graphs, $k \geq 3$. In [1] Barnette showed that there is a polyhedral graph with no 5-trestles. In [5] Gao proved that every 3-connected graph on the plane, projective plane, torus and Klein bottle has a 6-trestle.

Tkáč and Voss proved in [9] :

Theorem 4 *For every integer k, $2 \leq k \leq 6$, each chordal polyhedral graph G of toughness greater than $\frac{3+2^{k-2}}{2+2^{k-1}}$ has a k-trestle. There are chordal polyhedral graphs of toughness $\frac{21}{29} > 0,724$, or $\frac{51}{86} > 0,593$, or $\frac{139}{259} > 0,536$, without a 3-trestle, or a 4-trestle, or a 5-trestle, respectively.*

Kaiser, Tkáč and Voss [10] strengthened this result to sep-chordal polyhedral graphs, obtaining the same bounds for the toughness as in Theorem 4.

Theorem 5 *For every integer k, $2 \leq k \leq 6$, each sep-chordal polyhedral graph G of toughness greater than $\frac{3+2^{k-2}}{2+2^{k-1}}$ has a k-trestle. There are even chordal polyhedral graphs of toughness $\frac{21}{29} > 0,724$, or $\frac{51}{86} > 0,593$, or $\frac{139}{259} > 0,536$, without a 3-trestle, or a 4-trestle, or a 5-trestle, respectively.*

The bounds in Theorems 2 and 3 are sharp. The precise bounds in Theorems 1, 4 and 5 are unknown.

2 Sketch of the proof of Theorem 4

2.1 Construction of a spanning tree

Let G be a chordal polyhedral graph with $n \geq 5$ vertices. In [2] it was shown that G has at least two simplicial vertices (say v_1 and v_n). The vertex v_1 is adjacent with exactly three vertices (say v_2,v_3 and v_4) of a S3C Δ_{v_1} which separates the vertex v_1 from the rest of G. It is easy to see that $v_n \in V(G) \setminus \{v_1, v_2, v_3, v_4\}$ and that after removing an arbitrary simplicial vertex from a chordal polyhedral graph the resulting graph is again chordal and polyhedral.

We fix such simplicial vertex v_1 and its S3C Δ_{v_1} and delete step by step a simplicial vertex of G not in $\Delta_{v_1} \cup \{v_1\}$. This process stops with the complete graph on four vertices $H_4 := G[\Delta_{v_1} \cup \{v_1\}]$. Let $D(H_4)$ be the path (v_1, v_2, v_3, v_4). Let F denote the component of $G \setminus \Delta_{v_1}$ containing the $n-4$ remaining vertices of G. There is a unique contact vertex of Δ_{v_1}(say v_5) in F. Let $H_5 := G[H_4 \cup \{v_5\}]$. $D(H_5)$ is obtained from $D(H_4)$ by adding the vertex v_5 and joining it with the vertex v_4 of $D(H_4)$.

Let the vertices of $G \setminus H_4$ be denoted by $v_n, v_{n-1}, \ldots, v_5$ in the order of their elimination from G, so v_n is deleted firstly, v_{n-1} is deleted secondly, etca. We denote $H_i := G \setminus \{v_n, \ldots, v_{i+1}\} = G[H_4 \cup \{v_5, \ldots, v_i\}]$ for $5 \leq i \leq n$, and $H_n = G$.

The vertex v_{i+1} is the only one in $V(H_{i+1}) \setminus V(H_i) = \{v_{i+1}\}$. By construction it is a simplicial vertex of H_{i+1}; the three neighbours of v_{i+1} ly on a 3-cycle $\Delta_{v_{i+1}}$ which bounds a 3-face of H_i. In H_{i+1} (and also in G) the vertex v_{i+1} is a contact vertex of $\Delta_{v_{i+1}}$. The graph H_{i+1} can be constructed from H_i by inserting v_{i+1} in the unique 3-face bounded by $\Delta_{v_{i+1}}$ and joining v_{i+1} with all vertices of $\Delta_{v_{i+1}}$.

$D(H_{i+1})$ is obtained from $D(H_i)$ by adding v_{i+1} and joining v_{i+1} with the vertex v_{i+1}^- which is the last added vertex of Δv_{i+1} (all three vertices of Δv_{i+1} are in $D(H_i)$). The construction stops with $H_n = G$ and $D(G) := D(H_n)$. The tree $D(G)$ can be referred to as the *elimination tree* or *structure tree* of G.

It is easy to see that D(G) is a spanning tree of G with maximum degree 4 rooted at the vertex v_1. We orient the tree $D(G)$ from the root v_1 to the endvertices. Thus we can use the concepts *"predecessor"* and *"successor"* for the vertices in $D(G)$.

For convenience a vertex x of $D(G)$ is said to be an *i-vertex* if it has degree $deg_{D(G)}(x) = i$ in $D(G)$. Further, $D^k(G)$ denotes the subgraph of $D(G)$ induced by all vertices x of degree $deg_{D(G)}(x) \geq k$, i. e., it is induced by all i-vertices with $k \leq i \leq 4$. Obviously, $D^3(G)$ does not contain a vertex of H_4. Let U be a connected subtree of $D^2(G) \setminus H_4$. It has the root w_U, this is the vertex of U nearest to v_1 in $D(G)$. Observe, Δ_{w_U} is a S3C. Let U^* denote the subtree of $D(G) \setminus \{w_U^-\}$ containing w_U. Obviously, $U \subseteq U^*$. All vertices of U are in the same component of $G \setminus \Delta_{w_U}$. The properties of $D(G)$ imply that $\omega(G \setminus V(U \cup \Delta_{w_U})) = \omega(D(G) \setminus V(U))$, and we have the following upper bound for the toughness $\tau(G)$ of G:

(*A*) *Let U be a connected subgraph of the tree $D^2(G) \setminus H_4$. Then*

$$\tau(G) \leq \frac{|V(U)| + 3}{\omega(D(G) \setminus V(U))}.$$

2.2 The existence of 3-trestles in (sep)-chordal polyhedral graphs

We give a sketch of the proof of Theorem 4 for $k = 3$.

Theorem 6 *Each chordal polyhedral graph G of toughness $> \frac{5}{6}$ has a 3-trestle.*

The proof of Theorem 6: In the following let $G_4 := H_4$

In the structure tree $D(G)$ of G the subtrees of Fig. 1 are forbidden, because they cause a toughness $\leq \frac{5}{6}$ in G .
In (i) - (iv) we have formulated more extended results, derived from formula (A).

(i) *No two 4-vertices of $D(G)$ are adjacent.*

(ii) *Each 3-vertex of $D(G)$ has at most two 4-neighbours in $D(G)$.*

(iii) *Any two adjacent 3-vertices of $D(G)$ have at most two 4-neighbours in $D(G)$.*

(iv) *Any three 3-vertices of $D(G)$ inducing a 3-path of $D(G)$ have at most three 4-neighbours in $D(G)$.*

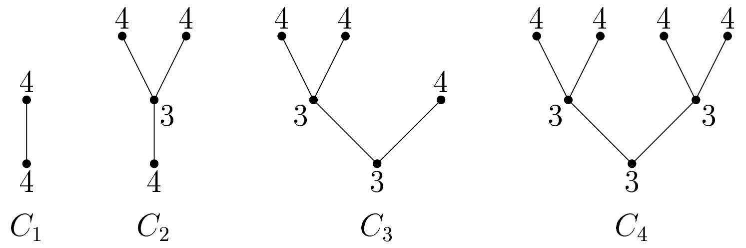

Figure 1.

With the help of the structure tree $D(G)$ we inductively construct a sequence of chordal subgraphs $G_1 := K_4, G_2, G_3, \ldots, G_r = G$, $1 \leq r \leq n-4$, and their 3-trestles $T_1, T_2, T_3, \ldots T_r$, respectively, where T_i is a 3-trestle of G_i such that each endvertex of $D(G_i)$ has degree 2 in T_i.

In the proof ([9]) we give Rules how to enlarge the 3-trestle T_i of G_i to a 3-trestle T_{i+1} of G_{i+1} using the results $(i) - (iv)$. Finally, T_r is a 3-trestle of $G_r = G$. $\square$

By Gerlach [6] in a sep-chordal polyhedral graph H the concept of a simplicial vertex has to be replaced by a simplicial subgraph S, where either S is a path P_t with t-vertices inducing with its contact 3-cycle Δ_{P_t} a wheel with $t+2$ spokes or S is a 3-cycle D inducing with its contact 3-cycle Δ_D a 3-sided prism, i.e., D and Δ_D are joined by a perfect matching. Thus H has a similar structure as G, and a similar proof can be given. The class of forbidden subgraphs of the corresponding structure tree $D(H)$ has to be enlarged by an infinite set of forbidden subgraphs.

The lower bounds in Theorems 4 and 5 can be obtained by constructing appropriate chordal polyhedral graphs.

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Binary Space Partition for Fat Orthogonal Hyperrectangles

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Abstract

We show that there is a *binary space partition* of size $O(n \operatorname{polylog} n)$ and depth $O(\operatorname{polylog} n)$ for n orthogonal fat hyperrectangles in $\mathbb{R}^d$ for any fix $d \in \mathbb{N}$. Specifically in 3-space, this improves earlier bounds of Agarwal et al.

1 Introduction

The *binary space partition* (*BSP*) is a data structure developed by the computer graphics community [10, 7] for fast rendering and shadow generation. It also found numerous applications in computational and combinatorial geometry. Combinatorial bounds on the worst case complexity of this data structure directly affects the complexity of the applications.

A BSP is obtained by a recursive cutting scheme in the d-dimensional Euclidean space ($\mathbb{R}^d$) for a set of disjoint (or non-overlapping) polyhedral objects. We split the bounding box of the polyhedra along a hyperplane into two parts and then we partition recursively the two subproblems corresponding to the two subcells until the interior of every subcell is disjoint from the boundary of all input polyhedra. This partitioning procedure can be represented by a binary tree (called *BSP tree*) where every intermediate node represents a convex cell along with splitting hyperplane and every leaf represents a convex cell empty of input boundaries.

The complexity of a BSP tree is measured by two parameters: The *size* $|P|$ of a BSP P is the number of nodes in the binary tree (and thus $(|P|-1)/2$ is the number of leaves of P, note that the collection of leaves corresponds to a convex subdivision of the space). The *depth* of P is the length of the longest path starting from the root of the tree.

We are interested in obtaining *combinatorial bounds* on the minimum size and depth of a BSP for selected families of n polyhedral scenes in $\mathbb{R}^d$ in terms of n and d. In the Euclidean plane, Paterson and Yao showed that there is always a $O(n \log n)$ size and $O(\log n)$ depth BSP for n disjoint line segments; the best known lower bound for the BSP size is $\Omega(n \log n / \log \log n)$ [11].

Paterson and Yao [8] also showed that for n quadrilaterals in $\mathbb{R}^3$, there is always a $O(n^2)$ size and $O(\log n)$ depth BSP. This upper bound is asymptotically optimal in the wost case by a construction of Eppstein [5]. For $(d-1)$-dimensional polytopes in R^d, the best known upper bound is $O(n^{d-1})$ whereas there is no better lower bound than $\Omega(n^2)$ (super-quadratic lower bounds are known for a special type of BSPs, for *autopartitions*, where every cell must be split by a hyperplane along one of the objects in that cell).

Research concentrated on finding families of objects for which smaller BSPs are possible: Two natural restrictions are (1) to consider *axis-parallel* objects or (2) to exclude long and skinny objects. In the plane, there is always a linear size BSP for line segments if the ratio of the longest and the shortest segment is bounded by a constant [4], or if the segments have a constant number of distinct orientations [12]. In any dimension $d \in \mathbb{N}$, De Berg [3] has generated linear size BSP for n *full-dimensional fat* objects in $\mathbb{R}^d$ (a full dimensional object is fat if the ratio of the radii of a circumscribed and an inscribed ball is bounded by a constant). This result does not apply to $(d-1)$-dimensional fat objects in $\mathbb{R}^d$.

¿From now on, we restrict our attention to *axis-parallel* rectangles in the Euclidean space. Dumitrescu et al. [6] studied k-dimensional *axis-parallel* rectangles in $\mathbb{R}^d$. Let us denote by $r(R_n(k,d))$ the minimum size of a BSP for a set $R_n(k,d))$ of n disjoint axis-parallel k-dimensional rectangles in $\mathbb{R}^d$. Let $r_n(d,k) = \max_{R_n(k,d)}\{r(R_n(k,d))\}$. It was shown in [6] that $r_n(k,d) = O(n^{d/(d-k)})$ for $1 \le k < d$ and $r_n(k,d) = \Omega(n^{d/(d-k)})$ for $1 \le k \le d/2$. The lower bound constructions are composed of long skinny rectangles in all cases. Thus $r_n(k,d) = \Theta(n^{d/(d-k)})$ for $1 \le k < d/2$, which coincides with the bound $r_n(1,d) = \Theta(n^{d/(d-1)})$, obtained earlier by Paterson and Yao [9]. Surprisingly for $k = d-1$ we have the same challenging gap between the best known lower bound $r_n(k,d) = \Omega(n^{d/(d-\lfloor (d-1)/2 \rfloor)})$ (where $n^{d/(d-\lfloor (d-1)/2 \rfloor)} = o(n^2)$) and the best known upper bound $r_n(k,d) = O(n^{d-1})$.

Rather than looking at full dimensional fat objects, we consider k-dimensional rectangles that are fat in their own dimension. We say that a (k-dimensional) rectangle r is *fat* (or *α-fat*) if the ratio between the length of any two sides of r is in the interval $[1/\alpha, \alpha]$, $\alpha \in \mathbb{R}^+$. Let $f(F_n(k,d))$ denote the minimum size of a BSP for a set $F_n(k,d)$ of n disjoint *fat* axis-parallel k-dimensional rectangles in $\mathbb{R}^d$, and let $f_n(d,k) = \max_{F_n(k,d)}\{f(F_n(k,d))\}$. Agarwal et al. [1] showed recently that $f_n(2,3) = n2^{O(\sqrt{\log n})}$, moreover their every BSP is $O(\log n)$ deep and can be generated in $n2^{O(\sqrt{\log n})}$ time. We improve upon the bound of Agarwal et al. and our results generalize to higher dimensions:

Theorem 1 *For any $d \in \mathbb{N}$, we have $f_n(d-1,d) = O(n \log^{6d-10} n)$, moreover every hyperrectangle is partitioned into $O(\log^{6d-10} n)$ pieces and the depth of every such BSP is $O(\log^{3d-5} n)$.*

We also give a lower bound for *orthogonal BSP*s where splitting hyperplanes are axis-parallel.

Theorem 2 *For any $n \in \mathbb{N}$, there are n disjoint orthogonal fat rectangles in $\mathbb{R}^3$ such that the size of every orthogonal BSP for it has size $\Omega(n \log n)$.*

In general we would like to determine the asymptotic behavior of $f_n(k,d)$ for every $d \in N$, $1 < k < d$. It is easy to see that we cannot hope for anything better than $f_n(k,d) = \Theta(r_n(1, d-k+1))$. Indeed, consider a worst case construction $R_n(1, d-k+1)$ for n line segments in $\mathbb{R}^{d-k+1}$. Embed $R_n(1, d-k+1)$ into $\mathbb{R}^d$ and draw a fat k-dimensional rectangle above every line segment. The size of the smallest BSP for this set $F_n(d,k)$ is at least $r_n(1, d-k+1)$. We conjecture that this bound can be attained apart from a polylogarithmic factor, that is, out of the k extents of a k-dimensional fat rectangle, $k-1$ extents are essentially redundant and allow for reducing the dimension of the space with $k-1$.

Conjecture 1 *For any $k, d \in \mathbb{N}$, $1 < k < d$,*

(i) $f_n(k,d) = O(r_n(1, d-k+1) \operatorname{polylog} n) = O(n^{(d-k+1)/(d-k)} \operatorname{polylog} n)$ *and the depth of every such BSP is* $O(\operatorname{polylog} n)$ *where the polylogarithmic factor depends on d and k;*

(ii) $f_n(k,d) = \Omega(r_n(1, d-k+1) \log n) = \Omega(n^{(d-k+1)/(d-k)} \log n)$.

2 Proof technique

Our proof is constructive and presents an explicit partitioning algorithm, where every splitting hyperplane is axis-parallel. The main difference compared with the approach of Agarwal et al. [1] is that we exploit more geometric properties of fat hyperrectangles, and this also makes the analysis of our algorithm considerably simpler. Our result implies that the function $n2^{O(\sqrt{\log n})}$ is not intrinsic to this problem (although it turns up miraculously in other geometric problems [13]).

2.1 Notations

Definition 1 *Given a full dimensional axis-parallel box* (cell) *C and a (hyper-)rectangle r intersecting the interior of C, we say that*

- *r is* long *with respect to C, if none of the vertices of r lies in the interior of C;*
- *a long r is a* free-cut *for C if none of the edges of r intersects the interior of C.*
- *a long r is a* shelf *for C if at least 1 but at most 2^{d-3} edges of r intersects the* $\operatorname{int}(C)$.
- *a long r is a* bridge *for C if exactly 2^{d-2} parallel edges of r intersect the interior of C.*

Our partitioning algorithm makes use of every free-cut, even if it is not always stated explicitly: Whenever a fragment $r \cap C$ is a free-cut for an intermediate cell C of the partitioning, we partition C along $r \cap C$. Therefore we may assume that every long rectangle w.r.t. a cell C is either a bridge or a shelf.

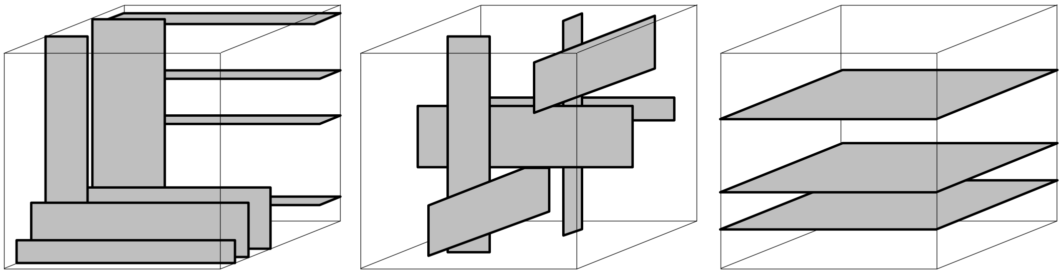

Figure 1: Shelves, bridges, and free-cuts for a cell C in $\mathbb{R}^3$.

The distinction between shelves and bridges allows us to exploit the geometric information they carry. The aspect ratio of the clipped bridge $r \cap C$ is constraint if r is an α-fat rectangle: The length of the edge of $r \cap C$ in the interior of C is at most α times longer than any of its edges along the boundary of C (we refer to this property as "*semi-fatness*"). The intersection $r \cap C$ of a shelf r and the cell C can have arbitrary aspect ratio, but it intersects at least three faces of C. We also distinguish d *classes* of long rectangles w.r.t. C: A long rectangle $r \cap C$ belongs to the x_i-class, $1 \leq i \leq d$, if the edges of r in $\text{int}(C)$ are parallel to the x_i-axis.

2.2 Overlay of BSPs

A powerful method to construct BSPs is a combination of several BSPs, which we call an *overlay BSP*. It allows us to partition the same region several times such that the size of the overlay BSP is at most the sum of the BSPs used. Consider the following setting: We are given a set F of objects in a cell C, a BSP P for F, and a subdivision $S(C)$ of C (which may be a result of a BSP for some other set of objects F' in C). We refine recursively the subdivision $S(C)$ according to the cuts made by the BSP P for F: Whenever P splits a subcell $C' \subset C$ into two parts, we split every region R of the current subdivision which lies along the splitting plane if the interior of R intersects an object of F and the splitting plane actually divides $F \cap R$. If F contains no objects in R or all objects of $F \cap R$ lie on one side of the splitting plane, then we keep R intact.

If we have k independent BSPs $P_1, P_2, \ldots, P_k$ for sets $F_1, F_2, \ldots, F_k$ of disjoint objects in the same region C, then we can obtain their overlay by starting with a trivial subdivision $S_0 = \{C\}$ and recursively forming the overlay of S_{i-1} and P_i ($i = 1, 2, \ldots, k$) to obtain a refined subdivision S_i. The resulting overlay partitioning is a BSP (since a splitting line of P_i splits –simultaneously– regions of S_i into two parts), and it is a BSP for each of $F_1, F_2, \ldots, F_k$. The depth of the overlay BSP is at most the sum of the depths of $P_1, P_2, \ldots, P_k$.

Given a set F of rectangles, a *mast* is an axis-parallel line segment clipped to a rectangle of F. If every P_i is orthogonal and cuts every mast into at most ℓ times, then the overlay P is also orthogonal and cuts every mast at most $k\ell$ times. So P partitions every rectangle $r \in F$ into at most $(k\ell + 1)^{d-1}$ fragments. Since we exploit every free-cut right after they are created, the number of fragments of the rectangle in disjoint subcells is at most $2(d-1)(k\ell)^{d-2}$.

2.3 Partitioning algorithm

Our partition algorithm is defined recursively. The base case is $d = 3$. We partition the rectangles in rounds. In every round, we consider one by one every cell of the current partition. In each cell C, we apply a BSP for the rectangles that are long w.r.t. C such that we cut every *mast* into $O(\text{polylog}\, n)$ pieces. This way, every round eliminates the rectangles that are long at the beginning of the round and may damage some other rectangles by splitting them into $O(\text{polylog}\, n)$ pieces. In the next round, every fraction of the a damaged rectangle r is either long w.r.t. one of the cells or is adjacent to one of the 2^{d-1} vertices of r. There are at most $\log n$ rounds if we start every round by splitting every cell along the median of all rectangle vertices in the interior. This scheme guarantees that in total every rectangle is partitioned into $O(\text{polylog}\, n)$ pieces.

One round takes care of all the rectangles long w.r.t. a cell C without making too much damage to other rectangles. Every round is composed of four hierarchically structured phases. The lowest level phase is a BSP for the shelves while cutting every mast at most $\log n$ times. The other three levels partition the bridges, but they also periodically eliminate all the newly created shelves. The second level phase partitions the set B of bridges such that in every subcell C' the portion of every bridge $b \in B$ within C' is either a shelf or belongs to one specific class. (There could actually be new bridges w.r.t. C', but we only care about the bridges in B.)

The third phase handles bridges belonging to the same class. Notice that bridges of the x_i-class traverse the cell in the x_i direction, therefore we can project them into a subspace R^{d-1} orthogonal to x_i. Then we can apply our $(d-1)$-dimensional algorithm if $d > 3$. If $d = 3$ then we apply a BSP for axis-parallel line segments in the plane while cutting every mast at most $O(\text{polylog}\, n)$ times.

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Open caps and cups in planar point sets*

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Abstract

A *configuration* is a finite set of points in the plane such that no 3 points lie on a line and no 2 points have the same x-coordinate. Let X be a configuration and let $p_1, \ldots, p_k$ be k points of X ordered according to the increasing x-coordinate. For $i = 1, \ldots, k-1$, let s_i be the slope of the line p_ip_{i+1}. The set $P = \{p_1, \ldots, p_k\}$ is a *k-cap* or a *k-cup*, if the sequence $s_1, s_2, \ldots, s_{k-1}$ is decreasing or increasing, respectively (see Fig. 1). The set $P = \{p_1, \ldots, p_k\}$ is *open (in X)*, if no point $p \in X$ with $x(p_1) < x(p) < x(p_k)$ lies above the polygonal line $p_1p_2 \ldots p_k$. We prove that for every $k, l \geq 2$ there is an integer $f(k,l)$ such that any configuration of size $\geq f(k,l)$ contains an open k-cap or an open l-cup. This can be seen as a generalization of the Erdős-Szekeres theorem. It implies results on empty polygons in k-convex configurations proved by Károlyi *et al.* [6], Kun and Lippner [7], and Valtr [11] (a configuration is *k-convex*, if it determines no triangle with more than k points in the interior). Another corollary is that for any $k, l \geq 2$ any sufficiently large configuration with no (open) k-cap contains an empty l-gon. We give double–exponential lower and upper bounds on $f(k,l)$. We also exactly determine the maximum size of a 1-convex set with no empty l-gon.

1 Introduction

A *configuration* is a finite set of points in the plane such that no 3 points lie on a line and no 2 points have the same x-coordinate. Let X be a configuration. We say that a subset of X is *in convex position*, if it is the vertex set of a convex polygon. Let $p_1, \ldots, p_k$ be k points of X ordered according to the increasing x-coordinate (i.e., $x(p_1) < x(p_2) < \cdots < x(p_k)$). For $i = 1, \ldots, k-1$, let s_i be the slope of the line p_ip_{i+1}, i.e. $s_i = (y(p_{i+1}) - y(p_i))/(x(p_{i+1}) - x(p_i))$. The set $P = \{p_1, \ldots, p_k\}$ is a *k-cap* or a *k-cup*, if the sequence $s_1, s_2, \ldots, s_{k-1}$ is decreasing or increasing, respectively (see Fig. 1). The set

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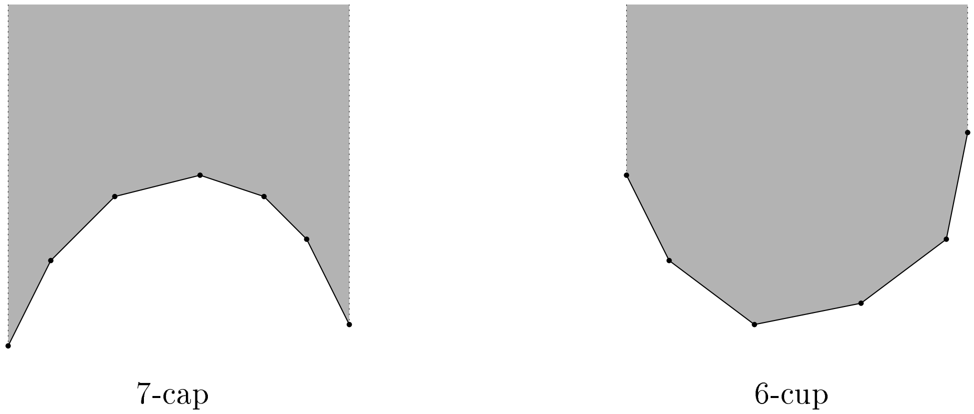

Figure 1: A 7-cap and a 6-cup; each of them is open if the unbounded shaded region contains no points of X.

$P = \{p_1, \ldots, p_k\}$ is *open (in X)*, if no point $p \in X$ with $x(p_1) < x(p) < x(p_k)$ lies above the polygonal line $p_1p_2 \ldots p_k$.

In 1935 Erdős and Szekeres proved the following classical result:

Theorem 1 (Erdős and Szekeres) *For any $n \geq 3$, there is an integer $F(n)$ such that any set of at least $F(n)$ points in general position in the plane contains n points in convex position.*

Erdős and Szekeres [2] proved Theorem 1 by proving that any sufficiently large configuration contains a k-cap or an l-cup. We show the following generalization of this result:

Theorem 2 *For any $k, l \geq 2$, there is a (least) integer $f(k,l)$ such that any configuration of size $\geq f(k,l)$ contains an open k-cap or an open l-cup.*

Theorem 2 has some interesting corollaries mentioned below. The function $f(k,l)$ is double–exponential, as shown in Theorem 3 below.

If X is a configuration, then a subset $P \subseteq X$ in convex position is called an *empty polygon (in X)*, if the interior of $\operatorname{conv} P$ contains no point of X. Answering a question of Erdős [1], Horton [4] constructed arbitrarily large configurations with no empty 7-gon. Harborth [3] showed that any configuration with more than 9 points contains an empty pentagon. It is a challenging open problem to prove or disprove that any sufficiently large configuration contains an empty hexagon. This problem is one of the motivations of our paper. Our results show some structural properties of configurations containing no empty polygons with many vertices. Thus, they might help in investigations of the empty–hexagon problem.

We say that a configuration X is *k-convex*, if the interior of every triangle determined by X contains at most k points of X. Theorem 2 implies the following result which shows that large k-convex configurations contain large empty polygons:

Corollary 1 (Valtr [11], Kun and Lippner [7]) *For any $k \geq 1$ and $l \geq 3$, there is a (least) integer $N(k,l)$ such that any k-convex configuration of size $\geq N(k,l)$ contains an empty l-gon.*

Corollary 1 in the special case $k = 1$ was first proved by Károlyi *et al.* [6]. The exact value of $N(1,l)$ is determined in Theorem 4 below. For general k and l, our proof of Corollary 1 gives the upper bound $N(k,l) \leq 2^{\binom{k+l}{k+2}-1}+1$, which is slightly better than the previous bound $N(k,l) \leq (k+2)^{(k+2)^l-1} \leq 2^{(k+2)^{l+1}}$ of Kun and Lippner [7].

We asked in [11] if also any large configuration with no k-cap contains a large empty polygon. This is answered affirmatively by the following direct consequence of Theorem 2:

Corollary 2 *For any $k,l \geq 2$, there is a least integer $m(k,l)$ such that any configuration of size $\geq m(k,l)$ with no open k-cap contains an empty l-gon.*

A *k-moon* is a k-point configuration M with a specific point $a(M) \in M$ (called *the apex of M*) such that any 4-point subset of M is in convex position if and only if it does not contain $a(M)$. The combinatorial structure of a k-moon is unique for each $k \geq 3$ (see Fig. 2). A k-moon M is *empty (in X)*, if the interior of the region $\operatorname{conv} M \setminus \operatorname{conv}(M \setminus \{a(M)\})$ contains no point of X (see Fig. 2). Theorem 2 is equivalent to the following corollary:

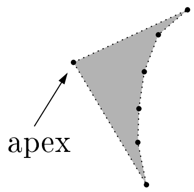

Figure 2: A 7-moon; it is empty in X if the shaded region contains no points of X.

Corollary 3 *For any $k,l \geq 2$, there is a least integer $z(k,l)$ such that any point p in any configuration X of size $\geq z(k,l)$ is the apex of an empty k-moon in X or it is one the vertices of an empty l-gon in X.*

We obtain double-exponential lower and upper bounds on $f(k,l)$ and on two related functions:

Theorem 3 *Let $k,l \geq 2$ and let $f(k,l)$ be the number given by Theorem 2. Then:*

(i) If $f'(k,l)$ denotes the minimum integer such that any configuration of size $\geq f'(k,l)$ contains a k-cap or an open l-cup, then

$$2^{\binom{\lfloor k/2\rfloor+\lfloor l/2\rfloor-2}{\lfloor k/2\rfloor-1}} \leq f'(k,l) \leq f(k,l) \leq 2^{\binom{k+l-2}{k-1}-1},$$

(ii) if $f''(k,l)$ denotes the minimum integer such that any configuration of size $\geq f''(k,l)$ contains an open k-cap or an l-cup, then

$$2^{\binom{\lfloor k/2\rfloor+\lfloor (l-1)/2\rfloor-2}{\lfloor k/2\rfloor-1}} \leq f''(k,l) \leq f(k,l).$$

If neither the k-cap nor the l-cup are required to be open, then the extremal function is exponential: If $f'''(k,l)$ denotes the minimum integer such that any configuration of size $\geq f'''(k,l)$ contains a k-cap or an l-cup, then $f'''(k,l) = \binom{k+l-4}{k-2} + 1$ (this was shown by Erdős and Szekeres [2] about 70 years ago).

Let $N(k,l)$ be the numbers given by Corollary 1 and let

$$M_l := \begin{cases} 2^{(l+1)/2} - 1, & \text{for } l \geq 3 \text{ odd}, \\ \frac{3}{2}2^{l/2} - 1, & \text{for } l \geq 4 \text{ even}. \end{cases}$$

The following bounds on $N(1,l)$ have been known:

$$M_l \leq N(1,l) \leq 2^{\lceil (2l+5)/3\rceil} - 1.$$

The lower bound follows from a construction of Károlyi [5] and Károlyi *et al.* [6], the upper bound was proved by Kun and Lippner [7]. Here we show that the lower bound is tight:

Theorem 4 *For any $l \geq 3$,*

$$N(1,l) = M_l.$$

Our proofs of Theorems 2 and 4 use methods similar to those of Kun and Lippner [7]. The proof of Theorem 2 has also some similarities with the original proof of Theorem 1.

OPEN PROBLEMS. Corollaries 1–3 give the (affirmative) answer to the following problem for some special types of subconfigurations:

Problem 1 *Which types of subconfigurations are contained in any sufficiently large configuration with no empty l-gon ($l \geq 6$ fixed)?*

If the sought type of subconfiguration is the empty hexagon then for each $l \geq 6$, Problem 1 is equivalent to the well–known empty–hexagon problem [1].

Problem 2 (empty–hexagon problem) *Does any sufficiently large configuration contain an empty hexagon?*

Since the so-called Horton sets (e.g. see [9, 10, 8]) contain no empty 7-gons, Problem 1 can have an affirmative answer for $l > 6$ only for types of configurations contained in all sufficiently large Horton sets.

2 Proofs

Proofs of Theorems 2–4 will appear in the full version of the paper (approx. 15 pages). Corrolaries 1–3 follow easily from Theorems 2 and 3:

Proof of Corollary 1. Let X be a configuration of size at least $f(k+3, l-1)+1$. Let p be a vertex of conv X, and let l be a line touching conv X at the point p. We transform X to a set X' by a projective transformation P which sends the point p to the "point" $(0, \infty)$ and the line l to the line at infinity. The configuration X' has at least $f(k+3, l-1)$ points in the real plane, thus it contains an open $(k+3)$-cap C or an open $(l-1)$-cup D. The first case cannot happen, since $P^{-1}(C) \cup \{p\}$ would be an (empty) $(k+4)$-moon and we would get a contradiction with the k-convexity of X. In the second case the set $P^{-1}(D) \cup \{p\}$ forms an empty l-gon. Thus, $N(k, l) \leq f(k+3, l-1)+1$. □

Proof of Corollary 2. Since any open l-cup is an empty l-gon, Theorem 2 immediately gives $m(k, l) \leq f(k, l)$. □

Proof of Corollary 3. If a configuration X has at least $2 \cdot f(k-1, l-1)$ points and $p \in X$ is any point in X, then p is a vertex of the convex hull of at least $f(k-1, l-1)+1$ points of X. We further continue analogously as in the proof of Corollary 1, obtaining $z(k, l) \leq 2 \cdot f(k-1, l-1)$. □

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Arc-coloring of directed hypergraphs and chromatic number of walls

Andrea Vietri [*]

Definition 1 *A directed hypergraph* $H = (V(H), E(H))$ *consists of a set* V *(the nodes) together with a set* $E \subseteq \mathcal{P}(V) \times V$*. Each element* $e := (A, z) \in E$ *is a hyperarc (or simply an arc).* A *and* z *are respectively the tail and the head of* e $(A = tail(e), z = head(e)\ )$*. The forward star of a node* z *is* $fstar(z) := \{e \in E(H) : z \in tail(e)\}$*, the backward star is* $bstar(z) := \{e \in E(H) : z = head(e)\}$*. The indegree of* z *is* $\delta^-(z) := |bstar(z)|$*, its outdegree is* $\delta^+(z) := |fstar(z)|$ *and its degree is* $\delta(z) := \max(\delta^-(z), \delta^+(z))$*. The degree of* H *is* $\Delta(H) := \max_{z \in V}(\delta(z))$*. The size of* H *is* $|H| := \sum_{e \in E} |tail(e)|$*.*

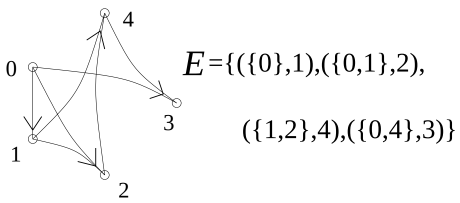

Figure 1: A directed hypergraph

Definition 2 *An arc-coloring of* H *is a map* $\gamma : E(H) \to \mathbf{N}$ *such that*

i) $(\ (A, x) \in E, (B, y) \in E, A \cap B \neq \emptyset\ ) \Rightarrow \gamma(A, x) \neq \gamma(B, y)$,

ii) $(\ (C, z) \in E, (D, z) \in E\ ) \Rightarrow \gamma(C, z) \neq \gamma(D, z)$,

provided $(A, x) \neq (B, y)$ *and* $C \neq D$*.*

H *is* k*-colorable if there exists an arc-coloring of* H *which requires* k *colors. The (directed) chromatic index of* H*, in symbols* $q(H)$*, is the least number* k *such that* H *is* k*-colorable.*

Property 1 *If* H *is a digraph, then* $q(H) = \Delta(H)$*.*

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Proof:[Hint of the proof] A suitable bipartite graph can be associated to H in a natural way. Then, König's Theorem easily implies the assertion. □

We rephrase the above property in the following fashion (Property 2).

Definition 3 *A pluriomino P is a finite set of unit squares in the plane, arranged like a chessboard having some squares cut out, possibly. Its degree, $\delta(P)$, is the greatest number of squares appearing in some row or column.*

Property 2 *The squares of a pluriomino P can be filled up with $\delta(P)$ distinct symbols in such a way that every row or column of P has no repetitions.*

Proof: We interpret P as the adjacency matrix of a digraph having as many nodes as the side of a chessboard containing P. Then we use Property 1. □

We generalize pluriominoes by associating a directed hypergraph H to an adjacency scheme P_H, as follows. If $V(H) = \{0, 1, \ldots, s\}$ and $E(H) = \{e_0, \ldots, e_t\}$, we consider a chessboard whose side is equal to $s + 1$; to each arc e_i we associate the row indexed by $head(e_i)$; on this row we choose the squares corresponding to the nodes of $tail(e_i)$. If two arcs have the same head, then the related squares have to be distinguished, as the squares of row 1 in the left side of Figure 2. *It may happen that some overlapping squares arise*, as shown in the right side of the same figure (the smaller squares are related to the arc $(\{3, 4\}, 0)$).

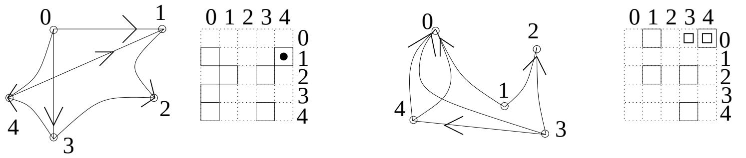

Figure 2: Directed hypergraphs and their representations

We formalize the above schemes in the non-overlapping case.

Definition 4 *A wall P is a pluriomino whose squares have been labeled under the condition that every label appears in a unique row of P. Its degree, $\delta(P)$, is the greatest number of distinct labels in the same row or column. Every maximal set of squares having the same label is a brick. A coloring of P is the assignment of a symbol to each square of P, in such a way that: 1) all the squares of any brick get the same symbol; 2) no symbol is present in more than one brick in the same row; 3) no symbol appears more than once in every column. The chromatic number of P, denoted by $\rho(P)$, is the least number of symbols required for a coloring of P.*

The next definition concerns all the hypergraphs whose adjacency schemes are actually walls.

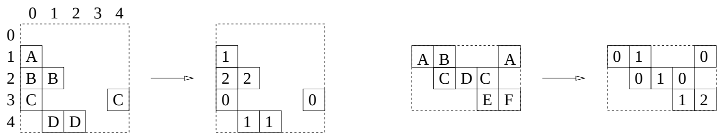

Figure 3: Walls and two colorings

Definition 5 *A directed hypergraph H is non-overlapping if there exists no pair of arcs of H sharing the same head and some node of the tails. Otherwise, H is overlapping and any pair of such arcs is termed overlapping as well.*

For a given non-overlapping hypergraph H we easily obtain: $\delta(P_H) = \Delta(H), \rho(P_H) = q(H)$. Furthermore, Property 2 can be rephrased as: H digraph $\Rightarrow \rho(P_H) = \delta(P_H)$. Finally, the trivial inequality $q(H) \geq \Delta(H)$ corresponds to $\rho(P_H) \geq \delta(P_H)$.

If H is overlapping, the interpretation by means of walls *fails.* Anyway, the next theorem explains how to manage the overlapping case.

Theorem 1 *To each overlapping hypergraph H one can associate a non-overlapping hypergraph H' such that*
1) There exists a bijection $f : E(H) \rightarrow E(H')$.
2) Every arc-coloring γ of H induces an arc-coloring γ' of H', defined as $\gamma \circ f^{-1}$, and conversely, using $\gamma' \circ f$ to obtain an arc-coloring γ of H from an arc-coloring γ' of H'. In particular, $q(H) = q(H')$.
3) The algorithm which transforms H into H' requires polynomial time with respect to $|H|$.

Proof:[Hint of the proof] All overlapping pairs are recursively eliminated by introducing some new nodes and modifying the tails of some arcs. □

A natural question is how large can be the chromatic index of a non-overlapping hypergraph whose degree has been fixed.

Definition 6 *A brick made of adjacent squares is termed connected. If a wall contains only connected bricks, then it is termed coherent.*

Theorem 2 *If P is a coherent wall, then $\rho(P) \leq 2\delta(P) - 1$ and the bound can be attained for every value of the degree. Furthermore, for every couple of integers $r \geq d \geq 2$ there exists a wall Q such that $\delta(Q) = d$, $\rho(Q) = r$.*

Proof:[Hint of the proof] Induction on the width of the wall. The bound is easily seen to be attained. To prove the second assertion, it is enough to provide a class of walls $\{Q_i\ ,\ i \geq 3\}$, such that $\delta(Q_i) = 2, \rho(Q_i) = i$, for all i. We first define the walls P_n, P_n^*, for $n \geq 2$. Then we put them side by side, so as to form the required walls (see Figure 4). □

Corollary 1 *For every couple of integers $r \geq d \geq 2$ there exists a directed hypergraph H such that $\Delta(H) = d$ and $q(H) = r$.*

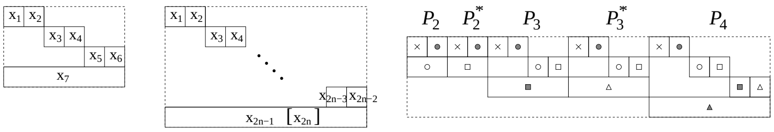

Figure 4: $P_4, P_n\ [P_n^*], Q_7$

If P is a wall of degree δ, we call *maximal* a column of P which contains δ squares. If $\delta > 1$, let C_δ denote a wall having $\delta - 1$ rows, each one with a unique label, and a further row made of δ bricks, each one meeting a maximal column. Further, given two walls P, Q, we say that P *contains* Q ($P \supset Q$) if Q can be obtained by removing some bricks of P. Now, let P be coherent. If P contains some $C_{\delta(P)}$, then of course $\rho(P) = 2\delta(P) - 1$, but the converse does not hold in general, as shown in Figure 5 (the three examples can be easily generalized to every degree $\delta > 1$).

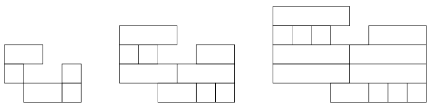

Figure 5: P coherent, $\rho(P) = 2\delta(P) - 1$, $C_{\delta(P)} \not\subset P$

Definition 7 *A wall is termed connected if it is coherent and the union of the closed squares of P, in the euclidean topology, is connected and cannot be disconnected by any finite set of points (that is, the underlying pluriomino must be a polyomino).*

Notice that the left wall in Figure 5 is not connected.

Theorem 3 *A connected wall P of degree 2, such that $\rho(P) = 3$, contains some C_2.*

Proof:[Hint of the proof] The negation of the thesis implies that $\rho(P) = 2$, provided P is connected and has degree 2. Indeed, the bricks of P are necessarily arranged like a stair. The coloring is then defined by induction on the number of rows. □

Among some results concerning **complexity**, we cite the

Theorem 4 *For all integers $k \geq d \geq 2$ with $k \geq 3$, the k-colorability of the arcs [resp. bricks] of a given non-overlapping hypergraph [resp. wall] having degree d is an NP-complete problem.*

Proof:[Hint of the proof] The 3-colorability of the edges of a cubic graph is an NP-complete problem [Ho]. We give a polynomial reduction from the edge-coloring of graphs (not necessarily cubic) to our problem. □

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